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The Rate of Convergence to Efficiency in the Buyer's Bid Double Auction as the Market Becomes Large

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A trader who privately knows his preferences may misrepresent them in order to influence the market price. This strategic behaviour may prevent realization of all gains from trade. In this paper, trade in a simple market with an explicit rule for price formation is modelled as a Bayesian game. We show that the difference between a trader's bid and his reservation value is maximally $O(1/m)$ where m is the number of traders on each side of the market. Competitive pressure as m increases thus quickly overcomes the inefficiency private information causes and forces the market towards an efficient allocation.

1. INTRODUCTION

A goal of a market is to implement a Pareto efficient allocation of resources. Classical general equilibrium theory focuses on the existence of prices that implement an efficient allocation. Solving for such prices requires information about traders' preferences. This information is typically not possessed by any one individual or institution, for each trader typically has some private information about his own preferences. The market must somehow elicit the necessary private information if it is to implement an efficient allocation.

A major obstacle to accomplishing this is the incentive that traders may have to misrepresent their private information. In a market with prices this strategic behaviour takes the form of distorting supply and demand in order to influence price. This behaviour may cause ex post inefficiency, i.e. all potential gains from trade may not be realized. Intuitively, strategic behaviour is only significant in small markets, for the likelihood that a trader can affect price decreases as the number of traders in the market becomes large. In the limiting case of a market with a continuum of traders strategic behaviour vanishes and traders willingly reveal their private information. Appropriate prices can then be calculated and efficiency results.

This paper develops the intuition that the number of traders is critical to the performance of a market by using Harsanyi's notion (1967-68) of a Bayesian game to model the impact of private information upon a simple market. We consider an independent, private values model. There are m sellers, each having a single item to sell, and m buyers, each wanting to buy at most one item. Each trader has a reservation value for the item that is independently drawn from the unit interval; a seller's value is drawn from distribution F_1 and a buyer's value is drawn from distribution F_2. A trader privately knows his own reservation value. Each trader is risk neutral.

The items are reallocated according to the following rules. Sellers and buyers simultaneously submit offers and bids. These offers and bids determine a closed interval in which a market-clearing price can be selected. We choose as the price the upper endpoint of this interval. Trade then occurs at this price between those buyers whose bids are at least as great as it and sellers whose offers are strictly less than it. We call this procedure the *buyer's bid double auction* (BBDA) because in the one seller-one buyer case the buyer's bid determines the price whenever trade occurs.

We consider Bayesian Nash equilibria in which all sellers use one strategy and all buyers use a second strategy. Each seller in the BBDA has a dominant strategy to set his offer equal to his reservation value because he can not influence the price when he trades. In response to these dominant strategies, each buyer has an incentive to bid less than his reservation value, which causes the BBDA to be ex post inefficient. We show, however, that the amount of misrepresentation by buyers must be small when the market is large. In fact, we prove that in any equilibrium strategy of the buyers the difference between a buyer's bid and his reservation value is $O(1/m)$, regardless of the distributions of the reservation values. Thus, as the market grows, competitive pressures quickly force buyers towards truthful revelation and the equilibrium outcome towards an ex post efficient, perfectly competitive allocation.

Three aspects of our result deserve emphasis. First, it applies to all Bayesian Nash equilibria in which the buyers adopt the same response to the sellers' dominant strategy of truthful revelation. Large misrepresentation simply cannot be equilibrium behaviour in a large market regardless of which equilibrium is chosen. Second, the rules of the BBDA give each seller the incentive to honestly report his reservation value. Under other rules for selecting a market clearing price both sellers and buyers misreport in equilibrium. Third, numerical examples suggest that for m as small as six almost all gains from trade are realized, i.e. essentially ex post efficient outcomes are achieved. Consequently, for all but small m, it may not be worthwhile to use Holmstrom and Myerson's (1983) information-based concepts of incentive efficiency to evaluate the BBDA's equilibria.

The questions that give rise to this work were articulated by Hayek (1945), Arrow (1959), Hurwicz (1972), and others. Hayek emphasized the importance of modelling the impact of private information upon an economy: the resource allocation problem

> ... is thus in no way solved if we can show that all the facts, if they were known to a single mind (as we hypothetically assume them to be given to the observing economist), would uniquely determine the solution; instead we must show how a solution is produced by the interactions of people each of whom possesses only partial knowledge (1945, p. 530).

Arrow (1959) criticized general equilibrium theory for failing to explain how Walrasian prices are formed. Hurwicz has been a pioneer in formally evaluating the informational and incentive feasibility of economic mechanisms:

> On the informational side, the question is whether the mechanism allows for the dispersion of information and limitations on the capacity of various units to process information. On the incentive side, there is the problem whether the rules prescribed by the mechanism are compatible with either individual or group incentives (1972, p. 298–299).

Because we model the BBDA as a Bayesian game, our result reflects the role that private information and individual incentives play within an explicit process of price formation. We do not, however, deal with limitations on rationality and information processing. Nevertheless, our result that all equilibrium strategies of the BBDA in a large market are close to truthful revelation suggests that cognitive limitations are unimportant in large markets.

Our result is reminiscent of a classic result in general equilibrium theory. Building on Debreu and Scarf's (1963) result on the convergence of the core to the Walrasian allocations, Debreu (1975) and Grodal (1975) showed that as a regular Arrow–Debreu economy is replicated, the maximum distance between a core allocation and its nearest Walrasian allocation is $O(1/m)$, where m is the number of replications.[1] Beyond the obvious fact that the same rate holds, both results show that in a large market equilibrium outcomes are close to Walrasian outcomes. Some differences, however, between these results should be kept in mind. First, they are based upon very different notions about what happens in a market. The core convergence results assume that the outcome of trade is efficient no matter how many traders are present, while in our model private information and individual incentives cause inefficiency in any finite market. Second, the core convergence results neither explain how a core allocation is achieved nor how prices are formed. Our result concerns an explicit procedure for price formation. If "price-taking behaviour" means accepting prices rather than trying to manipulate them in one's favour, then our result provides insight into this topic, while the core convergence results cannot. It is important to note, however, that the Arrow–Debreu framework is much richer than our elementary model.

A precursor of our result is Roberts and Postlewaite's (1976) study of the non-cooperative incentive that an agent within an Arrow–Debreu exchange economy has to act strategically. In their model each agent first reports an excess demand function, a competitive equilibrium is computed based on the reports, and finally goods are allocated according to the computed solution. They show that as a generic economy becomes large each agent's maximal gain from misreporting his excess demand function vanishes. Their result, while related, is different from ours because it does not concern equilibrium behaviour by the agents, it does not model agents' preferences as private information, and it does not state a rate at which misreporting vanishes.

Most directly our work stems from earlier research on Bayesian game models of double auctions. Chatterjee and Samuelson (1983) and Leininger *et al.* (1986) analyzed bilateral double auctions. The mathematical approach of this paper follows naturally from Satterthwaite and Williams's (1987) analysis of the bilateral case. Wilson (1985) showed that double auctions achieve Holmstrom and Myerson's (1983) standard of interim incentive efficiency when the market is sufficiently large. This paper complements Wilson's result by showing that markets also converge at a specified, rapid rate to ex post efficiency as they grow larger.

Our work also stems from the analysis of trading from the mechanism design viewpoint. Myerson and Satterthwaite (1983) developed techniques for computing the optimal revelation mechanism when reservation values are private on both sides of the market. For given distributions F_1 and F_2 the optimal mechanism maximizes the ex ante expected gains from trade subject to the constraints of private information and strategic behaviour. Gresik and Satterthwaite (1986, Theorem 5) showed that if the optimal mechanism is used, then the maximal gap between the reservation values of a buyer and a seller who are ex post inefficiently excluded from trade is at most $O((\ln m)^{1/2})/m)$.

1. See Hildenbrand (1982) and Dierker (1982) for surveys of this topic.

They conjectured that the tighter $O(1/m)$ rate of our result holds. Our convergence result improves upon theirs in two ways. First, it verifies their conjecture, for the order of the optimal mechanism's bound must be as small as the order of the BBDA's bound. Second, our result concerns a realistic trading procedure. The rules of the BBDA are stated in terms of the offers and bids; the Bayesian game framework is used not to define the BBDA but to analyze the outcome of trade under this procedure when there is private information. By contrast, an optimal mechanism is defined in terms of the distributions F_1 and F_2; changing the distributions changes the optimal mechanism's rules for allocating the items. As Wilson (1987) has emphasized, the rules of real-world trading mechanisms are independent of the underlying distributions.

Finally, McAfee and McMillan (1987) have surveyed the literature on one-sided and double auctions.[2] Their survey shows both the debt that our paper and other papers on double auctions owe to the literature on one-sided auctions and the distance that the double auction literature has to go before it reaches an equivalent level of sophistication. For example, our results are for the independent, private values model only. Consideration of Milgrom and Weber's (1982) more general model of affiliated values has as yet proved intractable.

2. NOTATION, MODEL, AND PRELIMINARY OBSERVATIONS

Consider a market with m buyers ($m \geqq 2$) and m sellers in which each seller wishes to sell an indivisible item and each buyer wishes to purchase at most one item.[3] Each seller has a reservation value independently drawn from the distribution F_1 and each buyer has a reservation value independently drawn from F_2. A trader's reservation value is his own private information. Each distribution F_i is a C^1 function whose density $f_i = F_i'$ is positive at every point in $(0, 1)$ and zero outside $[0, 1]$. The distributions F_1 and F_2 are common knowledge among the traders. We use v_1 to denote a seller's reservation value and v_2 to denote a buyer's reservation value. A seller's utility is zero if he fails to sell his item and $p - v_1$ if he does sell and the market price is p. Similarly a buyer's utility is zero if he fails to buy and $v_2 - p$ if he does buy.

These are the common knowledge rules of the BBDA. Every trader simultaneously submits an offer/bid. These offers/bids are arrayed in increasing order $s_{(1)} \leqq s_{(2)} \leqq \cdots \leqq s_{(2m)}$ and the price p is set at $s_{(m+1)}$. Trade occurs among sellers whose offers are strictly less than p and buyers whose bids are greater than or equal to p. When ties occur, p may not be a market-clearing price. In order to explain exactly who trades under the BBDA we refer to Table 2.1.

TABLE 2.1

Determination of the market price

	Sellers	Buyers
No. offers/bids $> s_{(m+1)}$	s	t
No. offers/bids $= s_{(m+1)}$	k	j
No. offers/bids $< s_{(m+1)}$	$m-s-k$	$m-t-j$

2. In a one-sided auction a seller with a known reservation value is attempting to maximize his revenue in selling an object(s) to a set of buyers whose reservation values are private. Thus the distinction between a one-sided auction and a double auction of the type we are studying is that in a double auction both buyers and sellers have private information while in a one-sided auction only the buyers have private information.

3. We have excluded the bilateral case ($m = 1$) because its analysis is different from the $m \geqq 2$ case. See Satterthwaite and Williams (1987).

Let s be the number of sellers whose offers exceed p, k the number of sellers whose offers equal p, t the number of buyers whose bids exceet p, and j the number of buyers whose bids equal p. There are $(m-s-k)$ offers and $(m-t-j)$ bids less than p. Note that $(s+k+t+j) \geqq m$ traders offer/bid at least as much as p, since $p = s_{(m+1)}$. Therefore

$$t+j \geqq m-s-k, \tag{2.1}$$

which means the demand $(t+j)$ at the price p is necessarily at least as large as the supply $(m-s-k)$.

Consider the case in which a single offer/bid uniquely determines $s_{(m+1)}$, i.e. $j+k=1$ and $t+s=m-1$. In (2.1) bring $s+k$ to the left-hand side; the left-hand side then sums exactly to m and (2.1) holds with equality. In this case, supply exactly equals demand and every buyer whose bid is at least p purchases an item and every seller whose offer is less than p sells his item. Next consider the remaining case in which at least two offer/bids equal $s_{(m+1)}$, i.e. $j+k \geqq 2$ and demand $t+j$ may strictly exceed supply $(m-s-k)$. The BBDA then prescribes that the supply of $(m-s-k)$ items is allocated beginning with the buyer who bid the most and working down the list of buyers whose bids are at least p. If in this process a point is reached where two or more buyers submitted identical bids and the remaining supply of unassigned items is insufficient to serve them, then the available supply is rationed among these bidders using a lottery that assigns each an equal chance of receiving an item. This completes the definition of the BBDA.

We adopt the Bayesian game framework to analyse the outcome of trade. Within this framework a trader's reservation value is his *type* and his *strategy* is a function that specifies an offer/bid for each of his possible types. An *equilibrium* consists of a strategy for each trader such that, for each of his possible reservation values, the offer/bid his strategy specifies maximizes his expected utility given the other traders' strategies and the distributions of their reservation values.

We now identify some basic properties of equilibria in the BBDA. The most fundamental property is that a seller can not influence the price p at which he trades by altering his offer. This follows from the BBDA's rule that a seller only sells if his offer is strictly less than the price $p \equiv s_{(m+1)}$. As noted by Wilson (1983), it follows that sellers have no incentive to act strategically, i.e. each seller's dominant strategy is to submit his reservation value as his offer.[4] Let $\tilde{S}$ denote this strategy:

$$\tilde{S}(v_1) = v_1 \tag{2.2}$$

for all $v_1 \in [0, 1]$.

Theorem 2.1. *In the BBDA*, $\tilde{S}$ is a dominant strategy for each seller.

Proof. Select a strategy for each buyer and for all but one of the sellers, and let v_1 be the reservation value of the exceptional seller. This seller would be no worse off by submitting an offer of $b = v_1$ rather than $b' > v_1$ because: (i) if he sells the item with the offer b' at a price $p > b'$, then he also sells it with the offer of $b = v_1$ at the unchanged price p; and (ii) if he fails to sell the item with the offer $b' \geqq p$, he can only gain if he instead offers $b = v_1$, for the price whenever he trades necessarily exceeds his offer. A similar analysis shows that the seller is no worse off with the offer of $b = v_1$ than an offer $b'' < v_1$. ||

We assume throughout this paper that all sellers adopt the strategy $\tilde{S}$. We also assume that all buyers use the same strategy. Let B denote the common strategy of buyers and let $\langle \tilde{S}, B \rangle$ denote a set of strategies in which each seller plays $\tilde{S}$ and each buyer plays B.

4. A stand-off equilibrium also exists in which all buyers bid zero, all sellers offer one, and no trade occurs.

In order to further establish the properties of equilibria $\langle \tilde{S}, B\rangle$ we need additional notation:

$\pi(v_2, b; B)$ = a buyer's expected utility when v_2 is his reservation value, b is his bid, and B is the common strategy of the other buyers;

$P(b; B)$ = the probability a buyer will trade when b is his bid and B is the common strategy of the other buyers;

$C(b; B)$ = the expected payment of a buyer when b is his bid and B is the common strategy of the other buyers.

Note that $\pi(v_2, b; B) = v_2 P(b; B) - C(b; B)$ and that $P(\cdot; B)$ is a probability distribution on the interval $[0, 1]$. Finally, $P(\cdot; B)$ is strictly increasing on this interval because (i) the density f_1 is positive on $(0, 1)$ and (ii) each seller uses his dominant strategy $\tilde{S}$.

Theorem 2.2. *If $\langle \tilde{S}, B\rangle$ is an equilibrium in the BBDA, then the function B has the following properties*: (i) $0 < B(v_2)$ *for all* $v_2 \in (0, 1]$; (ii) $B(v_2) \leqq v_2$ *for all* $v_2 \in [0, 1)$; (iii) $B(v_2)$ *is strictly increasing on* $[0, 1]$ *and differentiable almost everywhere.*

Proof. An important preliminary observation is this. Select a buyer. Suppose each seller uses $\tilde{S}$ and the other $m-1$ buyers use the strategy B, where no restriction is placed on B. For any $p \in (0, 1)$, if the selected buyer bids p, then there is a positive probability that the price will be p and the selected buyer will receive an item at this price. This is true because, given any array of bids from the $m-1$ buyers using B, a positive probability always exists that the offers of the m sellers will fall such that exactly m of the offers/bids of these $2m-1$ traders are strictly less than p, i.e. $p = s_{(m+1)}$.

This observation immediately implies (i) and (ii). If a buyer with reservation value $v_2 > 0$ bids $b'' \leqq 0$, then his expected utility is zero because no seller's offer will be less than b''. Bidding $b' \in (0, v_2)$, however, provides him with a positive probability of a profitable trade. This proves (i). If a type v_2 buyer ($v_2 < 1$) bids $b > v_2$, then a positive probability exists that the price will be in $(v_2, b]$ and he will trade at a loss. Reducing his bid to $b = v_2$ eliminates these losses without eliminating any profitable trades. This proves (ii).[5]

An argument from Chatterjee and Samuelson (1983, Theorem 1) shows that B must be non-decreasing. Let $v_2'' > v_2'$. Because $\langle \tilde{S}, B\rangle$ is an equilibrium,

$$\pi(v_2', B(v_2'); B) - \pi(v_2', B(v_2''); B) \geqq 0 \tag{2.3}$$

and

$$\pi(v_2'', B(v_2''); B) - \pi(v_2'', B(v_2'); B) \geqq 0. \tag{2.4}$$

Adding these inequalities produces

$$\pi(v_2'', B(v_2''); B) - \pi(v_2', B(v_2''); B) + \pi(v_2', B(v_2'); B) - \pi(v_2'', B(v_2'); B) \geqq 0. \tag{2.5}$$

Recall that $\pi(v_2, b; B) = v_2 P(b; B) - C(b; B)$. Using this formula, (2.5) reduces to

$$(v_2'' - v_2')P(B(v_2''); B) + (v_2' - v_2'')P(B(v_2'); B) \geqq 0. \tag{2.6}$$

or equivalently

$$(v_2'' - v_2')[P(B(v_2''); B) - P(B(v_2'); B)] \geqq 0. \tag{2.7}$$

5. We can not rule out extremely small bids (e.g. $b = -1000$) for a type-zero buyer and extremely large bids for a type-one buyer. These are probability zero cases that do not affect the expected utilities of other traders and therefore do not affect equilibrium calculations.

By assumption, $v_2'' > v_2'$; therefore, $P(B(v_2''); B) \geqq P(B(v_2'); B)$. Since $P(\cdot; B)$ is increasing, we conclude that $B(v_2'') \geqq B(v_2')$.

We now show by contradiction that B cannot be constant over any interval with non-empty interior. Suppose that $B(v_2) = b'$ for all v_2 in such an interval I. The bounds that we have derived upon B imply that $0 < b' < 1$. Our argument rests upon the following point: the probability of trade $P(b; B)$ is discontinuous at $b = b'$. This is true because the following events occur simultaneously with positive probability: (i) each buyer's reservation value is in I and therefore all buyers bid b', (ii) at least one seller's offer is less than b', and (iii) at least one seller's offer is greater than b'. Note that (ii) and (iii) require that $m \geqq 2$. Stipulations (i)-(iii) imply that the market price is b', the market fails to clear at this price, and the available units are allocated randomly among the buyers. Raising the selected buyer's bid from b' to $b'' > b'$ ensures that he receives an item with probability one in the stipulated situation rather than with some probability less than one under the random allocation rule. Therefore an $\varepsilon > 0$ exists such that $P(b''; B) > P(b'; B) + \varepsilon$ for all $b'' > b'$.

We next bound the change in the buyer's expected payment when he raises his bid from b' to b''. The change in his bid increases his payment only if either: (i) he trades with the bid b'' but would fail to trade with the bid b'; or (ii) the bid b' would be the market price, and he just drives up the price by raising his bid. In event (i), his payment is no more than b'', and in event (ii), the change in his payment is no more than $b'' - b'$. This implies the following bound on the change in the expected payment:

$$C(b''; B) - C(b'; B) \leqq b''[P(b''; B) - P(b'; B)] + (b'' - b'). \tag{2.8}$$

Because $B(v_2) \leqq v_2$ for all v_2, there exists a v_2' in I such that $B(v_2') = b' < v_2'$. To obtain a contradiction we show that the type v_2' buyer has an incentive to incrementally raise his bid above b'. The change in his expected payoff from increasing his bid from b' to some $b'' \in (b', v_2')$ is

$$\begin{aligned} \pi(v_2', b''; B) - \pi(v_2', b'; B) &= v_2'[P(b''; B) - P(b'; B)] + C(b'; B) - C(b''; B) \\ &\geqq (v_2' - b'')[P(b''; B) - (P(b'; B)] + (b' - b'') \\ &> (v_2' - b'')\varepsilon + (b' - b''). \end{aligned} \tag{2.9}$$

For b'' near b', (2.9) is positive, which contradicts the assumption that $\langle \tilde{S}, B \rangle$ is an equilibrium.

Finally, the existence of B' almost everywhere follows from the monotonicity of B by a well-known theorem in analysis (e.g. see Royden (1968, p. 96)). ||

Two points should be emphasized about the monotonicity of the buyers' strategy in an equilibrium $\langle \tilde{S}, B \rangle$. First, it implies that the probability of ties in the array of offers and bids is zero. Consequently we can ignore ties and the randomized allocations that they may necessitate. Second, the argument in Theorem 2.2 can be applied to double auctions besides the BBDA to show that when $m \geq 2$ an equilibrium common strategy of either side of the market must be increasing over all intervals in which the probability of trade is positive. Equilibrium strategies in the bilateral case may not be increasing; Leininger, Linhart, and Radnar (1986), for instance, have derived step-function equilibria in the bilateral split-the-difference double auction. Such equilibria, however, do not exist in this double auction when $m \geqq 2$.

3. THE FIRST-ORDER APPROACH

This section concerns a buyer's first-order condition for maximizing his expected utility conditional on his reservation value v_2, the use of a common strategy B by the other $m-1$ buyers, and the use of $\tilde{S}$ by each seller. If $\langle \tilde{S}, B \rangle$ is an equilibrium, then this conditional expected utility is maximized at $B(v_2)$. We interpret the first-order condition as a differential equation that must be satisfied almost everywhere by any function B that defines an equilibrium $\langle \tilde{S}, B \rangle$. Conversely, we show that any increasing function B defines an equilibrium $\langle \tilde{S}, B \rangle$ if (i) B satisfies the differential equation, (ii) B respects the bounds $0 < B(v_2) < v_2$ for all $v_2 \in (0, 1]$, and (iii) the distribution F_1 of each seller's reservation value satisfies a monotonicity condition.

The first-order condition is formally derived in the Appendix. Here we state the condition and explain it intuitively. In order to state it we must define three probabilities:

K_m = the probability that bid b lies between $s_{(m-1)}$ and $s_{(m)}$ in a sample of $m-1$ buyers using strategy B and $m-1$ sellers using $\tilde{S}$.

L_m = the probability that bid b lies between $s_{(m-1)}$ and $s_{(m)}$ in a sample of $m-2$ buyers using strategy B and m sellers using $\tilde{S}$.

M_m = the probability that the bid b lies between $s_{(m)}$ and $s_{(m+1)}$ in a sample of $m-1$ buyers using strategy B and m sellers using $\tilde{S}$.

Recall that $s_{(k)}$ is the kth-order statistic (i.e. the offer/bid that ranks kth from the bottom) in the specified sample.

Suppose a type v_2 buyer considers raising his bid by Δb above the value b, which may or may not equal the value $B(v_2)$. Assuming that $b = B(\bar{v}_2)$ for some $\bar{v}_2$ and that $B'(v_2)$ exists, his incremental expected utility is

$$\left[mf_1(b) K_m \Delta b + (m-1) \frac{f_2(\bar{v}_2)}{B'(\bar{v}_2)} L_m \Delta b \right] (v_2 - b - \Delta b) - M_m \Delta b. \tag{3.1}$$

The buyer has two considerations in raising his bid. First, it may increase his probability of obtaining an item and, second, it may increase by Δb the price he pays for an item that he would have received at price b. These two considerations correspond respectively to the two terms in (3.1), which we now explain in detail.

The term in brackets represents the probability that the selected buyer obtains an item by raising his bid. If initially he does not receive an item, then some buyer or seller's offer/bid above b determines the price p. If raising his bid is to benefit the buyer, then p must be in $(b, b+\Delta b)$, i.e. p must be just above b so that he surpasses it and becomes one of the buyers who purchases an item.

Select a seller in addition to the selected buyer. The probability that this seller's bid falls in the interval $(b, b+\Delta b)$ is $f_1(b)\Delta b$. Conditional on it falling in the interval and on the selected buyer bidding b, the probability that this offer determines the market price is K_m. Note that this probability is calculated on a sample of the remaining $m-1$ bids and $m-1$ offers because the selected buyer's bid and the selected seller's offer are fixed. Any of the m sellers could have been selected, so the probability that by increasing his bid the selected buyer jumps over a price-determining seller's offer is $mf_1(b)K_m\Delta b$. A similar argument shows that $(m-1)f_2(\bar{v}_2)L_m\Delta b/B'(\bar{v}_2)$ is the probability that the selected buyer jumps over a price-determining buyer's bid as he increases his bid. The density of a buyer's bids at b is $f_2(\bar{v}_2)/B'(\bar{v}_2)$, not $f_2(\bar{v}_2)$, because the distribution of a buyer's bids is different from the distribution of his reservation values. Finally, the selected buyer's expected gain from obtaining an item by raising his bid is the term in brackets times the gain $v_2 - b - \Delta b$ when this happens.

On the other side of the ledger is $M_m \Delta b$. If the buyer is the trader whose bid determines the price, then raising his bid Δb increases the price that he pays for an item by Δb. The expected cost of raising his bid is therefore Δb times the probability M_m that he is in fact the price-determining trader.

From (3.1) we obtain the formula for the marginal expected utility of a type v_2 buyer whose bid is b:

$$\frac{d\pi(v_2, b; B)}{db} = \left[mf_1(b)K_m + (m-1)\frac{f_2(\bar{v}_2)}{B'(\bar{v}_2)} L_m \right] (v_2 - b) - M_m. \tag{3.2}$$

If $\langle \tilde{S}, B \rangle$ is an equilibrium, then B satisfies the first-order condition $d\pi(v_2, B(v_2); B)/db = 0$ at all reservation values v_2 where B' exists.

To obtain a differential equation in the strategy B we must define the probabilities K_m, L_m, and M_m so that their values are functions only of the point (v_2, b):

$$K_m(v_2, b) = \textstyle\sum_{i=0}^{m-1} \displaystyle\binom{m-1}{i}^2 F_1(b)^{m-1-i}(1-F_1(b))^i F_2(v_2)^i (1-F_2(v_2))^{m-1-i}, \tag{3.3}$$

$$L_m(v_2, b) = \textstyle\sum_{i=1}^{m-1} \displaystyle\binom{m}{i}\binom{m-2}{i-1} F_1(b)^{m-i}(1-F_1(b))^i F_2(v_2)^{i-1}(1-F_2(v_2))^{m-i-1}, \tag{3.4}$$

$$M_m(v_2, b) = \textstyle\sum_{i=0}^{m-1} \displaystyle\binom{m-1}{i}\binom{m}{i} F_1(b)^{m-i}(1-F_1(b))^i F_2(v_2)^i (1-F_2(v_2))^{m-1-i}. \tag{3.5}$$

The probabilities K_m, L_m, and M_m in (3.1-3.2) are obtained by evaluating (3.3-3.5) at $v_2 = B^{-1}(b)$.

That $K_m(B^{-1}(b), b)$ is the probability that the bid b lies between $s_{(m-1)}$ and $s_{(m)}$ in a sample of $m-1$ buyers using strategy B and $m-1$ sellers using strategy $\tilde{S}$ can be seen as follows. The statement that b lies between $s_{(m-1)}$ and $s_{(m)}$ means that $m-1$ offers/bids are below b and that the remaining $m-1$ offers/bids in the sample are above b. We sum the probabilities of all possible events in which exactly $m-1$ offers/bids are less than b. A total of $m-1$ offers/bids less than b may be obtained by i bids and $(m-1-i)$ offers less than b. For a particular selection of i buyers and $(m-1-i)$ sellers, the probability that only their offers/bids are less than b is $F_1(b)^{m-1-i}(1-F_1(b))^i \times F_2(v_2)^i(1-F_2(v_2))^{m-1-i}$ where $v_2 = B^{-1}(b)$. $F_1(b)$ is the probability that a particular seller (using strategy $\tilde{S}$) offers less than b, and $F_2(v_2) = F_2(B^{-1}(b))$ is the probability that a particular buyer (using strategy B) bids less than b. The term

$$\binom{m-1}{i}^2 = \binom{m-1}{i}\binom{m-1}{m-1-i}$$

is the number of ways of simultaneously choosing i buyers from $m-1$ buyers and $(m-1-i)$ sellers from $m-1$ sellers. Similar arguments show that L_m and M_m are given by (3.4) and (3.5).[6]

A differential equation in the strategy B is obtained by setting (3.2) equal to zero and regarding K_m, L_m, and M_m as functions of v_2 and b. Suppose $\langle \tilde{S}, B \rangle$ is an equilibrium.

6. See David (1981, Chapter 2) for a discussion of this type of probability calculation.

Because B is necessarily increasing we can invert B and regard a buyer's reservation value v_2 as a function of his bid b, i.e. $v_2 = v_2(b) \equiv B^{-1}(b)$ and $\dot{v}_2 \equiv dv_2(b)/db = 1/B'(v_2)$. Substituting $\dot{v}_2$ into the differential equation and solving gives

$$\dot{v}_2 = \frac{M_m(v_2, b) - mf_1(b)K_m(v_2, b)(v_2 - b)}{(m-1)f_2(v_2)L_m(v_2, b)(v_2 - b)} \tag{3.6}$$

$$\dot{b} = 1 \tag{3.7}$$

where the tautology $\dot{b} \equiv db/db = 1$ has been added. Written in this form, the differential equation defines a vector field $(\dot{v}_2, \dot{b})$.

If $\langle \tilde{S}, B \rangle$ is an equilibrium, then (3.6-3.7) hold at every point $(v_2, B(v_2))$ at which $B'(v_2)$ exists. To establish a converse, we assume that the distribution F_1 of a seller's reservation value satisfies the following monotonicity property:

$$v_1 + F_1(v_1)/f_1(v_1) \text{ is increasing for } v_1 \in [0, 1]. \tag{3.8}$$

Given (3.8), if a solution curve to (3.6-3.7) defines an increasing function $b = B(v_2)$, then $\langle \tilde{S}, B \rangle$ is an equilibrium in the BBDA.

Theorem 3.1. *If $\langle \tilde{S}, B \rangle$ is an equilibrium in the BBDA, then $B(v_2) = b$ and $\dot{v}_2 = 1/B'(v_2)$ satisfy* (3.6-3.7) *at every $v_2 \in [0, 1]$ at which $B'(v_2)$ exists. Conversely, suppose* (3.8) *holds and B is a C^1 function on* $[0, 1]$ *such that (i) $B'(v_2) > 0$ and $0 < B(v_2) < v_2$ for all $v_2 \in (0, 1]$ and (ii) $B(v_2) = b$ and $\dot{v}_2 = 1/B'(v_2)$ satisfy* (3.6-3.7) *at every $v_2 \in (0, 1]$. Then $\langle \tilde{S}, B \rangle$ is an equilibrium of the BBDA.*

Proof. See Appendix. ||

We do not address the existence of equilibria here; Williams (1988) shows that an equilibrium $\langle \tilde{S}, B \rangle$ exists for a generic choice of the distributions F_1, F_2.

4. THE GEOMETRY OF SOLUTIONS

Theorem 2.2 states that if $\langle \tilde{S}, B \rangle$ is an equilibrium, then $0 \leqq B(v_2) \leqq v_2 \leqq 1$. The graph of an equilibrium strategy B therefore lies within the triangle $0 \leqq b \leqq v_2 \leqq 1$ (see Figure 4.1). Following an approach developed in Satterthwaite and Williams (1987), we describe the vector field (3.6-3.7) on this triangle in order to gain insight into the equilibria of the BBDA.

Formula (3.6) defines $\dot{v}_2$ as a real number at every point in the triangle except along the edges XY where $b = 0$ and XZ where $v_2 = b$. At points X and Z, $\dot{v}_2$ is indeterminate; between X and Y it is negative infinity and between X and Z it is positive infinity. To obtain well-defined values for the vector field $(\dot{v}_2, \dot{b})$ everywhere except X and Z we consider the field's normalization $\bar{v} = (\dot{v}_2, \dot{b})/|(\dot{v}_2, \dot{b})|$. Normalization does not alter the solution curves. Note that $\bar{v}$ is non-singular at every point in the triangle except X and Z.

Inspection of the field along the edges and at the vertices allows us to identify three sets where solution curves enter the triangle and one set where they leave the triangle. A solution curve enters at each point where the field points inward. Solutions may enter through X where $\dot{v}_2$ is indeterminate. The field $\bar{v}$ equals $(1, 0)$ and therefore points into the triangle along the edge XZ. It also points inward along the edge YZ at points where $F_1(b) > f_1(b)(1-b)$. A solution curve exits at any point where the field points outward. This occurs only on YZ (perhaps including vertex Z) at points where $F_1(b) < f_1(b)(1-b)$.

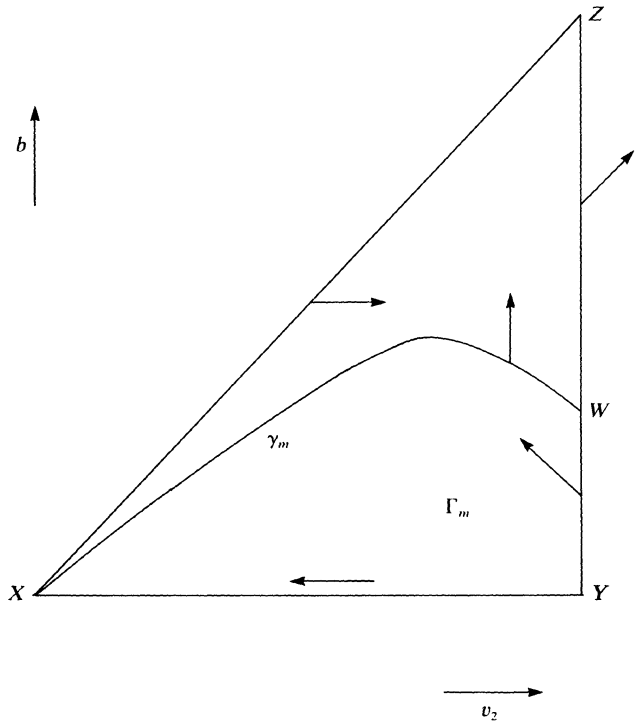

FIGURE 4.1

If $\langle \tilde{S}, B\rangle$ is an equilibrium then the graph of B lies in the triangle XYZ defined by the inequalities $0 \leqq b \leqq v_2 \leqq 1$. The arrows show the direction of the vector field $(\dot{v}_2, \dot{b})$ on the edges and at a point on γ_m

Figure 4.2 shows three solution curves for the case in which F_1 and F_2 are uniform and $m=2$. Curve ρ_1 enters from the edge XZ, ρ_2 enters from the vertex X, and ρ_3 enters from the lower half of the edge YZ. All exit along the upper half of edge YZ. Curve ρ_2 meets the conditions of Theorem 3.1 and therefore defines an equilibrium $\langle \tilde{S}, B\rangle$. Curve ρ_1 may be a segment of an equilibrium strategy B, but it is unclear how to complete its definition for reservation values that lie to the left of the point on XZ where it enters the triangle.[7] Finally, curve ρ_3 does not determine an equilibrium because it does not define the buyer's bid b as an increasing function of v_2, i.e. $\dot{v}_2$ is negative along a segment of ρ_3.

The failure of ρ_3 to determine an equilibrium illustrates an extremely important property of $\dot{v}_2$. Inside the triangle an open region necessarily exists where $\dot{v}_2$ is negative; formally we define this region as

$$\Gamma_m(F_1, F_2) \equiv \{(v_2, b): \dot{v}_2 < 0\}. \tag{4.1}$$

where the dependence in (3.6) of $\dot{v}_2$ on (v_2, b), F_1, F_2, and m is suppressed. Let γ_m denote the upper edge of Γ_m. The set Γ_m always contains the edge XY and some portion of the edge YZ. Figure 4.1 shows Γ_m as the region below the curve γ_m connecting X and W. The set Γ_m is important because the graph of any function B that defines an equilibrium $\langle \tilde{S}, B\rangle$ must lie outside Γ_m at every point where B is differentiable. In the

7. Extending B's graph down along the edge XZ towards X does not define an equilibrium. At each point on this extension B' exists and yet (3.5) is not satisfied. This violates Theorem 3.1.

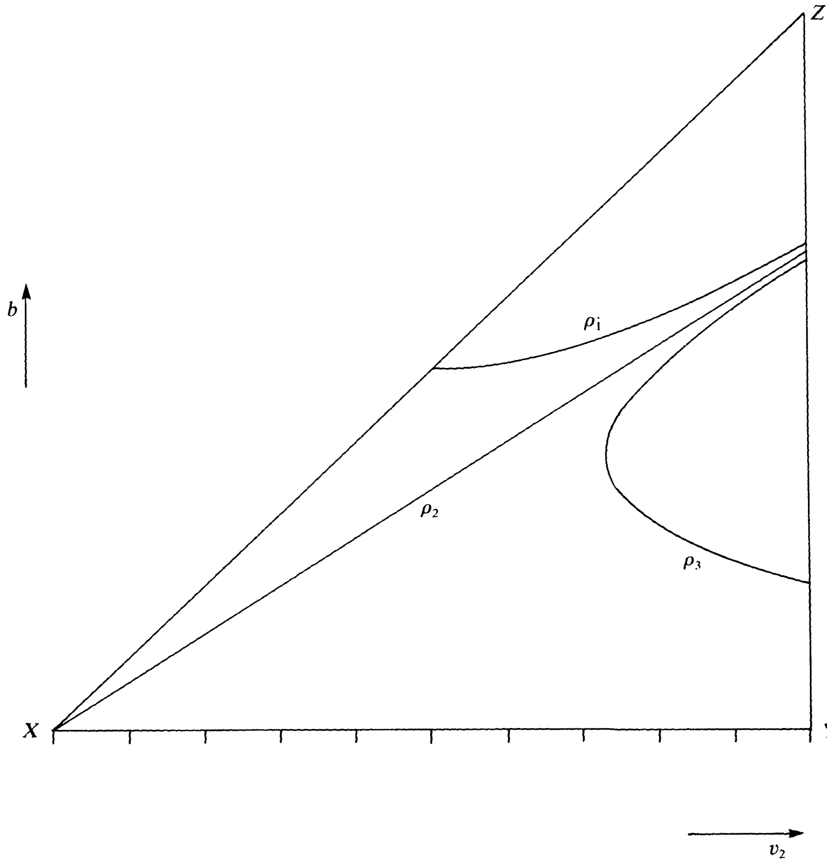

FIGURE 4.2

The curves ρ_1, ρ_2, and ρ_3 are solutions to the differential equation (3.6)-(3.7) when $m=2$ and reservation values are distributed uniformly. Only ρ_2 defines an equilibrium

next section we show that as m increases Γ_m grows and forces all equilibrium strategies towards edge XZ, which corresponds to truthful revelation. Figure 4.3 illustrates this. It graphs γ_m, the upper boundary of Γ_m, for values of m equal to one, eight, and sixteen when F_1 and F_2 are the uniform distribution. This property of Γ_m is the fundamental insight that underlies our convergence result.

The set Γ_m can be interpreted in terms of marginal expected utility. Choose a point (v_2, b) in Γ_m and suppose an equilibrium $\langle \tilde{S}, B\rangle$ did exist such that $B(v_2)=b$ and B is differentiable at v_2. Theorem 2.2 states that $B'(v_2)\geqq 0$. Select a buyer. If the other traders use their equilibrium strategies, formula (3.2) implies that the selected buyer's marginal expected utility is necessarily positive at (v_2, b) because by the definition of Γ_m, a negative number would be needed in place of $B'(v_2)$ in (3.2) in order to make his marginal expected utility zero. The selected buyer therefore has an incentive to raise his bid above $B(v_2)=b$, which contradicts the assumption that $\langle \tilde{S}, B\rangle$ is an equilibrium.

5. CONVERGENCE TO TRUTHFUL REVELATION

The complement of Γ_m in the triangle $0\leqq b\leqq v_2\leqq 1$ contains the edge XZ where the buyer's bid b equals his reservation value v_2. In this section we show that as m increases

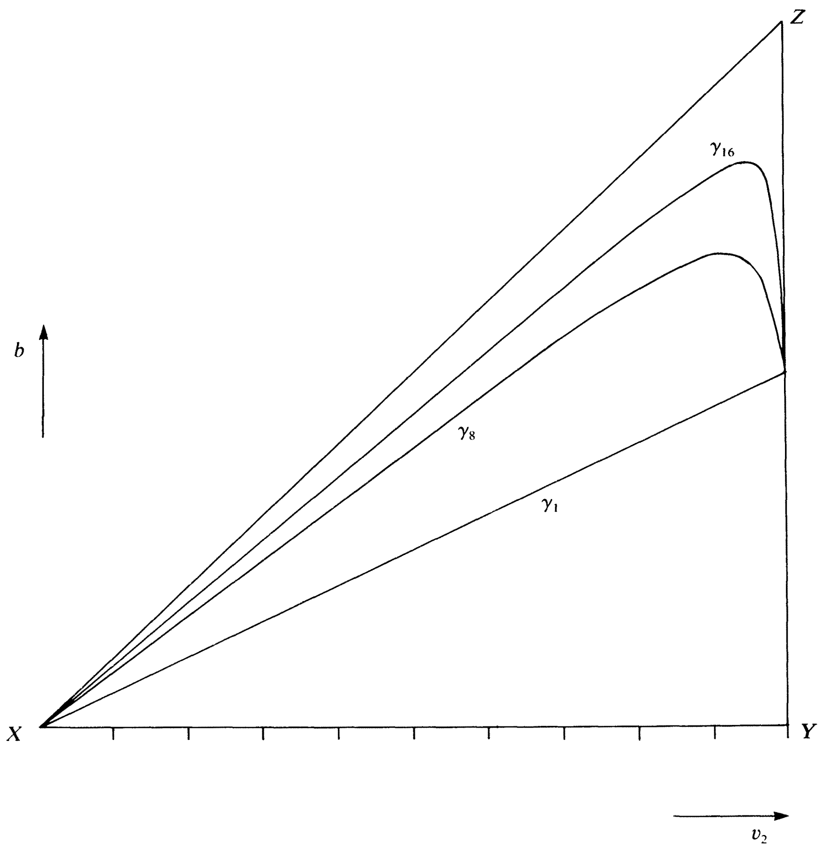

FIGURE 4.3

The boundaries γ_1, γ_8, and γ_{16} are shown for the uniform case. The graph of any equilibrium strategy B in a market with $2m$ traders must lie above γ_m almost everywhere. The edge XZ corresponds to the strategy of truthful revelation

the vertical distance between the boundary γ_m and the edge XZ is $O(1/m)$. The graph of an equilibrium strategy B must lie between γ_m and XZ at almost all values of v_2. This permits us to show that in equilibrium the difference between a buyer's reservation value and his bid is $O(1/m)$, no matter what his reservation value and no matter which equilibrium $\langle \tilde{S}, B\rangle$ is chosen.

Rearrangement of (3.6) produces an inequality defining the region in which $\dot{v}_2$ is non-negative:

$$\dot{v}_2 \geqq 0 \quad \text{if and only if } v_2 - b \leqq \frac{1}{f_1(b)} \times N_m(v_2, b) \tag{5.1}$$

where N_m is the ratio

$$N_m(v_2, b) \equiv \frac{M_m(v_2, b)}{mK_m(v_2, b)}. \tag{5.2}$$

The left-hand side of the second inequality in (5.1) is the amount by which the buyer's bid misrepresents his reservation value. Only the right-hand side depends on the number of traders. We therefore focus on the behaviour of N_m as m increases.

Recall that K_m is the probability that bid b lies between $s_{(m-1)}$ and $s_{(m)}$ in a sample of $m-1$ buyers and $m-1$ sellers, and M_m is the probability that bid b lies between $s_{(m)}$ and $s_{(m+1)}$ in a sample of $m-1$ buyers and m sellers. These two probabilities are almost the same; consequently one expects that, for each (v_2, b) pair, as m grows tht the ratio M_m/K_m approaches some constant. If this is so, then substitution into (5.2) gives our main result. Two theorems, whose proofs are in the Appendix, confirm this intuition.

Theorem 5.1 *For each pair of numbers* $0<b\leqq v_2<1$ *and all* $m\geqq 1$, *the ratio* $N_m(v_2, b)$ *is strictly decreasing in m.*

The functions K_m, L_m, M_m, and hence N_m are well-defined in the $m=1$ case, which permits us to state Theorem 5.1 using $m=1$. The statement of Theorem 5.2 uses the notation

$$z(v_2, b)=\frac{F_2(v_2)(1-F_1(b))}{F_1(b)(1-F_2(v_2))}. \tag{5.3}$$

Theorem 5.2. *If* $m\geqq 2$ *and* (v_2, b) *satisfies* $0<b\leqq v_2<1$, *then*

$$N_m(v_2; b)<\frac{2F_1(b)}{m}\max[1, z(v_2, b)]. \tag{5.4}$$

These theorems have the following interpretation. Consider $m'<m''$. If, for m', $\dot{v}_2$ is negative at some point (v_2, b), then Theorem 5.1 implies that it is also negative for m''. The region Γ_m therefore grows monotonically in m, i.e. for $m'<m''$, $\Gamma_{m'}\subset\Gamma_{m''}$. Theorem 5.2 describes the rate at which this region grows.

The main result of the paper follows from substituting the inequalities of Theorem 5.2 into (5.1).

Theorem 5.3. *Consider the BBDA when sellers' reservation values are drawn from* F_1 *and buyer's reservation values are drawn from* F_2. *A continuous function* $\kappa(v_2; F_1, F_2)$ *of* v_2 *exists such that, for any* $m\geqq 2$ *and any equilibrium* $\langle\tilde{S}, B\rangle$ *in a market of size m,*

$$v_2-B(v_2)\leqq\frac{\kappa(v_2; F_1, F_2)}{m} \tag{5.5}$$

at every v_2 *in the open interval* $(0, 1)$.

Proof. We first show that B satisfies (5.5) at all reservation values $v_2\in(0, 1)$ where $B'(v_2)$ exists. Fix v_2 and let $\bar{b}$ denote $B(v_2)$. From (5.1) and Theorem 5.2 we have

$$v_2-\bar{b}\leqq\frac{N_m(v_2, \bar{b})}{f_1(\bar{b})}<\frac{2}{m}\frac{F_1(\bar{b})}{f_1(\bar{b})}\max[1, z(v_2, \bar{b})]. \tag{5.6}$$

A finite bound on $v_2-\bar{b}$ that does not involve $\bar{b}$ is obtained by maximizing the right-hand side of (5.6) over a closed interval that contains $\bar{b}$. The bid $\bar{b}$ is bounded above by v_2 and below by zero. The right-hand side, however, may be infinite at $\bar{b}=0$. This complication is sidestepped by bounding $\bar{b}$ away from zero. The region Γ_2 is an open set that contains the triangle's lower edge XY. Theorem 5.1 implies that the point $(v_2, \bar{b})$ lies above the region Γ_2. Choose a continuous function μ on (0, 1) such that the graph of μ lies within Γ_2 and μ is greater than zero. The bid $\bar{b}$ therefore satisfies $\mu(v_2) \leqq \bar{b} \leqq v_2$. Define

$$\kappa(v_2) \equiv \max_{\mu(v2) \leqq b \leqq v_2} \frac{2F_1(b)}{f_1(b)} \max [1, z(v_2, b)]. \tag{5.7}$$

For convenience, we suppress the dependence of κ on F_1 and F_2. Note that κ is continuous in v_2 because μ is continuous. With this definition of κ, (5.5) holds at all points where B' exists.

We now show that (5.5) also holds at reservation values in (0, 1) where B' does not exist. Consider the set D_m of reservation values v_2 and bids b that violate (5.5):

$$D_m \equiv \{(v_2, b): 0 < b \leqq v_2 < 1 \text{ and } v_2 - \kappa(v_2)/m > b\}. \tag{5.8}$$

The set D_m is open because κ is continuous. Suppose, contrary to the theorem, that some $(v_2, B(v_2)) \equiv (v_2, b)$ is an element of D_m. A rectangle within D_m exists whose base is on the edge XY and whose upper right corner is (v_2, b). Because B is increasing, the graph of B must pass through the rectangle. Somewhere on this segment of the graph B' must exist, which contradicts the above result that (5.5) holds wherever B is differentiable. ||

As an illustration, we follow the proof of Theorem 5.3 to compute a suitable function κ for the case in which each trader's reservation values are uniformly distributed. To bound b away from zero, we choose a positive function $b = \mu(v_2)$ on (0, 1) whose graph lies within Γ_2. Formula (3.6) for $\dot{v}_2$ implies that

$$\gamma_1 = \{(v_2, b): v_2 = b + F_1(b)/f_1(b)\}. \tag{5.9}$$

In the uniform case, γ_1 is the graph of the function $b = v_2/2$. Let μ be this function. We now compute κ using (5.7). From (5.3), $z(v_2, b) = v_2(1-b)/(1-v_2)b$. Note that $z(v_2, b) \geqq 1$ for $v_2/2 \leqq b \leqq v_2$. Formula (5.7) therefore simplifies to

$$\kappa(v_2; F_1, F_2) = \max_{v_2/2 \leqq b \leqq v_2} \frac{2bv_2(1-b)}{(1-v_2)b} = \frac{v_2(2-v_2)}{(1-v_2)}. \tag{5.10}$$

which means that in the uniform case, the difference between a buyer's reservation value v_2 and his bid is less than or equal to $v_2(2-v_2)/(1-v_2)m$.

In the uniform case direct substitution shows that the strategy

$$B(v_2) = \frac{m}{m+1} v_2 \tag{5.11}$$

satisfies the first-order condition (3.6) and hence defines an equilibrium $\langle \tilde{S}, B \rangle$ for the market with m sellers and m buyers. Inspection shows that this equilibrium satisfies the bound $v_2(2-v_2)/(1-v_2)m$. While the bound in Theorem 5.3 is loose, this example shows that the rate of convergence $O(1/m)$ is sharp as a description of how fast all sequences of equilibria converge.

Curve ρ_2 in Figure 4.2 depicts this solution for $m = 2$. Substitution of $m = 1$ in (5.11) defines the equilibrium that Williams (1987) computed for the bilateral BBDA in the uniform case. Table 5.1 compares the total ex ante expected gains from trade that this sequence of equilibria generates with (i) the expected gains from the optimal mechanism (which was defined in the Introduction) and (ii) the expected gains that would be realized if all traders honestly reported their reservation values. The table shows the rapidity with which allocations in the BBDA approach ex post efficiency.

TABLE 5.1

Relative efficiency of the BBDA as the market grows: uniform case

m	T_B	T^*	T_0	$1-T_B/T_0$	$1-T^*/T_0$
2	0·37037	0·37746	0·39999	0·07415	0·05633
3	0·62227	0·62572	0·64286	0·03203	0·02666
4	0·87333	0·87527	0·88887	0·01748	0·01530
6	1·37421	1·37507	1·38462	0·00751	0·00690
8	1·87454	1·87504	1·88235	0·00415	0·00388
10	2·37470	2·37501	2·38095	0·00263	0·00249
12	2·87479	2·87501	2·88000	0·00181	0·00173

Notes. T_B is the total ex ante expected gains from trade that the equilibrium (5.11) of the BBDA generates, T^* is the total ex ante expected gains that the optimal mechanism generates, and T_0 is the total ex ante expected gains from trade that would be generated if traders honestly reported their reservation values. The values for T^* and T_0 are from Gresik and Satterthwaite (1986, Table 1).

6. ADDITIONAL COMMENTS

1. Insight into the effectiveness of the BBDA is obtained by comparing it to a plausible alternative, the fixed-price mechanism, that Hagerty and Rogerson (1985) and Gresik and Satterthwaite (1988) studied. This mechanism *a priori* fixes price at the value p that would be the competitive price if there were a continuum of sellers with reservation values distributed according to F_1 and a continuum of buyers with reservation values distributed according to F_2. In the uniform case, for instance, the price p is fixed at 0·5. In equilibrium each trader honestly reports his reservation value and sellers whose values are less than p trade at this price with buyers whose values are greater than p. If, as is likely for finite markets, supply does not equal demand, then whichever side of the market is long is randomly rationed. The combination of the fixed price and random rationing among all traders on the long side of the market is what makes honest reporting a dominant strategy for each trader.

This rationing creates inefficiency because a buyer whose gain from trade is large is just as likely to be excluded as a buyer whose gain is small. Table 6.1 illustrates the seriousness of this inefficiency by comparing this mechanism's performance with the BBDA's performance in the uniform case. Informally the reason for the poor performance of the fixed price mechanism is that it only uses traders' reports to determine who is willing to trade at the specified price p. The BBDA, on the other hand, extracts more information from the traders' reports by rank-ordering them according to their expressed desire to trade. Despite the misrepresentation that the BBDA induces, its more thorough use of the agents' reports results in dramatically better performance.

TABLE 6.1

Relative efficiency of the fixed price mechanism and the BBDA

m	2	4	6	8	10	12
$1-T_F/T_0$	0·2187	0·1826	0·1611	0·1462	0·1350	0·1262
$1-T_B/T_0$	0·0742	0·0175	0·0075	0·0042	0·0026	0·0018

Notes. T_B is the total ex ante expected gains from trade that the equilibrium (5.11) of the BBDA generates, T_F is the total ex ante expected gains that the fixed price mechanism generates, and T_0 is the total ex ante expected gains from trade that would be generated if traders honestly reported their reservation values. The values for T_0 and T_F are from Gresik and Satterthwaite (1988, Tables 1 and 2).

2. For finite markets the BBDA and other double auctions are ex post inefficient, i.e. when the market closes potential gains from trade may be "left on the table". Equilibrium strategies are fully revealing of traders' reservation values; consequently when the market closes the traders know if further gains from trade are possible. Cramton (1984) criticized one-shot double auctions on this point. Specifically, he argued that the use of a one-shot double auction implicitly assumes that traders pre-commit not to reopen the market even when it is common knowledge that further gains from trade exist. Such pre-commitment may be difficult or impossible to maintain. Our results suggest that this criticism of one-shot double auctions lacks force in large markets because the expected value of the unrealized gains from trade rapidly vanishes as the market grows.

3. A simple partial equilibrium calculation provides insight into our convergence result. It reveals that the driving force behind the $O(1/m)$ rate is the relative rates at which the likelihood of obtaining an item by increasing one's bid and the likelihood of simply driving up price go to zero as the number of traders increases. Consider the BBDA for a market with $2m$ traders in which $F \equiv F_1 \equiv F_2$ (with density f). Select a buyer with reservation value v_2. Suppose he believes the non-equilibrium conjecture that all other buyers will truthfully report their reservation values. In the sample of offers and bids from the $2m-1$ other traders, let g be the density of the critical offer/bid $s_{(m)}$ that the selected buyer must beat with his bid b in order to receive an item. As before, let M_m be the probability that the bid b lies between $s_{(m)}$ and $s_{(m+1)}$ in this sample and thus determines the market price. Adapting (3.2) to this simplified situation, the buyer chooses his bid to satisfy

$$(v_2 - b)g(b) = M_m. \tag{6.1}$$

When the buyer considers raising his bid, the left-hand side is his marginal expected gain from increasing his likelihood of receiving an item and the right-hand side is his marginal expected cost from driving up the price. Formulas in David (1981, p. 9) give

$$M_m = \binom{2m-1}{m} F(b)^m (1 - F(b))^{m-1} \tag{6.2}$$

$$g(b) = (2m-1)f(b)\binom{2m-2}{m-1} F(b)^{m-1}(1-F(B))^{m-1}. \tag{6.3}$$

Substitution into (6.1) implies

$$v_2 - b = \frac{F(b)}{mf(b)}. \tag{6.4}$$

which is the same rate that we obtained in Theorem 5.3.

4. A basic insight of the literature in social choice theory on strategy-proofness is that strategic behaviour is only avoidable in mechanisms in which individuals can not affect each other's allocations.[8] In the BBDA traders affect each other's allocations by affecting the expected price. The ability to affect price vanishes rapidly as the market grows. The social choice results therefore suggest that strategic behaviour should vanish as the market grows large. Our result shows that this in fact happens.

5. Williams (1988) generalized the results of this paper to the case in which the number of sellers may differ from the number of buyers. He showed that for any

8. See, for example, Satterthwaite and Sonnenschein (1981). The one important exception is the family of Groves mechanisms.

equilibrium $\langle \tilde{S}, B\rangle$ in the market with n sellers and q buyers

$$v_2 - B(v_2) \leqq \frac{\kappa(v_2; F_1, F_2)}{\min(n, q)}, \tag{6.5}$$

where $\kappa(v_2; F_1, F_2)$ is the same function as in (5.5). The proof of this more general result is obtained using the approach of our paper and its convergence result.

6. The BBDA is one example of the sealed-bid k-double auction. The more general formulation of the k-double auction is to set price equal to $(1-k)s_{(m)} + ks_{(m+1)}$ where k is a fixed parameter in the interval $[0, 1]$. The BBDA is the k-double auction in which $k = 1$. All the results of this paper have exact parallels for the seller's offer double auction in which $k = 0$. Our analysis of these two extreme cases is greatly facilitated because in each case traders on one side of the market truthfully reveal their reservation values. The analysis becomes more difficult when k is in the open interval $(0, 1)$ because then a trader on either side of the market can affect the price at which he trades; as a consequence all traders misrepresent their reservation values. As of this writing we have been unable to obtain the $O(1/m)$ convergence result for this more general case. We conjecture, however, that it is true for two reasons. First, regions analogous to Γ_m exist in the general case that bound equilibrium strategies. The key insight in our analysis of the BBDA thus generalizes to all k-double auctions. Second, numerical computation of equilibria in the general case supports the conjecture that all differentiable equilibria converge to truthful revelation as $O(1/m)$.

APPENDIX

Proof of Theorem 3.1. To prove the necessary part of the theorem, it is sufficient here to derive formula (3.2) for $d\pi/db$. The result in the theorem concerning (3.6–7) then follows from the discussion in the text. We derive the marginal expected utility at bid b of a type v_2 buyer who is bidding against m sellers, each using strategy $\tilde{S}$, and $m-1$ buyers, each using an increasing function B as his strategy. Let $x = s_{(m)}$ and $y = s_{(m+1)}$ in the array of $2m-1$ offers/bids received from the other traders and let $e(x, y)$ denote the joint density of x and y. Note that $e(x, y) = 0$ whenever $x > y$. Table A.1 catalogues the three distinct utility consequences of the bid b. For example, if b should be greater than y, then the selected buyer receives an item at price y and has utility $v_2 - y$.

The expected utility of bidding b is

$$\pi(v_2, b; B) = \int_b^1 \int_0^b (v_2 - b)e(x, y)dxdy + \int_0^b \int_0^y (v_2 - y)e(x, y)dxdy \tag{A.1}$$

where the first integral is the expected gain from the case II outcomes and the second integral is the expected gain from the case III outcomes. Differentiating with respect to b, produces

$$\begin{aligned}\frac{d\pi}{db} = &-\int_0^b (v_2 - b)e(x, b)dx + \int_b^1 (v_2 - b)e(b, y)dy \\ &- \int_b^1 \int_0^b e(x, y)dxdy + \int_0^b (v_2 - b)e(x, b)dx.\end{aligned} \tag{A.2}$$

TABLE A.1

Possible outcomes of a bid b

Case no.	Case definition	Ex post utility
I	$b < x < y$	0
II	$x < b < y$	$v_2 - b$
III	$x < y < b$	$v_2 - y$

Note: Ties are a probability zero event because all traders use increasing strategies.

The first and fourth terms cancel, (v_2-b) factors out of the second term, and the remaining integrals have straightforward probability interpretations:

$$d\pi/db=(v_2-b)g(b)-\Pr(x<b<y) \tag{A.3}$$

where $g(b)$ is the density of the order statistic x evaluated at b. This density can be shown to equal the term in brackets in (3.2) using the standard technique in David (1981, p. 9). Similarly, $\Pr(x<b<y)=M_m(B^{-1}(b), b)$, which completes our discussion of the theorem's necessary part.

Sufficiency of the first-order approach is proven as follows. Given a function B that meets the theorem's requirements, we must show that $\pi(v_2, b; B)$ is maximized at $b=B(v_2)$. Arguments in the proof of Theorem 2.2 show that we can restrict attention to $b\in(0, v_2]$. Two cases must be considered; $b\in(0, B(1)]$ and $b\in(B(1), v_2]$. The first case is facilitated by defining

$$J_m(v_2, b; B)\equiv mf_1(b)K_m(v_2, b)+(m-1)f_2(v_2)L_m(v_2, b)/B'(v_2). \tag{A.4}$$

Formula (3.2) then becomes

$$d\pi(v_2, b; B)/db=J_m(B^{-1}(b), b; B)(v_2-b)-M_m(B^{-1}(b), b) \tag{A.5}$$

and the differential equation (3.6) is equivalent to

$$J_m(v_2, B(v_2); B)(v_2-B(v_2))-M_m(v_2, B(v_2))=0. \tag{A.6}$$

Formula (A.5) can be rewritten as

$$\begin{aligned} d\pi(v_2, b; B)/db&=J_m(B^{-1}(b), b, B)(v_2-B^{-1}(b))\\ &\quad+J_m(B^{-1}(b), b; B)(B^{-1}(b)-b)-M_m(B^{-1}(b), b). \end{aligned} \tag{A.7}$$

If we evaluate the differential equation (A.6) at $v_2=B^{-1}(b)$, we obtain the last line in (A.7). We therefore have $d\pi/db$ equal to the top line. Note that (i) $J_m(B^{-1}(b), b; B)$ is positive for all $0<b<B(1)$, (ii) $d\pi(v_2, b; B)/db$ is zero at $b=B(v_2)$, and (iii) the function B^{-1} is increasing since B is increasing. The marginal expected utility $d\pi(v_2, b; B)/db$ therefore changes from positive to negative at $b=B(v_2)$, which establishes that $\pi(v_2, b; B)$ is maximized on $(0, B(1)]$ at $b=B(v_2)$.

Consider now the remaining case of $b\in(B(1), v_2]$. While the marginal expected utility $d\pi(v_2, b; B)/db$ is discontinuous at $b=B(1)$, the expected utility $\pi(v_2, b; B)$ is continuous in b on $[0, 1]$ because B is a C^1 function. It is therefore sufficient to prove that $d\pi(v_2, b; B)/db$ is negative over $(B(1), v_2]$. For a bid b in this interval (A.3) is

$$\begin{aligned} \frac{d\pi(v_2, b; B)}{db}&=(v_2-b)mf_1(b)K_m(1, b)-M_m(1, b)\\ &=(v_2-b)mf_1(b)[1-F_1(b)]^{m-1}-mF_1(b)[1-F_1(b)]^{m-1}\\ &=mf_1(b)[1-F_1(b)]^{m-1}\left[v_2-b-\frac{F_1(b)}{f_1(b)}\right]. \end{aligned} \tag{A.8}$$

Consider the last line of (A.8). The monotonicity property (3.8) implies that the expression in brackets is decreasing in b, and it is also increasing in v_2. Consequently if some line of (A.8) is negative at $v_2=1$ and $b=B(1)$, then each line is negative for any v_2 over the entire interval $(B(1), v_2]$.

We show that the first line is negative at $v_2=1$ and $b=B(1)$ by considering the solution B at that point. By hypothesis $\dot{v}_2$ is positive at all points $(v_2, B(v_2))$. The numerator of the right-hand side of (3.6) determines the sign of $\dot{v}_2$; at $(1, B(1))$ this numerator is $-[1-B(1)]mf_1(B(1))K_m(1, B(1))+M_m(1, B(1))>0$. The negative of this expression is the first line of (A.8) evaluated at $v_2=1$ and $b=B(1)$. ||

In proving Theorems 5.1 and 5.2 we use the following formula for $N_m/F_1(b)$:

$$\frac{N_m(v_2, b)}{F_1(b)}=\frac{\sum_{i=0}^{m-1}\binom{m-1}{i}\binom{m}{i}z^i}{\sum_{i=0}^{m-1}\binom{m-1}{i}\binom{m}{i}(m-i)z^i}. \tag{A.9}$$

The right-hand side has been derived from (5.2) by (i) factoring out F_1 from $M_m(v_2, b)$ and cancelling, (ii) dividing the numerator and denominator by $[F_1(b)(1-F_2(v_2))]^{m-1}$, and (iii) substituting

$$m\binom{m-1}{i}^2=\binom{m-1}{i}\binom{m}{i}(m-i) \tag{A.10}$$

into the denominator.

Proof of Theorem 5.1. It is sufficient to prove that N_m/F_1 is strictly decreasing in m. Substitute j for i as the index of the terms in the formula for N_{m+1}/F_1 that is given by (A.9). Next, compute the numerator of $(N_m-N_{m+1})/F_1$:

$$\left[\sum_{i=0}^{m}\binom{m-1}{i}\binom{m}{i}z^i\right]\left[\sum_{j=0}^{m}\binom{m}{j}\binom{m+1}{j}(m+1-j)z^j\right]$$
$$-\left[\sum_{i=0}^{m-1}\binom{m-1}{i}\binom{m}{i}(m-i)z^i\right]\left[\sum_{j=0}^{m}\binom{m}{j}\binom{m+1}{j}z^j\right]. \tag{A.11}$$

The proof will be completed by showing that all of the coefficients of this polynomial are non-negative, and some are strictly positive.

For $0\leqq k\leqq 2m-1$, the coefficient of z^k is

$$\sum_{i+j=k,\,0\leq i\leq m-1,\,0\leq j\leq m}\binom{m-1}{i}\binom{m}{i}\binom{m+1}{j}(i+1-j). \tag{A.12}$$

We now pair terms in this expression with the following formula: the $i=u$, $j=v$ term is paired with the $i=v-1, j=u+1$ term. Some terms may be left out by this pairing; there is no term to pair with the $i=k, j=0$ term (if such a term exists for the given value of k), and a term of the form $i=u$, $j=u+1$ is paired with itself. It is easy to see from (A.12), however, that a term with $j=0$ is positive, and a term with $i+1=j$ is zero. Except for these special cases, the formula pairs each term in (A.12) with a different term. This pairing is well-defined, i.e. if i', j' is assigned to i'', j'' by the formula, then i'', j'' is assigned to i', j'.

We now rewrite the sum of $i=u, j=v$ term and the $i=v-1, j=u+1$ term. Factoring out the $i=u, j=v$ term, we have

$$\binom{m-1}{u}\binom{m}{u}\binom{m}{v}\binom{m+1}{v}(u+1-v)+\binom{m-1}{v-1}\binom{m}{v-1}\binom{m}{u+1}\binom{m+1}{u+1}(v-u-1)$$
$$=\binom{m-1}{u}\binom{m}{u}\binom{m}{v}\binom{m+1}{v}(u+1-v)\left\{1-\left[\frac{v}{u+1}\right]^2\right\}. \tag{A.13}$$

The signs of the last two terms of the product on the second line of (A.13) are the same. The expression (A.13) is therefore positive except when $u+1=v$, which is a case that was discussed above. ‖

Proof of Theorem 5.2. The inequality (5.4) is equivalent to the following pair of inequalities: (i) if $z(v_2, b)\leqq 1$, then $N_m(v_2, b)/F_1(b)<2/m$, and (ii) if $z(v_2, b)\geqq 1$, then $N_m(v_2, b)/F_1(b)<2z(v_2, b)/m$. We begin by proving the first inequality. Using (A.9), it is sufficient to show that

$$\sum_{i=0}^{m-1}\binom{m-1}{i}\binom{m}{i}\left[1-\frac{2(m-i)}{m}\right]z^i \tag{A.14}$$

is negative for $0<z\leqq 1$. Multiplying through by m, we obtain

$$\sum_{i=0}^{m-1}\binom{m-1}{i}\binom{m}{i}(2i-m)z^i. \tag{A.15}$$

Note that the coefficient of z^i is positive if $i>m/2$, zero if $i=m/2$, and negative if $i<m/2$. Excluding the $i=m/2$ term (if it is present) and the $i=0$ term (which is clearly negative), we now pair the remaining terms with the following formula: for $1\leqq u<m/2$, the $i=u$ term is paired with the $i=m-u$ term. The sum of the $i=u$ and $i=m-u$ terms reduces as follows:

$$\binom{m-1}{u}\binom{m}{u}(2u-m)z^u+\binom{m-1}{m-u}\binom{m}{m-u}(m-2u)z^{m-u}$$
$$=\binom{m-1}{u}\binom{m}{u}(2u-m)z^u\left[1-\frac{u}{m-u}z^{m-2u}\right]. \tag{A.16}$$

Since $u < m/2$, and $z \leqq 1$, it is true that (i) $(2u - m) < 0$, (ii) $u/(m-u) < 1$, and (iii) $z^{m-2u} \leqq 1$. The second line of (A.16) is therefore negative, and it follows that (A.15) is also negative. This completes the proof of the first inequality.

We now turn to the second inequality. Again using (A.9), it is sufficient to show that

$$\sum_{i=0}^{m-1} \binom{m-1}{i}\binom{m}{i} z^i - \frac{2}{m}\sum_{i=0}^{m-1}\binom{m-1}{i}\binom{m}{i}(m-i)z^{i+1} \tag{A.17}$$

is negative when $z \geqq 1$. After reindexing the right-hand summation by replacing i with $i-1$ and then multiplying (A.17) by m, we obtain

$$m - 2mz^m + \sum_{i=1}^{m-1}\left[m\binom{m-1}{i}\binom{m}{i} - 2(m-i+1)\binom{m-1}{i-1}\binom{m}{i-1}\right]z^i. \tag{A.18}$$

Since $z \geqq 1$, $m - 2mz^m$ is negative. It is thus sufficient to focus on the remaining summation.

By factoring, this summation can be rewritten as

$$\sum_{i=1}^{m-1}\binom{m-1}{i}\binom{m}{i}\left[m - \frac{2i^2}{m-i}\right]z^i. \tag{A.19}$$

The coefficient of z^i is negative when $i > m/2$, zero when $i = m/2$, and positive when $i < m/2$. Excluding the $i = m/2$ term (if it exists), we pair terms as in the proof of the theorem's first part: for $1 \leqq u < m/2$, the $i = u$ term is paired with the $i = m - u$ term. The sum of these terms is

$$\binom{m-1}{u}\binom{m}{u}\left[m - \frac{2u^2}{m-u}\right]z^u + \binom{m-1}{m-u}\binom{m}{m-u}\left[m - \frac{2(m-u)^2}{u}\right]z^{m-u}. \tag{A.20}$$

The proof is completed by showing that the sum (A.20) is negative. Since $z \geqq 1$ and the $i = m - u$ term is negative, it is sufficient to show that

$$\binom{m-1}{u}\left[m - \frac{2u^2}{m-u}\right] + \binom{m-1}{m-u}\left[m - \frac{2(m-u)^2}{u}\right] \tag{A.21}$$

is negative. By factoring out $\binom{m-1}{u}/(m-u)$, this reduces to

$$\binom{m-1}{u}[m(m-u) - 2u^2 + mu - 2(m-u)^2]/(m-u). \tag{A.22}$$

The expression in brackets equals $-(m-2u)^2$, which shows that (A.20) is negative. ||

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