Dynamic Agency Problems: A Continuous-Time Approach

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Northwestern University Lectures

"Classic" Dynamic Agency Models

- Agent puts effort
  - in time periods 1, 2, 3… or continuously
- Probability of output depends on effort
- Principal observes output, not agent's effort
- Principal **commits** to a contract that specifies payments to the agent as a function of output history
- Problem: maximize principal's profit subject to giving agent a desired level of utililty

Applications

- macroeconomics (e.g. taxation, wealth distribution, economic growth)
- personnel economics (e.g. labor contracts)
- corporate finance (e.g. executive compensation, venture capital contracts, lending, capital structure and security design)
- health economics (e.g. reputation mechanisms for health professionals)

Basic Theory

- **Discrete Time:** Radner (*EMA* 1985), Rogerson (*EMA* 1985), Spear and Srivastava (*ReStud* 1987), Fudenberg, Holmstrom and Milgrom (*JET* 1990), Phelan and Townsend (*ReStud* 1991)
  - Patient agent → efficiency is attainable
  - Optimal contract based on agent's continuation value as state variable
- **Continuous Time:** Cvitanic, Wan and Zhang (2008), Westerfield (2008), Williams (2006a) and (2006b)
- **This Lecture:** Sannikov (2008) "A Continuous-Time Version of the Principal-Agent Problem," *Review of Economic Studies*
  - Optimal contract characterized by an ordinary differential equation, useful to compute the contract, understand its properties, do comparative statics analytically

Usefulness

- DeMarzo and Sannikov (*Journal of Finance* 2006):
  - Apply Sannikov (2007) to managerial incentives, capital structure
- Biais, Mariotti, Plantin and Rochet (*ReStud* 2007):
  - Show that the contract of DeMarzo-Sannikov arises in the limit of discrete-time settings
- Piskorski and Tchistyi (2007)
  - Adapt DeMarzo-Sannikov to study optimal mortgage design
- He (*JFE 2007)*
- Repeated Games: Sannikov (*EMA* 2006)
- Faingold and Sannikov (2007): games with reputation
- Fong (2007):
  - Application to health economics, allows adverse selection

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  - (Piskorski: job offers from Chicago, Columbia)
- **He** (*JFE 2007)*
  - (job offers from Princeton, Chicago)
- Repeated Games: Sannikov (*EMA* 2006)
- **Faingold** and Sannikov (2007): games with reputation
  - (Faingold: ReStud Tour 2006, job offers from Yale, UCLA)
- **Fong** (2007):
  - Application to health economics, allows adverse selection
  - (Fong: ReStud Tour 2008, job offers from Harvard, Stanford, Berkeley)

Plan

- The basic continuous-time model
- Discuss techniques used to derive the optimal contract
- Discuss properties of the contract

The setting

- Time $t \in [0, \infty)$
- Risk-neutral principal and risk-averse agent, common discount rate r
- Agent puts effort $A = \{A_t \in \mathcal{A}, 0 \leq t < \infty\}$
- Principal does not see effort, but observes output

$$dX_t = A_t\, dt + \sigma dZ_t$$

where Z is a Brownian motion

- Cost of effort $h : \mathcal{A} \to \Re$ continuous, increasing, convex with $h(0) = 0$ and $h(a) \geq \gamma_0 a$, $\gamma_0 > 0$
- Utility of consumption $u : [0, \infty) \to [0, \infty)$ continuous, increasing and concave with $u(0) = 0$ and $u'(c) \to 0 \text{ as } c \to \infty$

Problem Formulation

Find a contract $\{C_t, 0 \leq t < \infty\}$ and effort $A = \{A_t \in \mathcal{A}, 0 \leq t < \infty\}$ that maximizes the principal's profit

$$E\left[r\int_0^\infty e^{-rt}(A_t - C_t)dt\right]$$

subject to

$$W_0 = E\left[r\int_0^\infty e^{-rt}(u(C_t) - h(A_t))dt\right] \quad \text{given effort } \{A_t\}$$

$$\text{and} \quad W_0 \geq E\left[r\int_0^\infty e^{-rt}(u(C_t) - h(\hat{A}_t))dt\right] \quad \text{given effort } \{\hat{A}_t\}$$

5 steps to derive the optimal contract

1. Define the agent's continuation value $(W_t)_{t\geq 0}$ for any $\{C_t, 0 \leq t < \infty\}$ $\quad A = \{A_t \in \mathcal{A}, 0 \leq t < \infty\}$
2. Using the Martingale Representation Theorem, write an equation that describes the evolution of W_t
3. Necessary and sufficient conditions for the agent's effort to be optimal (sensitivity of W_t to X_t)
4. Using an HJB equation, conjecture an optimal contract
5. Verify that the optimal contract maximizes the principal's profit

(steps 4 and 5 rely on Ito's lemma)

The Agent's Continuation Value W_t

- **Step 1:** Given consumption $\{C_s, 0 \leq s < \infty\}$ and effort $\{A_s, 0 \leq s < \infty\}$, define the agent's continuation value

$$W_t = E_t\left[r\int_t^{\infty} e^{-r(s-t)}(u(C_s) - h(A_s))ds\right]$$

- **Step 2:** Proposition 1 W_t is the agent's continuation value if and only if

$$dW_t = r(W_t - u(C_t) + h(A_t))dt + rY_t(dX_t - A_t dt).$$

for some process $\{Y_t\}$ and $E[e^{-rt} W_t] \to 0$

Incentives

The agent maximizes

$$E[r(u(C_t) - h(A_t))dt + dW_t]$$

$$dW_t = \text{terms not affected by deviation} + rY_t dX_t$$

Agent will maximize $\quad rY_t A_t dt - rh(A_t)dt$

Step 3: Proposition 2 A contract is incentive-compatible iff

$$\forall t \geq 0,\ A_t \text{ maximizes } \ Y_t a - h(a) \qquad (*)$$

If the constraint (*) holds, we say Y_t enforces A_t

The Optimal Control Problem

$$dW_t = r(W_t - u(C_t) + h(A_t))dt + rY_t(dX_t - A_t dt)$$

- The principal
  - controls W_t with C_t, A_t and Y_t (which enforces A_t)
  - must honor promises, i.e. $E[e^{-rt}W_t] \to 0$
  - gets a flow of profit of $A_t - C_t$

Denote by $F(W_0)$ the maximal total profit that the principal can attain in this way

The Optimal Control Problem

$$dW_t = r(W_t - u(C_t) + h(A_t))dt + rY_t(dX_t - A_t dt)$$

If the principal chooses controls C_t, A_t and Y_t optimally, then

$$F(W_t) = E\left[\int_t^{t+s} e^{-r(u-t)}(A_u - C_u)du + e^{-rs}F(W_{t+s})\right]$$

The Optimal Control Problem

$$dW_t = r(W_t - u(C_t) + h(A_t))dt + rY_t(dX_t - A_t dt)$$

If the principal does not

$$F(W_t) > E\left[\int_t^{t+s} e^{-r(u-t)}(A_u - C_u)du + e^{-rs}F(W_{t+s})\right]$$

Drift of $\int_0^t e^{-rs}(A_s - C_s)ds + e^{-rt}F(W_t)$ is

$$a - c - F(W) + (W - u(c) + h(a))F'(W) + \frac{ry^2\sigma^2}{2}F''(W) \le 0$$

F(W) satisfies equation

$$\max_{c,a,y}\ a - c - F(W) + (W - u(c) + h(a))F'(W) + \frac{ry^2\sigma^2}{2}F''(W) = 0$$

subject to the constraint that y enforces a

When $a > 0$, this equation can be written as

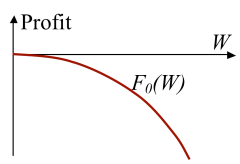
$$F''(W) = \min_{a>0,c} \frac{F(W) + c - a - (W - u(c) + h(a))F'(W)}{r\gamma(a)^2\sigma^2/2}$$

where $\gamma(a)$ is the minimal value of y that enforces a

Can solve this 2nd order ODE, if we know boundary conditions (where $a = 0$)

Effort 0

- Principal can offer constant consumption *c*
- Agent puts effort 0, gets utility *u(c)*
- Principal gets *retirement profit* $F_0(u(c)) = -c$



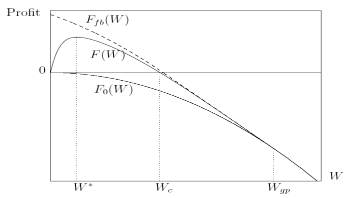
Solving for F(W)

Let F(W) ≥ F_0(W) be the unique function that satisfies

$$F''(W) = \min_{a>0,c} \frac{F(W) - a + c - F'(W)(W - u(c) + h(a))}{r\gamma(a)^2\sigma^2/2}.$$

with boundary conditions (for some $W_{gp} \geq 0$)

$$F(0) = 0,\ F(W_{gp}) = F_0(W_{gp}) \text{ and } F'(W_{gp}) = F_0'(W_{gp})$$



The Optimal Contract

$F(W_0)$ is the principal's profit in the optimal contract for $W_0 \in [0, W_{gp}]$. The agent's value in the optimal contract follows

$$dW_t = r\left(W_t - u(c(W_t)) + h(a(W_t))\right)\,dt + rY(W_t)\,\underbrace{(dX_t - a(W_t)dt)}_{\sigma dZ_t},$$

until the retirement time τ when W_t hits 0 or W_{gp}. For $t < \tau$ $C_t = c(W_t)$ and $A_t = a(W_t)$ are the maximizers in the ODE for F(W). After time τ, the agent receives constant consumption $C_t = -F(W_\tau)$ and puts effort 0.

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What if $W_0 < 0$?

Verification

Define F as above. Consider the process

$$G_t = r\int_0^t e^{-rs}(A_s - C_s)\,ds + e^{-rt}F(W_t)$$

When $dW_t = r(W_t - u(C_t) + h(A_t))dt + rY_t(dX_t - A_t dt)$

by Ito's lemma G_t has drift

$$re^{-rt}\left((A_t - C_t - F(W_t)) + F'(W_t)(W_t - u(C_t) + h(A_t)) + r\sigma^2 Y_t^2 \frac{F''(W_t)}{2}\right)$$

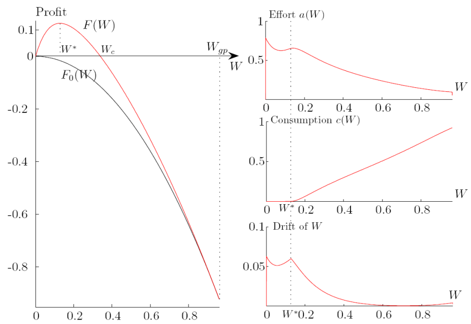
which is 0 in the conjectured optimal contract and ≤ 0 in any other incentive-compatible contract. Thus,

$$E\left[r\int_0^\infty e^{-rt}(A_t - C_t)dt\right] = E[G_\infty] \leq G_0 = F(W_0)$$

with equality in the optimal contract.

An Example

$u(c) = \sqrt{c}$, $h(a) = 0.5a^2 + 0.4a$, $r = 0.1$ and $\sigma = 1$.



Effort and consumption

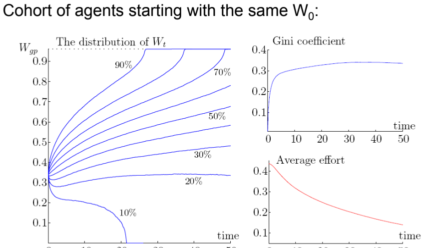
From the HJB equation, effort maximizes

$$\underbrace{ra}_{\text{output}} + \underbrace{h(a)F'(W)}_{\text{compensating for cost of effort}} + \underbrace{\frac{1}{2}r^2\gamma(a)^2\sigma^2F''(W)}_{\text{cost of providing incentives}}$$

- Effort is non-monotonic because of the cost of providing incentives
- **Theorem 5:** As r → 0, effort becomes monotonically decreasing in W_t.
- **Theorem 2:** The drift $r\left(W_t - u(c(W_t)) + h(a(W_t))\right)$ of W_t points in the direction where $F''(W_t)$ is increasing.

Example (continued)

- Cohort of agents starting with the same W_0:



- As income inequality ↑, average productivity ↓.
- Alesina and Rodrik (1994), Persson and Tabellini (1994), Perrotti (1996): inequality negatively related to economic growth.

Conclusions

- New method to study optimal dynamic incentives
- Clean characterization of an optimal contract with an ODE
- Linear (as in Holmstrom and Milgrom (1987)) over short periods, but nonlinear in the long run (retirement)
- Comparisons how the path of wages and the provision of incentives depend on the contractual environment
  - Agent's outside opportunities
  - Principal's outside opportunities
  - Promotion
- These have empirical relevance
- Applicability in many other settings (repeated games, finance, etc.)

DeMarzo-Sannikov, JF 2006

- Time $t \in [0, \infty)$
- Risk-neutral principal with discount rate r
- Risk-neutral agent with discount rate $\gamma > r$
- Firm produces cash flows $dY_t = \mu\, dt + \sigma\, dZ_t$
- Principal does observe Y_t, but sees agent's report
  - $d\hat{Y}_t = (\mu - a_t)\, dt + \sigma\, dZ_t$
- Principal's problem

$$\max_{C,\tau} E\left[\int_0^\tau e^{-rt}(\mu dt - dC_t) + e^{-r\tau} L\right]$$

$$\text{s.t.} \quad W_0 = E^{a=0}\left[\int_0^\tau e^{-\gamma t} dC_t + e^{-\gamma\tau} R\right] \qquad W_0 \geq E^{a}\left[\int_0^\tau e^{-\gamma t}(dC_t + \lambda a_t) + e^{-\gamma\tau} R\right]$$

Exercises

- **Step 1:** Given contract $\{C, \tau\}$, define the agent's continuation value when he tells the truth

$$W_t = E_t\left[\int_t^\tau e^{-\gamma(s-t)} dC_s + e^{-\gamma(\tau-t)} R\right]$$

- **Step 2: Represent the law of motion of W_t over time**

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- **Step 5: What do you think are the boundary conditons?**