JOURNAL OF ECONOMIC THEORY 48, 304–332 (1989)

The Rate at Which a Simple Market Converges to Efficiency as the Number of Traders Increases: An Asymptotic Result for Optimal Trading Mechanisms*

THOMAS A. GRESIK

John M. Olin School of Business, Washington University, St. Louis, Missouri 63130

AND

MARK A. SATTERTHWAITE

Kellogg Graduate School of Management, Northwestern University, Evanston, Illinois 60208

Received January 9, 1987; revised July 19, 1988

Private information in an independent, private-values market provides incentives to traders to manipulate equilibrium prices strategically. This strategic behavior precludes ex post efficient market performance. Increasing the number of traders improves the efficiency of some trading mechanisms by enabling them to better utilize the private information traders' bids and offers reveal. This paper shows that the expected inefficiency of optimally designed mechanisms, relative to ex post efficient allocations, decreases almost quadratically as the number of traders increases. *Journal of Economic Literature* Classification Numbers: 026, 022. © 1989 Academic Press, Inc.

1. INTRODUCTION

Private information prevents achievement of ex post efficiency within a small, private goods market. A trading mechanism's efficiency may improve as the number of traders increases because more traders may enable it to

* The participants of the Economic Theory Workshop at the University of Chicago helped this research materially by discovering a serious error in an early version of this work. We give special thanks to Larry Jones, Ken Judd, Ehud Kalai, Isaac Melijkson, Roger Myerson, and Roger Koenker for their help. This material is based upon work supported by the National Science Foundation under Grants SOC-7907542 and SES-8520247, the Center for Advanced Study in Managerial Economics and Decision Sciences at the Kellogg School, and the Health Care Financing Administration through a grant to Northwestern's Center for Health Services and Policy Research.

0022-0531/89 $3.00

utilize more efficiently the private information traders' bids and offers reveal. This paper shows that within a simple market an optimally designed mechanism rapidly overcomes the constraints of private information as the market becomes large: the expected inefficiency of the mechanism's allocations relative to ex post efficient allocations is, at worst, $O((\ln \tau)/\tau^2)$ where τ is an index of the number of traders. This result provides a benchmark against which the rate other trading mechanisms converge to ex post efficiency can be compared.

That private information is responsible for small markets' inefficiency may be seen by considering an example. Let a single buyer and a single seller bargain over an indivisible object. Suppose the reservation value of the seller is \$48, the reservation value of the buyer is \$52, and these reservation values are common knowledge. Ex post efficiency requires that trade occurs because the object is more valuable to the buyer than to the seller.[1] Given the absence of private information, a mediator can be appointed who can use the common knowledge of the traders' reservation values to set a take-it-or-leave-it price of \$50. The buyer and seller then agree to trade, and ex post efficiency is achieved.

If, however, each trader's reservation value is private to himself, then negotations on a satisfactory price may deadlock. If the buyer is confident that the seller's reservation value lies in the interval [25, 55], he may hold out for a price less than \$50. If the seller is confident that the buyer's value is in the interval in [45, 75], he may hold out for a price greater than \$50. Holding out is rational for each in terms of an expected utility calculation because not to hold out would allow the other trader to extract a disproportionate share of the expected gains from trade.

But if both hold out, no trade occurs and the outcome is ex post inefficient. Myerson and Satterthwaite [23] showed that, for bilateral trade in the presence of private information, this inefficiency is general: if trader participation is voluntary, no mechanism exists such that it always has an ex post efficient, Bayesian–Nash equilibrium. Thus incentives to engage in oppotunistic behavior are intrinsic to small markets whenever valuations are private.

In contrast to the small numbers case, this type of private information is not a problem within large markets. In the limit as a market becomes large each trader has no effect on the market clearing price. Therefore each trader reveals his true reservation value and an ex post efficient, competitive allocation results. These observations form the basis for economists' intuition that as a market grows in size the importance of private information as a source of inefficiency decreases.

[1] See Holmstrom and Myerson [17] for a definition and discussion of the three concepts: ex post efficiency, interim efficiency, and ex ante efficiency.

Our goal in this paper is to identify, as a function of the number of traders, an upper bound on the relative inefficiency in a market with private information. A reasonably precise statement of our result is this. The market we study consists of τN_0 sellers, each desiring to sell a single unit of traded good, and τM_0 buyers, each seeking to by a single unit of the good, where τ is an index of the market's size. Traders' preferences are fully described by their reservation values, which are independently drawn from the distribution F for buyers and the distribution H for sellers. Each trader's reservation value is private knowledge to him.[2]

For the moment, fix the value of τ. Construct a trading mechanism that is ex ante efficient in the sense that it

(a) satisfies individual rationality and

(b) maximizes the sum of buyers' and sellers' ex ante expected gains from trade.

Individual rationality means that the expected utility of a trader who knows his own reservation value but who does not yet know the reservation values of the other traders is nonnegative. Thus every trader, whatever his reservation value, wants to participate in the trading mechanism because participation offers him gain in expectation. The sum of the traders' ex ante expected utilities is the average gains from trade the mechanism would generate if

(a) it were utilized repeatedly and

(b) on each repetition every traders' reservation value were independently and freshly drawn from the distributions F and H.

For a given market size τ we define this optimal mechanism's relative inefficiency as follows. Compute the ex ante expected gains from trade that an ex post efficient mechanism would generate if it existed. While such mechanisms generally do not exist for the market we study, this measure is well defined and easily calculated. The optimal mechanism's relative efficiency is the ratio of its ex ante expected gains from trade to that of the ex post efficient mechanism. The optimal mechanism's relative inefficiency is then one minus its relative efficiency. Our main result is that, as τ becomes large, an upper bound on the optimal mechanism's relative inefficiency is $O((\ln \tau)/\tau^2)$.

Two caveats need emphasis. First, this is an optimal mechanism result. An optimal mechanism has the desirable property that it maximizes the sum of the traders' expected gains from trade. It has the undesirable property that the specific rules for trade vary as the distributions F and H

[2] This is the independent private values model that has been used in auction theory. See Milgrom and Weber [20].

vary. We do not observe trading institutions where rules vary in this manner; it is hard even to imagine such an institution. Trading rules that are invariant with respect to F and H are therefore unlikely to generate as great expected gains from trade as our optimal mechanism.[3] Nevertheless our conjecture is that realistic trading rules such as the sealed-bid double auction studied by Wilson [30, 31] may have relative inefficiencies of the same order as the optimal mechanism.[4]

The second caveat is that our results are derived in the context of a special model. Demand and supply are unitary, traders are risk neutral, and reservation values are independently distributed. It is hard to think of a market where this is a fully adequate abstraction of reality. For example, in many markets traders' reservation values are correlated with each other. This dependence enables each trader to infer information about the other traders' reservation values from his own reservation value. The extent to which our results hold when the specific assumptions of our model are relaxed is an open question.

This work is related to several sets of work in economic theory. First, and most directly related, is the work that Chatterjee and Samuelson [2], Myerson and Satterthwaite [23], Wilson [30, 31], Williams [29], Leininger *et al.* [19], and Satterthwaite and Williams [26] have done using the same basic model we use here. Chatterjee and Samuelson [2] showed with a bilateral example that one cannot expect ex post efficiency from the double auction. Myerson and Satterthwaite [23] showed that ex post efficiency is in general not achievable in bilateral trade if individual rationality is required. Williams [29] investigated ex ante efficient mechanisms where, instead of maximizing the expected gains from trade, the buyer and seller are assigned arbitrary welfare weights. Leininger *et al.* [19] and Satterthwaite and Williams [26] characterized the variety of equilibria that exist for the bilateral double auction. Wilson [31] studied double auctions with multiple buyers and sellers. He showed that if the number of traders is large enough and well-behaved equilibria exist, the double auction is interim efficient.

The second body of work to which this paper is related is auction theory. Auction theory is concerned with markets in which private information exists only on the buyer's side of the market, not on both sides as is the case in this paper. Our paper is most closely related to the normative work that Myerson [22] epitomizes. It is less closely related to the positive branch of auction theory such as Milgrom and Weber [20]. From the

[3] Satterthwaite and Wiliams [26] show that the bilateral sealed-bid double auction generically does not achieve ex ante efficiency.

[4] Satterthwaite and Williams [27] have recently shown this conjecture to be true for the buyers' bid double auction.

viewpoint of this work, auction theory is notable because it has been successful in relaxing the restrictive assumptions of private valuations and statistical independence. The presence of two-sided uncertainty creates technical difficulties that have prevented us from making a comparable relaxation of these assumptions.

The third body of work is general equilibrium theory. Roberts and Postlewaite [25] studied the noncooperative incentives that agents have to pursue strategic behavior within complete information exchange economies. They considered an exchange economy in which

(a) agents report preferences,

(b) a competitive equilibrium is computed based on the reported preferences, and

(c) goods are allocated as prescribed by the computed equilibrium.

They show that as the economy becomes large each agent's incentive to misreport his preferences in order to manipulate the calculated price becomes vanishingly small. This formalizes the idea that for large, perfectly competitive economies strategic behavior is unimportant. It, however, is not comparable with our result for three reasons:

(i) private information does not exist in their model,

(ii) each agent's equilibrium misrepresentation is not calculated, and

(iii) the rate at which the incentive to misrepresent vanishes is not calculated.

A number of authors including Hildenbrand [16], Debreu [5], and Dierker [6] have studied the rate at which core allocations within exchange economies converge to competitive allocations. Debreu, for example, showed that core allocations converge as the inverse of the number of agents. This can be interpreted as showing that the gains traders earn from engaging in strategic rather than price-taking behavior declines rapidly as the number of traders increase. Thus the spirit of these results is the same as in our results. The difference lies in the nature of the equilibrium concept used and the informational assumptions.

2. Preliminaries

Model

There are $M = \tau M_0$ buyers each of whom seeks to buy a single unit of the traded good and $N = \tau N_0$ sellers each of whom seeks to sell the single, indivisible unit he owns of the traded good. Denote the total number of traders by $n = M + N$. Let x_i and z_j represent buyer i and seller j's reservation values, respectively. Buyers' reservation values are independently drawn from the distribution F, and sellers' reservation values are independently drawn from the distribution H. Both distributions have positive densities (f and h) on the bounded interval $[a, b]$. The realization of each trader's reservation value is private to that trader and unverifiable by anyone else. The initial numbers of buyers and sellers and the distribution functions of their reservation values constitute the essential data of the trading problem. Therefore we call the quadruplet $\langle M_0, N_0, F, H \rangle$ the trading problem.

A trading problem $\langle M_0, N_0, F, H \rangle$ is *regular* if

(i) F and H have continuous and bounded first and second derivatives on (a, b) and

(ii) the functions $x_i + (F(x_i) - 1)/f(x_i)$ and $z_j + H(z_j)/h(z_j)$ are both nondecreasing over the interval (a, b).

The purpose of these regularity assumptions is to restrict the set of admissible trading problems sufficiently to permit us to construct ex ante efficient mechanisms.

Before proceeding we need additional notation. Let $x = (x_1, ..., x_M)$, $z = (z_1, ..., z_N)$, $x_{-i} = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_M)$, and $z_{-j} = (z_1, ..., z_{j-1}, z_{j+1}, ..., z_N)$. The density $g(x, z) = \prod_{i=1}^{M} f(x_i) \cdot \prod_{j=1}^{N} h(z_j)$ describes the joint density of all the reservation values, the density $g(x_{-i}, z) = g(x, z)/f(x_i)$ describes the density of reservation values buyer i perceives himself as facing, and the density $g(x, z_{-j}) = g(x, z)/h(z_j)$ describes the density of reservation values seller j perceives himself as facing.

For a particular trading problem $\langle M_0, N_0, F, H \rangle$, fix τ so that size of the market is $n = \tau(M_0 + N_0)$ traders. A trading mechanism consists of n probability schedules and n payment schedules that determine the final distribution of money and goods given the declared valuations of the traders. Let the probabilities of an object being assigned to buyer i and seller j in the final distribution of goods be $p_i^\tau(\hat{x}, \hat{z})$ and $q_j^\tau(\hat{x}, \hat{z})$, respectively, where $\hat{x}$ and $\hat{z}$ are the vectors of buyers and sellers' declared valuations. The declared valuation a trader reports need not be his true reservation value. Let the payments made to buyer i and seller j be $r_i^\tau(\hat{x}, \hat{z})$ and $s_j^\tau(\hat{x}, \hat{z})$, respectively. A negative value for r_i^τ indicates that buyer i plays $|r_i^\tau|$ units of money for the right to receive one unit of the traded object with probability p_i^τ. The r_i^τ and s_j^τ payments are not necessarily conditional on whether buyer i actually receives an object or seller j actually gives up his object.[5]

[5] We would like to regard the payments r_i and s_j to be certainty equivalents of payments that are made only when an individual is involved in a trade. Such a no-regret property seems desirable, but we have not identified the conditions under which it can be imposed.

We assume that the market size τ, the joint density of reservation values g, the probability schedules p and q, and the payment schedules r and s are common knowledge along all traders. The trading process is initiated when all players simultaneously declare reservation values in the interval $[a, b]$. Given these bids and offers, the N objects and money are reallocated as the trading mechanism (p, q, r, s) mandates.

Each trader has a von Neumann–Morgenstern utility function that is additively separable and linear both in money and in the reservation value of the traded object. Thus buyer i's expected utility, given that his true reservation value is x_i and the vectors of declared reservation values are $\hat{x}$ and $\hat{z}$, is

$$\bar{U}_i(x_i, \hat{x}, \hat{z}) = r_i^\tau(\hat{x}, \hat{z}) + x_i p_i^\tau(\hat{x}, \hat{z}). \tag{2.01}$$

Similarly, seller j's expected utility is

$$\bar{V}_j(z_j, \hat{x}, \hat{z}) = s_j^\tau(\hat{x}, \hat{z}) - z_j(1 - q_j^\tau(\hat{x}, \hat{z})). \tag{2.02}$$

Each trader's expected utility function is normalized so that if $(\hat{x}, \hat{z})$ are such that he is certain to neither trade an object nor make or receive a cash payment, then his expected utility is zero.

We constrain the trading mechanism in three ways to conform with our notions of voluntary trade among a set of independent buyers and sellers. First, in the final distribution of goods and money, the N objects are each assigned to a trader. Thus, necessarily, there is balance of goods in expectation:

$$\sum_{i=1}^{M} p_i^\tau(\hat{x}, \hat{z}) + \sum_{j=1}^{N} q_j^\tau(\hat{x}, \hat{z}) = N \tag{2.03}$$

for all $(\hat{x}, \hat{z})$. An allocation schedule achieves balance of goods in fact by making its assignment of objects correlated across traders.[6] Second, payments are constrained to offset receipts:

$$\sum_{i=1}^{M} r_i^\tau(\hat{x}, \hat{z}) + \sum_{j=1}^{N} s_j^\tau(\hat{x}, \hat{z}) = 0 \tag{2.04}$$

[6] Specifically for a given set of declared valuations, buyer 1 can be assigned an object with probability p_1 through an independent draw of a random number in the $[0, 1]$ interval. Buyer 2 can next be assigned an object with probability p_2 through a second independent draw, etc. This process of assigning objects through independent draws first to the M buyers and then to the N sellers can be continued until either (a) all N objects have been assigned or (b) K objects remain and exactly K buyers and sellers remain to have an object assigned to them. If eventually (a) occurs, then the remaining buyers and sellers should be excluded from receiving an object. If eventually (b) occurs, then the K remaining buyers and sellers should each receive an object. This rule guarantees that exactly N objects are distributed. The dependence that this rule induces between the probability of buyer 1 being assigned an object and seller N not being assigned an object has no effect on our results.

for all $(\hat{x}, \hat{z})$. The reason for this latter constraint is that trading connotes individuals freely cooperating with one another without subsidy or tax from a third party. Third, the mechanism must be individually rational.

In addition to these three constraints we also impose a fourth constraint on the mechanism: incentive compatibility. An incentive compatible mechanism never gives any trader an incentive to declare a reservation value different than his true reservation value, i.e., declaration of true values is a Bayesian–Nash equilibrium if the mechanism is incentive compatible.[7] Imposition of it is costless because the revelation principle states that for every mechanism an equivalent incentive compatible mechanism exists.[8] Therefore, even though we do not consider all conceivable mechanisms, we know that none of the mechanisms we overlook ex ante dominate the mechanisms we do consider.

Formalization of the individual rationality and incentive compatibility constraints requires additional notation and definitions. Let

$$\bar{p}_i^\tau(x_i) = \int \cdots \int p_i^\tau(x, z)\, g(x_{-i}, z)\, dx_{-i}\, dz, \tag{2.05}$$

and

$$\bar{r}_i^\tau(x_i) = \int \cdots \int r_i^\tau(x, z)\, g(x_{-i}, z)\, dx_{-i}\, dz. \tag{2.06}$$

Conditional on buyer i's reservation value being x_i, the quantities $\bar{p}_i^\tau(x_i)$ and $\bar{r}_i^\tau(x_i)$ are, respectively, his expected probability of receiving an object and his expected money receipts. The probabilities $\bar{q}_j^\tau$ and $\bar{s}_j^\tau$ have parallel definitions for seller j. The expected utilities of buyer i and seller j conditional on their reservation values are

$$U_i(x_i) = \bar{r}_i^\tau(x_i) + x_i \bar{p}_i^\tau(x_i) \tag{2.07}$$

and

$$V_j(z_j) = \bar{s}_j^\tau(z_j) - z_j(1 - \bar{q}_j^\tau(z_j)). \tag{2.08}$$

Individual rationality requires that, for all buyers i and all sellers j, $U_i(x_i) \geqslant 0$ for every $x_i \in [a, b]$ and $V_j(z_j) \geqslant 0$ for every $z_j \in [a, b]$. Incentive compatibility requires that, for every buyer i and all x_i and $\hat{x}_i$ in $[a, b]$,

$$U_i(x_i) \geqslant \bar{r}_i^\tau(\hat{x}_i) + x_i \bar{p}_i^\tau(\hat{x}_i) \tag{2.09}$$

[7] Harsanyi [15] introduced these concepts of Bayesian equilibrium.

[8] The revelation principle has its origins in Gibbard's paper [7] on straightforward mechanisms and was developed by Myerson [21, 22], Harris and Townsend [14], and Harris and Raviv [13].

and, for every seller j and all z and $\hat{z}$ in $[a, b]$,

$$V_j(z_j) \geqslant \bar{s}_j^\tau(\hat{z}_j) - z_j(1 - \bar{q}_j^\tau(\hat{z}_j)). \tag{2.10}$$

If (2.09) is violated for some x_i and $\hat{x}_i$, then buyer i has an incentive to declare $\hat{x}_i$ rather than his or her true reservation value, x_i. From this point forward we only consider incentive compatible mechanisms and we assume that traders always reveal their true reservation values.

Characterization of Incentive Feasible Mechanisms

A mechanism is called incentive feasible if it is both individually rational and incentive compatible. Theorem 1 characterizes all incentive feasible mechanisms. It exactly generalizes Myerson and Satterthwaite's [23] Theorem 1 from the bilateral case to the general case of multiple buyers and sellers.

THEOREM 1. *Consider a given replication τ of a trading problem $\langle M_0, N_0, F, H\rangle$. Let $p^\tau(\cdot, \cdot)$ and $q^\tau(\cdot, \cdot)$ be the buyers and sellers, probability schedules, respectively. Functions $r^\tau(\cdot, \cdot)$ and $s^\tau(\cdot, \cdot)$ exist such that $(p^\tau, q^\tau, r^\tau, s^\tau)$ is an incentive feasible mechanism if and only if $\bar{p}_i^\tau(\cdot)$ is a nondecreasing function for all buyers i, $\bar{q}_j^\tau(\cdot)$ is a nondecreasing function for all sellers j, and*

$$\sum_{i=1}^{M} \int \cdots \int \left(x_i + \frac{F(x_i) - 1}{f(x_i)}\right) p_i^\tau(x, z)\, g(x, z)\, dx\, dz$$
$$- \sum_{j=1}^{N} \int \cdots \int \left(z_j + \frac{H(z_j)}{h(z_j)}\right) [1 - q_j^\tau(x, s)]\, g(x, z)\, dx\, dz \geqslant 0. \tag{2.11}$$

Furthermore, given any incentive feasible mechanism, for all i and j, $U_i(\cdot)$ is nondecreasing, $V_j(\cdot)$ nonincreasing, and

$$\sum_{i=1}^{M} U_i(a_i) + \sum_{j=1}^{N} V_j(d_j)$$
$$= \sum_{i=1}^{M} \min_{x \in [a,b]} U_i(x) + \sum_{j=1}^{N} \min_{z \in [a,b]} V_j(z)$$
$$= \sum_{i=1}^{M} \int \cdots \int \left(x_i + \frac{F(x_i) - 1}{f(x_i)}\right) p_i^\tau(x, z)\, g(x, z)\, dx\, dz$$
$$- \sum_{j=1}^{N} \int \cdots \int \left(z_j + \frac{H(z_j)}{h(z_j)}\right) [1 - q_j^\tau(x, z)]\, g(x, z)\, dx\, dz. \tag{2.12}$$

This theorem is the key to constructing ex ante optimal mechanisms because it establishes that if the probability schedules (p^τ, q^τ) satisfy the relatively simple constraint (2.11), then payment schedules (r^τ, s^τ) exist such that the mechanism $(p^\tau, q^\tau, r^\tau, s^\tau)$ is an incentive feasible trading mechanism.[9]

Ex Ante Efficiency

A trader's ex ante expected utility from participating in trade is his expected utility evaluated before he learns his reservation value for the object. Thus $\tilde{U}_i = \int U_i(t) f(t)\, dt$ and $\tilde{V}_j = \int V_j(t) h(t)\, dt$ are buyer i and seller j's ex ante expected utilities, respectively. A trading mechanism is ex ante efficient if not trader's ex ante expected utility can be increased without either

(a) decreasing some other trader's ex ante expected utility or
(b) violating incentive feasibility.

We focus on a particular ex ante efficient mechanism: the one that places equal welfare weights on every trader and maximizes the sum of the traders' ex ante expected utilities. This maximiation is equivalent to maximizing the sum of all traders' expected gains from trade because each trader's utility function is separable in money and the object's reservation value. In contrast, ex post optimality requires that the mechanism, irrespective of incentive feasibility, exhausts the potential gains from trade by assigning the N objects to the N traders who have the highest reservation values.

Virtual Reservation Values and α-Schedules

Virtual reservation values play a crucial role in construction of ex ante efficient mechanisms.[10] Buyer i's virtual reservation value is

$$\psi^{\mathrm{B}}(x_i, \alpha) = x_i + \alpha \cdot \left(\frac{F(x_i) - 1}{f(x_i)} \right), \tag{2.13}$$

and seller j's virtual reservation value is

$$\psi^{\mathrm{S}}(z_j, \alpha) = z_j + \alpha \cdot \frac{H(z_j)}{h(z_j)} \tag{2.14}$$

where α is a nonnegative, scalar parameter. Let the vector of virtual reservation values be $\psi(x, z, \alpha) = [\psi^{\mathrm{B}}(x_1, \alpha), ..., \psi^{\mathrm{S}}(z_N, \alpha)]$.

[9] The assumption that trader's utility functions are linear in money is important in this simplification. Maximization of the expected gains from trade is dependent only on the final allocation of goods, not on the payments among the traders.

[10] Myerson [24] introduced the concept of virtual utility, A virtual reservation value is a special case of a virtual utility.

Define $R_i^{\mathrm{B}}(x, z, \alpha)$ to be the rank of the element $\psi^{\mathrm{B}}(x_i, \alpha)$ within ψ and define $R_j^{\mathrm{S}}(x, z, \alpha)$ to be the rank of the element $\psi^{\mathrm{S}}(z_j, \alpha)$ within ψ. For example, if $M = N = 1$ and $\psi = (0.4, 0.2)$, then $R_{i=1}^{B} = 2$ and $R_{j=1}^{\mathrm{S}} = 1$.[11] Given this notation, a trading problem $\langle M_0, N_0, F, H \rangle$, and a value τ, we define a class of buyer and seller probability schedules that are parameterized by α:

$$p_i^{\tau\alpha}(x, z) = \begin{cases} 1 & \text{if } \ R_i^{\mathrm{B}}(x, z, \alpha) > M, \\ 0 & \text{if } \ R_i^{\mathrm{B}}(x, z, \alpha) \leqslant M, \end{cases} \qquad i = 1, ..., M; \tag{2.15}$$

$$q_j^{\tau\alpha}(x, z) = \begin{cases} 1 & \text{if } \ R_j^{\mathrm{S}}(x, z, \alpha) > M, \\ 0 & \text{if } \ R_j^{\mathrm{S}}(x, z, \alpha) \leqslant M, \end{cases} \qquad j = 1, ..., N. \tag{2.16}$$

Let $p^{\tau\alpha} = (p_1^{\tau\alpha}, ..., p_M^{\tau\alpha})$ and $q^{\tau\alpha} = (q_1^{\tau\alpha}, ..., q_N^{\tau\alpha})$. This pair of probability schedules, which we call an α-schedule, assigns the N available objects to those N traders for whom the objects have the highest virtual reservation values.

Before proceeding further we should discuss virtual reservation values and α-schedules. If $\alpha = 0$, then $\psi^{\mathrm{B}}(x_i, 0) = x_i$ and $\psi^{\mathrm{S}}(z_j, 0) = z_j$. The virtual reservation values equal the true reservation values and the N objects are assigned to the N traders who have the highest reservation values. If, however, $\alpha > 0$, then $\psi_i(x_i, \alpha) < x_i$ and $\psi_j(z_j, \alpha) > z_j$ almost everywhere. Thus, for $\alpha > 0$, buyers' virtual reservation values are distorted downward to be below their true reservation values and sellers' virtual reservation values are distorted upward to be above their true reservation values. Intuitively these distortions express the strategic behavior that traders exhibit under a mechanism such as the double auction that is not incentive compatible. In this case the possibility exists that the objects will not be assigned to the N traders whose reservation values are highest. Specifically, if $\alpha > 0$, then pairs of reservation values (x_i, z_j) exist such that $x_i > z_j$ and $\psi^{\mathrm{B}}(x_i, \alpha) < \psi^{\mathrm{S}}(z_j, \alpha)$. Trade fails to occur in such a case even though it should because the buyer values the object more than the seller. For this reason an α-schedule does not necessarily achieve ex post optimality whenever $\alpha > 0$.

[11] If several elements of ψ have the same value so that it is ambiguous which buyers and sellers should be classified as having virtual reservation prices as ranking within the top N, then the probability schedules should randomize among the several candidates so as to guarantee that exactly N traders are assigned an object. Thus if seller 2 and buyer 3 are tied for rank M, then each should be given a nonindependent probability of 0.5 for receiving an object in the final allocation.

3. Results

Ex Ante Efficient Mechanisms and α^*-Schedules

Fix the value of the parameter $\alpha \geqslant 0$ and consider the α-schedule $(p^{\tau\alpha}, q^{\tau\alpha})$. Theorem 1 states necessary and sufficient conditions for payment schedules (r, s) to exist such that the trading mechanism $(p^{\tau\alpha}, q^{\tau\alpha}, r, s)$ is incentive feasible. Central to the theorem's requirements is inequality (2.11), the incentive feasibility (IF) constraint. For the case of an α-schedule, substitution of (2.13) and (2.14) into (2.11) yields the requirement

$$\begin{aligned} G(\alpha, \tau) = \int \cdots \int \Bigg\{ &\sum_{i=1}^{M} \psi^{\mathrm{B}}(x_i, 1)\, p_i^{\tau\alpha}(x, z) \\ &- \sum_{j=1}^{N} \psi^{\mathrm{S}}(z_j, 1)[1 - q_j^{\tau\alpha}(x, z)] \Bigg\}\, g(x, z)\, dx\, dz \\ \geqslant 0. \end{aligned} \tag{3.01}$$

This function $G(\alpha, \tau)$ plays a central role in the theorems that follow.

Let $\alpha^* = \min\{\alpha \in [0, 1) \mid G(\alpha, \tau) \geqslant 0\}$. An α-schedule $(p^{\tau\alpha}, q^{\tau\alpha})$ is an α^*-schedule if and only if $\alpha = \alpha^*$ and $\bar{p}_i^{\tau\alpha^*}(\cdot)$ and $\bar{q}_j^{\tau\alpha^*}(\cdot)$ are nondecreasing over $[a, b]$ for all buyers i and all sellers j. By definition, an α^*-schedule satisfies Theorem 1's requirements. Therefore payment schedules $(r^{\tau\alpha^*}, s^{\tau\alpha^*})$ exist such that the mechanism $(p^{\tau\alpha^*}, q^{\tau\alpha^*}, r^{\tau\alpha^*}, s^{\tau\alpha^*})$ is incentive feasible. We call this mechanism the α^*-mechanism for the trading problem $\langle M_0, N_0, F, H \rangle$ with a market size τ.

Theorem 2 states sufficient conditions for the α^*-mechanism—if it exists—to be an ex ante efficient mechanism. Theorem 3 states sufficient conditions for the α^*-mechanism to exist and be ex ante efficient for a trading problem with a given market size.

THEOREM 2. *Suppose an α^*-mechanism exists for market size τ of the trading problem $\langle M_0, N_0, F, H \rangle$. The α^*-trading mechanism $(p^{\tau\alpha^*}, q^{\tau\alpha^*}, r^{\tau\alpha^*}, s^{\tau\alpha^*})$ is ex ante efficient and has positive expected gains from trade.*

THEOREM 3. *If $\langle M_0, N_0, F, H \rangle$ is a regular trading problem, then, for every market size τ, the α^*-mechanism exists, is incentive feasible and ex ante efficient, and has positive expected gains from trade.*

Convergence to Ex Post Optimality

Before we determine the rate a which the ex ante optimal mechanism converges to ex post optimality, we need to show that it converges as

$\tau \to \infty$. Theorem 4 establishes this convergence both as it approaches the limit and in the limit. In order to understand the theorem, recall two facts. First, the closer the parameter α is to zero, the less virtual reservation values are distorted from true reservation values and the closer the α-schedule comes to achieving ex post optimal assignment of the objects. Second, for a given value of α and a given market size τ, if $G(\alpha, \tau) \geqslant 0$, then payment schedules (r, s) exist such that $(p^{\alpha\tau}, q^{\alpha\tau}, r, s)$ is incentive feasible.

THEOREM 4. *Pick an* $\alpha \in (0, 1)$. *If the trading problem* $\langle M_0, N_0, F, H \rangle$ *is regular, then a* $\tau' > 0$ *exists such that, for all market sizes* $\tau > \tau'$, $G(\alpha, \tau) \geqslant 0$. *Moreover* $\lim_{\tau \to \infty} G(0, \tau) = 0$.

The content of the theorem is that, no matter how close to zero we set α, if the market becomes large enough, then that α-schedule and its associated payment schedule is incentive feasible.

Rate of Convergence

We present two results. The first is an upper bound on the size of the parameter α^* as a function of τ. The magnitude of α^* as a function of τ is a measure of the mechanism's distance from ex post optimality.

THEOREM 5. *Consider a regular trading problem* $\langle M_0, N_0, F, H \rangle$. *The parameter* α^* *of the ex ante efficient* α^**-mechanism is at most* $O((\ln \tau)^{1/2}/\tau)$ *for large* τ.

The second result, which is our main result, states an upper bound on the expected proportion of the gains from trade that the optimal mechanism fails to realize. Let $T^*(\tau)$ represent the expected gains from trade that the ex ante efficient α^*-mechanism realizes for the trading problem $\langle M_0, N_0, F, H \rangle$ with market size τ and let $T^0(\tau)$ represents the expected gains from trade that an ex post efficient mechanism (if one existed) would realize for the same trading problem and market size. Let $W(\tau) = 1 - [T^*(\tau)/T^0(\tau)]$ be a measure of the market's relative inefficiency.

THEOREM 6. *Consider a regular trading problem* $\langle M_0, N_0, F, H \rangle$. *For the ex ante efficient,* α^**-mechanism,* $W(\tau)$ *is at most* $O(\ln \tau/\tau^2)$, *i.e., for large* τ, *a* K *exists such that*

$$W(\tau) = 1 - \frac{T^*(\tau)}{T^0(\tau)} \leqslant K \frac{\ln \tau}{\tau^2}. \tag{3.02}$$

Two comments about Theorem 6 are in order. First, the order of W as a function of τ indicates the mechanism's rate of convergence toward ex post optimality and is independent of the choice of the underlying distributions F and H. For a given value of τ, however, the absolute size of W is a function of F and H. Second, we conjecture that the bounds stated in Theorem 5 and 6 are not tight. Specifically, the example in the next section suggests that the true bounds are $O(1/\tau)$ for Theorem 5 and $O(1/\tau^2)$ for Theorem 6.

4. An Example

In this section we numerically calculate for varying market sizes τ the ex ante efficient α^*-trading mechanisms when $M_0 = N_0 = 1$ and traders' reservation values are uniformly distributed on the unit interval. This distributional assumption guarantees that the trading problem is regular as Theorem 3 requires. Therefore an α^*-mechanism exists for all market sizes τ.

The key step in constructing an efficient mechanism for a given number of traders is to calculate α^* as the solution of $G(\alpha, \tau) = 0$. Given that traders' reservation values are uniformly distributed over $[0, 1]$, $\psi^{B}(x_i, \alpha) = (1+\alpha)\,x_i - \alpha$ and $\psi^{S}(z_j, \alpha) = (1+\alpha)\,z_j$. Since $N_0 = M_0 = 1$, $G(\alpha, \tau) = 0$ reduces to

$$\begin{aligned} G(\alpha, \tau) &= \tau\left\{\int_0^1 \psi^{B}(x, 1)\,\bar{p}^{\tau\alpha}(x)\,f(x)\,dx \right. \\ &\qquad \left. - \int_0^1 \psi^{S}(z, 1)[1 - \bar{q}^{\tau\alpha}(z)]\,h(z)\,dz\right\} \\ &= \tau\left\{\int_0^1 (2x-1)\,\bar{p}^{\tau\alpha}(x)\,dx - \int_0^1 2z(1-\bar{q}^{\tau\alpha}(z))\,dz\right\} = 0, \quad (4.01) \end{aligned}$$

where all i and j subscripts have been suppressed because all traders are symmetric with each other. It may be rewritten as

$$\int_0^1 \{[2x-1]\,\bar{p}^{\tau\alpha}(x) - 2x[1-\bar{q}^{\tau\alpha}(x)]\}\,dx = 0. \qquad (4.02)$$

Calculation of the marginal probabilities $\bar{p}^{\tau\alpha}(x)$ and $\bar{q}^{\tau\alpha}(z)$ is messy, but straightforward.[12]

Table I presents the results. The values of α^* have the following interpretation. If buyer i with reservation value x_i and seller j with reservation value z_j are each the marginal trader on his side of the market, then

[12] Details are in Gresik and Satterthwaite [9].

TABLE I

Properties of the α^*-Mechanisms as the Number Traders Increases

τ	α^*	$\alpha^*/(1+\alpha^*)$	$1/\alpha^*$	$T^*(\tau)$	$T^0(\tau)$	$W(\tau)$
1	0.3333	0.2500	3.00	0.14060	0.16667	0.1564
2	0.2256	0.1841	4.43	0.37746	0.39999	0.0563
3	0.1603	0.1382	6.24	0.62572	0.64286	0.0267
4	0.1225	0.1091	8.17	0.87527	0.88887	0.0153
6	0.0827	0.0764	12.09	1.37507	1.38462	0.0069
8	0.0622	0.0586	16.08	1.87504	1.88235	0.0039
10	0.0499	0.0475	20.04	2.37501	2.38095	0.0025
12	0.0416	0.0399	24.04	2.87501	2.88000	0.0017

necessarily i's virtual reservation value is greater than j's virtual reservation value, i.e., $\psi^B(x_i, \alpha^*) > \psi^S(z_j, \alpha^*)$. Substitution of explicit forms for ψ^B and ψ^S into this inequality followed by some algebraic manipulation shows that necessarily

$$x_i - z_j > \frac{\alpha^*}{1+\alpha^*}. \tag{4.03}$$

This required, positive difference in reservation values is the wedge that privacy of traders' reservation values creates within finite-sized markets. Its presence makes achievement of ex post efficiency impossible. Note that as α^* becomes small, the size of this wedge becomes essentially equal to the value of α^* itself. The fourth column displays $1/\alpha^*$ and shows that α^* is apparently bounded from below by $1/2\tau$. The last column shows for that $W(\tau)$, the relative inefficiency of this market, vanishes as $(1/\tau^2)$ for larger τ. Finally, note that by the time the market reaches 12 traders $(\tau = 6)$ its relative inefficiency is down to the negligible level of less than 1%.

5. Comparison with a Fixed-Price Mechanism

Theorem 6 serves as a benchmark for evaluating how well other mechanisms elicit and use private information as the market becomes large. Here we make this comparison for the fixed-price mechanism.[13] It works as follows. Price is fixed at the competitive price c that would obtain if our simple market had a continuum of buyers with reservation value dis-

[13] William Rogerson suggested to us that the fixed-price mechanism is an interesting alternative to the double auction. See Hagerty and Rogerson [11] for a discussion of its properties in the bilateral case.

TABLE II

Comparative Inefficiences of the Fixed Price Mechanism (W') and the Optimal Mechanism (W)

τ	$W'(\tau)$	$W(\tau)$
1	0.2500	0.1564
2	0.2187	0.0563
3	0.1979	0.0267
4	0.1826	0.0153
6	0.1611	0.0069
8	0.1462	0.0039
10	0.1350	0.0025
12	0.1262	0.0017

tributed on F and a continuum of sellers with reservation values distributed on H. All buyers whose reservation values are greater than c indicate that they want to buy one unit and all sellers whose reservation values are less than c indicate that they want to sell one unit. The strategy of reporting honestly the desire to trade or not to trade is a dominant strategy for each trader because price is fixed. If the market does not clear, which is almost always the case, then rationing is done by random selection from among the traders on whichever side of the market is long.

The problem with random exclusion is that a buyer i whose gains from trade, $x_i - c$, are large is just as likely to be excluded as a buyer k whose gains from trade, $x_k - c$, are small. Therefore, as τ becomes large, the average loss per excluded trader remains a constant. This is unlike the optimal mechanism where, as τ becomes large, the average loss per unrealized trade declines rapidly because of the optimal mechanism's ability to use the private information it elicits.

Asymptotically for the fixed-price mechanism, the number of traders who wish to trade but who are excluded is $O(\tau^{1/2})$.[14] This is also the order of the gains from trade that the mechanism fails to realize. The number of traders who wish to trade at this fixed price c is $O(\tau)$. Therefore the gains from trade that a hypothetical ex post efficient mechanism would be expected to realize are $O(\tau)$. Dividing the order of the expected inefficiency by the order of the total gains available gives the result $W'(\tau) = O(1/\tau^{1/2})$ for the fixed-price mechanism, which contrasts starkly with $W(\tau) = O\{(\ln \tau)/\tau^2\}$ for the optimal trading mechanism. Further contrasting the

[14] This follows from the fact that the number of buyers who wish to trade at the fixed price c is a binomial variable that can be approximated asymptotically by a normal distribution with standard deviation $O(\tau^{1/2})$. This calculation is a special case of Bhattacharya and Majumdar's [1] Theorem 3.1.

performance of these two mechanisms is the fact that the magnitude of the inefficiency starts out larger for the fixed-price mechanism than for the α^*-mechanism. Table II, which shows $W(\tau)$ and $W'(\tau)$ for $M_0 = N_0 = 1$ and uniformly distributed reservation values, illustrates both these points. In fact, for the small values of τ shown on the table, $W'(\tau)$ converges to ex post efficiency at a rate noticeably slower than $O(1/\tau^{1/2})$. Both of these comparisons emphasize the benefit of eliciting valuation information from traders and—within the limits of incentive compatibility—using it to assign the objects appropriately.

6. Further Questions

Our results are only a starting for understanding how fast market mechanisms converge to perfect competition in the presence of private information. Four of the more important questions that need attention are as follows. First, are asymptotic results useful when studying trading problems? While the numerical results of Section 4 are supportive of the idea that even for small numbers the asymptotic rate is a good approximation, we cannot conclude without further investigation that it is an equally good, small number approximation for prior distributions other than the uniform. Second, if traders are risk averse, does the $O((\ln \tau)/\tau^2)$ result continue hold? A recent paper of Ledyard [18] emphasizes the importance of this question. He shows, within the context of a somewhat different model, how careful selection of utility functions for a fixed set of agents can lead to almost any desired equlibrium behavior.[15]

Third, if agents' reservation values are not independent of each other, but rather are correlated, then does our convergence result hold? Milgrom and Weber [20] have shown in their studies of auctions that such distinctions are important. Gresik [8] employs a discrete distribution model of bilateral trade to show that the introduction of affiliated valuations can result in the existence of ex post efficient trading mechanisms. Whether these results carry over to models with continuous distributions is an open question. Fourth, is our focus on optimal mechanisms constructed using the revelation principle appropriate? In practice direct revelation mechanisms are seldom used to allocate goods. The reason is that a direct revelation mechanism's allocation and payment rules must be changed each time the traders' prior distributions concerning other traders' reservation values change. This cannot be done practically because traders' priors are unobservable. Consequently, the rules of a real trading mechanism are kept

[15] Ledyard's argument as it stands not address the focus of this paper: how does a Bayesian equilibrium converge toward the competitive allocation as the initial set of traders is replicated repeatedly.

constant and not changed each time traders' expectations about each others' reservation values change. This makes the result of Wilson [31] concerning the interim efficiency properties of the double-auction mechanism desirable.

7. PROOFS

Preliminaries

Detailed proofs of Theorems 1, 2, and 3 are contained in Gresik and Satterthwaite [9] and in less detailed form in Gresik and Satterthwaite [10] and Wilson [30]. The techniques of the proofs are a straightforward generalizations of Myerson and Satterthwaite's [23] treatment of the bilateral case.

Proofs of Theorems 4, 5, and 6 require a detailed understanding of the asymptotic behavior of the marginal distributions $\bar{p}^{\tau\alpha}$ and $\bar{q}^{\tau\alpha}$. We defined $\bar{p}^{\tau\alpha}(x_i)$ to be the marginal probability that a buyer i with reservation value x_i receives an object.[16] Its interpretation in terms of a simple random trial is this. Fix α. Draw independently $M-1=\tau M_0-1$ buyers' reservation values from F and $N=\tau N_0$ sellers' reservation values from H. Transform these reservation values into virtual reservation values using $\psi^{\mathrm{B}}(\cdot, \alpha)$ and $\psi^{\mathrm{S}}(\cdot, \alpha)$, respectively. The probability $\bar{p}^{\tau\alpha}(x_i)$ is the probability that buyer i's virtual reservation value $\psi^{\mathrm{B}}(x_i, \alpha)$ is greater than the Mth-order statistic of the $M+N-1$ virtual reservation values of the other traders.[17] If $\psi^{\mathrm{B}}(x_i, \alpha)$ is less than the Mth-order statistic, then buyer i is not assigned an object. Denote with $\xi_{p\tau}$ this Mth-order statistic.[18] Then $\bar{p}^{\tau\alpha}(x_i)=\Pr\{\xi_{p\tau} \leqslant \psi^{\mathrm{B}}(x_i, \alpha)\}$. Thus, in order to understand $\bar{p}^{\tau\alpha}$ we must understand the Mth-order statistic $\xi_{p\tau}$.

A standard result is that the Mth-order statistic of a sample of $n=\tau(M_0+N_0)$ random variables independently drawn from a single distribution function is asymptotically normally distributed.[19] A second, less well-known result is that the expected value of the Mth-order statistic of a size n random sample drawn from a distribution converges asymptotically toward the population quantile of order $M_0/(M_0+N_0)$ at a rate $O((\ln \tau)/\tau^2)$.[20] Two reasons exist why these results cannot be applied

[16] The i subscript identifying the buyer is suppressed because, given our assumption that each buyer's reservation value is drawn from F and given our focus on α^*-mechanisms, every buyer's $\bar{p}^{\tau\alpha}$ distribution is identical.

[17] The first-order statistic is the smallest element of the sample, the second-order statistic is the second smallest element, etc.

[18] The meaning of the p subscript on $\xi_{p\tau}$ is made clear later in this section.

[19] See Theorem 9.2 in David [4, pp. 254–255] and Theorem A of Section 2.3.3 in Serfling [28, p. 77].

[20] See Hall [12], David and Johnson [3], and expression (4.6.3) in David [4, p. 80].

directly to our problem. The first is this. The $M-1$ buyers' reservation values are drawn from the distribution F and transformed into virtual reservation values by ψ^{B}. Similarly the N sellers' reservation values are drawn from the distribution H and transformed by ψ^{S}. Therefore the resulting sample of virtual reservation values are not drawn, as the standard theorems require, from a single distribution; it is a sample of nonidentically distributed random variables. The second problem is that $\bar{p}^{\tau\alpha}$ is the distribution for the Mth-order statistic of a sample of size $n-1$, not a sample of size n. In other words, as τ increases the ratio of buyers to sellers in the sample underlying $\bar{p}^{\tau\alpha}$ changes. Theorem 7 below resolves both problems.

In order to state Theorem 7 some additional notation is helpful. Let $\{\psi_1^{\mathrm{B}}, ..., \psi_{M-1}^{\mathrm{B}}, \psi_1^{\mathrm{S}}, ..., \psi_N^{\mathrm{S}}\}$ denote the vector of virtual reservation values where each virtual reservation value ψ_i^{B} is drawn independently from $\tilde{F}$ and each ψ_j^{S} is independently drawn from $\tilde{H}$. The distribution $\tilde{F}$ is the distribution that is obtained by drawing a reservation value from F and then transforming that value into a virtual reservation value by means of $\psi^{\mathrm{B}}(\cdot, \alpha)$. $\tilde{H}$ is similarly defined. Let $[a', b']$ be the union of the supports of $\tilde{F}$ and $\tilde{H}$. The dependence of $\tilde{F}$ on α is suppressed because we use only the asymptotic behavior of $\bar{p}^{\tau\alpha}$ for fixed values of α. Define, for any $t \in [a', b']$, the average distribution function to be $\Gamma(t) = p\tilde{F}(t) + (1-p)\tilde{H}(t)$ where $p = M_0/(M_0+N_0)$. The population quantile of order p is $\xi_p = \inf_y\{y: \Gamma(y) \geqslant p\}$. Finally, define $\sigma(t) = M_0\tilde{F}(t)[1-\tilde{F}(t)] + N_0\tilde{H}(t)[1-\tilde{H}(t)]$. It is the standard deviation of the random number of virtual reservation values that are no greater than t whenever the sample is M_0 buyers and N_0 sellers.

THEOREM 7. *Let $\tilde{\xi}_{p\tau}$ be the Mth-order statistic of a sample $(\psi_1^{\mathrm{B}}, ..., \psi_{M-1}^{\mathrm{B}}, \psi_1^{\mathrm{S}}, ..., \psi_N^{\mathrm{S}})$ where $M = \tau M_0$, $N = \tau N_0$, all ψ_i^{B} are drawn from the distribution $\tilde{F}$ and all ψ_j^{S} are drawn from the distribution $\tilde{H}$. Let $n = \tau(M_0+N_0)$ and $p = M_0/(M_0+N_0)$. In a neighborhood of ξ_p, Γ has positive continuous density Γ' and bounded second derivative Γ'', then, for any t,*

$$\lim_{\tau \to \infty} \Pr\left(\frac{[\tau(M_0+N_0)]^{1/2}(\tilde{\xi}_{p\tau}-\xi_p)}{\sigma(\xi_p)/\{(M_0+N_0)^{1/2}\Gamma'(\xi_p)\}} \leqslant t\right) = \Phi(t) \tag{7.01}$$

and, as $\tau \to \infty$,

$$|E(\tilde{\xi}_{p\tau}-\xi_p)| = O\left\{\frac{(\ln \tau)^{1/2}}{\tau}\right\}. \tag{7.02}$$

The theorem is stated from the buyer's point of view. A simple relabeling of the variables permits us to apply it to sellers. Its proof is found in Gresik and Satterthwaite [10, Th. 6.5]. The theorem is almost a restatement of the standard results for the special case in this paper. The aspect that differs from the standard results is that we have been unable to obtain the $O\{1/\tau\}$ bound on $|E(\bar{\xi}_{p\tau}-\xi_p)|$ that is found in the standard results and that, we conjecture, holds here. It is this inability that causes the bound in Theorem 6 to be $O((\ln \tau)/\tau^2)$ instead of $O(1/\tau^2)$.[21]

Proof of Theorem 4. As the initial step in this proof we must show how Theorem 7 applies to $\bar{p}^{\tau\alpha}$ and $\bar{q}^{\tau\alpha}$. Consider some buyer i. For i to be assigned an object his virtual reservation value must be greater than the Mth-order statistic of the virtual reservation values of the N sellers and the other $M-1$ buyers. Denote by $\psi^{B\alpha}_{(M)}$ this order statistic and let $\Lambda^{B\alpha}_{\tau}$ be its distribution function. *Theorem* 7 *applies to* $\psi^{B\alpha}_{(M)}$. It is asymptotic normal with an asymptotic expected value $\bar{\psi}^{B\alpha}_{(M)}$ and asymptotic variance σ^2_B/τ.

The density function $\bar{p}^{\tau\alpha}(\cdot)$ describes the distribution of the random variable $x(\alpha,\tau)=[\psi^B]^{-1}(\psi^{B\alpha}_{(M)})$ where $[\psi^B]^{-1}(\cdot)$ is the inverse of $\psi^B(\cdot,\alpha)$; it is the critical value that i's reservation value must exceed if i is to be assigned an object.[22] The variate $x(\alpha,\tau)$ is also asymptotically normal with asymptotic expectation $\bar{x}^*=[\psi^B]^{-1}(\bar{\psi}^{B\alpha}_{(M)})$ and asymptotic variance $J^2\sigma^2_B/\tau$ where $J=\partial[\psi^B]^{-1}/\partial x_i$ evaluated at $\bar{\psi}^{B\alpha}_{(M)}$. Consequently as τ becomes large the distribution of $x(\alpha,\tau)$ approaches a step function with the step at $\bar{x}^*$.

Define $\psi^{S\alpha}_{(M)}$, $\Lambda^{S\alpha}_{\tau}$, $\bar{\psi}^{S\alpha}_{(M)}$, σ^2_S, $z(\alpha,\tau)$, and $\bar{z}^*$ in parallel fashion. As τ becomes large the distribution $z(\alpha,\tau)$ approaches a step function with the step at $\bar{z}^*$ where $\bar{z}^*<\bar{x}^*$. The reason for the inequality $\bar{z}^*<\bar{x}^*$ is this. First, as τ becomes large, $|\bar{\psi}^{S\alpha}_{(M)}-\bar{\psi}^{B\alpha}_{(M)}|$ approaches zero because the samples that generate $\psi^{S\alpha}_{(M)}$ and $\psi^{B\alpha}_{(M)}$ become essentially identical as τ increases. Second, for all y in the ranges of $\psi^B(\cdot,\alpha)$ and $\psi^S(\cdot,\alpha)$, necessarily $[\psi^B]^{-1}(y)-[\psi^S]^{-1}(y)>0$ because $\psi^B(x,\alpha)-x<0$ and $\psi^S(x,\alpha)-x>0$. Third, (2.13) and (2.14) imply that if $\alpha>0$ and $w\in(a,b)$, then $\psi^S(w,\alpha)-\psi^B(w,\alpha)>0$.

We can now prove the theorem's second part: $\lim_{\tau\to\infty} G(0,\tau)=0$. One form in which (3.01), the IF constraint, can be written is

$$G(\alpha,\tau)=M\int_a^b \psi^B(x,1)\,\bar{p}^{\tau\alpha}(x)\,f(x)\,dx - N\int_a^b \psi^S(z,1)[1-\bar{q}^{\tau\alpha}(z)]\,h(z)\,dz \geqslant 0. \tag{7.03}$$

Theorem 7 implies that, as τ increases, the variances of $\bar{p}^{\tau\alpha}(\cdot)$ and $\bar{q}^{\tau\alpha}(\cdot)$

[21] See (7.23) and (7.24).

[22] The inverses exist because regularity implies monotonicity of ψ^B and ψ^S.

approach zero. This means that in the limit, if $\alpha = 0$, both distributions become step functions with the step at the competitive price c. Thus

$$\bar{p}^{\infty 0}(x) = \begin{cases} 0 & \text{if} \quad x \leqslant c, \\ 1 & \text{if} \quad x > c, \end{cases} \tag{7.04}$$

and

$$\bar{q}^{\infty 0}(z) = \begin{cases} 0 & \text{if} \quad z \leqslant c \\ 1 & \text{if} \quad z > c. \end{cases} \tag{7.05}$$

Substitution of these into (7.03) and integrating the resulting expression shows that, for $\alpha = 0$ and $\tau \to \infty$, the IF constraint is satisfied:

$$\begin{aligned} &\lim_{\tau \to \infty} G(0, \tau) \\ &\quad = M \int_a^b \psi^{\mathrm{B}}(x, 1)\, \bar{p}^{\infty 0}(x) f(x)\, dx - N \int_a^b \psi^{\mathrm{S}}(z, 1)[1 - \bar{q}^{\infty 0}(z)]\, h(z)\, dz \\ &\quad = M \int_c^b (xf(x) + F(x))\, dx - N \int_a^c (zh(z) + H(z))\, dz - M \int_c^b dx \\ &\quad = M \int_c^b d[xF(x)] - N \int_a^c d[zH(z)] - M \int_c^b dx = 0 \end{aligned} \tag{7.06}$$

because $H(a) = 0$, $F(b) = 1$, and $M(1 - F(c)) = NH(c)$. Therefore in the limit, when the number of traders becomes infinite, the competitive price c satisfies the IF constraint, describes the ex ante efficient mechanism, and is ex post efficient.

We now prove the first half of the theorem. Fix the value of α within $(0, 1)$. The resulting α-mechanism transforms the vector of traders' reservation values $(x_1, ..., x_M, z_1, ..., z_N)$ into a vector of virtual reservation values $(\psi^{\mathrm{B}}(x_1, \alpha), ..., \psi^{\mathrm{S}}(z_N, \alpha))$ and assigns the N objects to the N traders who have the highest virtual reservation values. Suppose, for some $\hat{\tau}$, $G(\alpha, \hat{\tau}) < 0$. As τ increases from $\hat{\tau}$ the distributions $\bar{p}^{\tau\alpha}$ and $\bar{q}^{\tau\alpha}$ approach step functions. Therefore, as with (7.06),

$$\begin{aligned} \lim_{\tau \to \infty} G(\alpha, \tau) &= \lim_{\tau \to \infty} \left\{ M \int_a^b \psi^{\mathrm{B}}(x, 1)\, \bar{p}^{\tau\alpha}(x)\, f(x)\, dx \right. \\ &\qquad \left. - N \int_a^b \psi^{\mathrm{S}}(z, 1)[1 - \bar{q}^{\tau\alpha}(z)]\, h(z)\, dz \right\} \\ &= \int_{\bar{x}^*}^b dxF(x) - N \int_a^{\bar{z}^*} dzH(z) - M \int_{\bar{x}^*}^b dx \\ &= M[bF(b) - \bar{x}^* F(\bar{x}^*)] - N\bar{z}^* H(\bar{z}^*) - M(b - \bar{x}^*) \\ &= \bar{x}^* M(1 - F(\bar{x}^*)) - \bar{z}^* NH(\bar{z}^*) \\ &= (\bar{x}^* - \bar{z}^*)\, M(1 - F(\bar{x}^*)) > 0 \end{aligned} \tag{7.07}$$

because:

(a) asymptotically $M(1-F(\bar{x}^*))$ is the expected number of buyers whose reservation values are greater than $\psi^{B\alpha}_{(M)}$ and are therefore assigned an object;

(b) asymptotically $NH(\bar{z}^*)$ is the expected number of sellers whose reservation values are less than $\psi^{S\alpha}_{(M)}$ and are therefore assigned to sell their objects;

(c) $M(1-F(\bar{x}^*))=NH(\bar{z}^*)>0$ because the blance of goods constraint requires that supply equal demand; and

(d) $\bar{x}^*-z^*>0$ is shown at the proof's beginning.

The asymptotic normality of $\Lambda^{B\alpha}_\tau$ and $\Lambda^{S\alpha}_\tau$ and the differentiability of $\psi^B(\cdot,\alpha)$ and $\psi^S(\cdot,\alpha)$ imply that, as τ increases, $G(\alpha,\tau)$ approaches $\lim_{\tau\to\infty} G(\alpha,\tau)$ continuously. Therefore, a τ' must exist such that, for all $\tau>\tau'$, $G(\alpha,\tau)\geqslant 0$.

Proof of Theorem 5. The proof is based on an analysis of the asymptotic properties of the IF constraint, $G(\alpha,\tau)=0$. Recall that, for a given τ, the ex ante efficient mechanism is the α^*-mechanism where α^* is the root of $G(\alpha,\tau)=0$. Rewriting (3.01) and reversing its order of integration gives

$$G(\alpha,\tau)=M\int_a^b I(t)\,\rho_B(t;\alpha,\tau)\,dt+N\int_a^b J(t)\,\rho_S(t;\alpha,\tau)\,dt-NK=0 \qquad (7.08)$$

where

$$I(t)=\int_t^b \psi^B(x,1)f(x)\,dx, \qquad J(t)=\int_t^b \psi^S(z,1)\,h(z)\,dz, \qquad (7.09)$$

$$\rho_B(x;\alpha,\tau)=d\bar{p}^{\tau\alpha}(x)/dx, \qquad \rho_S(z;\alpha,\tau)=d\bar{q}^{\tau\alpha}(z)/dz, \qquad (7.10)$$

$$\bar{p}^{\tau\alpha}(x)=\int_a^x \rho_B(t;\alpha,\tau)\,dt, \qquad \bar{q}^{\tau\alpha}(z)=\int_a^z \rho_S(t;\alpha,\tau)\,dt, \qquad (7.11)$$

$$K=\int_a^b \psi^S(z,1)\,h(z)\,dz=b. \qquad (7.12)$$

The functions ρ_B and ρ_S are probability density functions for $\bar{p}^{\tau\alpha}$ and $\bar{q}^{\tau\alpha}$, respectively. As the first part of the proof of Theorem 4 points out, $\bar{p}^{\tau\alpha}$ and

$\bar{q}^{\tau\alpha}$ are asymptotically normal distribution functions with variances that are $O(1/\tau)$; thus asymptotically ρ_B and ρ_S are normal densities.[23]

Taylor series expansions around c, the competitive price, may be taken of $I(t)$ and $J(t)$ and substituted into (7.08):

$$\begin{aligned} G(\alpha, \tau) = M \int_a^b \Big\{ & I(c) + I'(c)(t-c) \\ & + I''(c) \frac{(t-c)^2}{2} + R_B(t) \Big\} \rho_B(t; \alpha, \tau)\, dt \\ & + N \int_a^b \Big\{ J(c) + J'(c)(t-c) \\ & + J''(c) \frac{(t-c)^2}{2} + R_S(t) \Big\} \rho_S(t; \alpha, \tau)\, dt - NK \\ = 0 & \end{aligned} \tag{7.13}$$

where $I'(c)$ and $J'(c)$ are first derivatives of I and J evaluated at c, $I''(c)$ and $J''(c)$ are second derivatives, and $R_B(t)$ and $R_S(t)$ are the remainder terms for the expansions. Two sets of terms may be dropped. First, a derivation similar to that of Eq. (7.06) shows that, for large τ,

$$M \int_a^b I(c)\, \rho_B(t; \alpha, \tau)\, dt + N \int_a^b J(c)\, \rho_S(t; \alpha, \tau)\, dt - NK = 0; \tag{7.14}$$

therefore these three terms may be dropped.[24] Second, the two remainder terms R_B and R_S may be dropped because, for large τ, they are inconsequential in comparison with the remaining terms. This follows from three facts:

(i) both terms are $O[(t-c)^2]$,

(ii) the densities $\rho_B(\cdot; \alpha, \tau)$ and $\rho_S(\cdot; \alpha, \tau)$ become spikes centered on c as τ becomes large and α approaches zero, and

(iii) the region of integration is a bounded interval.

[23] See footnote 24 for a qualification of this statement.

[24] The reason that we must make (7.14) conditional on τ being large is that $\bar{q}^{\tau\alpha}(a) > 0$ and $\bar{p}^{\tau\alpha}(b) < 1$ for small τ, i.e., they are improper distribution functions for small τ. As τ becomes larger, $\bar{p}^{\tau\alpha}(a) \to 0$ and $\bar{p}^{\tau\alpha}(b) \to 1$ very quickly. Specifically, Theorem 6.1 in Gresik and Satterthwaite [10] implies that both $\bar{p}^{\tau\alpha}(a)$ and $1 - \bar{p}^{\tau\alpha}(b)$ are $O(e^{-\tau})$. For large τ these quantities are negligible.

Leonard J. Mirman
Dilip Mookherjee
K. S. Moorthy
Walter Muller
Barry Nalebuff
Peter Neary
David Newbery
Lars Tyge Nielsen
Shmuel Oren
Martin J. Osborne
Joseph M. Ostroy
Thomas Palfrey
David Pearce
James Peck
Bezalel Peleg
Motty Perry
Christophe Prechac
Garey Ramey
Assaf Razin
Jennifer Reinganum
Philip Reny
Rafael Repullo
Marcel K. Richter
Rafael Rob
John E. Roemer
William Rogerson
Tomasz Rolski
Alvin E. Roth
R. Robert Russell
Daniel Royer
Donald Saari
Hamid Sabourian
Raaj Kumar Sah
Maurice Salles
Dov Samet
Larry Samuelson
Thomas J. Sargent
Richard Schmalensee
David Schmeidler
Francoise Schoumaker
Manimay Sen
Wayne J. Shafer
Avner Shaked
William Sharkey
William Sharpe
K. A. Shepsle
Joaquim Silvestre
Leo K. Simon
Vernon E. Smith
Joel Sobel
Stephen E. Spear
Sanjay Srivastava
William Stanford
Ross Starr
Jeremy Stein
Maxwell Stinchcombe
Peter A. Streufert
Raghu Sundaram
Jan Svejnar
Bart Taub
William Thomson
Eric van Damme
Xavier Vives
Peter P. Wakker
Michael Waldman
Henry Y. Wan, Jr.
Birger Wernerfelt
David Wettstein
John A. Weymark
Steven R. Williams
Asher Wolinsky
Myrna Holtz Wooders
Michael Woodford
Randall Wright
William R. Zame
Itzhak Zilcha

Let $\tilde{z}(\alpha, \tau) = [\psi^{S}]^{-1}(\psi^{B\alpha}_{(M)})$. Therefore $\psi^{B}[x(\alpha, \tau), \alpha] = \psi^{S}[\tilde{z}(\alpha, \tau), \alpha] = \psi^{B\alpha}_{(M)}$. The standard result that the asymptotic expectation of a function of a random variable equals the function of the variable's asymptotic expectation applies to $x(\alpha, x)$ and $\tilde{z}(\alpha, \tau)$. Therefore, for large τ,

$$\psi^{B}[\bar{x}(\alpha, \tau), \alpha] = \psi^{S}[\bar{\tilde{z}}(\alpha, \tau), \alpha] \tag{7.18}$$

where $\bar{x}(\alpha, \tau)$ is the expected value of $x(\alpha, \tau)$, etc.

For any realization of reservation values, exactly M traders must have virtual utilities less than or equal to the realization of $\psi^{B\alpha}_{(M)}$. This means that the expected number of traders with virtual reservation values less than or equal to $\psi^{B\alpha}_{(M)}$ is M. Therefore, asymptotically,

$$(M-1)\,F[\bar{x}(\alpha, \tau)] + NH[\bar{\tilde{z}}(\alpha, \tau)] = M \tag{7.19}$$

where $F[\bar{x}(\alpha, \tau)]$ is the probability that a buyer will have a reservation value such that $\psi^{B}(x_i, \alpha) < \psi^{B\alpha}_{(M)}$, $(M-1)\,F[\bar{x}(\alpha, \tau)]$ is the expected number of the $M-1$ buyers who will not be assigned an object because their virtual utility values are too low, etc.

Equations (7.18) and (7.19) implicity define $\bar{x}(\alpha, \tau)$ and $\bar{\tilde{z}}(\alpha, \tau)$. Holding τ constant, they may be differentiated with respect to α:

$$\begin{gathered}(M-1)\,f\bar{x}_\alpha + Nh\bar{\tilde{z}}_\alpha = 0,\\ \bar{x}_\alpha + \frac{F-1}{f} + \alpha\frac{f^2\bar{x}_\alpha - (F-1)f'\bar{x}_\alpha}{f^2} = \bar{\tilde{z}}_\alpha + \frac{H}{h} + \alpha\frac{h^2\bar{\tilde{z}}_\alpha - Hh'\bar{\tilde{z}}_\alpha}{h^2}\end{gathered} \tag{7.20}$$

where $H = H(c)$, $F = F(c)$, $f = f(c)$, $h = h(c)$, $f' = df(c)/x_i$, $h' = dh(c)/dz_j$, $\bar{x}_\alpha = \partial\bar{x}(0, \tau)/\partial\alpha$, $\bar{\tilde{z}}_\alpha = \partial\bar{\tilde{z}}(0, \tau)/\partial\alpha$, and c is the competitive price. The derivatives are evaluated at $\alpha = 0$ and c because, as τ becomes large, $\alpha \to 0$, $\bar{x} \to c$, and $\bar{\tilde{z}} \to c$. Solving the system for $\bar{x}_\alpha$ and evaluating it for large τ at $\alpha = 0$ gives

$$\bar{x}_\alpha = \frac{N[fH - (F-1)h]}{Nhf + (M-1)\,f^2} \approx K' \tag{7.21}$$

where K' is some constant. Similar calculations show that $\bar{z}_\alpha = K''$, $\partial\sigma^2_{B}/\partial\alpha = O(1/\tau)$, and $\partial\sigma^2_{S}/\partial\alpha = O(1/\tau)$. The denominator of (7.17) is therefore dominated by constant terms and, for large τ, is $O(1)$.

For large τ both sides of (7.17) can be integrated because its denominator is essentially constant:

$$\int_{\infty}^{\tau} \alpha'(\tau)\, d\tau = -\int_{\infty}^{\tau} \frac{M_0 I' \bar{x}_\tau + N_0 J' \bar{z}_\tau + \frac{1}{2}\left(M_0 I'' \frac{\partial \sigma_{\mathrm{B}}^2}{\partial \tau} + N_0 J'' \frac{\partial \sigma_{\mathrm{S}}^2}{\partial \tau}\right)}{M_0 I' \bar{x}_\alpha + N_0 J' \bar{z}_\alpha + \frac{1}{2}\left(M_0 I'' \frac{\partial \sigma_{\mathrm{B}}^2}{\partial \alpha} + N_0 J'' \frac{\partial \sigma_{\mathrm{S}}^2}{\partial \alpha}\right)}\, d\tau$$

$$= -\frac{1}{K}\left\{M_0 I' \int_{\infty}^{\tau} \bar{x}_\tau\, d\tau + N_0 J' \int_{\infty}^{\tau} \bar{z}_\tau\, d\tau\right\}$$

$$-\frac{1}{2K}\left\{M_0 I'' \int_{\infty}^{\tau} \frac{\partial \sigma_{\mathrm{B}}^2}{\partial \tau}\, d\tau + N_0 J'' \int_{\infty}^{\tau} \frac{\partial \sigma_{\mathrm{S}}^2}{\partial \tau}\, d\tau\right\} \tag{7.22}$$

where $\bar{x}_\tau$, $\bar{z}_\tau$, $\partial\sigma_{\mathrm{B}}^2/\partial\tau$, and $\partial\sigma_{\mathrm{S}}^2/\partial\tau$ are evaluated at $\alpha = 0$ where $K = M_0 I' K' + N_0 J' K''$. Therefore, for large τ,

$$\alpha(\tau) = \alpha(\infty) - \frac{1}{K} M_0 I'(\bar{x}(0, \tau) - \bar{x}(0, \infty)) - \frac{1}{K} N_0 J'(\bar{z}(0, \tau) - \bar{z}(0, \infty))$$

$$-\frac{1}{2K} M_0 I''(\sigma_{\mathrm{B}}^2(0, \tau) - \sigma_{\mathrm{B}}^2(0, \infty))$$

$$-\frac{1}{2K} N_0 J''(\sigma_{\mathrm{S}}^2(0, \tau) - \sigma_{\mathrm{S}}^2(0, \infty))$$

$$= O\left(\frac{(\ln \tau)^{1/2}}{\tau}\right) + O\left(\frac{1}{\tau}\right)$$

$$= O\left(\frac{(\ln \tau)^{1/2}}{\tau}\right). \tag{7.23}$$

This follows from three facts. First, when $\alpha = 0$, $x(\alpha, \tau) = \bar{z}(\alpha, \tau) = \psi_{(M)}^{\mathrm{B}\alpha}$ and $\lim_{\tau \to \infty}(\psi_{(M)}^{\mathrm{B}\alpha}) = c$. Second, Theorem 7 implies that

$$|E(\psi_{(M)}^{\mathrm{B}\alpha} - c)| = O\left(\frac{(\ln \tau)^{1/2}}{\tau}\right). \tag{7.24}$$

Third, Theorem 4 states that $\alpha(\infty) = 0$.

Proof of Theorem 6. A Taylor series expansion of the ex ante expected gains from trade, $T[\alpha(\tau), \tau]$, that an α^*-mechanism realizes is

$$T(0, \tau) + \alpha(\tau)\frac{\partial T(0, \tau)}{\partial \alpha} + \frac{1}{2}[\alpha(\tau)]^2 \frac{\partial^2 T[\varepsilon(\tau), \tau]}{\partial \alpha^2} \tag{7.25}$$

where $\varepsilon(\tau) \in [0, \alpha(\tau)]$. Three facts allow us to evaluate (7.25). First, for large τ, the ex post optimal mechanism assigns the N objects to those N

agents whose reservation values are greater than c, the competitive price. Therefore

$$T(0, \tau) \approx \tau M_0 \int_c^b (t-c)\, f(t)\, dt + \tau N_0 \int_a^c (c-t)\, h(t)\, dt = O(\tau) \quad (7.26)$$

for large τ.

Second, the last two terms on the right-hand side of (7.25) represent the ex post gains from trade that the ex ante optimal mechanism fails to realize as a consequence of $\alpha(\tau)$ being greater than zero. Let $S(\alpha, \tau)$ represent these two terms. S may be evaluated, for large τ, as follows. Recall from the proof of Theorem 4 the meaning of $\bar{x}(\alpha, \tau)$ and $\bar{z}(\alpha, \tau)$. For large τ the expected number of buyers excluded from trading as α increases from zero to $\alpha(\tau)$ is

$$\tau M_0 \int_c^{\bar{x}(\alpha, \tau)} f(t)\, dt \quad (7.27)$$

and the gains from trade that are lost from this exclusion are

$$\tau M_0 \int_c^{\bar{x}(\alpha, \tau)} (t-c)\, f(t)\, dt. \quad (7.28)$$

A similar expression exists for the gains from trade that the α^*-mechanism fails to realize on the sellers' side. Consequently, for large τ,

$$-S(\alpha, \tau) = \tau N_0 \int_{\bar{z}(\alpha, z)}^{c} (c-t)\, h(t)\, dt + \tau M_0 \int_c^{\bar{x}(\alpha, \tau)} (t-c)\, f(t)\, dt. \quad (7.29)$$

Differentiation gives

$$-\frac{\partial S(\alpha, \tau)}{\partial \alpha} = -\tau N_0[c - \bar{z}(\alpha, \tau)]\, h[\bar{z}(\alpha, \tau)]\, \bar{z}_\alpha + \tau M_0[\bar{x}(\alpha, \tau) - c]\, f[\bar{x}(\alpha, \tau)]\, \bar{x}_\alpha \quad (7.30)$$

and

$$-\frac{\partial^2 S(\alpha, \tau)}{\partial \alpha^2} = -\tau N_0((c - \bar{z})[h\bar{z}_{\alpha\alpha} + h'(\bar{z}_\alpha)^2] - h(\bar{z}_\alpha)^2) + \tau M_0((\bar{x} - c)[f\bar{x}_{\alpha\alpha} + f'(\bar{x}_\alpha)^2] + f(\bar{x}_\alpha)^2) \quad (7.31)$$

where $\bar{z}=\bar{z}(\alpha, \tau)$, $h=h[\bar{z}(\alpha, \tau)]$, $\bar{z}_\alpha=\partial\bar{z}(\alpha, t)/\partial\alpha$, $\bar{z}_{\alpha\alpha}=\partial^2\bar{z}(\alpha, \tau)/\partial\alpha^2$, $h'=dh[\bar{z}]/dz$, etc. Evaluated for large τ and $\alpha=0$ these derivatives are

$$\frac{\partial T(0, \tau)}{\partial\alpha}=\frac{\partial S(0, \tau)}{\partial\alpha}=0 \tag{7.32}$$

and

$$-\frac{\partial^2 T(0, \tau)}{\partial\alpha^2}=-\frac{\partial^2 S(0, \tau)}{\partial\alpha^2}=\tau(N_0 h(c)(\bar{z}_\alpha)^2+M_0 f(c)(\bar{x}_\alpha)^2)=O(\tau) \tag{7.33}$$

because $\alpha(\tau)\to 0$, $\bar{x}(\alpha, \tau)\to c$, $\bar{z}(\alpha, \tau)\to c$, $\bar{x}_\alpha\to K'$, and $\bar{z}_\alpha\to K''$ as $\tau\to\infty$. Finally, the third fact is Theorem 5's result that for large τ, $\alpha(\tau)=O((\ln\tau)^{1/2}/\tau)$.

These facts are sufficient to evaluate the expression of interest:

$$\begin{aligned}1-\frac{T[\alpha(\tau), \tau]}{T(0, \tau)}&=1-\frac{T(0, \tau)+\alpha(\tau)\dfrac{\partial T(0, \tau)}{\partial\alpha}+\dfrac{1}{2}[\alpha(\tau)]^2\dfrac{\partial^2 T(\varepsilon(\tau), \tau)}{\partial\alpha^2}}{T(0, \tau)}\\&=\frac{1}{2}\frac{[\alpha(\tau)]^2}{T(0, \tau)}\frac{\partial^2 T(0, \tau)}{\partial\alpha^2}\\&=\frac{\{O(\ln\tau)^{1/2}/\tau)\}^2}{O(\tau)}O(\tau)=O\left\{\frac{\ln\tau}{\tau^2}\right\},\end{aligned} \tag{7.34}$$

which proves the theorem.

References

1. R. Bhattacharya and M. Majumdar, Random exchange economies, *J. Econ. Theory* **6** (1973), 37–67.
2. K. Chatterjee and W. Samuelson, Bargaining under incomplete information, *Oper. Res.* **31** (1983), 835–851.
3. F. N. David and N. L. Johnson, Statistical treatment of censored data, I. Fundamental formulae, *Biometrika* **41** (1954), 228–240.
4. H. A. David, "Order Statistics," 2nd ed., Wiley, New York, 1981.
5. G. Debreu, The rate of convergence of the core of an economy, *J. Math. Econ.* **2** (1975), 1–7.
6. E. Dierker, Gains and losses at core allocations, *J. Math. Econ.* **2** (1975), 119–128.
7. A. F. Gibbard, Manipulation of voting schemes: A general result, *Econometrica* **41** (1973), 587–602.
8. T. Gresik, Efficient bilateral trade mechanisms for dependent-type markets, Washington University, 1987.
9. T. Gresik and M. Satterthwaite, "The Number of Traders Required to Make a Market Competitive: The Beginnings of a Theory," CMSEMS DP. No. 551, Northwestern University, 1983.

10. T. GRESIK AND M. SATTERTHWAITE, "The Rate at Which a Simple Market Becomes Efficient as the Number of Traders Increases: An Asymptotic Result for Optimal Trading Mechanisms, CMSEMS DP No. 641, Northwestern University, 1985.
11. K. HAGERTY AND W. ROGERSON, Robust trading mechanisms, *J. Econ. Theory* **42** (1985), 94–107.
12. P. HALL, Some asymptotic expansions of moments of order statistics, *Stochastic Process. Appl.* **7** (1978), 265–275.
13. M. HARRIS AND A. RAVIV, A theory of monopoly pricing schemes with demand uncertainty, *Amer. Econ. Rev.* **71** (1981), 347–365.
14. M. HARRIS AND R. M. TOWNSEND, Resource allocation under asymmetric information, *Econometrica* **49** (1981), 33–64.
15. J. C. HARSANYI, Games with incomplete information played by Bayesian players, Parts I, II, and III. *Management Sci.* **14** (1967–68), 159–182, 320–334, 486–502.
16. W. HILDENBRAND, "Core and Equilibria of a Large Economy," Princeton University Press, Princeton, NJ, 1974.
17. B. HOLMSTROM AND R. B. MYERSON, Efficient and durable decision rules with incomplete information, *Econometrica* **51** (1983), 1799–1820.
18. J. LEDYARD, The scope of the hypothesis of Bayesian equilibrium. *J. Econ. Theory* **39** (1986), 59–82.
19. W. LEININGER, P. LINHART, AND R. RADNER, Equilibria of the sealed-bid mechanism for bargaining with incomplete information, *J. Econ. Theory* **48** (1989), 63–106.
20. P. MILGROM AND R. WEBER, A theory of auctions and competitive bidding, *Econometrica* **50** (1982), 1089–1122.
21. R. B. MYERSON, Incentive compatibility and the bargaining problem, *Econometrica* **47** (1979), 61–73.
22. R. B. MYERSON, Optimal auction design, *Math. Oper. Res.* **6** (1981), 58–73.
23. R. B. MYERSON AND M. A. SATTERTHWAITE, Efficient mechanisms for bilateral trading, *J. Econ. Theory* **29** (1983), 265–281.
24. R. B. MYERSON, Two-person bargaining problems with incomplete information, *Econometrica* **52** (1984), 461–488.
25. J. ROBERTS AND A. POSTLEWAITE, The incentive for price-taking behavior in large exchange economies, *Econometrica* **44** (1976), 115–128.
26. M. SATTERTHWAITE AND S. WILLIAMS, Bilateral trade with the sealed-bid k-double auction: Existence and efficiency, *J. Econ. Theory* **48** (1989), 107–133.
27. M. SATTERTHWAITE AND S. WILLIAMS, Rate of convergence to efficiency in the buyers bid double auction as the market becomes large, *Rev. Econ. Studies*, in press.
28. R. J. SERFLING, "Approximation Theorems of Mathematical Statistics," Wiley, New York, 1980.
29. S. WILLIAMS, Efficient performance in two agent bargaining, *J. Econ. Theory* **41** (1987), 154–172.
30. R. WILSON, Efficient trading, *in* "Issues in Contemporary Microeconomics and Welfare" (G. Feiwel, Ed.), Macmillan, London, 1985.
31. R. WILSON, Incentive efficiency of double auctions, *Econometrica* **53** (1985), 1101–1115.