Political Competition and the Dynamics of Parties and Candidates

Preliminary Draft

Stefan Krasa*

December 7, 2016

Abstract

Standard models of political competition do not differentiate between parties and the candidates selected by those parties, nor between electoral competition and competition between parties for support and influence among the public. This paper introduces a framework that distinguishes between parties and their candidates, creating roles for both the importance of a party's power (size) and of party leaders' preferences for policy, which have been separately emphasized in the existing literature. For this interplay between party size and policy concerns to work, our model is dynamic. The paper provides results on existence and uniqueness of equilibria, investigates the impacts of party extremism on candidate choice and political polarization, a party's decision whether or not to compete in an election, and conditions under which a dynamic median voter theorem can be obtained. We also show that if aggregate preferences are not transitive, political competition may lead to extremism.

*University of Illinois, Department of Economics, Urbana, IL 61801 USA; skrasa@illinois.edu

1 Introduction

The classic models of political competition assume that parties are teams of individuals who either "seek office to enjoy income, prestige, and power" (Downs, 1957) or attempt to maximize their utilities (Wittman, 1973). These papers neither differentiate between parties and the candidates selected by those parties, nor between electoral competition and party competition for support and influence among the public. Further, parties are static (Wittman, 1977): Elections matter only for the current period, and do not affect party composition, availability of competitive candidates, and preferences of party leaders in the future. The objective of this paper introduce a framework that distinguishes between parties and their candidates, creating a role for both the importance of a party's power (size) and party leaders' preferences for policy, which have been emphasized separately in the existing literature. Our approach is able to resolve difficulties in reconciling predictions of existing models with empirical evidence.

For example, standard models with parties that seek to maximize their vote share cannot explain the divergence of party platform or changes in political polarization in the U.S that has been observed in the data (e.g., Poole and Rosenthal (2000)). If parties care about policy outcomes, and median voter positions are uncertain, then standard models predict polarized policy platforms. In these models, polarization only increases if parties become more uncertain about the position of the median voter. It seems unlikely that the reported increase of political polarization in the US over the last decades was caused by less accurate polling data.

Furthermore, if party leaders maximize utility and thus care only about policy, then according to standard models parties should enter every election, because this would force the opposing party to moderate. In practice, many elections at the state level in the U.S. are not contested. For example, in one third of all House and Senate elections in North Carolina in 2016, only a single candidate will be on the general election ballot. If there is a Democratic candidate in a North Carolina district, the candidate may choose not to enter the campaign even if that candidate's preferences match those of the district's median voter. Krasa and Polborn (2016) argue this would be because the median voter dislikes the positions of Democrats from other districts and therefore wants Democrats to remain in the minority. However, this argument does not apply because both the House and Senate in North Carolina have very large Republican majorities, and thus small party changes in individual Senate and House districts would not affect control of the Legislature. Similarly, this argument fails to explain why Republicans do not contest some districts in North Carolina.

To create a model whose results can be better reconciled with observations, we introduce a framework that distinguishes between parties and their candidates, creating a role for both the importance of a party's power (size) and party leaders' preferences for policy, which have been emphasized separately in the existing literature. For this interplay between party size and policy concerns to work, our model must be dynamic.

In our model, parties select candidates to compete in an election. As in Osborne and Slivinski (1996) and Besley and Coate (1997), candidates are unable to commit ex-ante to policies, and if elected, will make policy choices that maximize their own utility. We assume that each party must select a candidate from the set of its members. As in Snyder and Ting (2002) or Osborne and Tourky (2008), we can think of parties as providing information about the candidate types that otherwise would not be known to voters, and parties have this information about their members only. The assumption also can be justified if individuals do not want to be candidates for a party, unless their preferences are compatible with those of other party members. This reflects the experience of many Southern Democrats who became Republicans when they felt that their views were no longer welcome in the Democratic Party (Strom, 1990).

In the model, we assume candidates are chosen by party leaders, whose identity in turn depends on the party membership. In general, we describe a mapping from the set of parties into the space of citizen types, which we also can interpret as an aggregation of preferences, as in Baron (1993) or Caplin and Nalebuff (1997). For example, if the space of citizen types is one-dimensional, and the candidate is determined in a primary in which all party members participate, then the median party member would be the leader. If, instead, partisans participate at higher rates in the primary, then party leaders would be more extreme than the party medians.

Although party leaders care about policy, they also seek to attract more citizens who identify with the party, because this provides the party with a larger set of individuals from which candidates can be selected in the future. Thus, both policy and party support are modeled, combining the two key features of Downs (1957) and Wittman (1973). The choice of candidate determines party membership in the following model period. Hence party leaders may face a tradeoff between growing the party to include more moderates and choosing more partisan policies that leaders may prefer, but which in turn may alienate moderates. A further tradeoff arises because the leader in the current period may not be the leader in the next period: If the party attracts more moderate members then the leadership may become more moderate and vice versa if the party loses the support of moderates.

Our model describes a dynamic game in which the state of the game at each point in time is given by the allocation of citizens into parties. We first provide existence result for Markov perfect equilibria and investigate model equilibria when the type and policy space are one dimensional, as in many current models. We show that equilibrium policies are more volatile than the median voters positions, even without any uncertainty about the median voter's position. Further, policy divergence is a general feature of the equilibrium of our model. In contrast, in the standard model with policy motivated candidates (without uncertainty) policies converge and track the position of the median voter.

The model also differentiates between extremism of candidates and extremism of parties. The latter would for example arise if an increasing number of partisans participate in a party's primary. We show that if a party is more extremist it will be more concerned about its ability to win with a more extreme candidate in the current period, than about the party's future, thus alienating its more moderate supporters.

We also show that parties may choose not compete in an election. If initially one of the parties is too dominant, the model indicates that the other party will not compete in the election, because it does not have a candidate who is attractive in the general election and because it would take too long to grow party membership to the point where such a candidate can be found. Instead, the dynamic model indicates that party leaders prefer that their former members participate in the other party's selection process, providing a moderating influence on candidate choice. However, not competing become less attractive if the opposing party is more extreme. In other words, the model provides a very simple explanation of why it may not be optimal for a party to contest an election.

The paper also provides condition under which equilibria in an infinite horizon converge to the steady state in which the median voter theorem applies. In the steady state the median voter also becomes the cutoff between the party on the left and the party on the right.

We then apply our model to situations in which aggregate preferences are not transitive and Condorcet cycles exist. As already noted by Wittman (1973), the presence of policy-motivated parties can lead to existence of pure strategy equilibria, and the same is true for our model. However, a key difference in our model is that the identity of the party leaders is determined endogenously. We first show in a simple model with three preferences types and no Condorcet winner that the most frequent preference type will be elected, illustrating that this model comes up with a unique pure strategy equilibrium. In contrast, Downs (1957) would predict randomization between all policies, while in Wittman (1973) any equilibrium outcome can be generated, depending on the exogenous policy preferences of each party. It is advantageous to have a model that gives sharp predictions to make it empirically testable.

We then consider a situation in which there are two policy issues. A majority of individuals care primarily about the first issue (say the economy), while a minority of voters strongly prefer a position on the second issue (say opposition to gun control), which the majority opposes. In a model with policy motivation as in Wittman (1973), opposition to gun control would only be in the wining platform if it is also the preferred position of one of the parties. In contrast, in our model, the steady state equilibrium involves both parties choosing candidates who oppose gun control, despite the fact that the party leaders would prefer stricter gun laws. Thus, candidates have politically more extreme positions than party leaders and average party supporter, contrasting with results from standard models in which political competition forces parties to moderate. The difference in predictions arises from the fact that parties and candidates are separate entities in our model, and that parties are limited to choosing candidates from among their members.

A standard explanation of why candidates adopt positions that are unpopular with a majority of the electorate is that outside lobbyists or donors force candidates into these positions. The problem with this explanation is that voters have to be somewhat ignorant about the impact of those policies, otherwise a candidate who chooses the popular position would win, and the lobbyists' or donors' expenses would not yield any payoff. In contrast, our model provides a simpler explanation in which political competition by itself is sufficient to produce this effect.

An important aspect of our model is the link between candidate policies and party membership in subsequent time periods. Poutvaara (2003) provides a dynamic model in which such a link exists, but parties are myopic and policy choice is unrestricted as in the standard models. As a consequence, parties are policy motivated and size or party power is irrelevant. Gomberg et al. (2004) and Gomberg et al. (2016) consider static models, in which parties aggregate preferences and propose policies, and individuals sort themselves into parties as a function of these policies. Unlike in our model, policy choice is not driven by political competition.

In our model parties select candidates who implement their most preferred policy if elected. We could also assume that parties choose platforms, but that these platforms are restricted to the Pareto set of all party members, as in Levy (2004).[1] Such an alternative approach would not affect the results in the one-dimensional case, because we show that parties are intervals and therefore any policy in a party's Pareto set, can be implemented as the preferred policy of one party member. Similarly, our results for discrete type and policy spaces would remain essentially unchanged.

The paper is organized as follows. Section 2 describes the model. The definition and existence of equilibria is discussed in section 3. Section 4 analyzes the the case where the policy space is the

[1]Similarly, in Morelli (2004) parties also impose restrictions on the policies that candidates can select.

interval $[0, 1]$, proves existence of equilibria and provides comparative static results. Section 5.1 derives a dynamic median voter theorem. Sections 5.2 and 5.3 investigate convergence to steady state equilibria when aggregate preferences are not transitive. Section 6 concludes.

2 Model

Time is indexed by $t = 0, \dots, T$, where T can be either finite or infinite. Let X be the policy space at each time t. The preferences of a citizen of type $\theta \in \Theta$ are described by a utility function $u_\theta \colon X \to \mathbb{R}$. We assume that there is a unique policy $x(\theta) \in X$ that maximizes the citizen's utility, i.e., $x(\theta)$ solves $\max_x u_\theta(x)$. The population of citizens at time t is described by a probability distribution ϕ_t on Θ. Each citizen discounts future utility at a rate β, where $0 < \beta < 1$.

There are two parties indexed by $i = 1, 2$. The set of party members at time t are given by $S_{1,t}, S_{2,t} \subset \Theta$. We allow for the possibility that $S_{i,t} = \emptyset$, which will imply that a party does not compete in the election.

Each party has a leader at time t who selects the candidate for the election. The identity of the party leader can change over time as the party changes. To described this dependency formally, let $\mathscr{S}$ be the set of all subsets of Θ. Then the leader of party i is determined by a function $m_i \colon \mathscr{S} \to \Theta$. If $\Theta \subset \mathbb{R}$, then the party leader could, for example, be the median of S_i. We require that the party leader is always a member of the party, i.e., $m_i(S) \in S$ for all $S \in \mathscr{S}$ with $S \neq \emptyset$.

At each time t, the leader of each party i selects a candidate $\theta_{i,t} \in S_{i,t}$. All citizens vote for one of the candidates, and the candidates with the majority of votes wins. Winning candidates select policies that maximizes their utility, i.e., a candidate of type θ implements policy $x(\theta)$. A party can also choose not to compete at period t, in which case $S_{i,t} = \emptyset$.

In the subsequent periods, party membership may change, depending on policies chosen in each period. Party membership in period $t + 1$ is described by a function $\psi \colon \Theta^2 \times \mathscr{S}^2 \to \mathscr{S}^2$, mapping the candidate types θ^i, $i = 1, 2$ and the party structure at time t into a party structure at time $t + 1$. We assume that if $S_{i,t} = \emptyset$ then $S_{i,t+1} = \emptyset$

We will consider alternative specifications for ψ, which will be governed by the basic principle that if individual choose to be affiliated with a party, then they choose the party whose policy they prefer. We refer to this property as consistency.

Definition 1 *Function $\psi = (\psi_1, \psi_2)$ is consistent if and only if for all for all θ_1, $\theta_2 \in \Theta$ and for all $S_1, S_2 \neq \emptyset$, the sets $S'_i = \psi_i(\theta_1, \theta_2, S_1, S_2)$ have the following property:*

If $\theta \in S_i'$ *then* $u_\theta(x(\theta_i)) \geq u_\theta(x(\theta_{-i}))$

Consistency allows for the possibility that some individuals, θ are "independent", i.e., that they are not affiliated with any party. If all individuals belong and thus could be candidate of one of the two parties, then the the following condition must be satisfied.

$$u_\theta(x(\theta_{i,t})) > u_\theta(x(\theta_{-i,t})) \text{ implies } \theta \in S_{i,t+1} \text{ and } \theta \notin S_{-i,t+1}. \tag{1}$$

So far we allow for arbitrary party structures. However, in practice the set of parties that can occur in a game may be a strict subset $\mathscr{K}$ of $\mathscr{S} \times \mathscr{S}$.

For example, suppose that the policy and type space is $[0,1]$ and that utility is of the form $u_\theta(x) = -(x-\theta)^2$. Suppose that the parties select candidates of type $\theta_1 = 1/3$ and $\theta_2 = 2/3$ and that there are no independents, i.e., (1) holds. Then the new parties are given by $S_1' = [0, 0.5]$ and $S_2' = [0.5, 1]$. Moreover, for these utility functions, any policies $\theta_1 \neq \theta_2$ would lead to parties that are intervals of the form $[0, s]$ and $[s, 1]$.

Formally we describe such a set $\mathscr{K}$ as attainable.

Definition 2 *A set of party allocations* $\mathscr{K} \subset \mathscr{S} \times \mathscr{S}$ *is attainable if and only if* $\psi(\theta_1, \theta_2, S_1, S_2) \in \mathscr{K}$ *for all* $(S_1, S_2) \in \mathscr{S} \times \mathscr{S}$ *and* $\theta_i \in S_i$, $i = 1, 2$.

There is a large literature that uses dynamic games to investigate how policies are determined. Most prominently the literature on legislative bargaining extends the sequential bargaining model of Baron and Ferejohn (1989) to the case where changes to policy can be enacted repeatedly over time, with a deterministic status quo policy as in Baron (1996) and Kalandrakis (2004), or a stochastic status quo as in Duggan and Kalandrakis (2012). There are a number of differences between these previous models and our model. First, we have restrictions on the action space that change endogenously over time, because of our assumption that candidates must be selected from their respective parties. Second, we always have two simultaneous rather than sequential proposers, who are the leaders of the two parties. Third, there is no status quo policy, because in every period a new policy can be enacted. Finally, the state is the party allocation, rather than the previous policy, and it depends deterministically on the actions of players in previous periods.

3 Equilibrium: Definition and Existence

A sub game at time t is determined by the sets $S_{i,t}$, $t = 1, 2$ that represent party membership. Because party membership in period $t+1$ is determined by the candidate positions and not by the outcome of the election, the election outcome only determines current but not future payoffs. Thus the voting stage reduces to a one-period problem. Given a set of candidates $\theta_t^i \in S_{i,t}$ at t and citizens using weakly dominant strategies, it follows that type θ votes for candidate i if $u_\theta(x(\theta_{i,t})) > u_\theta(x(\theta_{-i,t}))$. Hence, the median voter, $\theta_{m,t}$, is decisive.

Let $x(\theta_i)$ be the policy chosen by candidate θ_i, $i = 1, 2$. Then the probability that party i wins the election at time t is given by $\pi_{i,t}(\theta_1, \theta_2)$ where

$$\pi_{i,t}(\theta_1, \theta_2) = \begin{cases} 1 & \text{if } \phi_t\Big(\{\theta \,\big|\, u_\theta(x(\theta_i)) > u_\theta(x(\theta_{-i}))\}\Big) > 0.5; \\ 0 & \text{if } \phi_t\Big(\{\theta \,\big|\, u_\theta(x(\theta_i)) > u_\theta(x(\theta_{-i}))\}\Big) < 0.5; \\ q \in [0,1] & \text{otherwise;} \end{cases} \tag{2}$$

Clearly, $\pi_{1,t}(\theta_1, \theta_2) + \pi_{2,t}(\theta_1, \theta_2) = 1$

In contrast to voting by citizens, the selection of candidates by party leaders impacts payoffs in future time periods, because it influences future party composition. We consider Markov perfect equilibria. The state at time t is given by the party composition $(S_1, S_2) \in \mathscr{K}$. A Markov strategy maps states, i.e., the composition of parties, into a probability distribution over candidates. That is, let $\mathscr{K}$ be an attainable set of party allocations, and $\mathscr{P}(\Theta)$ the set of all probabilities on Θ. Then a (mixed) Markov strategy for party i is given by a transition probability $\mu_i\colon \mathscr{P}(\Theta) \times \mathscr{K} \to [0,1]$. That is, $\mu_i(\cdot, S_1, S_2)$ is a probability distribution over candidate positions θ for all $K = (S_1, S_2) \in \mathscr{K}$. Because candidates are selected from parties, $\theta_i \in S_i$, the support of $\mu_i(\cdot, S_1, S_2)$ must be contained in S_i, i.e., $\mu_i(S_i, S_1, S_2) = 1$.

The payoff of an individual of type θ from strategies $\mu_{i,t} = 1, 2$ at time t given party allocation $K = (S_1, S_2) \in \mathscr{K}$ is given by

$$\begin{aligned} U_t(K, \mu_{1,t}, \mu_{2,t}, \theta) = \int \Big(& \sum_{i=1}^{2} \pi_{i,t}(\theta_1, \theta_2) v_\theta(x(\theta_i)) \\ & + \beta U_{t+1}(\psi(\theta_1, \theta_2, K), \mu_{1,t+1}, \mu_{2,t+1}, \theta) \Big) d\mu_{1,t}(d\theta_1, K) d\mu_{2,t}(d\theta_2, K) \end{aligned} \tag{3}$$

Before candidates are chosen parties can choose wether or nor to to compete. If party i does not compete then $S_i = \emptyset$.

In a Markov perfect equilibrium, the candidate choices must be optimal for each leader, given the candidate chosen by the other party.

Definition 3 *$(\mu_{1,t}, \mu_{2,t})$, $\pi_{i,t}$, $t = 1, \ldots, T$ is a Markov perfect equilibrium if and only if*

1. *$\pi_{i,t}$ satisfies* (2) *for all $t = 1, \ldots, T$;*

2. *For all $t = 1, \ldots, T$, and for all $K \in \mathscr{K}$ and all probabilities ν with support on S_i, where $K = (S_1, S_2)$,*

$$U_t(K, \mu_{1,t}, \mu_{2,t}, m_i(S_i)) \geq U_t(K, \mu_{i,t}, \nu, m_i(S_i)), \tag{4}$$

for parties $i = 1, 2$.

3. *If party i competes at time t then*

$$U_t(K, \mu_{1,t}, \mu_{2,t}, m_i(S_i)) \geq U_t(\emptyset, S_{-i}, \mu_{1,t}, \mu_{2,t}, m_i(S_i)), \tag{5}$$

where $K = (S_1, S_2)$. If party i does not compete, then the inequality in (5) *is reversed.*

If the type space Θ is finite and there are finitely many time periods T, then existence of Markov perfect equilibria is standard (c.f., Theorem 1.3.1 in Fudenberg and Tirole (1991)). In particular, at each time t the players are $\theta_{i,t} = m_i(S_{i,t})$. These players choose from a finite set of strategies $\theta_{i,t} \in S_{i,t}$. The state at period $t + 1$ is given by $\psi(\theta_{1,t}, \theta_{2,t}, S_{i,t}, S_{2,t})$. We can translate this game into strategic form. If T is finite, then the set of players, $\theta_{i,t}$, as well as the set of strategies is finite, and existence of mixed strategy Nash equilibria follows immediately. By construction, i.e., because each agent at time t is identified as a separate player, the equilibrium is also Markov perfect.

The existence result generalizes immediately to the case with infinitely many time periods.

Proposition 1 *Let Θ be finite and time T finite or infinite. Then there exist a Markov perfect equilibrium (in mixed strategies).*

In the final sections of this paper, we will consider models with finitely many types, where Proposition 1 is applicable. However, we also want to investigate the party dynamics when the type space and policy space are the interval $[0, 1]$ to compare our result to the classic case.

4 The Model with Two Time Periods

4.1 Existence of Pure Strategy Equilibria

Proposition 1 does not apply if the type space is continuous. In fact, it is well known that subgame perfect equilibria may not exist even in a two-stage game, when actions spaces are no longer finite (Harris et al., 1995). One possibility to derive general existence results it to introduce noise in the process that affects the future state (the allocation of parties) in the next period. More specifically, in order to get existence with a finite time horizon, the transition probability for the states must be setwise continuous (Rieder, 1979). Alternatively, if the time horizon is infinite, norm-continuity is required (see Theorem 2 of Jaśkiewics and Nowak (2016)). Neither property holds if the transition function is deterministic as in our case, but the properties can be satisfied if sufficient noise is added (Duggan, 2012). For example, we could assume that after ψ selects an interim party allocation $K \in \mathcal{K}$, a random shock, modeled as a transition probability $q_t\colon \mathcal{P}(\mathcal{K})\times\mathcal{K} \to [0,1]$, determines the final party allocation according to the probability distribution $q_t(\cdot, K)$.

We choose to work with a deterministic setting instead, because this allows us to get a sharper result on convergence of policies. As a consequence, we will have to make more specific assumption on the mapping, ψ, that determines party affiliations and on preferences.

If all citizens belong to one of the parties and a single-crossing property holds, then we can show that parties can be represented as intervals.

Lemma 1 *Let $X = \Theta = [0,1]$ and suppose that ψ_1 and ψ_2 satisfy condition 1, and are compact valued. Further, suppose utility satisfies the single-crossing property that $\frac{\partial u_\theta(x)}{\partial \theta}$ is strictly monotone (increasing or decreasing) in x. Let $\theta_1 \neq \theta_2$ be the previous period's policies. Then there exists $s \in [0,1]$ such that $\psi_i(\theta_1, \theta_2, S_1, S_2) = [0, s]$ and $\psi_{-1}(\theta_1, \theta_2, S_1, S_2) = [s, 1]$.*

For example, if we consider Euclidean preferences, $u_\theta(x) = -(x-\theta)^2$ and the previous period's candidates are $\theta_1 < \theta_2$, then the next period party cutoff is $s = 0.5(\theta_1 + \theta_2)$, and parties will be $S_1 = [0, s]$ and $S_2 = [s, 1]$. [2]

We next determine what happens if one of the parties does not compete. Recall that the choice of not competing is made before parties select their candidates. As a consequence, citizens who would otherwise have been affiliated with the opposing party may now choose to be candidates or participate in the election of the candidate who will be unopposed in the main election. We therefore assume the that party i consists of all types $\Theta = [0,1]$ if the opposing party does not compete.

Assumption 1 *Suppose that $S_i \neq \emptyset$ and $S_{-i} = \emptyset$. Then $\psi_i(\theta_1, \theta_2, S_1, S_2) = [0,1]$.*

Finally, we describe the functions m_i that determine the party leader who strategically selects a candidate. In view of Lemma 1 we can restrict attention to parties that are given by intervals

[2]More generally, we can allow for the possibility that there are independents, i.e., some types that are not affiliated with one of the two parties. For example, suppose that currently parties are given by $S_1 = [0, s_1]$ and $S_2 = [s_2, 1]$ with $s_1 < s_2$, and that policies are $\theta_1 < \theta_2$. If party membership is slower to adjust to changes, then $\psi_1(\theta_1, \theta_2, S_1, S_2) = [0, \alpha s_1 + (1-\alpha)0.5(\theta_1+\theta_2)]$ and $\psi_2(\theta_1, \theta_2, S_1, S_2) = [\alpha s_2 + (1-\alpha)0.5(\theta_1+\theta_2)]$, where $0 \leq \alpha \leq 1$. If $\alpha = 0$ then we have again the case without independents, where adjustments are immediate, while $\alpha = 1$ means that party composition never changes. Note that this definition of ψ_i may not satisfy consistency.

of the form $[0, s]$ or $[s, 1]$. As a consequence, with a slight abuse of notation we can describe the party leader by functions m_1 and m_2 that only depend on s. That is, $m_1(s)$ is the party leader if the party is given by $[0, s]$, while $m_2(s)$ is the party leader if the party is given by $[s, 1]$. We make the following assumptions.

Assumption 2

1. *$m_i\colon [0, 1] \to \mathbb{R}$ are continuous for $i = 1, 2$.*
2. *$m_1(s) \leq s$ and $m_2(s) \geq s$.*
3. *m_1 is increasing and m_2 is decreasing in s.*
4. *$m_1(1) \leq \theta_m \leq m_2(0)$.*

For example, if party 1 is $[0, s]$ we could define $m_1(s) = 0.5s$ or $m_2(s) = 1 - 0.5s$ if party 2 is given by $[s, 1]$. This would correspond to the case where citizens are uniformly distributed on $[0, 1]$ and the "party leader" is the median party member, a situation that would arise if all party members participate in the primary.

We can also model the case where party leaders are more extreme than the median party members. For example, if the candidate is selected in a primary at which a higher fraction of partisan members participates, then $m_1(s)$ would always be to the left of the party median and $m_2(s)$ to the right. Linear functions $m_i(s)$ that satisfy assumption 2 and have this property are given by

$$m_1(s) = \xi_1 s, \quad m_2(s) = 1 - \xi_2(1 - s), \text{ where } 0 \leq \xi_1, \xi_2 \leq 0.5. \tag{6}$$

We will use this representation of m_i in Proposition 5 to perform comparative statics with respect to "party extremism" — party i becomes more extreme if ξ_i is decreased. If m_i changes over time then we write $m_{i,t}$.

Next, we start by analyzing the case with two time periods, and prove existence of pure strategy subgame perfect equilibria. Even if there are multiple equilibria, payoffs are always the same.

Proposition 2 *Let $T = 2$. Suppose that condition* (1)*, and assumptions 1, and 2 hold. If preferences satisfy the single crossing property of Lemma 1 then there exists a Markov perfect equilibrium in pure strategies.*

All pure strategy subgame perfect equilibria are payoff equivalent. Further, if parties are not indifferent between competing and not competing then equilibrium strategies are unique.

4.2 Non Centrist Policies and Policy Divergence

We now use the fact that equilibrium payoffs are unique, which we established in Proposition 2, to analyze how equilibrium policies change in response to movements of the median voter's position. We show that if different parties win in the two periods, then equilibrium policies can move by more than the position of the median voter. In contrast, if the same party wins twice, policies may not respond at all to changes of the electorate. This is substantially different from the standard one dimensional case without uncertainty, where policy positions shift exactly by the same amount as the median voter. The results are summarized in Proposition 3 below.

To determine equilibria, we proceed by backward induction, following the argument in the proof of Proposition 2. If $\theta_{m,2}$ is the position of the median voter at $t = 2$, then party 1 wins if $\theta_{m,2} < s_2$. If $\theta_{m,2} > s_2$ then party 2 wins. If $\theta_{m,2} = s_2$ then either candidate can win.

If party 1's candidate wins at $t = 2$ then the candidate's position is $\theta_{1,2} = \max\{m_1(1), 2\theta_{m2} - s_2\}$, where $m_1(1)$ is the position of the leader of party 1 if party 2 does not compete, and $\theta_{m,2}$ is the position of the median voter at $t = 2$. If party 2 competes, then the party chooses a candidate $\theta_{2,2} = s_2$. If party 2's candidate wins, then the position of that candidate is $\theta_{2,2} = \min\{m_2(0), 2\theta_{m2} - s_2\}$. If party 1 competes then the position is $\theta_{1,2} = s_2$. The winning policy at $t = 2$ is therefore

$$x_2 = h(s) = \begin{cases} m_2(0) & \text{if } s < 2\theta_{m,2} - m_2(0); \\ 2\theta_{m,2} - s & \text{if } 2\theta_{m,2} - m_2(0) \leq s \leq 2\theta_{m_2} - m_1(1); \\ m_1(1) & \text{if } s > 2\theta_{m_2} - m_1(1). \end{cases} \tag{7}$$

At $t = 1$ party 1 wins if $\theta_{m,1} < s_1$ and party 2 wins if $\theta_{m,1} > s_1$. Further, at $t = 1$ it is optimal for the losing party to select the most moderate candidate possible, provided party 1 competes. In other words, if the party cutoff is s_1 and party 2 loses then $\theta_{2,1} = s_1$. Party 1 selects a candidate with policy, x_1, that solves

$$\max_{x_1 \in [0, s_1]} u_\theta(x_1) + \beta u_\theta\left(h\left(\frac{x_1 + s_1}{2}\right)\right) \text{ s.t. } x_1 \geq 2\theta_{m,1} - s_1, \tag{8}$$

where $\theta = m_1(s_1)$ is the position of the leader of party 1.

Similarly, if $\theta_{m,1} > s_1$ then party 2 wins, and the winning policy x_2 solves

$$\max_{x_1 \in [s_1, 1]} u_\theta(x_1) + \beta u_\theta\left(h\left(\frac{x_1 + s_1}{2}\right)\right) \text{ s.t. } x_1 \leq 2\theta_{m,1} - s_1, \tag{9}$$

where $\theta = m_2(s_1)$ is the position of the leader of party 2.

Consider the case where party 1 wins in the first period and hence Problem 8 applies. If the constraint in the problem binds, then it is immediate that policies move more than the median if party 2 wins in the second period. In particular, the winning policy in the first period is $x_1 = 2\theta_{m,1} - s_1$, resulting in a party cutoff $s_2 = \theta_{m,1}$. If $\theta_{m,2} > \theta_{m,1}$ then party 2 wins in the next period, and the winning policy $x_2 = h(s_1) \geq \theta_{m,1}$. Thus $|x_2 - x_1| > |\theta_{m,2} - \theta_{m,1}|$.

If $\theta_{m,2} < \theta_{m,1} < s_1$ then party 1 wins in both periods, and if the constraint of Problem 8 binds, then the winning policy in both periods is strictly to the left of the median voter (assuming that $m_1(1) < \theta_{m,2}$). In this case it is possible that the policy changes less than the median voter.

For example, suppose that $\theta_{m,1} = 0.5$, $\theta_{m,2} = 0.45$ and $s_1 = 0.6$. If the constraint of Problem 8 binds then $x_1 = 2\theta_{m,1} - s_1 = 0.4$. In the next period, the party cutoff is $s_1 = 0.5$. The winning policy at $t = 2$ is therefore $2\theta_{2,m} - s_2 = 0.4$ and hence the policy does not respond to changes in the electorate's preferences. Note that in both periods policies are strictly to the left of the median voters' ideal points.

We now prove more generally, that the policies move more than the median voters if different parties win, while they move less if the same party wins twice.[3]

Proposition 3 *Suppose that utility is of the form $u_\theta(x) = -|x - \theta|^\gamma$, $\gamma \geq 1$. Let $m_1(1) + s_1 < 2\theta_{m,1}$, $\theta_{m,1} < s_1$ and $\beta > 0$. Then:*

1. *Policies change more than the median voters, i.e., $|x_1 - x_2| > |\theta_{m,1} - \theta_{m,2}|$; party 1 wins in period 1; party 2 in period 2, and there is policy divergence in both periods if one of the following two conditions holds:*
   - *(a) $s_1 < (\theta_{m_1} + \theta_{m,2})/2$;*
   - *(b) $(\theta_{m_1} + \theta_{m,2})/2 \leq s_1 < \theta_{m,2}$ and $\beta < \min\left\{2^{2-\gamma}, 1\right\}$.*
2. *Let $\theta_{m,1} < \theta_{m,2} < s_1$. Then party 1 wins in both periods. If $\gamma \to \infty$ then in the limit winning policies do not change in response to changes of the median voter's position, i.e., $|x_1 - x_2| \to 0$ as $\gamma \to \infty$.*

Intuitively, why does the restriction that candidates are selected from parties increase the volatility of policies? Suppose that party 1 wins in the first period and party 2 in the second period. Party 1 could constrain party 2 at $t = 2$ by nominating a more moderate candidate at $t = 1$. The new party cutoff s_2 would then be closer to the median voter, which in turn means that party 1's candidate is more competitive, which prevents party 2 from nominating a candidate who is too extreme. However, the new party cutoff is $s_2 = (s_1 + \theta_{1,1})/2$. Hence nominating a candidate

[3] The proposition is written for the case where $\theta_{m,1} < s_1$. The case $\theta_{m,1} > s_1$ is analogous.

who is more moderate by "one unit" only results in the party cutoff moving to the right by 1/2 of a unit. If utility is $u_\theta(x) = -|x - \theta|$ it follows that moderation at $t = 1$ is not worthwhile.

If the curvature of utility and if β are increased, parties may lower the volatility of policies, but at least if $\beta < 2^{2-\gamma}$ policies still move by more than the median voter. For example, if utility is $u_\theta(x) = -(x - \theta)^2$, then the result holds for any $\beta < 1$.

However, even if the curvature of utility goes to infinity, volatility does not go to zero when different parties win, and we can still find cases in which policies move more than the median voters as condition (a) of the proposition indicates. Policy x_1 must be in $[0, s_1]$, and if s_1 is closer to $\theta_{m,1}$ than to $\theta_{m,2}$, party 2 uses its electoral advantage in the second period to nominate a more partisan candidate, resulting in a large policy change $|x_1 - x_2|$.

Proposition 3 also shows that policies may change very little if the same party wins in both periods. If the curvature of utility is sufficiently large, and the same party has an electoral advantage in both periods, the winning party will attempt to minimize policy changes.

4.3 The Effect of Current Candidates on the Party's Future

Consider a situation in which one party selects an extremist candidate. Such a candidate will have a negative impact on the future composition of the party, resulting in a tradeoff between winning with a more extreme candidate or improving the party's future electoral prospects.

To better understand this tradeoff, consider the following example. Suppose that the current party cutoff is $s_1 = 0.6$: All types $\theta \leq 0.6$ belong to party 1, and all types $\theta \geq 0.6$ to party 2. Because parties can only select one of their members as a candidate, party 2's most competitive candidate is located at 0.6. If the median voter is at $\theta_{m,1} = 0.5$, party 1 can win by nominating any candidate θ between 0.4 and 0.6. However, a more extreme candidate nominated by party 1 alienates more party members, and the new party cutoff becomes $s_2 = (\theta + 0.6)/2$. If s_2 remains to the right of the median voter, $\theta_{m,2}$, then party 1 can still win at $t = 2$, but it must choose a more centrist candidate, given that party 2 has now access to more competitive candidates. If $\theta_{m,2} > s_2$ then party 1 will lose in the next round, because party 2 can select a candidate who is more attractive to the median voter. Thus, party leaders in our model face the tradeoff of selecting more partisan candidates in the current period versus retaining the party's size and dominance in the future.

We want to investigate how this tradeoff between choosing more extreme candidates in the current period versus retaining the party's size in the future is affected by party extremism.

Consider the following example. Suppose that utility is given by $u_\theta(x) = |x - \theta|$, that $\theta_{m,1} = \theta_{m,2} = 0.5$ and $s_1 = 0.6$. Let $m_1(s_1) > 0.4$.[4] Then party 1 nominates a candidate at position x_1 that solves problem 8. It follows that $x_1 = m_1(s_1)$. If function h defined in (7) is locally independent of the next period's party cutoff, s_2, then this is immediate. Otherwise, if $h(s_2) = 2\theta_{m,2} - s_2 = 1 - s_2$, then the fact that $s_2 = (s_1 + x_1)/2$ implies $x_1 = m_1(s_1)$ is optimal.

The party cutoff in the next period is therefore $s_2 = (s_1 + m_1(s_1))/2 = 0.3 + 0.5m_1(s_1)$. If $h(s_2) = 2\theta_{m,2} - s_2$, then the winning policy in the next period is $x_2 = 2\theta_{m,2} - s_2 = 0.7 - 0.5m_1(s_1)$. The distance between x_2 and the median voter is $\left|\theta_{m,2} - x_2\right| = 0.5m_1(s_1) - 0.2$. Thus, a more moderate party, characterized by a higher value of $m_1(s_1)$, will be able to use its electoral advantage in the next period in order to implement a more extreme position. In contrast, a more extreme party wants to take advantage of a current electoral advantage, thereby hurting their prospects in the future.

We get a similar result if party 2 wins in the next period. For example, suppose that $\theta_{m,2} = 0.55$ and that $0.4 < m_1(s_1) < 0.5$. The party cutoff in the next period is again $s_2 = 0.3 + 0.5m_1(s)$, but now $s_2 > \theta_{m,2}$. Thus party 2 wins, and the winning position is $x_2 = 0.7 - 0.5m_1(s_1)$. The distance of this policy from the median voter is $\left|\theta_{m,2} - x_2\right| = 0.15 - 0.5m_1(s_1)$. Thus, a more moderate party 1 will now result in a more moderate winning position, i.e., a policy x_2 that is closer to the median voter at $\theta_{m,2}$. The reason is that the more moderate party 1 nominates a more moderate candidate at $t = 1$, which result in a larger party at $t = 2$. This larger, more competitive party 1 limits party 2's ability to win with more extreme candidates.

The insights from this example also hold more generally.

Proposition 4 *Suppose that utility is of the form $u_\theta(x) = |x - \theta|^\gamma$ for $\gamma \geq 1$. Let x_t denote the winning policies in periods $t = 1, 2$. Suppose that the winning party i at $t = 1$ becomes more extreme, i.e., the distance between the party leader and the median voter, $\left|m_i(s) - \theta_{m,1}\right|$, increases, resulting in policies y_t, $t = 1, 2$. Then the winning policy is more extreme in period 1, i.e., $\left|y_1 - \theta_{m,1}\right| \geq \left|x_1 - \theta_{m,1}\right|$, but more moderate in period 2, i.e., $\left|y_2 - \theta_{m,2}\right| \leq \left|x_2 - \theta_{m,2}\right|$, if party i wins again, and more extreme, i.e., $\left|y_2 - \theta_{m,2}\right| \geq \left|x_2 - \theta_{m,2}\right|$ if party i loses at $t = 2$. The inequalities are strict for some values of $m_i(s)$ and γ.*

This result show that if the decision makers in a party become more extreme, for example, because partisans start participating more in primaries compared to moderates, then these decision makers become more concerned about electing a partisan than about damaging the party's future electoral prospects. A "weaker" party i in the second period will have to nominate a more moderate

[4] We assume for the moment that the identity of the party leader is not a function of time.

candidate and will be less able to prevent the opposing party from winning.

4.4 The Choice not to Compete

As mentioned in the introduction, in many state level elections in the US only one major-party candidate is on the ballot. The decision by a party not to compete is also a feature of our model, and is largely determined by two parameters: (i) the extremism of the party leaderships, and (ii) a party's current electoral advantage. We will see that a more extreme leadership of one party makes it more likely that both parties compete. The comparative statics with respect to a party's electoral advantage, i.e., the party cutoff s_1, is more subtle. That is, increasing the electoral disadvantage may not make a party less likely to compete.

In section 4.2 we have analyzed the equilibrium for subgames starting at $t = 2$, and the winning policy is characterized in (7). These results immediately imply when it is optimal for a party to compete in the second period.

Suppose that $\theta_{2,m} = 0.5$ and that $m_1(s) = \xi_1 s$, and $m_2(s) = 1 - \xi_2(1 - s)$ as in (6). If $s_2 > 0.5$ and party 2 competes, then party 1 will select a candidate at $x_2 = \max\{1 - s_2, \xi_1 s_2\}$; position $1 - s_2$ is the most liberal position that can win, while $\xi_1 s_2$ is the ideal point of party 1's leader. If party 2 did not compete, then the resulting policy is $m_1(1) = \xi_1$. Hence, party 2 competes if $\xi_1 \leq \bar{\xi}_1 = 1 - s_2$. In other words, party 2 is more likely to compete if party 1 is more extreme, or if party 1's electoral advantage, s_2, is not too large. Note that the extremism of party 2, i.e., the value ξ_2, is irrelevant for party 2's choice whether or not to compete.

If we consider the initial model period, $t = 1$, then it is still true that a more extreme party 1 makes party 2 more willing to compete. However, the cutoff, $\bar{\xi}_1$ now depends also on ξ_2, and a lower value of ξ_2 (a more extreme party 2) will in general raise the cutoff $\bar{\xi}_1$. The effect of a party's competitiveness, s_1 becomes ambiguous.

We now state the formal result and then provide some intuition.

Proposition 5 *Suppose utility is of the form $u_\theta(x) = -|x - \theta|^\gamma$ and that $\theta_{1,m} = \theta_{2,m} = 0.5$. Further, assume that $m_1(s) = \xi_1 s$, and $m_2(s) = 1 - \xi_2(1 - s)$ as in (6) where $0 \leq \xi_1, \xi_2 \leq 0.5$. Let $s_1 > 0.5$*

Then:

1. *There exists $\bar{\xi}_1$ such that party 2 competes at $t = 1$ if and only if party 1 is sufficiently extreme, i.e., $\xi_1 \leq \bar{\xi}_1$.*

2. *The cutoff $\bar{\xi}_1$ is non-increasing in ξ_2, i.e., a more extreme party 2 competes if a less extreme version of party 2 would also compete.*

3. *If the constraint of problem 8 binds, i.e, if $\xi_1 \leq (2-2s_1-(0.5\beta)^{\frac{1}{\gamma-1}})/(2(1-(0.5\beta)^{\frac{1}{\gamma-1}})s_1)$, then increasing the party cutoff s_1 decreases $\bar{\xi}_1$, i.e., if party 1's electoral advantage increases, party 2 is less likely to compete.*

4. *If the constraint of problem 8 is slack, then $\bar{\xi}_1$ decreases when s_1 is increased if $(0.5\beta)^{1/(\gamma-1)}$ is sufficiently close to 1.*

The cutoff $\bar{\xi}_1$ is obtained at the point where party 2's leader at $\theta = 1 - \xi_2(1 - s_1)$ is indifferent between competing and not competing, i.e., where

$$-(\theta - x_1)^\gamma - \beta(\theta - x_2)^\gamma = -(\theta - \bar{\xi}_1)^\gamma - \beta(\theta - \bar{\xi}_1)^\gamma, \tag{10}$$

and where x_1 and x_2 are party 1's winning polices, who may themselves depend on $\bar{\xi}_1$. If party 2 does not compete at $t = 1$, a candidate at $m_1(1) = \xi_1$ is nominated.[5] Recall that $x_2 \geq \xi_1$, because by not competing party 2 can ensure that no policy to the left of ξ_1 wins. Thus, if $\xi_1 = \bar{\xi}_1$ then $x_1 \leq \bar{\xi}_1$ and $x_2 \geq \bar{\xi}_2$. In fact both equalities must be strict.

Now suppose that party 2 becomes more extreme, which implies that θ in (10) increases. Because $x_1 < x_2$ it must be the case that $x_1 < \bar{\xi}_1 < x_2$. Consider a lottery with outcomes x_1 and x_2 that occur with probabilities $1/(1+\beta)$ and $\beta/(1+\beta)$, respectively. Then (10) implies that $\bar{\xi}_1$ is the lottery's certainty equivalent. The degree of absolute risk aversion for utility $u(x) = -(\theta - x)^\gamma$ is $r_A = -u''(x)/u'(x) = (\gamma - 1)(\theta - x)$. Thus, if θ increases, absolute risk aversion decreases. This implies that raising θ raises the lottery's certainty equivalent $\bar{\xi}_1$.

Intuitively, a party leader has the choice between not competing, and receiving policy $\bar{\xi}_1$ in both periods, or competing and receiving a less preferred policy $x_1 < \bar{\xi}_1$ at $t = 1$ and a more preferred policy $x_2 > \bar{\xi}_1$ at $t = 2$. An extreme leader of party 2, dislikes both policies more than a moderate party leader. However, the more extreme leader's tradeoff between the policies differs: He receives relatively more utility from x_2 compared to x_1 and is therefore more willing to compete at $t = 1$ in order to get the more moderate policy at $t = 2$.

Numerically, the effect of party 2 extremism on $\bar{\xi}$ is limited as figure 1 indicates. Recall that $\xi_2 = 0$ corresponds to the most extreme possible leader of party 2, located at $\theta = 1$, whereas $\xi_2 = 0.5$ moves the party leader to $\theta = 0.8$ (when $s_1 = 0.6$). Party 2 competes as long as $\xi_1 \leq \bar{\xi}_1$,

[5]We assume that party 2 cannot reenter in the second period, and as a consequence the winning policy at $t = 2$ is again $\bar{\xi}_1$.

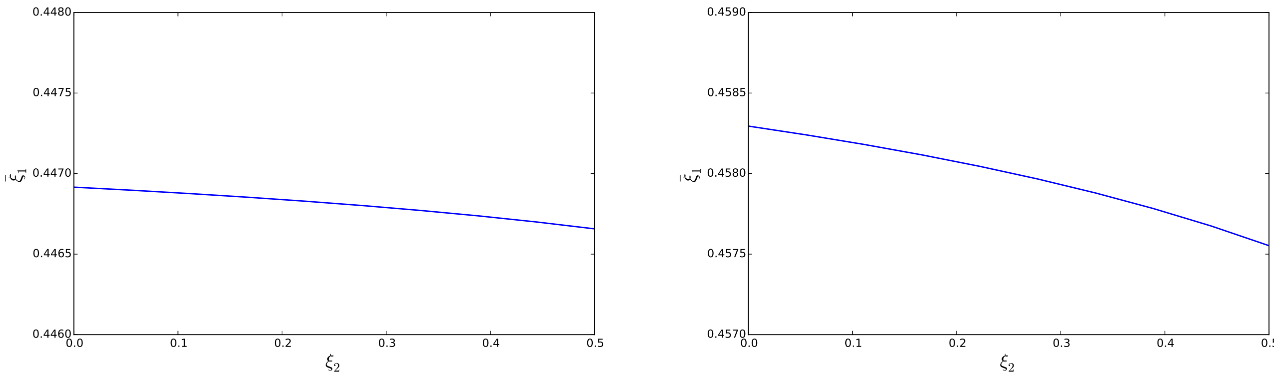

Figure 1: The impact of party 2 extremism, ξ_2, on the cutoff value $\bar{\xi}_1$, for $s = 0.6$, $\beta = 0.9$, $\gamma = 1.2$ (left panel) and $\gamma = 5$ (right panel)

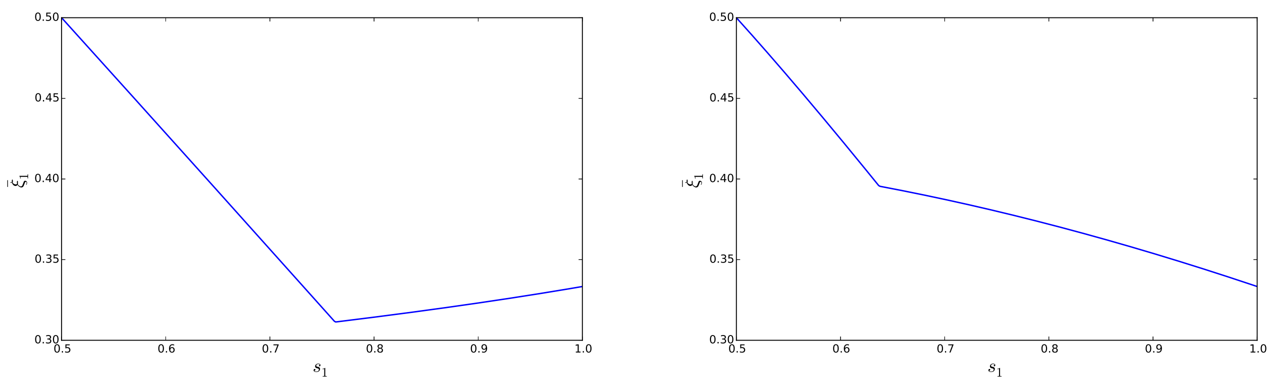

Figure 2: The impact of party 1's electoral advantage, s_1, on the cutoff value $\bar{\xi}_1$, for $\xi_2 = 0.1$, $\beta = 0.4$, $\gamma = 1.2$ (left panel) and $\gamma = 3$ (right panel)

and ξ_2 influences this cutoff only minimally. The impact of ξ_2 on the cutoff is even smaller if γ is closer to 1. Thus, party 2's decision whether or not to compete is primarily determined by party 1's extremism.

Now consider the case where s_1 and hence party 1's electoral advantage increases. If the constraint of problem 8 binds then x_1 decreases, and x_2 increases, which makes the leader of party 2 worse off. This would make it more attractive for party 2 not to compete, but s_1 also raises the position of party 2's leader, θ, which (as we have just shown) makes party 2 more willing to compete. It turns out, however, that this latter effect is dominated by the former, i.e., the overall effect of raising party 1's electoral advantage is a reduction in party 2's willingness to compete.

If we raise s_1 to a level at which the constraint of problem 8 is slack, the overall effect of s_1 on

$\bar{\xi}_1$ becomes ambiguous. In particular, raising s_1 results in a more moderate leader of party 1, who may prefer a less extreme position at $t = 1$, i.e., a higher x_1. This increase of x_1 may also make the leader of party 2 better off, who as a consequence would be more likely to compete. Proposition 5 indicates that an increase of s_1 results in an increase of $\bar{\xi}_1$ if either β and γ are not too large.

Figure 2 shows how $\bar{\xi}_1$ changes in response to s_1. Initially $\bar{\xi}_1$ decreases because the constraint of Problem 8 binds for these values. The lines have a kink at the point where the constraint becomes slack. In the left-panel γ is sufficiently small so that $\bar{\xi}_1$ rises when s_1 is sufficiently large. With the somewhat higher value of γ in the right-panel, $\bar{\xi}_1$ is strictly decreasing.

Figures 1 and 2 also show that $\bar{\xi}_1$ responds much more strongly to changes in s_1 than to changes in ξ_2. In other words, unlike extremism of party 2's leader, the level of party 2's electoral disadvantage, measured by parameter s_1, strongly influences the party's decision whether or not to compete.

5 The Model with Infinitely Many Time Periods

5.1 Dynamic Median Voter Theorem

We now address the question whether we get convergence to the median voter's ideal position if the position of the median voter does not change over time. Proposition 5 indicates that if β is small then convergence will not occur if one party is too dominant, in the sense that the party cutoff s is too far from the median voter, θ_m. In such a case, the minority party does not compete, and the majority party will nominate a non-median candidate. Thus, we can only expect a dynamic median voter theorem to hold if β is sufficiently close to 1.

We will construct an equilibrium in which the minority party always competes by nominating a candidate at the party cutoff. The majority party selects more extreme candidates, but after finitely many periods convergence to the median occurs. To determine the equilibrium we proceed recursively.

Without loss of generality suppose that the party cutoff $s > \theta_m$. Let $x(s)$ be the policy selected by the winning party's candidate, and let $U(s, \theta)$ be the continuation utility of an individual of type θ if the current state is s. Then

$$x(s) \in \arg\max_{x \in [0,s]} u_{m_1(s)}(x) + \beta U\left(\frac{x(s) + s}{2}, m_1(s)\right) \quad \text{s.t.} \quad x \geq 2\theta_m - s, \tag{11}$$

and

$$U(s, \theta) = u_\theta(x(s)) + \beta U\left(\frac{x(s) + s}{2}, \theta\right). \tag{12}$$

Condition (11) ensures that the candidate choice is optimal for the current leader of party 1, given that party 2's candidate is located at s. In order to ensure that it is not optimal for party 2 to deviate by selecting a candidate at s',

$$U\left(\frac{x(s)+s}{2}, m_2(s)\right) \geq U\left(\frac{x(s)+s'}{2}, m_2(s)\right), \tag{13}$$

for any $s' \geq s$. Finally, it must be better for party 2 to compete for any cutoff s, i.e.,

$$U\left(s, m_2(s)\right) \geq \frac{u_{m_2(s)}(m_1(1))}{1-\beta}, \tag{14}$$

for all $s > \theta_m$.

For the case where $u_\theta(x) = -|x-\theta|$ and functions m_i are linear as in (6) we can use (11) to (14) to find a closed form solution for the equilibrium strategy.

Proposition 6 *Suppose that utility is $u_\theta(x) = -|x-\theta|$ and that the median voter is at $\theta_m = 0.5$ in all time periods. Further, suppose that party leaders are given by $m_1(s) = \xi_1 s$ and $m_2(s) = 1-\xi_2(1-s)$. Then for $\xi_i \leq \min\left\{\sqrt{5}-2, \frac{\beta}{2-\beta}\right\}$ there exist a Markov perfect equilibrium in pure strategies with the following properties:*

If $s > \theta_m$ then party 1 wins with a candidate at position $x(s) = \max\{2\theta_m - s, m_1(s)\}$. If $s < \theta_m$ then party 2 wins with a candidate at position $x(s) = \min\{2\theta_m - s, m_2(s)\}$. Starting at any $s \in [0,1]$, the equilibrium policies converge to a steady state after at most two periods. In the steady state the median voter is the party cutoff, and the median voter's preferred policy is implemented in equilibrium.

An interesting insight of Proposition 6 is that the policy $x(s)$ is not monotone in the party cutoff s. Consider the case where $s > 0.5$ and party 1 wins. Initially raising s results in a more extreme policy, i.e., $x(s)$ is decreasing. However, once $s + m_1(s) \geq 2\theta_m$, then $x(s)$ is increasing in s. Intuitively, party 1 first uses an electoral advantage to win with a more partisan candidate. However, as s is increased further, the party leadership also becomes more moderate. For sufficiently large s this latter effect dominates and results in the selection of more moderate candidates.

5.2 Steady States Without a Condorcet Winner

In the previous section we showed that in a model with a one dimensional policy space, after finitely many periods a steady state is reached. Unless one party chooses not to compete, the steady state is the Condorcet winner, i.e., the median voter. We now investigate what happens in a model, in which no Condorcet winner exists.

The simplest such model is as follows: Suppose there are three different types $\Theta = \{A, B, C\}$, and three policies, $X = \{x_A, x_B, x_C\}$. Utilities are as follows:

1. $u_A(x_A) = 2$, $u_A(x_B) = 1$, $u_A(x_C) = 0$;
2. $u_B(x_B) = 2$, $u_B(x_C) = 1$, $u_B(x_A) = 0$;
3. $u_C(x_C) = 2$, $u_C(x_A) = 1$, $u_C(x_B) = 0$.

Further, consider a distribution for which no single type by themselves can determine the outcome of the election, i.e., $\phi(\{\theta\}) < 0.5$, while $\phi(\Theta \setminus \{\theta\}) > 0.5$ for any type $\theta \in \Theta$. Further, assume that $\phi(\{A\}) > \phi(\{B\}) > \phi(\{C\})$. As a consequence of these assumptions, a majority of the electorate prefers A to B, prefers B to C, and prefers C to A, and we have a Condorcet cycle.

Consider first the classic model of Wittman (1973), where party (or equivalently candidate) preferences are fixed exogenously. Suppose that candidate 1 has the preferences of the type A voter, while candidate 2's preferences correspond to those of type B. Unlike in our model, both candidates can commit ex-ante to a policy.

Then the game's payoff matrix is as follows:

		Cand. 2		
	Policy	A	B	C
Cand. 1	A	2,0	2,0	0,1
	B	2,0	1,2	1,2
	C	0,1	1,2	0,1

As Wittman (1973) already noted, his approach, unlike that of Downs (1957), allows for the existence of pure strategy equilibria even in cases where aggregate preferences violate transitivity. Indeed a unique pure strategy equilibrium exists in this game, while office motivated candidates would mix between all three different policies. In the unique pure strategy equilibrium, candidate 1 selects policy B, and candidate 2 chooses C. Of course, B wins. A dynamic version as in Wittman (1977) would obviously result in B, C as a steady state, with B winning in every period. Of course, the equilibrium changes if we change the types of the policy motivated candidates.

Now consider again our model with infinitely many time periods. Rather than being fixed, the types that can select the candidates are determine endogenously, for given selection functions m_i, which we now specify.

We assume that the function m_i is the same for both parties, and we therefore drop the index i. Let $m(S) = \arg\max_{\theta \in S} \phi(\{\theta\})$, i.e., the most frequent type can choose the policy. If S consists of one or two types, then $m(S)$ simply corresponds to majority rule for determining a party's candidate in a primary.

If parties cannot exit, then the following are the equilibrium strategies:

1. If parties are $\{A, B\}$ and $\{C\}$, then B is nominated and wins against C.

2. If parties are $\{B, C\}$ and $\{A\}$, then C is nominated and wins against A.

3. If parties are $\{A, C\}$ and $\{B\}$, then A is nominated and wins against B.

Suppose that at time t the party composition is $\{A, B\}$, and $\{C\}$. In the proposed equilibrium, B is nominated and wins. The party composition remains unchanged, because both A and B prefer x_B to x_C. Hence, on the equilibrium path the policy will remain at x_B. Now suppose that party $\{A, B\}$ nominates candidate A. Both B and C prefer x_C to x_A. Hence, A loses and the parties change to $\{A\}$, $\{B, C\}$. In the proposed equilibrium, C wins and implements x_C in all future time periods. Because both A and B prefer x_B to x_C, this deviation is not optimal.

Now, suppose that we start with parties $\{A\}$, $\{B, C\}$ at time t. In equilibrium C is nominated, wins against A and policy x_C is implemented. If party $\{B, C\}$ nominates B then A wins, with the votes of A and C types. At $t + 1$ the parties are then $\{A, C\}$ and $\{B\}$ and the policy remains at A. Both B and C are better off with policy x_C than policy x_A, and hence the deviation is not optimal. The remaining case, where parties are $\{A, C\}$, and $\{B\}$ is similar. One can also verify that these strategies are unique.

Now consider the case where parties can exit. Suppose that we are in the situation where parties are $\{A, B\}$ and $\{C\}$. In this case B wins, resulting in policy x_B. If party C exits in the current period, then in the next period the new single party is $S = \{A, B, C\}$. Because $m(S) = \{A\}$, the party leader is of type A and "wins" (there is no competition), and would select policy x_A, which also makes type C better off.

Similarly, if parties are $S_1 = \{B, C\}$ and $S_2 = \{A\}$, then it is optimal for party 1 to exit, because this would again move the policy to x_A. Finally, when $S_1 = \{A, C\}$ and $S_2 = \{B\}$, then A wins. Party $S_2 = \{B\}$ has no incentive to drop out because the equilibrium policy would not change. Thus, in all cases A wins, and policy x_A is implemented.

Note that this result would not change if parties were allowed to reenter, i.e., if starting with a single party $S = \{A, B, C\}$, one type is allowed to separate from S to form its own party.

For example, suppose that we are in a situation with only a single party $\{A, B, C\}$, in which case type A chooses policy x_A. Clearly, it is not optimal for A to form its own party, because this would result in policy x_C. If B separates, then $S_1 = \{A, C\}$ and $S_2 = \{B\}$ in which case the optimal policy remains at x_A. Finally, if C separates, then parties are $S_1 = \{A, B\}$ and $S_2 = \{C\}$ in which case B wins, which makes C worse off.

5.3 Minorities with Strong Preference Intensities

Suppose there are two issues $i = 1, 2$, and two possible policy choices on each issue, which we denote by 0 and 1.[6] For example, we can think of issue 1 as a tax policy (e.g., keeping taxes the same or lowering taxes) and issue 2 relating to gun control (e.g., for or against background checks). There are four types of voters, A, B, C, and D, their utilities are as follows, and the fraction of these types in the population are as follows:

Type A $u(x_1, x_2) = -2|x_1| - |x_2|$. Thus, $(0,0) \succ (0,1) \succ (1,0) \succ (1,1)$; 35%

Type B $u(x_1, x_2) = -2|x_1 - 1| - |x_2|$. Thus, $(1,0) \succ (1,1) \succ (0,0) \succ (0,1)$; 25%

Type C $u(x_1, x_2) = -|x_1| - 2|x_2 - 1|$. Thus, $(0,1) \succ (1,1) \succ (0,0) \succ (1,0)$; 22%

Type D $u(x_1, x_2) = -|1 - x_1| - 2|x_2 - 1|$. Thus, $(1,1) \succ (0,1) \succ (1,0) \succ (0,0)$; 18%

Suppose that the policy on, say issue 1, is fixed to some value x_1 and cannot be affected by candidates. Then a majority of citizens would prefer 0 on the second issue, i.e., a majority prefers $(x_1, 0)$ to $(x_1, 1)$. Similarly, if the position on issue 2 is fixed to x_2, citizens would also prefer 0 on the first issue — again a majority prefers $(0, x_2)$ to $(1, x_2)$. However, types C and D care very strongly about the second issue, and a consequence $(0, 0)$ is not a Condorcet winner. To see this, consider two candidates who choose one of the four policies $x \in X$. Then we get the following vote shares:

		Cand. 2			
	Policy	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
Cand. 1	$(0,0)$	0.5,0.5	0.57,0.43	0.6,0.4	0.35,0.65
	$(1,0)$	0.43,0.57	0.5,0.5	0.25,0.75	0.6,0.4
	$(0,1)$	0.4,0.6	0.75,0.25	0.5,0.5	0.57,0.43
	$(1,1)$	0.65,0.35	0.4,0.6	0.43,0.57	0.5,0.5

It is easy to see that $(0, 0)$ would be defeated by policy $(1, 1)$, and that there is no Condorcet winner. If there are two parties (or candidates) as in Downs (1957), who maximize vote shares,

[6] See Krasa and Polborn (2010) for a general description of models where policies on each issue are binary.

only a mixed strategy equilibrium exists. In this mixed strategy equilibrium, $(0, 0)$ is played by both parties with probability $7/32$, $(0, 1)$ with probability $15/32$, and $(1, 1)$ with probability $10/32$ — $(1, 0)$ is never chosen. As a result, policy outcomes $(0, 0)$ and $(1, 1)$ will occur with probabilities of about $1/4$ each, and $(0, 1)$ with probability $1/2$.

In the equilibrium, parties randomize over positions that cover both extremes of the political spectrum, position $(0, 0)$, and its complete opposite $(1, 1)$. In other words, we get the counterfactual result that one and the same party may adopt with strictly positive probability a platform that calls for both higher taxes and gun control or a platform that promises low taxes and no restrictions on guns. The separation of parties and candidates in our model prevents this from happening. If one party in our model attracts all low-tax and pro-gun types, $(1, 1)$, then the other party will attract all the $(0, 0)$ types, who hold the liberal position on both issues. Hence it is not possible for one and the same party to have both one candidate who could credibly commit to $(1, 1)$ and another candidate who could commit to $(0, 0)$.

We next determine equilibria for the case of policy-motivated candidates.

5.4 Policy Motivated Candidates

Suppose that candidate 1's preferences are those of type A voters, while candidate 2's are those of type B voters. Then we get the following payoff matrix.

		Cand. 2			
	Policy	$(0, 0)$	$(1, 0)$	$(0, 1)$	$(1, 1)$
Cand. 1	$(0, 0)$	0,−2	0,−2	0,−2	−3,−1
	$(1, 0)$	0,−2	−2,0	−1,−3	−2,0
	$(0, 1)$	0,−2	−1,−3	−1,−3	−1,−3
	$(1, 1)$	−3,−1	−2,0	−1,−3	−3,−1

The unique pure strategy equilibrium is $(0, 1)$, $(0, 0)$, and $(0, 0)$ wins.

In order for the minority positions to get adopted, at least one of the candidate themselves must prefer the position on the second issue. For example, suppose that candidate 1 still has the preferences of a type A voter, while candidate 2 has those of a type D voter. Then we get the following payoff matrix.

		Cand. 1			
	Policy	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
Cand. 2	$(0,0)$	0,−3	0,−3	0,−3	−3,0
	$(1,0)$	0,−3	−2,−2	−1,−1	−2,−2
	$(0,1)$	0,−3	−1,−1	−1,−1	−1,−1
	$(1,1)$	−3,0	−2,−2	−1,−1	−3,0

Now the unique pure strategy equilibrium is $(0,1)$, $(1,1)$, where candidate 1 wins with $(0,1)$.

Thus, whether or not the minority is chosen in equilibrium depends on the policy motivation of the candidates themselves. In other words, unless candidates themselves want policy 1 on the second issue, it will not be chosen in equilibrium. We now compare this prediction to that of our model.

5.5 Party Equilibrium

We now determine steady states using our model where parties and thus the individuals who chose policy are chosen endogenously. As in section 5.2 assume that m_i is the same for both parties, and as a consequence we drop the index i. We assume that the type whose policy is preferred by a majority of the party members is the leader. Thus, if a party consists of a single type or two types, then the most frequent type is the leader, i.e., $m(S) = \arg\max_{\theta \in S} \phi(\{\theta\})$. Further, this implies $m(\{A,B,C\}) = A$, $m(\{A,B,D\}) = B$, $m(\{A,C,D\}) = C$, and $m(\{B,C,D\}) = D$. We can define $m(\{A,B,C,D\})$ arbitrarily, because this does not affect the equilibrium.

Consider first the equilibrium from section 5.4 with two policy motivated candidates who prefer position 0 on the second issue. In this equilibrium $x_1 = (0,1)$ and $x_2 = (0,0)$, and policy x_2 wins. Given these policies, parties are uniquely determined (by consistency) as $S_1 = \{C,D\}$ and $S_2 = \{A,B\}$. The party leaders are therefore $m(S_1) = C$ and $m(S_2) = A$. However, type C would be strictly better off choosing D as the candidate. In particular, D would choose policy $(1,1)$ if elected, and a majority of the electorate prefers $(1,1)$ to $(0,0)$ (see the vote-share table in section 5.3). Type C prefers $(1,1)$ to $(0,0)$, and hence $x_1 = (0,1)$, $x_2 = (1,1)$ cannot be a steady state.

Up to relabeling of parties, there are three steady state equilibria, both of which entail the minority position being adopted on the second issue. The first steady state equilibrium is as follows:

Parties are given by $S_1 = \{A\}$ and $S_2 = \{B,C,D\}$. Party 1 selects a candidate of type A by default, while party 2 chooses type D by default. Type D's policy of $(1,1)$ wins agains $(0,0)$. The leader of party 2, $D = m(S_2)$ receives his most preferred policy, and has therefore no incentive to deviate, while party ' does not have the choice of selecting a different candidate.

The second steady state is similar:

Parties are given by $S_1 = \{A, C, D\}$ and $S_2 = \{B\}$, while C and B are selected as candidates. Candidate C wins with policy $(0, 1)$.

In the third steady state, parties are given by $S_1 = \{A, C\}$ and $S_2 = \{B, D\}$, respectively. Because there are more A than C types, candidate selection in party 1 is done by type A. Similarly, there are more B than D types in party 2, and thus, B types select a candidate for party 2. As a consequence, the situation resembles that discussed in section 5.4 with two policy motivated candidates of type A and B respectively. The key difference, however, is that party 1 must select one of its members, i.e., either a type A or a type C, as a candidate, which would result in policies $(0, 0)$ and $(0, 1)$, respectively, if party 1 wins. Party 2 can choose between B and D, with would result in policies $(1, 0)$, and $(0, 1)$, respectively.

The equilibrium is for both parties to choose candidates that favor the minority position, 1, on the second issue. In particular, party 1 selects a type C individual, i.e., policy $(0, 1)$, while party 2 selects type a D individual, which corresponds to policy $(1, 1)$. Thus, the vote-share table in section 5.3 reveals that party 1 and policy $(0, 1)$ wins. This happens despite the fact that both party leaders would prefer policy 0 on the second issue. However, if party 1 nominated its leader's most preferred candidate, A, then party 1 would lose, because A's policy $(0, 0)$ loses agains D's policy $(1, 1)$. Similarly, party 2 does not benefit from nominating candidate B. Candidate B would choose policy $(1, 0)$, and hence in the next period, parties are given by $S_1 = \{A, C, D\}$ and $S_2 = \{D\}$. As a consequence type $C = m(S_1)$ will select the candidate in the next period. Hence, we are in the first type of steady state with policy $(0, 1)$ winning.

Thus, political competition may force parties to choose candidates whose preferences on some issues neither represent those of a majority of the party and of the electorate. In the first two types of steady steady state equilibria one party attracts all members of the minority, who then being a majority within the party determine the policy within the party. In the third type of steady state, party leaders share the majority position on the second issue, but political competition forces them to choose a candidate who prefers the minority position. In all cases, "vocal minorities" have an outsize influence on policy.

As mentioned above, a majority of citizens separately prefers position 0 on both issue, which is the policy a type A candidate would implement, if elected. It is indeed possible for candidate A to win in a steady state, but we need a very particular choice of party leaders, which are specified in Proposition 7 below.

Suppose for example that parties are given by $S_1 = \{A, C\}$, and $S_2 = \{B, D\}$, and party leaders are the majority types A and B. In order for A to win at time t, party 2 must nominate B. However, D would win agains A, which would increase the party leader's payoff in the current period. In the next period, $t + 1$, parties are given by $S_1 = \{A\}$ and $S_2 = \{B, C, D\}$. If, in this case party 2 selects either a candidate B or D, then the deviation at t would be optimal. The only deterrent to such a deviation is that C becomes the party leader and is able to move the parties to a situation in which C always wins, which can be the case when parties at a future point in time are given by $S_2 = \{A, C, D\}$ and $S_1 = \{B\}$. Of course, this requires players to be sufficiently patient. In other words, if β is too small, A can never win in a steady state equilibrium.

Intuitively, what happens is that party 2 at time period t could win by adopting the minority position on issue 2. However, this would lead extremists, who care primarily about issue 2, to join the party, and change its direction by changing the leadership. The final outcome, policy $(0, 1)$ is disliked by the original party leader of type B, however, these types have lost control over party 1 at $t + 1$, and in the long run they join party 2.

Conditions for existence of steady states in which the minority positions win are easier to specify. For example, if parties are $S_1 = \{A, C, D\}$ and $S_2 = B$ and C is the leader of party 1, it is immediate that C is nominated, wins and that we have a steady state. Note that C can be interpreted as a "median" type. That is, if voters were separately asked for the favorite position on each issue, 0 and 1 would be the majority position, which is the preferred policy of type C.

Similarly, if parties are $S_1 = \{B, C, D\}$, $S_2 = \{A\}$, and D is the party leader, then D is nominated, wins, and we have again a steady state. In this case D would be the "median" type.

In the one-dimensional case we have seen that parties and policies always converge to a steady state. This is no longer true when there is no Condorcet winner. That is, we can show that in some cases, parties as well policies oscillate, but never converge. We now state the formal result, which is proved in Lemma 2 and 3 in the Appendix.

Proposition 7

1. *A steady state equilibrium in which candidate A wins can only exist if β is sufficiently close to 1, if $m_i(\{A, C\}) = A$, $m_{-i}(\{B, D\}) = B$, $m_{-i}(\{B, C.D\}) = C$, and if there exists another steady state in which C wins.*

2. *If $m_i(\{B, C, D\}) = C$ for party i, then there exists a steady state in which C wins. If $m_i(\{A, C, D\}) = D$ then there exists a steady state in which D wins.*

3. *For any β there exist choices of the leadership functions m_i, and initial party allocations such*

that policies and parties do not converge to a steady state.

6 Conclusion

The objective of this paper is to introduce a model that distinguishes between parties and their candidates, creating roles for both the party's size and party leaders' preferences for policy. These roles have been separately emphasized in the existing literature, starting with the seminal contributions of Downs (1957) and Wittman (1973). Formally, the model is a dynamic game, in which the state in each period is given by the allocation of individuals into parties.

We show that policy divergence is natural result of the model, generated by the fact that parties may not be able to choose a candidate that perfectly targets the median voter. The model also predicts that the winning policies overshoot changes in the media voter's position. As a consequence, policies are more volatile than the the median voter. The model also distinguishes between extremisms of parties and candidates, and shows that when parties become more extreme, more seats should be contested.

Finally, we apply the model to situations in which no Condorcet winner exists. Unlike existing models, we derive sharp predictions about equilibria, and we show that candidates may be more extreme than both party members and party leaders. In such a case, political competition may lose its moderating influence on policy.

7 Appendix

Proof of Proposition 1. For $T < \infty$ the result follows immediately. Suppose that time is infinite. Because Θ is finite, the number of players in the game remains finite. Theorem 2 of Jaśkiewics and Nowak (2016) or the related theorem in Mertens and Parthasarathy (2003) implies existence if the transition rule satisfies norm continuity. The transition rule in our game is deterministic and given by function ψ. However, since the action space of the game, i.e., the set of candidate types Θ, is finite, norm continuity is immediate. ■

Proof of Lemma 1. Let x and y denote the policies that would be implemented by the candidates of parties 1 and 2, respectively. We can assume that $x \neq y$.

Suppose that there exist $\theta_1 < \theta_2 < \theta_3$ with $u_{\theta_i}(x) - u_{\theta_i}(y) \geq 0$ for $i = 1, 3$ and $u_{\theta_2}(x) - v_{\theta_2}(y) < 0$. Then the function $f(z) = u_z(y) - v_z(x)$ assumes a local minimum at some point $\bar{z}$ in the open interval (θ_1, θ_3). As consequence, $0 = f'(z) = \frac{\partial u_z(y)}{\partial \theta} - \frac{\partial u_z(x)}{\partial \theta}$. This, however, is a contradiction because $\frac{\partial u_z(x)}{\partial \theta}$ is strictly monotone in x, which together with the fact that $x \neq y$ implies that $\frac{\partial u_z(y)}{\partial \theta} \neq \frac{\partial u_z(x)}{\partial \theta}$.

As a consequence, there are three cases:

1. $u_\theta(x) - u_\theta(y) > 0$ for all θ,

2. $u_\theta(x) - u_\theta(y) < 0$ for all θ,

3. There exits a unique $\bar{\theta}$ such that $u_{\bar{\theta}}(x) - u_{\bar{\theta}}(y) = \alpha$.

In the first case, all types prefer party 2, and in the second case all types prefer party 1, and we clearly have the desired structure. In the third case, define $s = \bar{\theta}$. Because $\bar{\theta}$ is unique if follows that the set $\{\theta | u_\theta(x) - u_\theta(y) \geq 0\}$ is an interval of the form $[0, s]$ or $[s, 1]$, and similar for $\{\theta | u_\theta(x) \geq u_\theta(y)\}$.
■

Proof of Proposition 2. By assumption each type θ has a unique most preferred policy $x(\theta)$. Further, if $\theta \neq \theta'$ then the single crossing property implies that $x(\theta) < x(\theta')$ for $\theta < \theta'$.

In particular if $y = x(\theta)$ is the individuals optimal policy then $\frac{\partial u_\theta(x)}{\partial x} = 0$. By assumption $\frac{\partial^2 u_\theta(x)}{\partial x \partial \theta} > 0$. As a consequence, $\frac{\partial u_{\theta'}(x)}{\partial x} > 0$. Since utility is strictly concave it follows that $\frac{\partial u_{\theta'}(y)}{\partial x} = 0$ for some $y > 0$.

Without loss of generality we can therefore rescale Θ such that type θ's most preferred policy is $\theta = x(\theta)$. After this rescaling it is possible that Θ is a strict subset of $[0, 1]$. We can define arbitrary preferences on $[0, 1] \setminus \Theta$, because types in $[0, 1] \setminus \Theta$ are never picked as party leaders and therefore do not affect the game.

Suppose that at $t = 2$ parties are given by $S_1 = [0, s]$ and $S_2 = [s, 1]$, and let $\theta_{m,2}$ be the median type at $t = 2$. There are two cases to consider:

Case 1: $\theta_{m,2} \leq s$.

If party 2 competes, then party 1 selects candidate $\theta_1 = \max\{m_1(s), 2\theta_{m,2} - s\}$; party 2 selects candidate $\theta_2 = s$. The median voter at $\theta_{m,2}$ and all types $\theta < \theta_{m,2}$ prefer candidate 1. Hence, electing candidate 1 is an equilibrium.

If party 2 does not compete then party 1 is given by $[0, 1]$. Monotonicity of m_1 therefore implies that $m_1(1) \geq m_1(s)$. If the other party does not compete then the party leader $m_1(1)$ will nominate a candidate with at $\tilde{\theta} = m_1(1)$. Assumption 2 implies $\tilde{\theta} = m_1(1) \leq \theta_{m,2} \leq m_2(s)$. Thus, party 2 will exit if $\theta_1 < \tilde{\theta}$. The winning candidate at $t = 2$ is therefore given by $\bar{\theta} = \max\{m_1(1), 2\theta_{m,2} - s\}$. The winning policy, $x_2(s)$ at time $t = 2$ is therefore an increasing, continuous function of s.

Case 2: $\theta_{m,2} \geq s$.

In this case party 2 wins. The analysis is analogous to that of case 1. Specifically, the winning candidate is given by $\bar{\theta} = \min\{m_2(0), 2\theta_{m,2} - s\}$, where $m_2(0)$ is the leader of party 2 if party 1 does not compete.

If follows immediately that the equilibrium at each subgame at $t = 2$ is unique.

Now consider the first stage of the game. Suppose that both parties compete and parties are given by $[0, s]$ and $[s, 1]$. Assume without loss of generality that the median voter at time $t = 1$ satisfies $\theta_{m,1} \leq s$. Then party 1 wins if $\theta_1 \geq 2\theta_{m,2} - s$. As a result, the party leader, type $\theta = m_1(s)$ solves

$$\max_{x \in [0,s]} u_\theta(x) + \beta u_\theta\left(2\theta_{m,2} - \frac{x + s}{2}\right) \text{ s.t. } x \geq 2\theta_{m,1} - s. \tag{15}$$

The second derivative of the object with respect to x is

$$u''_\theta(x) + \frac{\beta}{4} u''_\theta\left(2\theta_{m,2} - \frac{x + s}{2}\right) < 0.$$

Hence, (15) has a unique solution, x_1, which is party 1's equilibrium strategy. It is not optimal for party 2 to deviate, because this would raise the party cutoff at $t = 2$ which, as shown above, lowers or at best keeps the policy at $t = 2$ the same.

Finally, if party 2 does not compete, then party 1 will nominate a candidate at $m_1(1)$ in both periods. Party 2 will therefore not compete if the payoff $u_\theta(m_1(1)) + \beta u_\theta(m_1(1))$ strictly exceeds the payoff from competing, where $\theta = m_2(s)$. Hence, an equilibrium exists.

We next prove uniqueness.

Given that the solution of (15) is unique for any s, it is sufficient to prove that there does not exist an equilibrium in the subgame starting after party 2's decision to compete in which party 2's candidate $\theta_2 > s$.

If party 2 deviates by changing θ_2, then as show above the winner in the second period changes unless $\bar{\theta} = m_1(1)$ or $\bar{\theta} = m_2(0)$, depending on whether party 1 or 2 wins. Given that local changes of θ_2 do not change the identity of the winning candidate at $t = 2$, the same is true for small changes of the type of party 1's candidate.

Next, note that the constraint of optimization problem 15 must be slack. Else, lowering θ_2 would result in party 1 winning the election. Thus, the leader of party 1's optimal candidate is locally unconstrained, and therefore $\theta_1 = m_1(s)$. However, this means that party 2 would be strictly better off not competing, as this would result in a candidate of type $\theta = m_1(1) > m_1(s) = 1$ winning. This contradiction proves that there does not exist an equilibrium in a subgame starting at $t = 1$ with both parties competing, where the losing party selects a candidate $\theta \neq s$. Hence equilibrium payoffs are unique.

Finally, note that the only case in which strategies are not unique is when one of the parties is indifferent between competing and not competing. ■

Proof of Proposition 3. First, suppose that $\theta_{m,1} < s_1 < (\theta_{m_1} + \theta_{m,2})/2$. If party 2 does not compete, the winning policy is $m_1(1)$. If party 2 competes, then the equilibrium policy solves problem 8. Thus, $x_1 \geq 2\theta_{m,1} - s_1 > m_1(1)$, where the last inequality follows because by assumption $m_1(1) + s_1 < 2\theta_{m,1}$. Thus, it is optimal for party 2 to compete, and party 1 wins with a policy $x_1 \leq s_1$. The winning policy in the next period is $x_2 = h(x_1) = 2\theta_{m,2} - 0.5(x_1 + s_1) \geq 2\theta_{m,2} - s_1$. Thus, $|x_1 - x_2| = x_2 - x_1 \geq 2(\theta_{m,2} - s_1) > \theta_{m,2} - \theta_{m,1}$, because $s_1 < (\theta_{m_1} + \theta_{m,2})/2$.

To consider the remaining case, we determine the first-order condition for policy x_1.

Suppose that at either $s_2 < 2\theta_{m,2} - m_2(0)$ or $s_2 > 2\theta_{m_2} - m_1(1)$. Then $\frac{\partial h(0.5(x_1+s_1))}{\partial x_1} = 0$. The objective or problem 8 is strictly concave because $u'' < 0$.

If the constraint is slack then $u'_\theta(x_1) = 0$, which implies $x_1 = \theta$ (recall that we assume $u'_\theta(\theta) = 0$). The constraint must be satisfied and hence $m_1(s_1) = \theta = x_1 \geq 2\theta_{m,1} - s_1$. This, and monotonicity of m_1 (from assumption 2) imply $m_1(1) \geq 2\theta_{m,1} - s_1$. This contradicts the assumption that $m_1(1) + s_1 <$

$2\theta_{m,1}$.

Thus, $2\theta_{m,2} - m_2(0) \leq s_1 \leq 2\theta_{m_2} - m_1(1)$ and hence $(\partial/\partial x_1)h(0.5(s_1 + x_1)) = -0.5$. Further, we must have $\theta = m_1(s_1) < x_1$. The derivative of the object of problem 8 is therefore

$$-\gamma(x_1 - \theta)^{\gamma-1} + \frac{\gamma\beta}{2}\left|2\theta_{m,2} - \frac{x_1 + s_1}{2} - \theta\right|\left(2\theta_{m,2} - \frac{x_1 + s_1}{2} - \theta\right)^{\gamma-2}. \tag{16}$$

If follows immediately that the second derivative is strictly negative.

Note that if $x_1 < \frac{1}{3}(2\theta_{m,2} + 2\theta_{m,1} - s_1)$, then $x_2 > \frac{1}{3}(5\theta_{m,2} - \theta_{m,1} - s_1)$ and $|x_1 - x_2| > |\theta_{m,2} - \theta_{m,1}|$. It is therefore sufficient to prove that (16) is strictly negative at $x_1 = \frac{1}{3}(2\theta_{m,2} + 2\theta_{m,1} - s_1)$, i.e., that

$$-\gamma\left(\frac{2}{3}\theta_{m,2} + \frac{2}{3}\theta_{m,1} - \frac{1}{3}s_1 - \theta\right)^{\gamma-1} + \frac{\gamma\beta}{2}\left(\frac{5}{3}\theta_{m,2} - \frac{1}{3}\theta_{m,1} - \frac{1}{3}s_1 - \theta\right)^{\gamma-1} < 0. \tag{17}$$

Note that (17) holds if and only if

$$\left(\frac{2}{3}\theta_{m,2} + \frac{2}{3}\theta_{m,1} - \frac{1}{3}s_1 - \theta\right) > \left(\frac{\beta}{2}\right)^{\frac{1}{\gamma-1}}\left(\frac{5}{3}\theta_{m,2} - \frac{1}{3}\theta_{m,1} - \frac{1}{3}s_1 - \theta\right). \tag{18}$$

Next, $\theta \leq m_1(1) < 2\theta_{m,1} - s_1$. Thus, (18) holds for any such θ if

$$\left(\frac{2}{3}\theta_{m,2} - \frac{4}{3}\theta_{m,1} + \frac{2}{3}s_1 - \theta\right) \geq \left(\frac{\beta}{2}\right)^{\frac{1}{\gamma-1}}\left(\frac{5}{3}\theta_{m,2} - \frac{7}{3}\theta_{m,1} - \frac{1}{3}s_1 - \theta\right). \tag{19}$$

Finally, if (19) holds for $s = 0.5(\theta_{m,1} + \theta_{m,2})$ then it holds for any $s \geq 0.5(\theta_{m,1} + \theta_{m,2})$. As a consequence, (19) holds if

$$(\theta_{m,2} - \theta_{m,1}) \geq \left(\frac{\beta}{2}\right)^{\frac{1}{\gamma-1}}(2\theta_{m,2} - 2\theta_{m,1}). \tag{20}$$

Thus, (20) holds if $\beta \leq 2^{2-\gamma}$, in which case $|x_1 - x_2| > |\theta_{m,1} - \theta_{m,2}|$.

Finally, suppose that $\theta_{m,1} < \theta_{m,2} < s_1$. If we set (16) equal to zero, then

$$x_1 - \theta = \left(\frac{\beta}{2}\right)^{\frac{1}{\gamma-1}}\left(2\theta_{m,2} - \frac{x_1 + s_1}{2} - \theta\right). \tag{21}$$

If $\gamma \to \infty$ then (21) implies that $x_1 \to (4\theta_{m,2} - s_1)/3$.

Next, we show that the constraint of problem 8 is slack for large γ. It is sufficient to show that $(4\theta_{m,2} - s_1)/3 > 2\theta_{m,1} - s_1$. This inequality is equivalent to $2\theta_{m,2} > 3\theta_{m,1} - s_1$, which holds because $s_1 > \theta_{m,1}$ and $\theta_{m,1} < \theta_{m,2}$. Because $s_1 > \theta_{m,2}$ it also follows that $(4\theta_{m,2} - s_1)/3 < s_1$. Thus, for large γ the solution to problem 8 is characterized by (21). It is also optimal for party 2 to compete, otherwise the equilibrium policy becomes $m_1(1)$ and by assumption $m_1(1) < 2\theta_{m,1} - s_1$, which we have shown to be strictly less than x_1. Thus, x_1 given by (21) is the equilibrium policy.

Finally, it follows that $h((4\theta_{m,2} - s_1)/3) = 4(\theta_{m,2} - s_1)/3$. Therefore $|x_1 - x_2| \to 0$ as $\gamma \to \infty$. ■

Proof of Proposition 4. Without loss of generality consider the case where party 1 wins in the first period. If function h defined in (7) is locally independent of s, then problem 8 implies that $x_1 = \theta = m_1(s_1)$, i.e., the party chooses its most preferred candidate in the current period. In the second period, party 1 either chooses again the leader's most preferred candidate or party 2 does not compete. In both cases this means that party 2 would be better off not competing and thus the winning policies do no change.

Now let $h(s) = 2\theta_{m,2} - s$. If the constraint of problem 8 binds, then $x_1 = 2\theta_{m,1} - s$. In this case, changing $m_1(s)$ does not change the solution. The same is true if $x_1 = s_1$.

Next, suppose that there exists an interior solution. Such an interior solution exists at least if $m_1(s) > 2\theta_{m,1} - s$. Let $\theta = m_1(s)$. Then the first order condition of problem 8 with a slack constraint is given by

$$v'(x - \theta) = \frac{\beta}{2} v' \left(2\theta_{m,2} - \frac{x + s_1}{2} - \theta \right), \tag{22}$$

where $v'(z) = \gamma z |z|^{\gamma - 2}$. In order for (22) to have a solution, $x - \theta$ and $2\theta_{m,2} - \frac{x+s_1}{2} - \theta$ must have the same sign. Both cases yield the same solution

$$x = \frac{\left(\frac{\beta}{2}\right)^{\frac{1}{\gamma-1}} (4\theta_{m,2} - 2\theta - s_1) + 2\theta}{\left(\frac{\beta}{2}\right)^{\frac{1}{\gamma-1}} + 2}. \tag{23}$$

Equation (23) implies that a more extreme party leader, i.e., a lower θ results in a more extreme policy in the current period. This, however, means that in the next period, the winning candidate's policy is more moderate. If party 1 wins again in the next period, which is the case if $\theta_{m,2} < (s_1 + x_1)/2$, then this means that party 1 must nominate a more moderate candidate. If party 2 wins at $t = 2$, then they will have a more partisan candidate.

■

Proof of Proposition 5. First suppose that the constraint of problem 8 is slack. Then x_1 is given by (23) and $x_2 = 2\theta_m - (x_1 + x_2)/2$. Because $\theta_{m,1} = \theta_{m,2} = 0.5$ and $\theta = \xi_1 s$, we get

$$x_1 = \frac{\left(\frac{\beta}{2}\right)^{\frac{1}{\gamma-1}} (2 - s_1(1 + 2\xi_1)) + 2\xi_1 s_1}{\left(\frac{\beta}{2}\right)^{\frac{1}{\gamma-1}} + 2}, \text{ and } x_2 = \frac{\left(\frac{\beta}{2}\right)^{\frac{1}{\gamma-1}} \xi_1 s_1 + 2 - (1 + \xi_1) s_1}{\left(\frac{\beta}{2}\right)^{\frac{1}{\gamma-1}} + 2}. \tag{24}$$

The constraint of problem 8 would bind if $x_1 \leq 1 - s_1$, i.e., if and only if

$$\xi_1 \leq \frac{2 - 2s_1 - \left(\frac{\beta}{2}\right)^{\frac{1}{\gamma-1}}}{2\left(1 - \left(\frac{\beta}{2}\right)^{\frac{1}{\gamma-1}}\right)s_1}. \tag{25}$$

Case 1: Condition (25) holds, i.e., the constraint of problem 8 binds.

In this case party 1's candidate is at positions $1 - s$ and 0.5 in periods $t = 1, 2$. If $\xi_1 = 0.5$ it is immediate that party 2 is at least as well off from competing in period $t = 2$, while party 2 is strictly better off competing if $\xi_1 < 0.5$.

Party 2 is better off competing in period $t = 1$ if

$$-(\theta - (1 - s_1))^\gamma - \beta(\theta - 0.5)^\gamma \geq -(1 + \beta)(\theta - \xi_1)^\gamma, \tag{26}$$

where $\theta = m_2(s_1) = 1 - \xi_2(1 - s_1)$ is the leader of party 2. Inequality (26) holds if and only if $\xi_1 = \bar{\xi}_1(s_1, \theta, \gamma)$, where

$$(\theta - (1 - s_1))^\gamma + \beta(\theta - 0.5)^\gamma = (1 + \beta)(\theta - \bar{\xi}_1(s_1, \theta, \gamma))^\gamma, \tag{27}$$

and hence

$$\bar{\xi}_1(s_1, \theta, \gamma) = \theta - \left(\frac{(\theta - (1 - s_1))^\gamma + \beta(\theta - 0.5)^\gamma}{1 + \beta}\right)^{\frac{1}{\gamma}}. \tag{28}$$

We first show that $(\partial/\partial s_1)\bar{\xi}_1(s_1, \theta, \gamma) < 0$. Recall that $\theta = 1 - \xi_2(1 - s_1)$. Thus,

$$\begin{aligned}\frac{\partial\bar{\xi}_1(s_1, \theta, \gamma)}{\partial s_1} &= \xi_2 - \frac{1}{(1 + \beta)^{\frac{1}{\gamma}}}\frac{(1 + \xi_2)(\theta - (1 - s_1))^{\gamma-1} + \beta\xi_2(\theta - 0.5)^{\gamma-1}}{\left((\theta - (1 - s_1))^\gamma + \beta(\theta - 0.5)^\gamma\right)^{\frac{\gamma-1}{\gamma}}} \\ &= \xi_2 - \frac{(1 + \xi_2)(\theta - (1 - s_1))^{\gamma-1} + \beta\xi_2(\theta - 0.5)^{\gamma-1}}{(1 + \beta)(\theta - \bar{\xi}_1(s_1, \theta, \gamma))^{\gamma-1}} \\ &< \xi_2 - \frac{(1 + \xi_2)(\theta - (1 - s_1))^{\gamma-1}}{(1 + \beta)(\theta - \bar{\xi}_1(s_1, \theta, \gamma))^{\gamma-1}} \leq \xi_2 - \frac{1 + \xi_2}{1 + \beta} < 0,\end{aligned} \tag{29}$$

where the second inequality follows because $\bar{\xi}_1 > 1 - s_1$, and that last inequality because $\beta\xi_2 < 1$.

Next, we show that $\bar{\xi}_1$ is non-increasing in ξ_2. Because θ is strictly decreasing in ξ_2, it is sufficient to show that $(\partial/\partial\theta)\bar{\xi}_1(s_1, \theta, \gamma) \geq 0$.

Note that

$$\begin{aligned}\frac{\partial\bar{\xi}_1(s_1, \theta, \gamma)}{\partial\theta} &= 1 - \frac{1}{(1 + \beta)^{\frac{1}{\gamma}}}\frac{(\theta - (1 - s_1))^{\gamma-1} + \beta(\theta - 0.5)^{\gamma-1}}{\left((\theta - (1 - s_1))^\gamma + \beta(\theta - 0.5)^\gamma\right)^{\frac{\gamma-1}{\gamma}}} \\ &= 1 - \frac{\frac{1}{1+\beta}(\theta - (1 - s_1))^{\gamma-1} + \frac{\beta}{1+\beta}(\theta - 0.5)^{\gamma-1}}{(\theta - \bar{\xi}_1(s_1, \theta, \gamma))^{\gamma-1}}\end{aligned} \tag{30}$$

Let $0 \leq \alpha \leq 1$ and $A, B \geq 0$. Then Hölder's inequality implies that

$$\left(\alpha A^{\gamma-1} + (1-\alpha)B^{\gamma-1}\right)^{\frac{1}{\gamma-1}} \leq \left(\alpha A^{\gamma} + (1-\alpha)B^{\gamma}\right)^{\frac{1}{\gamma}}. \tag{31}$$

In particular, Hölder's inequality states that

$$E[|XY|] \leq \left(E[|X|^{p}\right)^{\frac{1}{p}} \left(E[|Y|^{q}\right)^{\frac{1}{q}}, \tag{32}$$

for any random variables X and Y, where $\frac{1}{p} + \frac{1}{q} = 1$. Substituting $X = Z^{\gamma-1}$, $Y = 1$, $p = \gamma/(\gamma-1)$ and $q = 1/\gamma$ into (32) yields

$$E[|Z|^{\gamma-1}] \leq \left(E[|Z|^{\gamma}]\right)^{\frac{\gamma-1}{\gamma}},$$

which implies

$$\left(E[|Z|^{\gamma-1}]\right)^{\frac{1}{\gamma-1}} \leq \left(E[|Z|^{\gamma}]\right)^{\frac{1}{\gamma}},$$

Substituting for Z the random variables that takes values A and B with probabilities α and $1-\alpha$, respectively, implies. (31).

Substituting $\alpha = 1/(1+\beta)$, $A = \theta - (1-s_1)$, and $B = \theta - 0.5$ into (31), and using (27) yields

$$\begin{aligned} \frac{1}{1+\beta}(\theta-(1-s_1))^{\gamma-1} + \frac{\beta}{1+\beta}(\theta-0.5)^{\gamma-1} &\leq \left(\frac{1}{1+\beta}(\theta-(1-s_1))^{\gamma} + \frac{\beta}{1+\beta}(\theta-0.5)^{\gamma}\right)^{\frac{\gamma-1}{\gamma}} \\ &= \left(\theta - \bar{\xi}_1(s_1,\theta,\gamma)\right)^{\gamma-1}. \end{aligned} \tag{33}$$

which implies that (30) is non negative. Hence, $(\partial/\partial\theta)\bar{\xi}_1(s_1,\theta,\gamma) \geq 0$. Also note $(\partial/\partial\theta)\bar{\xi}_1(s_1,\theta,\gamma)$ is strictly positive given that Hölders inequality can be strict.

Case 2: Condition (25) is violated and $x_2 \leq \xi_1$.

If condition (25) is violated then x_1 is given by (24) If $x_2 < \xi_1$ then party 2 does not compete in the second period. Thus, party 2 will compete in the first period if and only if $x_2 \geq \xi_1$. Thus,

$$\xi_1 \leq \frac{\left(\frac{\beta}{2}\right)^{\frac{1}{\gamma-1}}(2-s_1)}{\left(\frac{\beta}{2}\right)^{\frac{1}{\gamma-1}}(1+2s_1)+2-2s_1}. \tag{34}$$

It is immediate that increasing s_1 tightens constraint (34) (as long as the right-hand side of (34) is non-negative). Further, changing ξ_2 does not affect the constraint.

Case 3: Condition (25) is violated and $x_2 > \xi_1$.

It is better for party 2 to compete if

$$-\frac{(\theta-x_1)^{\gamma}}{1+\beta} - \frac{\beta\left(\theta-1+\frac{x_1+s_1}{2}\right)^{\gamma}}{1-\beta} \geq -(\theta-\xi_1)^{\gamma}, \tag{35}$$

where x_1 is given by (24).

Note that

$$\frac{\partial x_1}{\partial \xi_1} = \frac{2s_1\left(1-\left(\frac{\beta}{2}\right)^{\frac{1}{\gamma-1}}\right)}{2+\left(\frac{\beta}{2}\right)^{\frac{1}{\gamma-1}}}.$$

Hence $0 < \frac{\partial x_1}{\partial \xi_1} < 1$. The derivative of the left-hand side of (35) with respect to ξ_1 is therefore

$$\begin{aligned}\left(\frac{\gamma(\theta-x_1)^{\gamma-1}}{1+\beta} - \frac{\beta\left(\theta-1+\frac{x_1+s_1}{2}\right)^{\gamma-1}}{2(1+\beta)}\right)\frac{\partial x_1}{\partial \xi_1} &< \left(\frac{\gamma(\theta-x_1)^{\gamma-1}}{1+\beta} + \frac{\beta\left(\theta-1+\frac{x_1+s_1}{2}\right)^{\gamma-1}}{1+\beta}\right)\frac{\partial x_1}{\partial \xi_1} \\ < \left(\frac{\gamma(\theta-x_1)^{\gamma}}{1+\beta} + \frac{\beta\left(\theta-1+\frac{x_1+s_1}{2}\right)^{\gamma}}{1+\beta}\right)^{\frac{\gamma-1}{\gamma}} \frac{\partial x_1}{\partial \xi_1} &\leq \gamma(\theta-\xi_1)^{\gamma-1}\frac{\partial x_1}{\partial \xi_1} < \gamma(\theta-\xi_1)^{\gamma-1},\end{aligned} \tag{36}$$

where the third inequality follows from (31). Inequality (36) shows that the derivative of the left-hand side of (35) is strictly less than the derivative of the right-hand side. Thus, if (36) holds for some $\xi_1 > 0$, it also holds for all $0 \leq \xi_1' < \xi_1$.

It follows immediately that (35) is satisfied for $\xi_1 = 0$. Thus, there either exists a unique value $\bar{\xi}$ for which (25) does not hold and

$$(\theta-x_1)^{\gamma} + \beta\left(\theta-1+\frac{x_1+s_1}{2}\right)^{\gamma} = (1+\beta)(\theta-\bar{\xi}_1(s_1,\theta))^{\gamma}, \tag{37}$$

or it is better for party 2 to compete for all ξ_1 for which (25) does not hold. In this latter case we define $\bar{\xi}_1$ such that (25) holds with equality, i.e.,

$$\bar{\xi}_1 = \frac{2-2s_1-\left(\frac{\beta}{2}\right)^{\frac{1}{\gamma-1}}}{2\left(1-\left(\frac{\beta}{2}\right)^{\frac{1}{\gamma-1}}\right)s_1}. \tag{38}$$

If $\bar{\xi}_1$ is defined by (38) then it follows immediately that $\frac{\partial \bar{\xi}_1}{\partial s_1} < 0$, i.e., raising s_1 decreases the set of ξ_1 for which party 2 is willing to compete. Further, party 2's ideal point $\theta = m_2(s_1)$ does not affect $\bar{\xi}_1$.

Now consider the case where $\bar{\xi}_1$ is defined by (37). To show that $\bar{\xi}_1$ is non-increasing in ξ_2, it is again sufficient to show that $\frac{\partial \bar{\xi}_1}{\partial \theta} \geq 0$.

If (37) is satisfied then (35) holds with equality. Taking the derivative of the left-hand side of (35) with respect to θ and using (31) implies

$$\begin{aligned}-\gamma\frac{(\theta-x_1)^{\gamma-1}}{1+\beta} - \gamma\frac{\beta\left(\theta-1+\frac{x_1+s_1}{2}\right)^{\gamma-1}}{1-\beta} &\geq -\gamma\left(\frac{(\theta-x_1)^{\gamma}}{1+\beta} + \frac{\beta\left(\theta-1+\frac{x_1+s_1}{2}\right)^{\gamma}}{1-\beta}\right)^{\frac{\gamma-1}{\gamma}} \\ &= -\gamma(\theta-\bar{\xi}_1)^{\gamma-1}\end{aligned}$$

Thus, the derivative of left-hand side of (35) is strictly greater than that of the right-hand side. Raising θ therefore slackens the constraint. As we have shown above, lowering ξ_1 also slackens (35). To equalize to the sides of the inequality, we must therefore raise $\bar{\xi}$. Thus, raising $\bar{\xi}$ is non-decreasing θ, i.e., $\frac{\partial \bar{\xi}_1}{\partial \theta} \geq 0$.

Finally, the change of $\bar{\xi}_1$ when s_1 is ambiguous unless $\frac{\partial x_1}{\partial s_1}$ is sufficiently negative.

Note that

$$\frac{\partial x_1}{\partial s_1} = \frac{2\xi_1 - (1 + 2\xi_1)\left(\frac{\beta}{2}\right)^{\frac{1}{\gamma-1}}}{2 + \left(\frac{\beta}{2}\right)^{\frac{1}{\gamma-1}}},$$

and hence the derivative is negative if β or γ are large.

It also follows that $\frac{\partial x_1}{\partial s_1} + 1 > 0$. Suppose that $\frac{\partial x_1}{\partial s_1} \leq 0$. Keep θ fixed. Then the derivative of the left-hand side of (35) with respect to s_1 is

$$\frac{\gamma(\theta - x_1)^{\gamma-1}}{1+\beta} \frac{\partial x_1}{\partial s_1} - \frac{\beta\left(\theta - 1 + \frac{x_1+s_1}{2}\right)^{\gamma-1}}{2(1+\beta)} \left(\frac{\partial x_1}{\partial s_1} + 1\right) < 0 \tag{39}$$

Thus, in order for (35) to hold with equality, ξ_1 must decrease. Thus, $\bar{x}_1$ is decreasing in s_1. However, as s_1 is increased, $\theta = 1 - \xi_2(1 - s_1)$ increases. We have shown the increasing θ increases $\bar{x}_1$. The overall effect is therefore only negative if $\frac{\partial x_1}{\partial s_1}$ is sufficiently small, i.e., if β and γ are small.
■

Proof of Proposition 6. Note that $m_1(s_t) = \xi_1 s$ implies that $\bar{s}_1 = 2\theta_m/(1 + \xi_1)$ solves $2\theta_m - \bar{s}_1 = m_1(\bar{s}_1)$. Since $\xi_1 < 1$ it follows that $\bar{s}_1 > \theta_m$. If $s \in [\theta_m, \bar{s}_1]$ then the new party cutoff is θ_m, and in equilibrium both candidates nominate $x = \theta_m$ in any subsequent period.

Let $\bar{s}_k = 2^k\theta_m/(1 + \xi_1)^k$. Then $\bar{s}_k$ satisfies $m_1(\bar{s}_k) + \bar{s}_k = 2\bar{s}_{k-1}$ for $k > 1$. Further, there exists K such that $\bar{s}_K \geq 1$.

Let $\bar{s}_0 = \theta_m$. If the party cutoff $s \in [\bar{s}_{k-1}, \bar{s}_k]$ and both parties follow the proposed strategies, then the party cutoff in the next period is in $[\bar{s}_{k-2}, \bar{s}_k]$ if $k > 1$ or $s = \theta_m$ if $k = 1$. Thus, the equilibrium policies converge to θ_m after at most K time periods. For $\theta_m = 0.5$ it follows that $\bar{s}_3 \geq 1$ for all $\xi_1 < 0.5$. Thus, it takes at most three periods to converge to θ_m as the equilibrium policy.

We next show that it is not optimal for party 1 to deviate, i.e., (11) holds. If $s_t \in [\bar{s}_0, \bar{s}_1]$ this follows immediately. In particular, for any x with $2\theta_m - s \leq x \leq \theta_m$ it follows that the party cutoff s' at $t + 1$ is again in the interval $[\bar{s}_0, \bar{s}_1]$. Raising x by some amount δ at t lowers current utility by

δ, but raises the party cutoff to 0.5δ. This in turn, lowers the equilibrium policy and raises utility in the next period by only 0.5δ. Hence, a deviation is not optimal.

Now suppose that $s_t \in [\bar{s}_1, \bar{s}_2]$ for $k > 1$. Changing x by δ in the current period would lower utility by δ because the equilibrium is at the party leader's ideal point $x = m_1(s)$. We show that the benefits of such a deviation in future periods are strictly less than δ. In particular the new party cutoff s'_{t+1} differs from s_{t+1} by $\delta/2$. Note that $s'_{t+1} \in [\bar{s}_0, \bar{s}_2]$. If $s'_{t+1} \in [\bar{s}_0, \bar{s}_1]$ then the equilibrium policy changes by $\delta/2$ and hence the maximum possible gain to the leader of party 1 at t is $\beta\delta/2 < \delta$, i.e., the deviation is not optimal. If $s'_{t+1} \in [\bar{s}_1, \bar{s}_2]$ then the equilibrium policy changes by $\xi_1\delta/2$. At $t+2$ the party cutoff s'_{t+2} differs from the proposed equilibrium cutoff s_{t+2} by $\delta/4 + \xi_1\delta/4$. Given that $s_{t+2} \in [\bar{s}_0, \bar{s}_1]$ it follows that the equilibrium policy changes by the same amount. Thus, the possible gains at $t + 1$ and $t + 2$ do not exceed the initial loss of the deviation at t if

$$\frac{\beta\xi_1}{2} + \frac{\beta^2(1 + \xi_1)}{4} \leq 1. \tag{40}$$

Finally, suppose that $s_t \in [\bar{s}_2, 1]$ and that the policy at t is changed by δ, again resulting in a utility loss of δ. If as a result of the deviation the new party cutoff $s'_{t+1} \in [\bar{s}_0, \bar{s}_2]$ then the argument is the same as above. Thus consider the case where $s_{t+1} > \bar{s}_2$. As a result s'_{t+1} differs from s_{t+1} by $\delta/2$ and the policy is changed by $\xi_1\delta/2$. The party cutoff at $t + 2$ is changed by $\delta/4 + \xi_1\delta/4$ and the policy by $\delta\xi_1/4 + \xi_1^2\delta/4$. Finally at $t + 3$ the party cutoff after the deviation, s'_{t+3} differs from s_{t+3} by $\delta/8 + \xi_1\delta/4 + \xi_1^2\delta/8$, which is the same as the move change of the policy at $t + 3$. Hence the possible gains at $t + 1$, $t + 2$, and $t + 3$ do not exceed the initial loss of the deviation at t if

$$\frac{\beta\xi_1}{2} + \frac{\beta^2(1 + \xi_1)}{4} + \frac{\beta^3(1 + \xi_1)^2}{8} \leq 1. \tag{41}$$

It follows immediately that (40) and (41) hold for any $\xi_1 \leq 0.5$ and any $\beta \leq 1$. Thus, it is not optimal for party 1 to deviate, i.e., (11) holds.

Next, we must show that it is not optimal for party 2 to deviate, i.e., (13) holds.

It follows immediately that a deviation by party 2 at $s_t \in [\bar{s}_0, \bar{s}_1]$ to some policy $y > s_t$ is not optimal. The deviation has no impact on the current policy, but changes the policy in the next period from θ_m to $\theta_m - \delta/2$, making party 2 worse off. Now suppose that $s_t \in [\bar{s}_1, \bar{s}_2]$. If after the deviation, $s'_{t+1} \in [\bar{s}_0, \bar{s}_1]$ then the resulting policy is shifted by $\delta/2$ to the left, making party 2 worse off. Thus, assume that $s'_{t+1} > \bar{s}_1$. If $s'_{t+1} \in [\bar{s}_1, \bar{s}_2]$ then the policy at $t + 1$ is moved to the right by $\xi_1\delta/2$. At $t+2$ the cutoff s'_{t+2} differs from the proposed equilibrium cutoff s_{t+2} by $\delta/4 + \xi_1\delta/4$, and the policy is shifted to the left by this amount, making party 2 worse off. Hence, the deviation is not optimal if

$$2\xi_1 \leq \beta(1 + \xi_1). \tag{42}$$

Now suppose that $s'_{t+1} > \bar{s}_2$. As in the previous case, the policy at $t + 1$ is moved to the right by $\xi_1\delta/2$ and the cutoff at $t+2$ is moved by $(1+\xi_1)\delta/4$. As a consequence, the policy at $t+2$ is moved to the right by $(1+\xi_1)\xi_1\delta/4$. The party cutoff at $t+3$ changes by $(1+\xi_1)\delta/8+(1+\xi_1)\xi_1\delta/8$, which is the amount by which the winning policy is moved to the left in period $t + 3$. The benefit of the deviation to party 2 in $t + 1$ and $t + 2$ is therefore dominated by the loss in period $t + 3$ if

$$4\xi_1 + 2\beta(1 + \xi_1)\xi_1 \leq \beta^2(1 + \xi_1)^2. \tag{43}$$

Equations (42) and (43) are both satisfied if and only if

$$\beta \geq \frac{\xi_1 + \sqrt{\xi_1^2 + 4\xi_1}}{1 + \xi_1}. \tag{44}$$

The right-hand side of (44) is greater or equal to one if $\xi_1 \geq \sqrt{5} - 2$. However, for all such values ξ_1 it follows that $\bar{s}_2 > 1$. Hence, it takes only two periods to converge. This, however, means that (42) is sufficient to prevent deviations by party 2.

Finally, party 2's payoff from competing must exceed the payoff from not competing. It is sufficient to consider the case where $s_t = 1$. The resulting equilibrium policies are ξ_1 and $2\theta_m-\xi_1 = 1 - \xi_1$ and then convergence to $\theta_m = 0.5$ is obtained. Alternatively, if party 1 exits, the equilibrium policy is ξ_1 in all periods, which makes the leader of party 2 worse off. Thus, a pure strategy equilibrium with convergence to policy θ_m exists if (42) holds and $\bar{s}_2 \geq 1$. Both inequalities are equivalent to $\xi_i \leq \min\left\{\sqrt{5} - 2, \frac{\beta}{2-\beta}\right\}$.

■

Lemma 2 *Consider the model of section 5.3. A steady state equilibrium in which candidate A wins can only exist if β is sufficiently close to 1, if $m_i(\{A, C\}) = A$, $m_{-i}(\{B, D\}) = B$, $m_{-i}(\{B, C.D\} = C$, and if there exists another steady state in which C wins.*

Proof of Lemma 2. Consider the following eight different party configurations, which are the states of the game. In states $s_{1,i}$, $i = 1, 2$ parties are $S_i = \{A, C, D\}$, $S_{-i} = \{(B\}$; in states $s_{2,i}$ parties are $S_i = \{B, C, D\}$, $S_{-i} = \{A\}$; in states $s_{3,i}$ we have $S_i = \{A, B\}$, $S_{-i} = \{C, D\}$; and finally, in states $s_{4,i}$ we have $S_i = \{A, C\}$, $S_{-i} = \{B, D\}$.

We first show that there does not exist an equilibrium in which B is nominated by one of the parties and wins. It is only possible for B to win against D. Thus, the state must be $s_{3,1}$ or $s_{3,2}$ to support these choice of candidate in a steady state. Without loss of generality suppose that the state is $s_{3,1}$. Suppose that party 2 deviates at t by nominating C as candidate. Then the state in the next period, t' is $s_{1,2}$, i.e., party 2 consists of all types A, C and D. If C is nominated in all future period, then we would have reached a steady state. The original deviation at t would then optimal for all members of party 2. As a consequence, either A or D must be nominated. If D is nominated, then B wins an we are back in state $s_{3,1}$. However, the deviation leading to policy $(0, 1)$ in period t made all members of the party better off. Thus, assume that A is nominated. A wins against B, leading to policy $(0, 0)$. In the next period, the state is $s_{4,2}$, i.e., party 2 is $S_2 = \{A, C\}$.

Next, suppose that there exists a steady state in which type A is nominated by one of the parties, say party 1, and wins. Then the state must be either $s_{3,1}$ or $s_{4,1}$.

Suppose that the state is $s_{3,1}$. Thus, parties 1 and 2 must nominate candidates of type A and D. Suppose that party 2 deviates, by nominating a candidate of type D. Candidate D would win, leading to state $s_{2,2}$ in the next period. If $m_2(\{B, C, D\}) \in \{C, D\}$, then this deviation is optimal. In particular if D is the party leader, then we have a steady state with policy $(1, 1)$ implemented in every period, compared to $(0, 0)$ which makes both type C and D straitly better off. If the party leader is of type C, then C's payoff must be at least as good as that from receiving $(1, 1)$ in every period, otherwise, the party leader could nominated D. Further, type D is strictly better off with any policy that is not $(0, 0)$ in every period. Now suppose that $m_2(\{B, C, D\}) = B$. For type B policy $(1, 1)$ is the second ranked choice after $(1, 0)$. In order to get to a situation in which B can win, B must be paired against a candidate D. This is only possible in states $s_{3,1}$ and $s_{3,2}$. By assumption, candidates A and D are nominated in state $s_{3,1}$. Thus, we can assume that the state is $s_{3,2}$. The only way to get to state $s_{3,2}$ from $s_{2,2}$ is via state $s_{1,2}$, i.e., where party 2 consists of types A, C and D. Party 2 would have to nominate candidate D who would lose against B. In state $s_{3,2}$ candidates D and B would have to be again nominated. However, note that party 2 can ensure that policy $(0, 1)$ is a steady state by nominating a candidate of type C. This would make members of party 2 strictly better off, and hence state $s_{3,2}$ cannot be reached. This proves that a state $s_{3,1}$ cannot support a steady state in which A wins.

Now suppose that the state is $s_{4,1}$. In order for a type A candidate to win, party 2 must nominate at type B candidate. Suppose that the leader of party 2 is of type D. Note that party 2 could win by nominating a type D candidate. The state in the following period would be $s_{2,2}$. Independent of the identity of the leader of party 2, the type D is better off then receiving $(0, 0)$ in all periods, and hence it would be optimal to deviate. Thus, assume that B is the leader of party 2 in state $s_{4,1}$.

Note that B strictly prefers $(1, 1)$ to $(0, 0)$, but is strictly worse off if the policy is $(0, 1)$, i.e., if a candidate of type C wins. This is only possible if $m_2(\{B, C.D\} = C$ and C is nominated in next period, t'. In this case C loses against A and the resulting state is $s_{3,1}$. As shown above, we cannot have a steady state with candidates A and C being nominated. Further, if candidates are B and D are nominated then B would win, and we would have a steady state. However, in this case it would have been better at t' for type C to nominate type B, who would would have won. Thus, there would have to be a mixed strategy equilibrium in state $s_{3,1}$. If types C and B are nominated, then we would get to state $s_{1,2}$, i.e., party 2 is consists of types $\{A, C, D\}$. Further, party 2 must nominate C, else, it wold have been better for type C at time t' to nominate a type B candidate. Hence, $s_{3,1}$ is a steady state. As a consequence, type A can win in equilibrium if the discount rate β is not too small, and if both $m_2(\{B, D\}) = B$ and $m_2(\{B, C.D\} = C$. However, in this case there also exists a steady state in which type C wins with policy $(0, 1)$. ■

Lemma 3 *For any β there exist choices of m_i such that neither parties nor policies converge to a steady state.*

Proof of Lemma 3. Let state $s_{1,1}$ and $s_{4,1}$ defined as in the proof of Lemma 2, i.e., in state $s_{1,1}$ parties are $S_1 = \{A, C, D\}$, $S_2 = \{B\}$; and in states $s_{4,1}$ we have $S_1 = \{A, C\}$, $S_2 = \{B, D\}$.

Suppose that $m_1(\{A, C, D\}) = A$, $m_1(\{A, C\}) = C$ and $m_2(\{B, D\}) = B$. Then the following is an equilibrium. In state $s_{1,1}$ parties 1 and 2 nominate candidates A and B, respectively. A wins, and the state in the following period is $s_{4,1}$. Now parties nominate candidates C and B. Candidate C wins, and the state reverts back to $s_{1,1}$. Neither candidate can improve. In particular, party 1's ideal candidate always wins in each period. The party 1 could only improve by ensuring that in the subsequent period the policy remains at the current ideal point, which is not possible. Party 2 can only influence the outcome by changing the candidate in state $s_{4,1}$ to D. Candidate C still wins, but the state remains at $s_{4,1}$. However, the party leader $m_2(\{B, D\}) = B$ is worse off given that C wins in the next period instead of A. ■

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