DeMarzo-Sannikov, JF 2006

- Time $t \in [0, \infty)$
- Risk-neutral principal with discount rate r
- Risk-neutral agent with discount rate $\gamma > r$
- Firm produces cash flows $dY_t = \mu\, dt + \sigma\, dZ_t$
- Principal does observe Y_t, but sees agent's report
  - $d\hat{Y}_t = (\mu - a_t)\, dt + \sigma\, dZ_t$
- Principal's problem

$$\max_{C,\tau} E\left[\int_0^\tau e^{-rt}(\mu dt - dC_t) + e^{-r\tau} L\right]$$

$$\text{s.t.}\quad W_0 = E^{a=0}\left[\int_0^\tau e^{-\gamma t} dC_t + e^{-\gamma\tau} R\right] \qquad W_0 \geq E^{a}\left[\int_0^\tau e^{-\gamma t}(dC_t + \lambda a_t) + e^{-\gamma\tau} R\right]$$

Exercises

- **Step 1:** Given contract {C, τ}, define the agent's continuation value when he tells the truth

$$W_t = E_t\left[\int_t^\tau e^{-\gamma(s-t)} dC_s + e^{-\gamma(\tau-t)} R\right]$$

- **Step 2: Represent the law of motion of W_t over time**

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$$\beta_t \geq \lambda$$

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- **Step 4: Derive the HJB equation for the principal's profit b(W)**

$$\text{look at drift of } \int_0^t e^{-rs}(\mu dt - dC_t) + e^{-rt} b(W_t)$$

$$\text{when } dC_t = 0,\ rb(W) = \mu + \gamma W b'(W) + \frac{1}{2}\lambda^2\sigma^2 b''(W)$$

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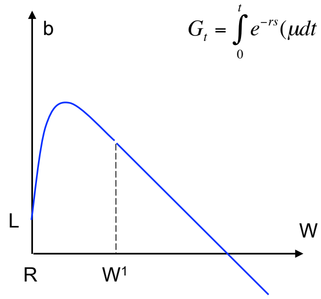
- **Step 5: What are the boundary conditions?**

$$b(R) = L, \quad b'(W^1) = -1.$$

Verification

$$G_t = \int_0^t e^{-rs}(\mu dt - dC_t) + e^{-rt} b(W_t)$$



Implementation

Implementing the Optimal Contract

- Take $\lambda = 1$

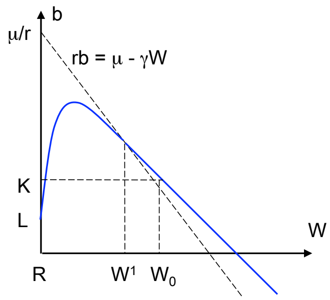
$$dW_t = \gamma W_t\, dt + (d\hat{Y}_t - \mu\, dt) - dC_t$$

- $[R, W^1]$ credit line
- $M_t = (W^1 - W_t)$ balance
- $W^1 - R$ credit limit

$$dM_t = \gamma M_t\, dt + (\mu - \gamma W^1)\, dt + dC_t - d\hat{Y}_t$$

- $\mu - \gamma W^1$ coupon payments on debt
- Let $(\mu - \gamma W^1)/r$ be the face value of debt

Implementation



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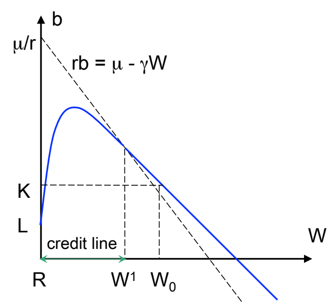

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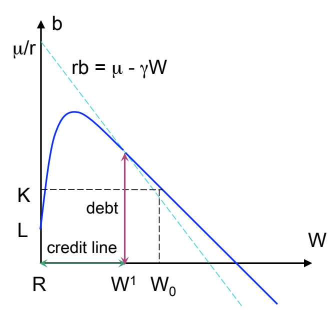
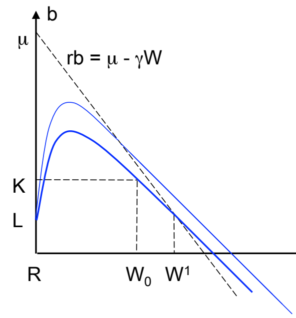


Increase risk



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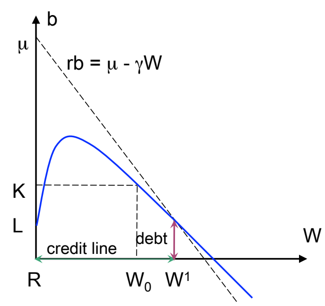
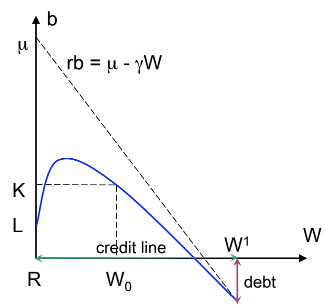


More risk



Tailoring credit line to a project

- Credit limit involves this trade-off
  - longer credit line → flexibility for potential losses
  - delays consumption
- As $\sigma\uparrow$: credit limit $\uparrow$, debt $\downarrow$

Comparative Statics with Algebra

- Recall that $\mu - rb(W^1) - \gamma W^1 = 0$ determines W^1, so we can find out how W^1 (credit line) depends on a parameter θ (one of parameters L, μ, γ, σ or λ) if we know how the profit function $b_\theta(W)$ depends on θ
- Similarly, if we use $b(W_0) = K$ to determine W_0, we can find out how parameters affect W_0 if we know how they affect $b_\theta(W)$

Comparative Statics with Algebra

- *Suppose θ is one of parameters L, μ, γ, σ or λ and denote by $b_\theta(W)$ profit for that parameter value. Start with*

$$rb_\theta(W) = \mu + \gamma W b_\theta'(W) + \frac{1}{2}\lambda^2\sigma^2 b_\theta''(W), \quad b_\theta(R) = L, \quad b_\theta(W^1) = -1$$

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- Differentiate with respect to θ
- Get equation for $\frac{db_\theta(W)}{d\theta}$ and boundary conditions.

Comparative Statics with Algebra

- **Lemma.** *Suppose θ is one of parameters L, μ, γ, σ or λ and denote by $b_\theta(W)$ profit for that parameter value. Then*

$$\frac{db_\theta(W)}{d\theta} =$$
$$E\left[\int_0^\tau e^{-rt}\left(\frac{d\mu}{d\theta} + \frac{d\gamma}{d\theta} W_t b_\theta'(W_t) + \frac{1}{2}\frac{d\lambda^2\sigma^2}{d\theta} b_\theta''(W_t)\right)dt + e^{-r\tau}\frac{dL}{d\theta} \mid W_0 = W\right]$$

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- Example

$$\frac{db(W)}{dL} = E\left[e^{-r\tau} \mid W_0 = W\right] > 0 \quad \Rightarrow \quad \frac{dW^1}{dL} = -\frac{rE\left[e^{-r\tau} \mid W_0 = W\right]}{\gamma - r} < 0$$

Table of Results

	d credit limit	*d* debt	dW^0	$db(W^*)$
dL	-	+	+	+
dR	-	-	-	-
$d\gamma$	-	±	-	-
$d\mu$	+	+	+	+
$d\sigma^2$	+	-	-	-

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Papers that use this model

- Biais, Mariotti, Plantin and Rochet (ReStud 2007):
  – Show that this contract arises in the limit of discrete-time settings
  – Propose a different implementation of the optimal contract
- Piskorski and Tchistyi (2007)
  – Adapt the model to study optimal mortgage design
  – Assume i.i.d. income of homeowner (see today's seminar for a model that can allow for correlated income)
- He (JFE *2007)*: extends the model to geometric Brownian motion
- Sannikov (2007) "Agency Problems, Screening and Increasing Credit lines"
  – Allows for adverse selection – what if the principal does not know μ
- DeMarzo, Fishman, He and Wang (2007)
- Hoffmann and Pheil (2008)
  – Allow μ to change at random and show that agent gets rewarded for luck in an optimal contract