

# Dynamic Agency and the $q$ Theory of Investment\*

Peter DeMarzo<sup>†</sup>      Michael Fishman<sup>‡</sup>      Zhiguo He<sup>§</sup>      Neng Wang<sup>¶</sup>

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## Abstract

We introduce dynamic agency into the neoclassical  $q$  theory of investment. Costly external financing arises endogenously from dynamic agency, and affects firm value and investment. Agency conflicts drive a history-dependent wedge between average  $q$  and marginal  $q$ , and make the firm's investment dependent on realized profits. Higher profits lead to higher investment, though there is always underinvestment relative to the first best. Also, optimal investment is decreasing in firm-specific risk and positively serially correlated over time. The agent's optimal compensation is deferred when past profits are low, whereas cash compensation is optimal when past profits are high. Moreover, the higher the firm-specific risk, the more the optimal contract relies on deferred compensation. In deriving the optimal contract, the key state variable is the agent's continuation payoff, which depends (positively) on the firm's profit history. We show how this state variable can be interpreted as a measure of financial slack, allowing us to relate the results to the firm's financial slack. To study the effect of persistent shocks on investment, we extend our model to allow the firm's output price to be stochastic. In contrast to static agency models, the agent's compensation in the optimal dynamic contract depends not only on the firm's past performance, but also on the output price, even though the output price is observable, contractible, and beyond the agent's control. Agent compensation increases (decreases) when the output price increases (decreases). Moreover, the optimal contract may call for the agent to be fired in the event of an output price decrease. In this context of persistent shocks, we also show that holding average  $q$  fixed, investment is increasing in financial slack.

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<sup>†</sup>Graduate School of Business, Stanford University.

<sup>‡</sup>Kellogg School of Management, Northwestern University.

<sup>§</sup>Booth School of Business, University of Chicago.

<sup>¶</sup>Columbia Business School.

# 1 Introduction

This paper integrates dynamic agency theory into the neoclassical  $q$  theory of investment. Doing so allows us to endogenize the cost of external financing and explore the effect of these costs on the relation between investment decisions and Tobin's  $q$ . With optimal contracts, we show that in contrast to the neoclassical model, investment is sensitive to the firm's past profitability even after controlling for Tobin's  $q$ . Moreover, we link this result to the dynamics of the firm's financial slack and management compensation.

Following the classic investment literature, e.g. Hayashi (1982), we endow the firm with a constant-returns-to-scale production technology; output is proportional to the firm's capital stock but is subject to productivity shocks. The firm can invest/disinvest to alter its capital stock, but this investment entails a convex adjustment cost which is homogenous of degree one in investment and capital stock. With no agency problem, we have the standard predictions that average  $q$  equals marginal  $q$ , and with quadratic adjustment costs, the investment-capital ratio is linear in average  $q$  (Hayashi (1982)).<sup>1</sup>

To the neoclassical setting we add a dynamic agency problem. At each point in time, the agent chooses an action which together with the (unobservable) productivity shock determines output. Our agency model matches a standard principal-agent setting in which the agent's action is unobserved costly effort, and this effort affects the mean rate of production. Alternatively, we can interpret the agency problem as one in which the agent can divert output for his private benefit. The agency side of our model builds on the discrete-time models of DeMarzo and Fishman (2007a, b) and the continuous-time formulation of DeMarzo and Sannikov (2006).

The optimal contract specifies, as a function of the history of the firm's profits, (i) the agent's compensation; (ii) the level of investment in the firm; and (iii) whether the contract is terminated. Termination could involve the replacement of the agent or the liquidation of the

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<sup>1</sup>Abel and Eberly (1994) extend neoclassical investment theory to allow for fixed adjustment costs as well as a wedge between the purchase and sale prices of capital.

firm. Going forward we will use the terms termination and liquidation interchangeably. Through the contract, the firm's profit history determines the agent's current discounted expected payoff, which we refer to as the agent's "continuation payoff,"  $W$ , and current investment which in turn determines the current capital stock,  $K$ . These two state variables,  $W$  and  $K$ , completely summarize the contract-relevant history of the firm. Moreover, because of the size-homogeneity of our model, the analysis simplifies further and the agent's continuation payoff per unit of capital,  $w = W/K$ , becomes sufficient for the contract-relevant history of the firm.<sup>2</sup>

Because of the agency problem, investment is below the first-best level. The degree of underinvestment depends on the firm's realized past profitability, or equivalently, through the contract, the agent's continuation payoff (per unit of capital),  $w$ . In particular, investment is increasing in  $w$ . To understand this linkage, note that in a dynamic agency setting, the agent is rewarded for high profits, and penalized for low profits, in order to provide incentives. As a result, the agent's continuation payoff,  $w$ , is increasing with past profitability. In turn, a higher continuation payoff for the agent relaxes the agent's incentive-compatibility (IC) constraints since the agent now has a greater stake in the firm (in the extreme, if the agent owned the entire firm there would be no agency problem). Finally, relaxing the IC constraints raises the value of investing in more capital.

In the analysis here, the gain from relaxing the IC constraints comes by reducing the probability, within any given amount of time, of termination. If profits are low the agent's continuation payoff  $w$  falls (for incentive reasons) and if  $w$  hits a lower threshold, the contract is terminated. We assume termination entails costs associated with hiring a new manager or liquidating assets, and show that even if these costs appear small they can have a large impact on the optimal contract and investment.

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<sup>2</sup>We solve for the optimal contract using a recursive dynamic programming approach. Early contributions that developed recursive formulations of the contracting problem include Green (1987), Spear and Srivastava (1987), Phelan and Townsend (1991), and Atkeson (1991), among others. Ljungqvist and Sargent (2004) provide in-depth coverage of these models in discrete-time settings.

We also show that in an optimal contract the agent's payoff depends on persistent shocks to the firm's output price even though the output price is observable, contractible, and beyond the agent's control. When the output price increases the contract gives the agent a higher continuation payoff. The intuition is that the marginal cost of compensating the agent is lower when the output price is high because relaxing the agency problem is more valuable when the output price is high. This result may help to explain the empirical importance of absolute, rather than relative, performance measures for executive compensation. This result also implies that the agency problem generates an amplification of output price shocks. An increase in output price has a direct effect on investment since the higher output price makes investment more profitable. There is also an indirect effect. With a higher output price, it is optimal to offer the agent a higher continuation payoff which, as discussed above, leads to further investment. This is in contrast to models with exogenous costs of external financing, where financing costs would tend to dampen the impact of price shocks on investment.

As in DeMarzo and Fishman (2007a, b) and DeMarzo and Sannikov (2006), we show that the state variable,  $w$ , which represents the agent's continuation payoff, can also be interpreted as a measure of the firm's financial slack. More precisely,  $w$  is proportional to the size of the current cash flow shock that the firm can sustain without liquidating, and so can be interpreted as a measure of the firm's liquid reserves and available credit. The firm accumulates reserves when profits are high, and depletes its reserves when profits are low. Thus, our model predicts an increasing relation between the firm's financial slack and the level of investment.

The agency perspective leads to important departures from standard  $q$  theory. First, we demonstrate that both average and marginal  $q$  are increasing with the agent's continuation payoff,  $w$ , and therefore with the firm's financial slack and past profitability. This effect is driven by the nature of optimal contracts, as opposed to changes in the firm's investment opportunities. Second, we show that despite the homogeneity of the firm's production technology (including

agency costs), average  $q$  and marginal  $q$  are no longer equal. Marginal  $q$  is below average  $q$  because an increase in the firm's capital stock reduces the firm's financial slack (the agent's continuation payoff) per unit of capital,  $w$ , and thus tightens the IC constraints and raises agency costs. The wedge between marginal and average  $q$  is largest for firms with intermediate profit histories. Very profitable firms have sufficient financial slack that agency costs are small, whereas firms with very poor profits are more likely to be liquidated (in which case average and marginal  $q$  coincide). These results imply that in the presence of agency concerns, standard linear models of investment on average  $q$  are misspecified, and variables such as managerial compensation, financial slack, past profitability, and past investment will be useful predictors of current investment.

Our paper builds on DeMarzo and Fishman (2007a). The current paper provides a closer link to the theoretical and empirical macro investment literature. Our analysis is also directly related to other analyses of agency, dynamic contracting and investment, e.g., Albuquerque and Hopenhayn (2004), Quadrini (2004) and Clementi and Hopenhayn (2006). We use the continuous-time recursive contracting methodology developed in DeMarzo and Sannikov (2006) to derive the optimal contract. Philippon and Sannikov (2007) analyze the optimal exercise of a growth option in a continuous-time dynamic agency environment. The continuous-time methodology allows us to derive a closed-form characterization of the investment Euler equation, optimal investment dynamics, and compensation policies.<sup>3</sup>

Lorenzoni and Walentin (2007) and Schmid (2008) develop discrete-time models of the relation between investment and  $q$  in the presence of agency problems. In these models the agent must be given incentives to not default and abscond with the assets, and whether he complies is observable. In our analysis, whether the agent takes appropriate actions is unobservable, hence the termination threat is invoked in equilibrium.

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<sup>3</sup>In addition, our analysis owes much to the recent dynamic contracting literature, e.g., Gromb (1999), Biais et al. (2007), DeMarzo and Fishman (2007b), Tchistyi (2005), Sannikov (2006), He (2007), and Piskorski and Tchistyi (2007) as well as the earlier optimal contracting literature, e.g., Diamond (1984) and Bolton and Scharfstein (1990).

A growing literature in macro and finance introduces more realistic characterizations for firms' investment and financing decisions. These papers often integrate financing frictions such as transaction costs of raising funds, financial distress costs, and tax benefits of debt, with a more realistic specification for the production technology such as decreasing returns to scale. See Gomes (2001), Cooper and Ejarque (2003), Cooper and Haltiwanger (2006), Abel and Eberly (2005), and Hennessy and Whited (2006), among others, for recent contributions. For a survey of earlier contributions, see Caballero (1999).

We proceed as follows. In Section 2, we specify our continuous-time model of investment in the presence of agency costs. In Section 3, we solve for the optimal contract using dynamic programming. Section 4 analyzes the implications of this optimal contract for investment and firm value. Section 5 provides an implementation of the optimal contract using standard securities and explores the link between financial slack and investment. In Section 6, we consider the impact of output price variability on investment, firm value, and the agent's compensation. Section 7 concludes. All proofs appear in the Appendix.

## 2 The Model

We formulate an optimal dynamic investment problem for a firm facing an agency problem. First, we present the firm's production technology. Second, we introduce the agency problem between investors and the agent. Finally, we formulate the optimal contracting problem.

### 2.1 Firm's Production Technology

Our model is based on a neoclassical investment setting. The firm employs capital to produce output, whose price is normalized to 1 (Section 6 considers a stochastic output price). Let  $K$  and  $I$  denote the level of capital stock and gross investment rate, respectively. As is standard in capital accumulation models, the firm's capital stock  $K$  evolves according to

$$dK_t = (I_t - \delta K_t) dt, \quad t \geq 0, \tag{1}$$

where  $\delta \geq 0$  is the rate of depreciation.

Investment entails adjustment costs. Following the neoclassical investment with adjustment costs literature, we assume that the adjustment cost  $G(I, K)$  satisfies  $G(0, K) = 0$ , is smooth and convex in investment  $I$ , and is homogeneous of degree one in  $I$  and the capital stock  $K$ . Given the homogeneity of the adjustment costs, we can write

$$I + G(I, K) \equiv c(i)K, \tag{2}$$

where the convex function  $c$  represents the total cost per unit of capital required for the firm to grow at rate  $i = I/K$  (before depreciation).

We assume that the incremental gross output over time interval  $dt$  is proportional to the capital stock, and so can be represented as  $K_t dA_t$ , where  $A$  is the cumulative productivity process.<sup>4</sup> We model the instantaneous productivity  $dA_t$  in the next subsection, where we introduce the agency problem. Given the firm's linear production technology, after accounting for investment and adjustment costs we may write the dynamics for the firm's cumulative (gross of agent compensation) cash flow process  $Y_t$  for  $t \geq 0$  as follows:

$$dY_t = K_t (dA_t - c(i_t)dt), \tag{3}$$

where  $K_t dA_t$  is the incremental gross output and  $K_t c(i_t)dt$  is the total cost of investment.

The contract with the agent can be terminated at any time, in which case investors recover a value  $lK_t$ , where  $l \geq 0$  is a constant. We assume that termination is inefficient and generates deadweight losses. We can interpret termination as the liquidation of the firm; alternatively, in Section 3, we show how  $l$  can be endogenously determined to correspond to the value that shareholders can obtain by replacing the incumbent management (see DeMarzo and Fishman (2007b) for additional interpretations). Since the firm could always liquidate by disinvesting it is natural to specify  $l \geq c'(-\infty)$ .

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<sup>4</sup>We can interpret this linear production function as a reduced form for a setting with constant returns to scale involving other *flexible* factors of production; i.e. because  $\max_N \epsilon_t K_t^{\alpha_t} N^{1-\alpha_t} - \omega_t N - \phi_t K_t = K_t f(\epsilon_t, \alpha_t, \omega_t, \phi_t)$ , the productivity shock  $dA_t$  can be thought of as fluctuations in the underlying parameters  $\epsilon_t, \alpha_t, \omega_t$ , or  $\phi_t$ .

## 2.2 The Agency Problem

We now introduce an agency conflict induced by the separation of ownership and control. The firm's investors hire an agent to operate the firm. In contrast to the neoclassical model where the productivity process  $A$  is exogenously specified, the productivity process in our model is affected by the agent's unobservable action. Specifically, the agent's action  $a_t \in [0, 1]$  determines the expected rate of output per unit of capital, so that

$$dA_t = a_t \mu dt + \sigma dZ_t, \quad t \geq 0, \quad (4)$$

where  $Z = \{Z_t, \mathcal{F}_t; 0 \leq t < \infty\}$  is a standard Brownian motion on a complete probability space, and  $\sigma > 0$  is the constant volatility of the cumulative productivity process  $A$ . The agent controls the drift, but not the volatility of the process  $A$ .

When the agent takes the action  $a_t$ , he enjoys private benefits at rate  $\lambda(1 - a_t) \mu dt$  per unit of the capital stock, where  $0 \leq \lambda \leq 1$ . The action can be interpreted as an effort choice; due to the linearity of private benefits, our framework is also equivalent to the binary effort setup where the agent can *shirk*,  $a = 0$ , or *work*,  $a = 1$ . Alternatively, we can interpret  $1 - a_t$  as the fraction of cash flow that the agent diverts for his private benefit, with  $\lambda$  equal to the agent's net consumption per dollar diverted. In either case,  $\lambda$  represents the severity of the agency problem and, as we show later, captures the minimum level of incentives required to motivate the agent.

Investors are risk-neutral with discount rate  $r > 0$ , and the agent is also risk-neutral, but with a higher discount rate  $\gamma > r$ . That is, the agent is impatient relative to investors. This impatience could be preference based or may derive indirectly because the agent has other attractive investment opportunities. This assumption avoids the scenario where investors indefinitely postpone payments to the agent. The agent has no initial wealth and has limited liability, so investors cannot pay negative wages to the agent. If the contract is terminated, the agent's reservation value, associated with his next best employment opportunity, is normalized to zero.

### 2.3 Formulating the Optimal Contracting Problem

We assume that the firm's capital stock,  $K_t$ , and its (cumulative) cash flow,  $Y_t$ , are observable and contractible. Therefore investment  $I_t$  and productivity  $A_t$  are also contractible.<sup>5</sup> To maximize firm value, investors offer a contract that specifies the firm's investment policy  $I_t$ , the agent's cumulative compensation  $U_t$ , and a termination time  $\tau$ , all of which depend on the history of the agent's performance, which is given by the productivity process  $A_t$ .<sup>6</sup> The agent's limited liability requires the compensation process  $U_t$  to be non-decreasing. We let  $\Phi = (I, U, \tau)$  represent the contract and leave further regularity conditions on  $\Phi$  to the appendix.

Given the contract  $\Phi$ , the agent chooses an action process  $\{a_t \in [0, 1] : 0 \leq t < \tau\}$  to solve:

$$W(\Phi) = \max_{\{a_t \in [0, 1] : 0 \leq t < \tau\}} \mathbb{E}^a \left[ \int_0^\tau e^{-\gamma t} (dU_t + \lambda(1 - a_t)\mu K_t dt) \right], \quad (5)$$

where  $\mathbb{E}^a(\cdot)$  is the expectation operator under the probability measure that is induced by the action process. The agent's objective function includes the present discounted value of compensation (the first term in (5)) and the potential private benefits from taking action  $a_t < 1$  (the second term in (5)).

We focus on the case where it is optimal for investors to implement the efficient action  $a_t = 1$  all the time and provide a sufficient condition for the optimality of implementing this action in the appendix. Henceforth the expectation operator  $\mathbb{E}(\cdot)$  is under the measure induced by  $\{a_t = 1 : 0 \leq t < \tau\}$ , unless otherwise stated. We call a contract  $\Phi$  *incentive compatible* if it implements the efficient action.

At the time the contract is initiated, the firm has  $K_0$  in capital. Given an initial payoff of

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<sup>5</sup>Based on the growth of the firm's capital stock, the firm's investment process can be deduced from (1), and hence the firm's productivity process  $A_t$  can be deduced from (3) using  $I_t$  and  $Y_t$ .

<sup>6</sup>As we will discuss further in Section 5, the firm's access to capital is implicitly determined given the investment, compensation, and liquidation policies. Note also that given  $A$  and the investment policy, the variables  $K$  and  $Y$  are redundant and so we do not need to contract on them directly. In principle, the contract could also depend on public randomization. But as we will verify later, the optimal contract with commitment does not entail public randomization. The optimal contract without commitment (i.e., the optimal renegotiation-proof contract) may rely on public randomization; see Appendix C.

$W_0$  for the agent, the investors' optimization problem is

$$P(K_0, W_0) = \max_{\Phi} \mathbb{E} \left[ \int_0^{\tau} e^{-rt} dY_t + e^{-r\tau} lK_{\tau} - \int_0^{\tau} e^{-rt} dU_t \right] \quad (6)$$

*s.t.*  $\Phi$  is incentive-compatible and  $W(\Phi) = W_0$ .

The investors' objective is the expected present value of the firm's gross cash flow plus termination value less the agent's compensation. The agent's expected payoff,  $W_0$ , will be determined by the relative bargaining power of the agent and investors when the contract is initiated. For example, if investors have all the bargaining power, then  $W_0 = \arg \max_{W \geq 0} P(K_0, W)$ , whereas if the agent has all the bargaining power, then  $W_0 = \max\{W : P(K_0, W) \geq 0\}$ . More generally, by varying  $W_0$  we can determine the entire feasible contract curve.

### 3 Model Solution

We begin by determining optimal investment in the standard neoclassical setting without an agency problem. We then characterize the optimal contract with agency concerns.

#### 3.1 A Neoclassical Benchmark

With no agency conflicts – corresponding to  $\lambda = 0$  in which case there is no benefit from shirking and/or  $\sigma = 0$  in which case there is no noise to hide the agent's action – our model specializes to the neoclassical setting of Hayashi (1982), a widely used benchmark in the investment literature. Given the stationarity of the economic environment and the homogeneity of the production technology, there is an optimal investment-capital ratio that maximizes the present value of the firm's cash flows. Because of the homogeneity assumption, we can equivalently maximize the present value of the cash flows *per unit of capital*. In other words, we have the Hayashi (1982) result that the marginal value of capital (marginal  $q$ ) equals the average value of capital (average or Tobin's  $q$ ), and both are given by

$$q^{FB} = \max_i \frac{\mu - c(i)}{r + \delta - i} \quad (7)$$

That is, a unit of capital is worth the perpetuity value of its expected free cash flow (expected output less investment and adjustment costs) given the firm's net growth rate  $i - \delta$ . So that the first-best value of the firm is well-defined, we impose the parameter restriction

$$\mu < c(r + \delta). \quad (8)$$

Inequality (8) implies that the firm cannot profitably grow faster than the discount rate. We also assume throughout the paper that the firm is sufficiently productive so termination/liquidation is not efficient, i.e.,  $q^{FB} > l$ .

From the first-order condition for (7), first-best investment is characterized by

$$c'(i^{FB}) = q^{FB} = \frac{\mu - c(i^{FB})}{r + \delta - i^{FB}} \quad (9)$$

Because adjustment costs are convex, (9) implies that first-best investment is increasing with  $q$ . Adjustment costs create a wedge between the value of installed capital and newly purchased capital, in that  $q^{FB} \neq 1$  in general. Intuitively, when the firm is sufficiently productive so that investment has positive NPV, i.e.  $\mu > (r + \delta)c'(0)$ , investment is positive and  $q^{FB} > 1$ . In the special case of quadratic adjustment costs,

$$c(i) = i + \frac{1}{2}\theta i^2, \quad (10)$$

we have the explicit solution:

$$q^{FB} = 1 + \theta i^{FB} \quad \text{and} \quad i^{FB} = r + \delta - \sqrt{(r + \delta)^2 - 2\frac{\mu - (r + \delta)}{\theta}}.$$

Note that  $q^{FB}$  represents the value of the firm's cash flows (per unit of capital) prior to compensating the agent. If investors promise the agent a payoff  $W$  in present value, then absent an agency problem the agent's relative impatience ( $\gamma > r$ ) implies that it is optimal to pay the agent  $W$  in cash immediately. Thus, the investors' payoff is given by

$$P^{FB}(K, W) = q^{FB}K - W.$$

Equivalently, we can express the agent's and investors' payoff on a per unit of capital basis, with  $w = W/K$  and

$$p^{FB}(w) = P^{FB}(K, W)/K = q^{FB} - w.$$

In the neoclassical setting, the time-invariance of the firm's technology implies that the first-best investment is constant over time, and independent of the firm's history or the volatility of its cash flows. As we will explore next, agency concerns significantly alter these conclusions.

### 3.2 The Optimal Contract with Agency

We now solve for the optimal contract when there is an agency problem, that is, when  $\lambda\sigma > 0$ . Recall that the contract specifies the firm's investment policy  $I$ , payments to the agent  $U$ , and a termination date  $\tau$ , all as functions of the firm's profit history. The contract must be incentive compatible (i.e., induce the agent to choose  $a_t = 1$  for all  $t$ ) and maximize the investors' value function  $P(K, W)$ . Here we outline the intuition for the derivation of the optimal contract, leaving formal details to the appendix.

Given an incentive compatible contract  $\Phi$ , and the history up to time  $t$ , the discounted expected value of the agent's future compensation is given by:

$$W_t(\Phi) \equiv \mathbb{E}_t \left[ \int_t^\tau e^{-\gamma(s-t)} dU_s \right]. \quad (11)$$

We call  $W_t$  the agent's continuation payoff as of date  $t$ .

The agent's incremental compensation at date  $t$  is composed of a cash payment  $dU_t$  and a change in the value of his promised future payments, captured by  $dW_t$ . To compensate for the agent's time preference, this incremental compensation must equal  $\gamma W_t dt$  on average. Thus,

$$\mathbb{E}_t (dW_t + dU_t) = \gamma W_t dt. \quad (12)$$

While (12) reflects the agent's average compensation, in order to maintain incentive compatibility, his compensation must be sufficiently sensitive to the firm's incremental output  $K_t dA_t$ .

Adjusting output by its mean and using the martingale representation theorem (details in the appendix), we can express this sensitivity for any incentive compatible contract as follows:<sup>7</sup>

$$dW_t + dU_t = \gamma W_t dt + \beta_t K_t (dA_t - \mu dt) = \gamma W_t dt + \beta_t K_t \sigma dZ_t. \quad (13)$$

To understand the determinants of the incentive coefficient  $\beta_t$ , suppose the agent deviates and chooses  $a_t < 1$ . The instantaneous cost to the agent is the expected reduction of his compensation, given by  $\beta_t (1 - a_t) \mu K_t dt$ , and the instantaneous private benefit is  $\lambda (1 - a_t) \mu K_t dt$ . Thus, to induce the agent to choose  $a_t = 1$ , incentive compatibility is equivalent to:

$$\beta_t \geq \lambda \text{ for all } t.$$

Intuitively, incentive compatibility requires that the agent have a sufficient exposure to the firm's realized output. We will further show that this incentive compatibility constraint binds. To see why, note that the agent has limited liability,  $W_t \geq 0$ . As a result, termination must occur when  $W_t = 0$ , as incentive compatibility cannot be maintained if the firm continues with  $W_t = 0$  (the agent has no downside). The agent's exposure to  $dZ_t$  in (13) then implies that termination will occur with positive probability. An optimal contract will therefore set the agent's sensitivity to  $\beta_t = \lambda$  to reduce the cost of liquidation while maintaining incentive compatibility. Intuitively, incentive provision is necessary, but costly due to the reliance on the threat of *ex post* inefficient liquidation. Hence, the optimal contract requires the minimal necessary level of incentive provision.

Whatever the history of the firm up to date  $t$ , the only relevant state variables going forward are the firm's capital stock  $K_t$  and the agent's continuation payoff  $W_t$ . Therefore the payoff to investors in an optimal contract after such a history is given by the value function  $P(K_t, W_t)$ , which we can solve for using dynamic programming techniques. As in the earlier analysis of the

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<sup>7</sup>Intuitively, the linear form of the contract's sensitivity can be understood in terms of a binomial tree, where any function admits a state-by-state linear representation. Note also that this representation does not allow for additional public randomization, which we show in the proof would not be optimal.

first-best setting, we use the scale invariance of the firm's technology to write  $P(K, W) = p(w)K$  and reduce the problem to one with a single state variable  $w = W/K$ .

We begin with a number of key properties of the value function  $p(w)$ . Clearly, the value function cannot exceed the first best, so  $p(w) \leq p^{FB}(w)$ . Also, as noted above, to deliver a payoff to the agent equal to his outside opportunity (normalized to 0), we must terminate the contract immediately as otherwise the agent could consume private benefits. Therefore,

$$p(0) = l. \tag{14}$$

Next, because investors can always compensate the agent with cash, it will cost investors at most \$1 to increase  $w$  by \$1. Therefore,  $p'(w) \geq -1$ , which implies that the total value of the firm,  $p(w) + w$ , is weakly increasing with  $w$ . In fact, when  $w$  is low, firm value will strictly increase with  $w$ . Intuitively, a higher  $w$  – which amounts to a higher level of deferred compensation for the agent – reduces the probability of termination (within any given amount of time). This benefit declines as  $w$  increases and the probability of termination becomes small, suggesting that  $p(w)$  is concave, a property we will assume for now and verify shortly.

Because there is a benefit of deferring the agent's compensation, the optimal contract will set cash compensation  $du_t = dU_t/K_t$  to zero when  $w_t$  is small, so that (from Eq. (13))  $w_t$  will rise as quickly as possible. However, because the agent has a higher discount rate than investors,  $\gamma > r$ , there is a cost of deferring the agent's compensation. This tradeoff implies that there is a compensation level  $\bar{w}$  such that it is optimal to pay the agent with cash if  $w_t > \bar{w}$  and to defer compensation otherwise. Thus we can set

$$du_t = \max\{w_t - \bar{w}, 0\}, \tag{15}$$

which implies that for  $w_t > \bar{w}$ ,  $p(w_t) = p(\bar{w}) - (w_t - \bar{w})$ , and the compensation level  $\bar{w}$  is the smallest agent continuation payoff with

$$p'(\bar{w}) = -1. \tag{16}$$

When  $w_t \in [0, \bar{w}]$ , the agent's compensation is deferred ( $du_t = 0$ ). The evolution of  $w = W/K$  follows directly from the evolutions of  $W$  (see (13)) and  $K$  (see (1)), noting that  $dU_t = 0$  and  $\beta_t = \lambda$ :

$$dw_t = (\gamma - (i_t - \delta))w_t dt + \lambda(dA_t - \mu dt) = (\gamma - (i_t - \delta))w_t dt + \lambda\sigma dZ_t. \quad (17)$$

Equation (17) implies the following dynamics for the optimal contract. Based on the agent's and investors' relative bargaining power, the contract is initiated with some promised payoff per unit of capital,  $w_0$ , for the agent. This promise grows on average at rate  $\gamma$  less the net growth rate  $(i_t - \delta)$  of the firm. When the firm experiences a positive productivity shock, the promised payoff increases until it reaches the level  $\bar{w}$ , at which point the agent receives cash compensation. When the firm has a negative productivity shock, the promised payoff declines, and the contract is terminated when  $w_t$  falls to zero.

Having determined the dynamics of the agent's payoff, we can now use the Hamilton-Jacobi-Bellman (HJB) equation to characterize  $p(w)$  for  $w \in [0, \bar{w}]$ :

$$rp(w) = \sup_i (\mu - c(i)) + (i - \delta)p(w) + (\gamma - (i - \delta))wp'(w) + \frac{1}{2}\lambda^2\sigma^2p''(w). \quad (18)$$

Intuitively, the right side is given by the sum of instantaneous expected cash flows (the first term in brackets), plus the expected change in the value of the firm due to capital accumulation (the second term), and the expected change in the value of the firm due to the drift and volatility (using Ito's lemma) of the agent's continuation payoff  $w$  (the remaining terms). Investment  $i$  is chosen to maximize investors' total expected cash flow plus "capital gains," which given risk neutrality must equal the expected return  $rp(w)$ .

Using the HJB equation (18), we have that the optimal investment-capital ratio  $i(w)$  satisfies the following Euler equation:

$$c'(i(w)) = p(w) - wp'(w). \quad (19)$$

The above equation states that the marginal cost of investing equals the marginal value of investing from the investors' perspective. The marginal value of investing equals the current per

unit value of the firm to investors,  $p(w)$ , plus the marginal effect of decreasing the agent's per unit payoff  $w$  as the firm grows.

Equations (18) and (19) jointly determine a second-order ODE for  $p(w)$  in the region  $w_t \in [0, \bar{w}]$ . We also have the condition (14) for the liquidation boundary as well as the “smooth pasting” condition (16) for the endogenous payout boundary  $\bar{w}$ . To complete our characterization, we need a third condition to determine the optimal level of  $\bar{w}$ . The condition for optimality is given by the “super contact” condition:<sup>8</sup>

$$p''(\bar{w}) = 0. \quad (20)$$

We can provide some economic intuition for the super contact condition (20) by noting that, using (18) and (16), (20) is equivalent to

$$p(\bar{w}) + \bar{w} = \max_i \frac{\mu - c(i) - (\gamma - r)\bar{w}}{r + \delta - i}. \quad (21)$$

Eq. (21) can be interpreted as a steady-state valuation constraint. The left side represents the total value of the firm at  $w = \bar{w}$  while the right side is the discounted value of the firm's cash flows given the cost of maintaining the agent's continuation payoff at  $\bar{w}$  (because  $\gamma > r$ , there is a cost to deferring the agent's compensation  $\bar{w}$ ).

We now summarize our main results on the optimal contract in the following proposition.<sup>9</sup>

**Proposition 1** *The investors' value function  $P(K, W)$  is proportional to capital stock  $K$ , in that  $P(K, W) = p(w)K$ , where  $p(w)$  is the investors' scaled value function. For  $w_t \in [0, \bar{w}]$ ,  $p(w)$  is strictly concave and uniquely solves the ODE (18) with boundary conditions (14), (16), and (20). For  $w > \bar{w}$ ,  $p(w) = p(\bar{w}) - (w - \bar{w})$ . The agent's scaled continuation payoff  $w$  evolves according to (17), for  $w_t \in [0, \bar{w}]$ . Cash payments  $du_t = dU_t/K_t$  reflect  $w_t$  back to  $\bar{w}$ , and the*

<sup>8</sup>The super contact condition essentially requires that the second derivatives match at the boundary. See Dixit (1993).

<sup>9</sup>We provide necessary technical conditions and present a formal verification argument for the optimal policy in the appendix.

contract is terminated at the first time  $\tau$  such that  $w_\tau = 0$ . Optimal investment is given by  $I_t = i(w_t) K_t$ , where  $i(w)$  is defined in (19).

The termination value  $l$  could be exogenous, for example the capital's salvage value in liquidation. Alternatively  $l$  could be endogenous. For example, suppose termination involves firing and replacing the agent with a new (identical) agent. Then the investors' termination payoff equals the value obtained from hiring a new agent at an optimal initial continuation payoff  $w_0$ . That is,

$$l = \max_{w_0} (1 - \kappa)p(w_0), \tag{22}$$

where  $\kappa \in [0, 1)$  reflects a cost of lost productivity if the agent is replaced.

## 4 Model Implications and Analysis

Having characterized the solution of the optimal contract, we first study some additional properties of  $p(w)$  and then analyze the model's predictions on average  $q$ , marginal  $q$ , and investment.

### 4.1 Investors' Scaled Value Function

Using the optimal contract in Section 3, we plot the investors' scaled value function  $p(w)$  in Figure 1 for two different termination values. The gap between  $p(w)$  and the first-best value function reflects the loss due to agency conflicts. From Figure 1, we see that this loss is higher when the agent's payoff  $w$  is lower or when the termination value  $l$  is lower. Also, when the termination value is lower, the cash compensation boundary  $\bar{w}$  is higher as it is optimal to defer compensation longer in order to reduce the probability of costly termination.

The concavity of  $p(w)$  reveals the investor's induced aversion to fluctuations in the agent's payoff. Intuitively, a mean-preserving spread in  $w$  is costly because the increased risk of termination from a reduction in  $w$  outweighs the benefit from an increase in  $w$ . Thus, although investors are risk neutral, they behave in a risk-averse manner toward idiosyncratic risk due to the agency friction. This property fundamentally differentiates our agency model from the

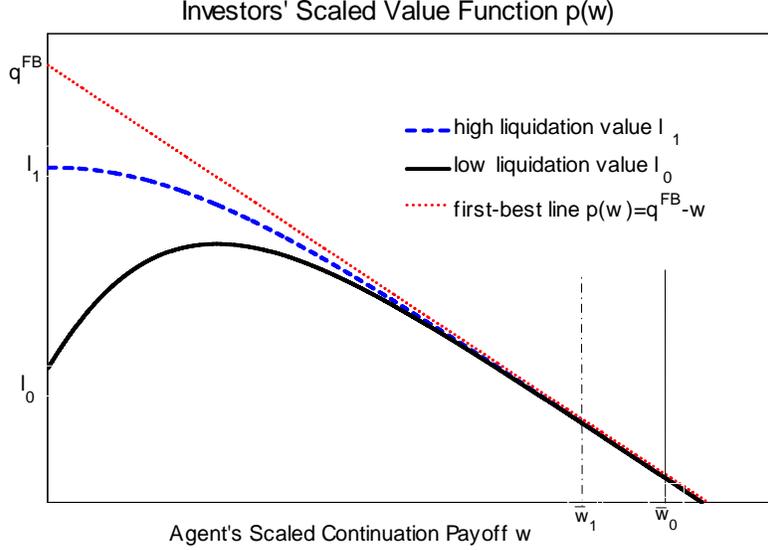


Figure 1: Investors' scaled value function  $p(w)$  as a function of agent's scaled continuation payoff  $w$ . We illustrate scenarios where the liquidation value is high ( $l_1$ ) and low ( $l_0$ ).

neoclassical Hayashi (1982) result where volatility has no effect on investment and firm value. The dependence of investment and firm value on idiosyncratic volatility in our model arises from investors' inability to distinguish the agent's actions from luck.

While  $p(w)$  is concave, it need not be monotonic in  $w$ , as shown in Figure 1. The intuition is as follows. Two effects drive the shape of  $p(w)$ . First, as in the first-best neoclassical benchmark of Section 3.1, the higher the agent's claim  $w$ , the lower the investors' value  $p(w)$ , holding the total surplus fixed. This is just a wealth transfer effect. Second, increasing  $w$  allows the contract to provide incentives to the agent with a lower risk of termination. This "incentive alignment effect" creates wealth, raising the total surplus available for distribution to the agent and investors. As can be seen from the figure, the wealth transfer effect dominates when  $w$  is large, but the incentive alignment effect can dominate when  $w$  is low and termination is sufficiently costly.

If the liquidation value is sufficiently low so that the value function  $p$  is non-monotonic, then while termination is used to provide incentives *ex ante*, it is inefficient *ex post*. However, inefficient termination provides room for renegotiation, since both parties will have incentives

to renegotiate to a Pareto-improving allocation. Thus, with liquidation value  $l_0$ , the optimal contract depicted in Figure 1 is not renegotiation-proof (while the contract is renegotiation-proof with liquidation value  $l_1$ ). In Appendix C, we show that the main qualitative implications of our model are unchanged when contracts are constrained to be renegotiation-proof. Intuitively, renegotiation weakens incentives and has the same effect as increasing the agent's outside option (which reduces the investors' payoff).

Alternatively, if the agent can be fired and costlessly replaced, so that the liquidation value is endogenously determined as in Eq. (22) with  $\kappa = 0$ , then  $p'(0) = 0$  and the optimal contract will be renegotiation-proof. We can also interpret the case with  $l_1$  in Figure 1 in this way.

## 4.2 Average and Marginal $q$

Now we use the properties of  $p(w)$  to derive implications for  $q$ . Total firm value, including the claim held by the agent, is  $P(K, W) + W$ . Therefore, average  $q$ , defined as the ratio between firm value and capital stock, is denoted by  $q_a$  and given by<sup>10</sup>

$$q_a(w) = \frac{P(K, W) + W}{K} = p(w) + w. \quad (23)$$

Marginal  $q$  measures the incremental impact of a unit of capital on firm value. We denote marginal  $q$  as  $q_m$  and calculate it as:

$$q_m(w) = \frac{\partial (P(K, W) + W)}{\partial K} = P_K(K, W) = p(w) - wp'(w). \quad (24)$$

While average  $q$  is often used in empirical studies due to the simplicity of its measurement, marginal  $q$  determines the firm's investment via the standard Euler equations.

One of the most important and well-known results in Hayashi (1982) is that marginal  $q$  equals average  $q$  when the firm's production and investment technologies exhibit homogeneity as shown in our neoclassical benchmark case. While our model also features these homogeneity

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<sup>10</sup>Note that excluding the agent's future compensation  $w$  in the definition of  $q$  does not give the prediction that Tobin's  $q$  equals marginal  $q$  even in the neoclassical benchmark (Hayashi (1982)) setting (see Section 3.1). Thus, empirical measures of  $q$  that ignore managers' claims may be misspecified.

properties, agency costs cause the marginal value of investing,  $q_m$ , to differ from the average value of capital stock,  $q_a$ . In particular, comparing (7), (23) and (24) and using the fact that  $p'(w) \geq -1$ , we have the following inequality:

$$q^{FB} > q_a(w) \geq q_m(w). \quad (25)$$

The first inequality follows by comparing (21) and the calculation of  $q^{FB}$  in (7). Average  $q$  is above marginal  $q$  because, for a given level of  $W$ , an increase in capital stock  $K$  lowers the agent's scaled continuation payoff  $w$ , which lowers the agent's effective claim on the firm, and hence induces a more severe agency problem. The wedge between average and marginal  $q$  is non-monotonic. Average and marginal  $q$  are equal when  $w = 0$  and the contract is terminated. Then  $q_a > q_m$  for  $w > 0$  until the cash payment region is reached,  $w = \bar{w}$ . At that point, the incentive benefits of  $w$  are outweighed by the agent's impatience, so that  $p'(\bar{w}) = -1$  and again  $q_a = q_m$ .

Both average  $q$  and marginal  $q$  are functions of the agent's scaled continuation payoff  $w$ . Because  $p'(w) \geq -1$ , average  $q$  is increasing in  $w$  (reflecting the incentive alignment effect noted earlier). In addition, the concavity of  $p(w)$  implies that marginal  $q$  is also increasing in  $w$ . In Figure 2, we plot  $q_a$ ,  $q_m$ , and the first-best average (also marginal)  $q^{FB}$ .

It is well understood that marginal and average  $q$  are forward looking measures and capture future investment opportunities. In the presence of agency costs, it is also the case that both marginal and average  $q$  are positively related to the firm's profit history. Recall that the value of the agent's claim  $w$  evolves according to (17), and so is increasing with the past profits of the firm, and that both marginal and average  $q$  increase with  $w$  for incentive reasons. Unlike the neoclassical setting in which  $q$  is independent of the firm's history, both marginal and average  $q$  in our setting are history dependent.

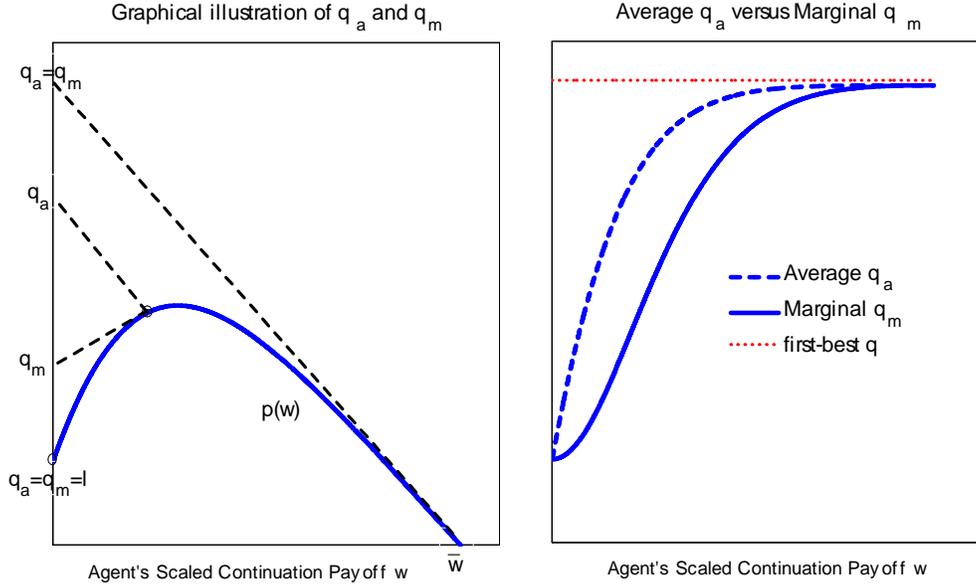


Figure 2: Average  $q_a$  and marginal  $q_m$ . The left panel shows a geometrical illustration of the determination of  $q_a$  and  $q_m$ . The right panel plots  $q_a$  and  $q_m$  with the first best  $q^{FB}$ .

### 4.3 Investment and $q$

We now turn to the model's predictions on investment. First, note that the investment-capital ratio  $i(w)$  in our agency model naturally depends on  $w$ . Indeed, the first-order condition for optimal investment (19) can be written in terms of marginal  $q$ :

$$c'(i(w)) = q_m(w) = p(w) - wp'(w). \quad (26)$$

The convexity of the investment cost function  $c$  and the monotonicity of  $q_m$  imply that investment increases with  $w$ . Specifically,

$$i'(w) = \frac{q'_m(w)}{c''(i(w))} = -\frac{wp''(w)}{c''(i(w))} \geq 0, \quad (27)$$

where the inequality is strict except at termination ( $w = 0$ ) and the cash payout boundary ( $p''(\bar{w}) = 0$ ).

Intuitively, when  $w$  is low, inefficient termination becomes more likely. Hence, investors optimally invest less. In the limiting case where termination is immediate ( $w = 0$ ), the marginal

benefit of investing is just the liquidation value  $l$  per unit of capital. Thus, the lower bound on the firm's investment is given by  $c'(i(0)) = l$ . Assuming  $c'(0) > l$ , the firm will disinvest near termination.

Now consider the other limiting case, when  $w$  reaches the cash payout boundary  $\bar{w}$ . Then because  $q_m(w) < q^{FB}$  from (25), we have  $i(\bar{w}) < i^{FB}$ . Thus, even at this upper boundary, there is underinvestment – the strict relative impatience of the agent, i.e.  $\gamma > r$ , creates a wedge between our solution and first-best investment. In the limit, when  $\gamma$  is sufficiently close to  $r$ , the difference between  $i(\bar{w})$  and  $i^{FB}$  disappears. That is, the underinvestment at the payout boundary is due to the model's assumption  $\gamma > r$ , not agency costs.

To summarize, in addition to costly termination as a form of underinvestment, the investment/capital ratio is lower than the first-best level, that is  $i(w) < i^{FB}$  always. Thus, our model features underinvestment at all times. Figure 3 shows the investors' value function and the investment-capital ratio for two different volatility levels. Note the increasing relation between investment and the agent's continuation payoff  $w$ . Hence investment is positively related to past performance. Moreover, given the persistence of  $w$ , investment is positively serially correlated. By contrast, in the first-best scenario, investment is insensitive to past performance.

Figure 3 also shows that the value of the firm and the rate of investment are lower with a higher level of idiosyncratic volatility,  $\sigma$ . With higher volatility, firm profits are less informative regarding agent effort, and incentive provision becomes more costly. This effect reduces the value of the firm and the return on investment.<sup>11</sup> The same comparative statics would result from an increase in the rate  $\lambda$  at which the agent accrues private benefits (exacerbating the agency problem). In fact, from Proposition 1, firm value and the level of investment depend only on the product of  $\lambda$  and  $\sigma$  – the extent of the agency problem is determined by both firm volatility and the agent's required exposure to it.

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<sup>11</sup>Panousi and Papanikolaou (2008) present evidence that investment is lower for firms with higher idiosyncratic risk.

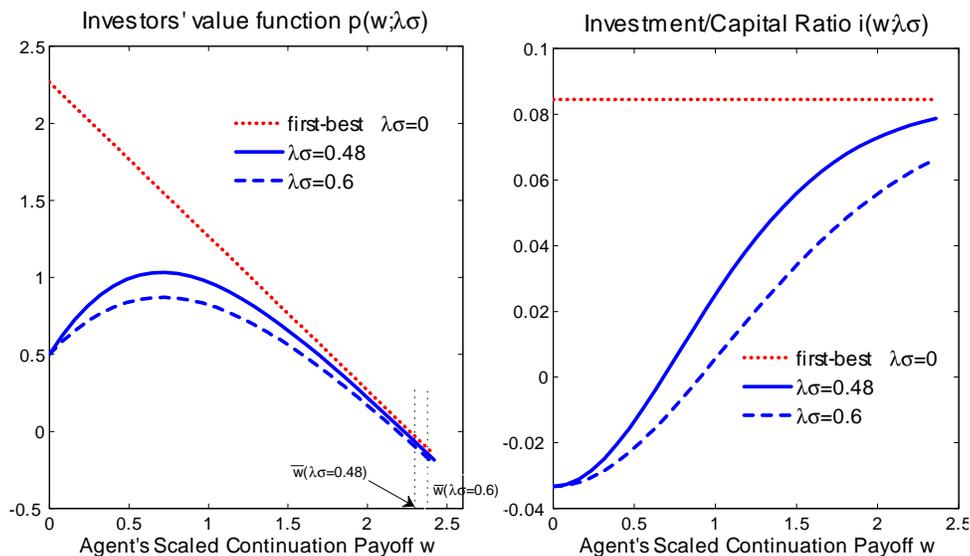


Figure 3: The effect of volatility,  $\sigma$ , and the severity of the agency problem,  $\lambda$ , on the investors' scaled value function  $p(w)$  and the investment/capital ratio  $i(w)$ .

Note also that the cash payout boundary  $\bar{w}$  increases with the severity of the agency problem. As  $\lambda\sigma$  increases, so does the volatility of the agent's continuation payoff  $w$ . To reduce the risk of inefficient termination, it is optimal to allow for a higher level of deferred compensation.

#### 4.4 A calibration exercise

While our model utilizes simplifying assumptions such as constant returns to scale as in Hayashi (1982), the model nonetheless provides some guidance on the potential quantitative impact of agency on investment and firm value. Eberly, Rebelo and Vincent (2008) provide empirical evidence in support of Hayashi (1982). Following their work, we calibrate our model using the following parameters: the annual interest rate is  $r = 4.6\%$ , the expected productivity  $\mu = 20\%$ , annual volatility  $\sigma = 10\%$ , and the agent's discount rate  $\gamma = 4.7\%$ . For the full sample of large firms in Compustat from 1981 to 2003, Eberly, Rebelo, and Vincent (2008) document that the average  $q$  is 1.3, and the investment-capital ratio 15%. Calibrating the first-best  $q^{FB}$  and the first-best investment-capital ratio  $i^{FB}$  to these sample averages, we obtain  $\theta = 2$  and  $\delta = 12.5\%$

for our model. We choose per unit capital liquidation value  $l = 0.9$  (Hennessy and Whited (2005)) and  $\lambda = 0.75$  as baseline parameters.

We then simulate our model at monthly frequency, generating a sample path with 200 years. We compute the sample average of  $q$  and investment-capital ratio  $i(w)$ . We then repeat the simulation 200 times and report the average here. The top panel of Table 1 reports the impact of the agency parameter  $\lambda$  on average  $q$  and the investment-capital ratio, given the baseline parameters described above. Note that the degree of underinvestment is significant. Even for  $\lambda = .25$ , there is an average 8.18% reduction in the investment-capital ratio. For  $\lambda = 1$ , the investment-capital ratio is 19.36% lower than the first-best level. Agency lowers firm value (average  $q$ ) about 3.63% from the first-best level when agency is severe ( $\lambda = 1$ ) and about 1.43% even when agency is relatively less severe ( $\lambda = 0.25$ ).

The bottom panel shows the impact of liquidation (measured by  $l$ ) on firm value and investment (fixing the agency parameter  $\lambda = 0.75$ ). Note that the degree of investment distortion is relatively insensitive to the liquidation value. Varying  $l$  from 0 to 0.9 only mildly changes the degree of underinvestment from 16.72% to 20.10%. Naturally, the impact on firm value is also relatively small, ranging from 3.06% (when  $l = 1$ ) to 3.38% (when  $l = 0$ ). Intuitively, liquidation happens rarely on the equilibrium path, and hence the present discounted value of liquidation is low and the impact of liquidation on investment and firm value is thus of the second order.

Table 1: The impact of agency friction on  $q_a$  and  $i$

	Panel A: Agency parameter $\lambda$			
	0.25	0.5	0.75	1
mean $(1 - q_a/q^{FB})$ %	1.43	2.42	3.20	3.63
mean $(1 - i/i^{FB})$ %	8.18	13.80	17.75	19.36
	Panel B: Liquidation value $l$			
	0	0.5	0.9	1
mean $(1 - q_a/q^{FB})$ %	3.38	3.24	3.20	3.06
mean $(1 - i/i^{FB})$ %	20.10	18.60	17.75	16.72

Baseline parameters:  $r = 4.6\%$ ,  $\mu = 20\%$ ,  $\sigma = 10\%$ ,  $\gamma = 4.7\%$ ,  $\theta = 2$ ,  $\delta = 12.5\%$ ,  $l = 0.9$ , and  $\lambda = 0.75$ . In calculating the sample average, we simulate the model at monthly frequency for 200 years, with 200 repetitions.

## 5 Implementing the Optimal Contract

In Section 3, we characterized the optimal contract in terms of an optimal mechanism. In this section, we consider implications of the optimal mechanism for the firm’s financial slack, and explore the link between financial slack and investment.

Recall that the dynamics of the optimal contract are determined by the evolution of the agent’s continuation payoff  $w_t$ . Because termination occurs when  $w_t = 0$ , we can interpret  $w_t$  as a measure of the firm’s “distance” to termination. Indeed, from (17), the largest short-run shock  $dA_t$  to the firm’s cash flows that can occur without termination is given by  $w_t/\lambda$ . This suggests that we can interpret  $m_t = w_t/\lambda$  as the firm’s available financial slack, i.e. the largest short-run loss the firm can sustain before the agent is terminated and a change of control occurs.

We can formalize this idea in a variety of ways. Financial slack may correspond to the firm’s cash reserves (as in Biais et al. (2007)), or a line of credit (as in DeMarzo and Sannikov (2006) and DeMarzo and Fishman (2007b)), or be a combination of the firm’s cash and available credit. Payments to investors may correspond to payouts on debt, equity, or other securities. Rather than attempt to describe all possibilities, we’ll describe one simple way to implement the optimal contract, and then discuss which of its features are robust.

Specifically, suppose the firm meets its short-term financing needs by maintaining a cash reserve. (Recall that the firm will potentially generate operating losses, and so needs access to cash or credit to operate.) Let  $M_t$  denote the level of cash reserves at date  $t$ . These reserves earn interest rate  $r$ , and once they are exhausted, the firm cannot operate and the contract is terminated.

The firm is financed with equity. Equity holders require a minimum payout rate of<sup>12</sup>

$$dD_t = [K_t (\mu - c(i_t)) - (\gamma - r) M_t] dt.$$

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<sup>12</sup>If  $dD_t < 0$  we interpret this as the maximum rate that the firm can raise new capital, e.g, by issuing equity. We can show, however, that if  $\lambda = 1$ , then  $dD_t > 0$ .

The first component of the dividend,  $K_t(\mu - c(i_t))$ , corresponds to the firm's expected free cash flow. The second component,  $(\gamma - r)M_t$ , adjusts for the relative impatience of the agent, and is negligible when  $\gamma \approx r$ . If the agent fails to meet this minimal payout rate, the contract is terminated. Other than this constraint, the agent has discretion to choose an effort level  $a_t$ , the firm's investment-capital ratio  $i_t$ , as well as additional payout  $X_t$  in excess of the minimum payout rate described above. The agent is compensated by receiving a fraction  $\lambda$  of any "special" dividends  $X_t$ .

Under this implementation, the firm's cash reserves will grow according to

$$dM_t = rM_t dt + dY_t - dD_t - dX_t. \quad (28)$$

The value of the firm's equity is given by

$$S_t = \mathbb{E}_t \left[ \int_t^\tau e^{-r(s-t)} (dD_t + (1 - \lambda)dX_t) + e^{-r(\tau-t)} lK_t \right], \quad (29)$$

where  $\tau$  is the first stochastic (hitting) time such that  $M_t = 0$ . The expected payoff to the agent is given by

$$W_t = \mathbb{E}_t \left[ \int_t^\tau e^{-\gamma(s-t)} \lambda dX_t \right]. \quad (30)$$

The following proposition establishes that the above specification implements the optimal contract.

**Proposition 2** *Suppose the firm has initial cash reserves  $M_0$  and can operate as long as  $M_t \geq 0$  and it maintains the minimum payout rate  $dD_t$ . Then it is optimal for the agent to choose effort  $a_t = 1$  and to choose the investment-capital ratio  $i_t$  given in Proposition 1. The agent accumulates cash reserves  $M_t$  until  $m_t \equiv M_t/K_t = \bar{w}/\lambda$ , and pays out all cash in excess of this amount. Given this policy, the agent's payoff  $W_t = \lambda M_t$ , and coincides with the continuation payoff of Proposition 1. Finally, the firm's stock price satisfies  $S_t = (p(\lambda m_t) + m_t) K_t$ .*

In this implementation, regular dividends are relatively “smooth” and approximately correspond to the firm’s *expected* rate of free cash flow. The cash flow fluctuations induced by the firm’s productivity shocks are absorbed by the firm’s cash reserves until the maximal level of reserves is achieved or the firm runs out of cash. Also, because the above financial policy implements the optimal contract, there is no *ex ante* change to the policy (such as issuance of alternative securities) that will make shareholders better off.<sup>13</sup>

The above implementation is not unique. For example, the minimum dividend payouts could be equivalently implemented as required coupon payments on long-term debt or preferred stock. (In fact, such an implementation may be more natural if termination is interpreted as liquidating the firm, as opposed to firing the manager.) Also, rather than solely use cash reserves, the firm may maintain its financial slack through a combination of cash and credit. In that case the measure  $M_t$  of financial slack will correspond to the firm’s cash and available credit, and the contract will terminate once these are exhausted.<sup>14</sup> Again, financial slack  $M_t$  will be proportional to the manager’s continuation payoff  $W_t$  in the optimal contract. In fact because  $W_t$  is a measure of the firm’s “distance to termination” in the optimal contract, its relation to the firm’s financial slack is a robust feature of *any* implementation.

In our implementation, financial slack per unit of capital is given by  $m = w/\lambda$ . Therefore, we can reinterpret some earlier results in terms of this measure of financial slack. Specifically, we have the following results:

- Financial slack is positively related to past performance.
- Average  $q$  (corresponding to enterprise value plus agent rents) and marginal  $q$  increase with financial slack.

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<sup>13</sup>If the optimal contract is not renegotiation-proof, then not surprisingly there may be *ex post* improvements available to shareholders. See the appendix for further discussion.

<sup>14</sup>The implementation developed here using cash reserves is similar to that in Biais et. al (2007). In DeMarzo and Fishman (2007b) and DeMarzo and Sannikov (2006), the firm’s financial slack is in the form of a credit line. With a credit line, the minimum dividend payout rate would be reduced by the interest due on the credit line and the risk-free interest rate on the firm’s unused credit.

- Investment increases with financial slack.
- Expected agent cash compensation (over any time interval) increases with financial slack.
- The maximal level of financial slack is higher for firm's with more volatile cash flows and firms with lower liquidation values.<sup>15</sup>

The investment literature often focuses on the positive relation between firms' cash flow and investment; see for example, Fazzari, Hubbard, and Petersen (1988), Hubbard (1998) and Stein (2003). While our results are consistent with this effect, our analysis suggests that financial slack (a stock, rather than flow measure) has a more direct influence on investment. It is also worth noting that our dynamic agency model does not yield a sharp prediction on the sensitivity of  $di/dm$  with respect to financial slack. That is, as shown by Kaplan and Zingales (1997) it is difficult to sign  $d^2i/dm^2$  without imposing strong restrictions on the cost of investment. Their result can be understood in the context of our model from Eq. (27), where it is clear that  $i'(w)$ , and therefore  $di/dm$ , depends on the convexity of the investment cost function  $c''$ . Thus,  $d^2i/dm^2$  will depend on the third-order derivatives of the cost function.

## 6 Persistent Price Shocks

The only shocks in our model thus far are the firm's idiosyncratic, temporary productivity shocks. While these shocks have no effect in the neoclassical setting, they obscure the agent's actions to create an agency problem. The optimal incentive contract then implies that these temporary, idiosyncratic shocks have a persistent impact on the firm's investment, growth, and value.

In this section, we extend the model to allow for persistent observable shocks to the firm's profitability. As an example, we consider fluctuations in the price of the firm's output. These

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<sup>15</sup>Unlike our earlier results, this result for volatility does not extend to the level of private benefits  $\lambda$ . While the optimal contract (in terms of payoffs and net cash flows) only depends on  $\lambda\sigma$ , in this implementation the maximal level of financial slack is given by  $\bar{m} = \bar{w}/\lambda$ , so that although  $\bar{w}$  increases with  $\lambda$ , the maximal level of financial slack  $\bar{m}$  will tend to decrease with the level of private benefits.

price shocks differ in two important ways from the firm’s temporary productivity shocks. First, it is natural to assume that these price shocks are observable and can be contracted on. Second, because these price shocks are persistent they will affect the firm’s optimal rate of investment even in the neoclassical setting.

Our goal is to explore the interaction of public, persistent shocks with the agency problem and the consequences for investment, financial slack, and managerial compensation. As a benchmark, if the price shocks were purely transitory, they would have no effect on the firm’s investment or the agent’s compensation, with or without an agency problem. Investors would simply absorb the price shocks, insulating the firm and the agent. As we will show, however, if the price shocks are persistent they will affect both the optimal level of investment and the agent’s compensation, with the latter effect having an additional feedback on the firm’s investment.

We extend our baseline model with i.i.d. productivity shocks to a setting where output price is stochastic in Section 6.1. Then, we analyze the interaction effects in Section 6.2. Finally, in Section 6.3 we examine the relation between investment and financial slack controlling for average  $q$ .

## 6.1 The Model and Solution Method

We extend the basic model by introducing a stochastic price  $\pi_t$  for the firm’s output. To keep the analysis simple, we model  $\pi_t$  as a two-state Markov regime-switching process.<sup>16</sup> Specifically,  $\pi_t \in \{\pi^L, \pi^H\}$  with  $0 < \pi^L < \pi^H$ . Let  $\xi^n$  be the transition intensity out of regime  $n = L$  or  $H$  to the other regime. Thus, for example, given a current price of  $\pi^L$  at date  $t$ , the output price changes to  $\pi^H$  with probability  $\xi^L dt$  in the time interval  $(t, t + dt)$ . The output price  $\pi_t$  is observable to investors and the agent and is contractible. The firm’s operating profit is then

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<sup>16</sup>Piskorski and Tchisty (2007) consider a model of mortgage design in which they use a Markov switching process to describe interest rates. They were the first to incorporate such a process in a continuous-time contracting model, and our analysis follows their approach.

given by the following modification of Eq. (3):

$$dY_t = K_t(\pi_t dA_t - c(i_t)dt). \quad (31)$$

Let  $P(K, W, \pi)$  denote the investors' value function, when capital stock is  $K$ , the agent's continuation payoff is  $W$ , and the output price is  $\pi$ . Again, using the scale invariance of the firm's technology, we conjecture that for  $n = L$  or  $H$  and  $w = W/K$ , we can write the value function as

$$P(K, W, \pi^n) = Kp_n(w), \quad (32)$$

where  $p_n(w)$  represents the investors' scaled value per unit of capital in state  $n$ .

To determine the dynamics of the agent's scaled continuation payoff  $w$ , we must first consider how the agent's payoff is affected if the output price changes. For example, suppose the output price jumps from  $\pi^L$  to  $\pi^H$ . How should the agent's scaled continuation payoff  $w$  respond to this exogenous price shock?

In designing the optimal contract, investors optimally adjust the agent's continuation payoff to minimize agency costs. When the output price  $\pi_t$  switches from  $\pi^L$  to the higher value  $\pi^H$ , the firm becomes more profitable. In Figure 4, this is captured by the expansion of the investors' value function, i.e.,  $p_H(w) \geq p_L(w)$  for any  $w$ . Because firm value is higher, the benefit of avoiding termination/liquidation is also higher, and we will show that this increases the marginal benefit of increasing the agent's payoff, i.e.,  $p'_H(w) \geq p'_L(w)$  for any  $w$ . This observation suggests that it is optimal to increase the agent's continuation payoff  $w$ , and thus the firm's financial slack, when the output price increases in order to reduce agency costs.

To formalize this effect, let  $\psi_{nm}(w)$  denote the endogenous adjustment of  $w$  conditional on a jump of the output price from state  $\pi^n$  to the alternative state  $\pi^m$ , so that the agent's scaled continuation payoff changes from  $w$  just prior to the jump to  $w + \psi_{nm}(w)$  immediately after. The optimal adjustment should equate the marginal cost of compensating the agent before and after the jump. Given that investors have to deliver an additional dollar of compensation to the

agent, what is their marginal cost of doing so in each state? The marginal cost is captured by the marginal impact of  $w$  on the investors' value function, i.e.,  $p'_n(w)$ . Therefore, the compensation adjustment  $\psi_{nm}$  is given by

$$p'_n(w) = p'_m(w + \psi_{nm}(w)) \quad (33)$$

which is feasible as long as  $p'_n(w) \leq p'_m(0)$ . If  $p'_n(w) > p'_m(0)$ , the agent's payoff jumps to zero ( $\psi_{nm}(w) = -w$ ) and the contract terminates in order to minimize the difference in the marginal cost of compensation. See Figure 4; there, if the output price is high and  $w < w^c$ , where  $w^c$  is determined by

$$p'_H(w^c) = p'_L(0),$$

a drop in the output price triggers termination.

The above discussion leads to the following dynamics for the agent's continuation value  $w$ . Like before, cash compensation is deferred up to a threshold  $\bar{w}^n$ , but now the threshold depends on the output price. Letting  $N_t$  denote the cumulative number of regime changes up to time  $t$ , the dynamics for the agent's scaled continuation payoff with output price  $\pi^n$  and  $w_t \in [0, \bar{w}^n]$  is given by

$$dw_t = (\gamma - (i_t - \delta)) w_t dt + \lambda (dA_t - \mu dt) + \psi_{nm}(w_t) (dN_t - \xi^n dt), \quad (34)$$

Like before, the diffusion martingale term  $\lambda (dA_t - \mu dt)$  describes the agent's binding incentive constraint, implied by the concavity of investors' scaled value functions in both regimes (see the Appendix).<sup>17</sup> The jump martingale term  $\psi_{nm}(w_t) (dN_t - \xi^n dt)$  has a zero drift, and this guarantees that the agent's continuation payoff  $W$  grows at  $\gamma$  on average, taking into account the net capital accumulation rate,  $i_t - \delta$ , along the equilibrium path. In the appendix, we provide a formal characterization of the optimal contract, and derive the following key properties:

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<sup>17</sup>The incentive provision  $\lambda K_t (dA_t - \mu dt)$  does not scale with output price  $\pi_t$ . This is consistent with the interpretation of the agent's action as effort, as the agent's shirking benefit (personal cost of effort) is independent of the output price. In contrast, the incentive provision would scale with  $\pi_t$  under the interpretation that the agent's action involves cash diversion. Otherwise, the qualitative conclusions of the model would remain unchanged.

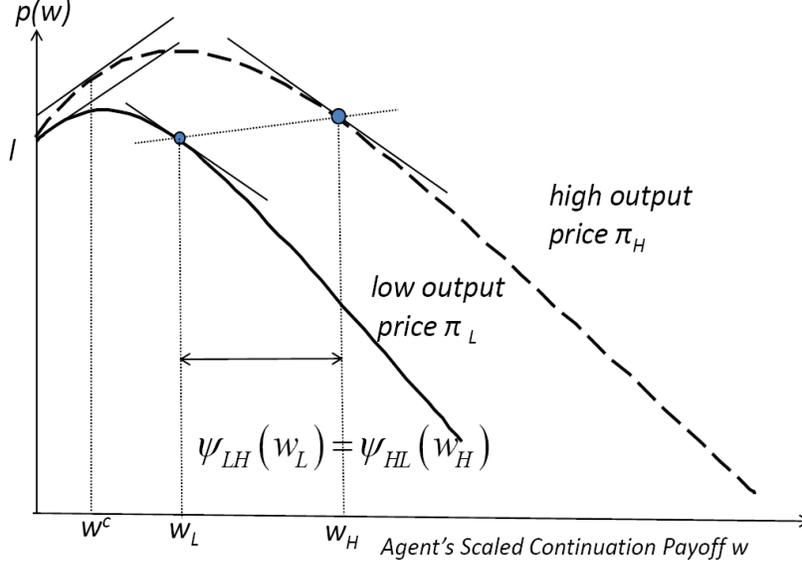


Figure 4: Agent's continuation payoff (compensation) adjustment  $\psi_{HL}(w)$  (or  $\psi_{LH}(w)$ ) given exogenous price changes. As shown, the optimality of compensation policy implies that in the interior region,  $p'_L(w_L) = p'_H(w_L + \psi_{LH}(w_L))$ .

**Proposition 3** *With output price  $\pi^n$ , the agent's continuation payoff evolves according to (34) until  $w_t = 0$  and the contract is terminated or  $w_t = \bar{w}^n$  and the agent receives cash. The investors' value functions  $p_L, p_H$  are concave, with  $p_L < p_H$  for  $w > 0$ , and  $p'_L < p'_H$  for  $w \leq \bar{w}^H$ . Thus, the compensation adjustment  $\psi_{LH}$  is positive, and  $\psi_{HL}$  is negative. Moreover, if the output price is high and  $w_t$  is low enough such that  $p'_H(w_t) \geq p'_L(0)$ , then  $\psi_{HL}(w_t) = -w_t$  and the contract is immediately terminated if the output price falls.*

See Figure 4 for an illustration of the investor's value function and the compensation adjustment functions in this setting.

## 6.2 Model Implications

Here we discuss a number of implications from our model for the impact of output price fluctuations on the agent's compensation, the firm's financial slack, and the optimal level of investment.

**Agent Compensation.** An important implication of our results is that the agent's compensation will be affected by persistent shocks to the firm's profitability, even when those shocks

are publicly observable and unrelated to the agency problem. Specifically, the agent is rewarded when the output price increases,  $\psi_{LH} > 0$ , and is penalized – and possibly immediately terminated – when the output price declines,  $\psi_{HL} < 0$ . This result is in contrast with conventional wisdom that optimal contracts should insulate managers from exogenous shocks and compensate them based solely on relative performance measures. Rather, in our dynamic model managerial compensation will *optimally* be sensitive to the *absolute* performance of the firm.

The intuition for this result is that an increase in the firm’s profitability makes it efficient to reduce the likelihood of termination by increasing the level of the agent’s compensation. Thus, the optimal contract shifts the agent’s compensation from low price states to high price states. More generally, in a dynamic agency context, the optimal contract smoothes the *marginal cost of compensation*. Unlike static models, in a dynamic setting the marginal cost of compensation is generally state dependent, as it is cheaper to increase the agent’s rents (and thus align incentives) in states where the incentive problem is more costly.

We assumed here that the liquidation/termination value of the firm is independent of the current output price, thus making termination relatively more costly in the high price state. If termination corresponds to firing and replacing the agent, then although  $l_H > l_L$  (firm value upon replacement is higher when the output price is high), the qualitative results discussed above remain unchanged. Indeed, as long as it is costly to replace the agent ( $\kappa > 0$  in (22)), then  $w^c > 0$  and the agent may be fired and replaced if the output price falls.

However, if  $l_H$  and  $l_L$  are specified in some alternative manner, it is possible that termination would be sufficiently less costly when the output price were high so that it would be optimal to reduce the agent’s compensation (and thereby increase the risk of termination) when the output price increased. But while these specific results could change depending on such assumptions, the more important qualitative result – that the agent’s compensation is affected by persistent observable shocks – would remain.

**Hedging and Financial Slack.** We can also interpret these results in the context of the firm’s financial slack. Because  $\psi_{LH} > 0$ , it is optimal to increase the firm’s available slack when the output price rises, and decrease slack when the output price decreases. This sensitivity could be implemented, for example, by having the firm hold a financial derivative that pays off when the output price increases. A similar adjustment to financial slack might be implemented with convertible debt – when the output price and firm value increases, bondholders convert their securities, and financial slack increases. Whatever the specific implementation, it is optimal for the firm to *increase* the sensitivity of its cash position to the output price. Notably, it is *not* optimal for the firm to hedge its output price. Rather, the firm’s hedging policy should smooth the changes in the marginal value of financial slack.<sup>18</sup>

**Amplification of Investment.** The interaction between the agency problem and investment policy creates an amplification effect when the output price changes. Specifically, investment increases if the output price increases for two reasons. There is a direct effect in that investment is more profitable with a higher output price. There is also an indirect effect. Because the agent’s continuation payoff  $w$ , and thus the firm’s financial slack, will increase with an increase in the output price, the agency problem is reduced, and again, investment is more profitable.

Figure 5 plots the changes in the investment-capital ratio when the output price increases and decreases. First, consider the left panel. The solid line corresponds to the total change in the investment-capital ratio when the output price rises, i.e.,  $i_H(w + \psi_{LH}(w)) - i_L(w)$ . The dashed line depicts the direct effect of a regime switch on the investment-capital ratio,  $i_H(w) - i_L(w)$ . The difference between the solid and dashed lines is  $i_H(w + \psi_{LH}(w)) - i_H(w)$ , the indirect effect resulting from the upward adjustment of the firm’s financial slack and the agent’s compensation. Interestingly, when the output price jumps up, the “direct effect” understates the impact of the

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<sup>18</sup>This motive for hedging is related to that of Froot, Scharfstein, and Stein (1993), who suggest that firms should hedge to fund their investment needs when external capital is costly. Here, firms smooth the agency costs that underlie their cost of capital.

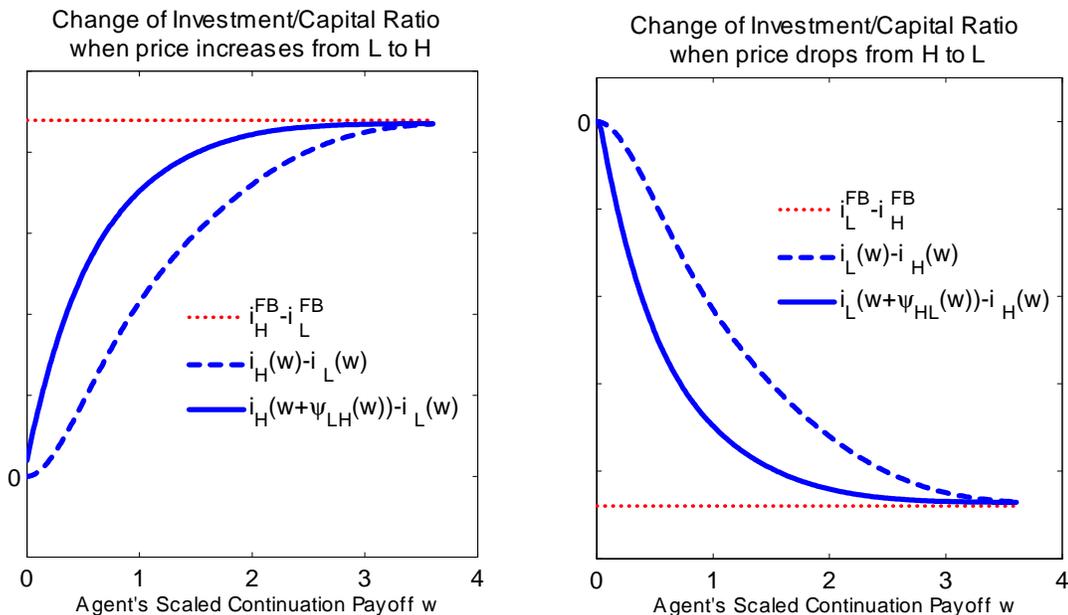


Figure 5: Amplification of investment.

regime switch on the investment-capital ratio, because – given the positive relation between investment  $i_H(w)$  and the agent’s payoff / financial slack  $w$  – the additional upward adjustment of slack / compensation ( $\psi_{LH}(w) > 0$ ) amplifies the increase in investment. Conversely, as seen in Figure 5, a drop in the output price leads to a direct and indirect effect which both reduce the level of investment. In both cases the indirect effect vanishes when the agent’s continuation payoff is high: Intuitively, the impact of financial slack on investment decreases when financial slack is sufficiently high.

In summary, the adjustment of the firm’s financial slack under the optimal contract amplifies the response of investment to output price fluctuations. This additional agency channel may play a role in amplifying and propagating output price shocks and contributing to business cycle fluctuations.

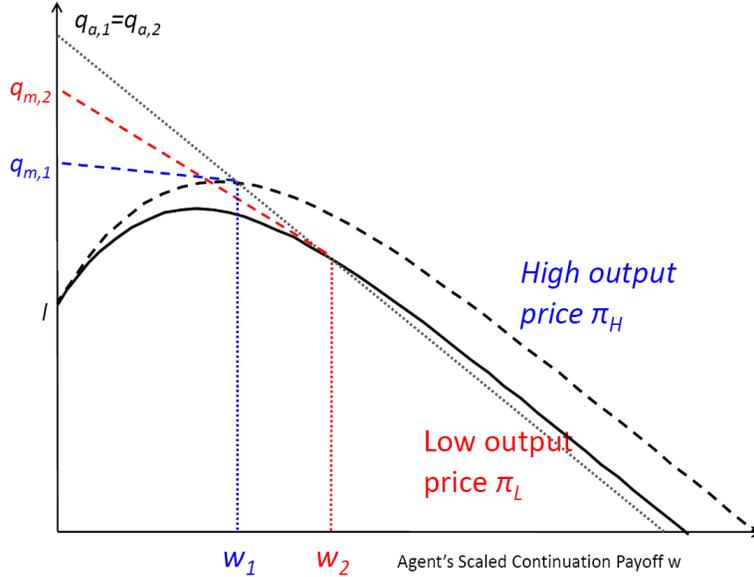


Figure 6: Comparison of marginal  $q$ , holding average  $q$  fixed.

### 6.3 Investment, Financial Slack, and Average $q$

In Section 5, we noted the positive relation between investment and financial slack that results from the dynamics of the agency problem. More interesting empirically, however, is the relation between investment and financial slack that survives after controlling for average (or Tobin's)  $q$ . We could investigate this effect in our baseline model by considering heterogeneity across firms in other model parameters, such as the firm's average profitability. With the stochastic output price model developed here, however, we can make an even stronger point by considering the relation between investment, financial slack, and average  $q$  for a single firm. Specifically, for a given firm, and a given average  $q$ , we will show that investment is increasing with financial slack.

Figure 6 shows the investor value functions  $p_L$  and  $p_H$  for each output price regime. Consider two situations that lead to the same average  $q$ : a high output price with low financial slack  $w_1$ , and a low output price with high financial slack  $w_2$ . These points can be seen to have the same average  $q = p(w) + w$ , as they lie along a line with slope -1. Thus, the figure shows that financial

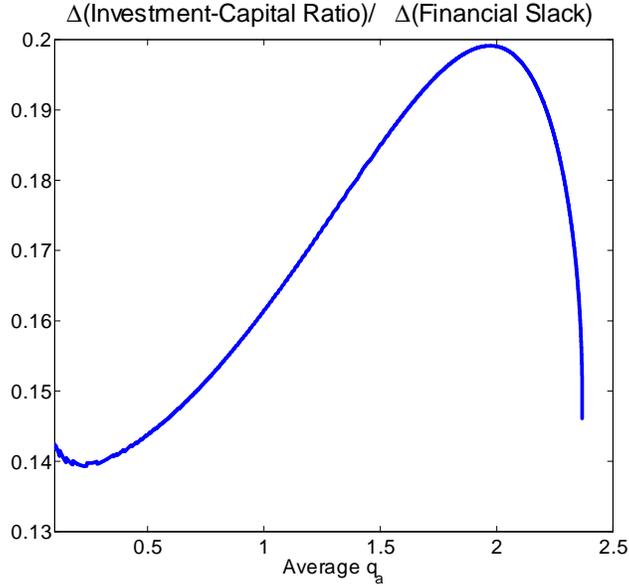


Figure 7: The ratio of the investment difference to the financial slack difference, when holding average  $q$  fixed.

slack and profitability are substitutes in the determination of average  $q$ .

The figure also shows that despite average  $q$  being equal, marginal  $q$  differs for these two points. Marginal  $q = p(w) - wp'(w)$  can be determined as the intercept with the vertical  $p$ -axis of the line tangent to the value function from each point. As shown in the Figure 6, the situation with the higher  $w_2$  (and low output price) has a higher marginal  $q_2$ . Because investment is increasing in marginal  $q$ , even after controlling for average  $q$ , higher financial slack leads to a higher investment rate.

In Figure 7, we illustrate this result for the whole range of average  $q$  in our example. For each level of average  $q$ , we compute the difference in financial slack ( $\Delta w$ ) and investment ( $\Delta i$ ) across the two regimes, and plot the ratio  $\Delta i / \Delta w$ . The figure thus shows that more financial slack will be associated with higher investment, holding average  $q$  fixed.

## 7 Concluding Remarks

This paper integrates dynamic agency into a neoclassical model of investment. Using a continuous-time recursive contracting methodology, we characterize the impact of dynamic agency on firm value and optimal investment dynamics. Agency costs introduce a history-dependent wedge between marginal  $q$  and average  $q$ . Even under the assumptions which imply homogeneity (e.g. constant returns to scale and quadratic adjustment costs as in Hayashi (1982)), investment is no longer linearly related to average  $q$ . Because of the agency problem, investment is increasing in past profitability. The key state variable in the analysis is the agent's continuation payoff. We show how this state variable can be interpreted as a measure of financial slack, allowing us to relate the results to the firm's financial slack.

To understand the potential importance of productivity shocks on firm value and investment dynamics in the presence of agency conflicts, we extend our model to allow the firm's output price to vary stochastically over time. Here we find that investment increases with financial slack after controlling for average  $q$ . In addition, the agent's compensation will depend not only on the firm's realized productivity, but also on realized output prices, even though output prices are beyond the agent's control. This result may help to explain the empirical relevance of absolute performance evaluation. Moreover, this result on compensation also suggests that the agency problem provides a channel through which the response of investment to output price shocks is amplified and propagated. A higher output price encourages investment for two reasons. First, investment becomes more profitable. Second, the optimal compensation contract rewards the agent with a higher continuation payoff, which in turn relaxes the agent's incentive constraints and hence further raises investment. In ongoing research, we continue to explore the sensitivity of managerial compensation to external shocks in a range of dynamic agency contexts.

Our model has a constant returns to scale technology and the wedge between  $q_a$  and  $q_m$  results from the agency problem. Decreasing returns to scale would also lead to  $q_a > q_m$ .

Incorporating an agency problem into a model with decreasing returns to scale should lead to investment dynamics that are similar to those obtained here. A difference, though, is that as the firm grows,  $q_m$  declines and so we would expect investment to be increasing with financial slack but decreasing with firm size.

# Appendices

## A Proof of Proposition 1

We impose the usual regularity condition on the payment policy

$$\mathbb{E} \left( \int_0^\tau e^{-\gamma s} dU_s \right)^2 < \infty. \quad (\text{A.1})$$

And, we require that

$$\mathbb{E} \left[ \int_0^T (e^{-rt} K_t)^2 dt \right] < \infty \text{ for all } T > 0. \quad (\text{A.2})$$

and

$$\lim_{T \rightarrow \infty} \mathbb{E} (e^{-rT} K_T) = 0. \quad (\text{A.3})$$

Both regularity conditions place certain restrictions on the investment policies. Since the project is terminated at  $\tau$ , throughout we take the convention that  $H_T \mathbf{1}_{\{T > \tau\}} = H_\tau$  for any stochastic process  $H$ .

Throughout the proof, we take the adjustment cost  $c(i)$  as the commonly used quadratic form.

**Lemma 1** *For any contract  $\Phi = \{U, \tau\}$ , there exists a progressively measurable process  $\{\beta_t : 0 \leq t < \tau\}$  such that the agent's continuation value  $W_t$  evolves according to*

$$dW_t = \gamma W_t dt - dU_t + \beta_t K_t (dA_t - \mu dt) \quad (\text{A.4})$$

under  $a_t = 1$  all the time. The contract  $\Phi$  is incentive-compatible, if and only if  $\beta_t \geq \lambda$  for  $t \in [0, \tau)$ .

In the optimal contract,  $\beta_t = \lambda$ , so the above dynamics in (A.4) is the one in (13). The proof is similar to DeMarzo and Sannikov (2006); we omit the proof here.

Now we verify that the contract and the associated investment policy derived from the Hamilton-Jacobi-Bellman equation are indeed optimal. The evolution of  $w = \frac{W}{K}$  follows easily from the evolutions of  $W$  and  $K$ . The key ODE (18), under the quadratic adjustment cost and the associated optimal investment policy, is

$$(r + \delta) p(w) = \mu + \frac{(p(w) - wp'(w) - 1)^2}{2\theta} + p'(w) (\gamma + \delta) w + \frac{\lambda^2 \sigma^2}{2} p''(w). \quad (\text{A.5})$$

**Lemma 2** *The scaled investors' value function  $p(w)$  is concave on  $[0, \bar{w})$ .*

**Proof.** By differentiating (A.5) we obtain

$$(r + \delta) p' = -\frac{(p - wp' - 1) wp''}{\theta} + (\gamma + \delta) wp'' + (\gamma + \delta) p' + \frac{\lambda^2 \sigma^2}{2} p'''. \quad (\text{A.6})$$

Evaluating (A.6) at the upper-boundary  $\bar{w}$ , and using  $p'(\bar{w}) = -1$  and  $p''(\bar{w}) = 0$ , we find

$$\frac{\lambda^2 \sigma^2}{2} p'''(\bar{w}) = \gamma - r > 0;$$

therefore  $p''(\bar{w} - \epsilon) < 0$ . Now let  $q(w) = p(w) - wp'(w)$ , and we have

$$(r + \delta) q(w) = \mu + \frac{(q(w) - 1)^2}{2\theta} + (\gamma - r) wp'(w) + \frac{\lambda^2 \sigma^2}{2} p''.$$

Suppose that there exists some  $\tilde{w} < \bar{w}$  such that  $p''(\tilde{w}) = 0$ . Choose the largest  $\tilde{w}$  s.t.  $p''(w + \epsilon) < 0$ . Evaluating the above equation at  $\tilde{w}$ , we have

$$(r + \delta)q(\tilde{w}) = \mu + \frac{(q(\tilde{w}) - 1)^2}{2\theta} + (\gamma - r)\tilde{w}p'(\tilde{w}).$$

Since  $q(\tilde{w}) < q^{FB}$ , and  $(r + \delta)q^{FB} = \mu + \frac{(q^{FB} - 1)^2}{2\theta}$ , we have  $p'(\tilde{w}) < 0$ . Therefore evaluating (A.6) at point  $\tilde{w}$ , we obtain

$$(r + \delta)p'(\tilde{w}) = p'(\tilde{w})(\gamma + \delta) + \frac{\lambda^2\sigma^2}{2}p'''(\tilde{w}),$$

which implies that  $p'''(\tilde{w}) = \frac{2(r-\gamma)}{\lambda^2\sigma^2}p'(\tilde{w}) > 0$ . It is inconsistent with the choice of  $\tilde{w}$  where  $p''(\tilde{w}) = 0$  but  $p''(\tilde{w} + \epsilon) < 0$ . Therefore  $p(\cdot)$  is strictly concave over the whole domain  $[0, \bar{w})$ . Q.E.D. ■

Take any incentive-compatible contract  $\Phi = (I, U, \tau)$ . For any  $t \leq \tau$ , define its auxiliary gain process  $\{G\}$  as

$$\begin{aligned} G_t(\Phi) &= \int_0^t e^{-rs}(dY_s - dU_s) + e^{-rt}P(K_t, W_t) \\ &= \int_0^t e^{-rs} \left( K_s dA_s - I_s ds - \frac{\theta I_s^2}{2K_s} - dU_s \right) + e^{-rt}P(K_t, W_t), \end{aligned} \quad (\text{A.7})$$

where the agent's continuation payoff  $W_t$  evolves according to (13). Under the optimal contract  $\Phi^*$ , the associated optimal continuation payoff  $W_t^*$  has a volatility  $\lambda\sigma K_t$ , and  $\{U^*\}$  reflects  $W_t^*$  at  $\bar{W}_t^* = \bar{w}K_t$ .

Recall that  $w_t = W_t/K_t$  and  $P(K_t, W_t) = K_t p(w_t)$ . Ito's lemma implies that, for  $t < \tau$ ,

$$e^{rt}dG_t = K_t \left\{ \left[ \begin{aligned} -rp(w_t) + \mu - I_t/K_t - \frac{\theta}{2}(I_t/K_t)^2 + (I_t/K_t - \delta)(p(w_t) - w_t p'(w_t)) \\ + \gamma w_t p'(w_t) + \frac{\beta_t^2 \sigma^2}{2} p''(w_t) \end{aligned} \right] dt \right. \\ \left. + [-1 - p'(w_t)] dU_t/K_t + \sigma [1 + \beta_t p'(w_t)] dZ_t \right\}.$$

Under the optimal investment policy  $I_t^*/K_t$  and optimal incentive policy  $\beta_t = \lambda$ , the first piece in the bracket—which is just our (18)—stays at zero always; and other investment policies and incentives provision will make this term nonpositive (here we use the concavity of  $p$ ). The second piece captures the optimality of the cash payment policy. It is nonpositive since  $p'(w_t) \geq -1$ , but equals zero under the optimal contract.

Therefore, for the auxiliary gain process we have

$$dG_t(\Phi) = \mu_G(t) dt + e^{-rt}K_t\sigma [1 + \beta_t p'(w_t)] dZ_t,$$

where  $\mu_G(t) \leq 0$ . Let  $\varphi_t \equiv e^{-rt}K_t\sigma [1 + \beta_t p'(w_t)]$ . The condition (A.1) and the condition A.2 imply that  $\mathbb{E} \left[ \int_0^T \varphi_t dZ_t \right] = 0$  for  $\forall T > 0$  (note that  $p'$  is bounded). And, under  $\Phi$  the investors' expected payoff is

$$\tilde{G}(\Phi) \equiv \mathbb{E} \left[ \int_0^\tau e^{-rs} dY_s - \int_0^\tau e^{-rs} dU_s + e^{-r\tau} l K_\tau \right].$$

Then, given any  $t < \infty$ ,

$$\begin{aligned}
\tilde{G}(\Phi) &= \mathbb{E}[G_\tau(\Phi)] \\
&= \mathbb{E}\left[G_{t \wedge \tau}(\tilde{\Phi}) + \mathbf{1}_{t \leq \tau} \left[ \int_t^\tau e^{-rs} dY_s - e^{-rs} dU_s \right] + e^{-r\tau} lK_\tau - e^{-rt} P(K_t, W_t) \right] \\
&= \mathbb{E}\left[G_{t \wedge \tau}(\tilde{\Phi})\right] + e^{-rt} \mathbb{E}\left\{ \left[ \int_t^\tau e^{-r(s-t)} (dY_s - dU_s) + e^{-r(\tau-t)} lK_\tau - P(K_t, W_t) \right] \mathbf{1}_{t \leq \tau} \right\} \\
&\leq G_0 + (q^{FB} - l) \mathbb{E}[e^{-rt} K_t].
\end{aligned}$$

The first term of third inequality follows from the negative drift of  $dG_t(\Phi)$  and martingale property of  $\int_0^{t \wedge \tau} \varphi_s dZ_s$ . The second term is due to the fact that

$$\mathbb{E}_t \left[ \int_t^\tau e^{-r(s-t)} (dY_s - dU_s) + e^{-r(\tau-t)} lK_\tau \right] \leq q^{FB} K_t - w_t K_t$$

which is the first-best result, and

$$q^{FB} K_t - w_t K_t - P(K_t, W_t) < (q^{FB} - l) K_t$$

as  $w + p(w)$  is increasing ( $p' \geq -1$ ). But due to (A.3), we have  $\tilde{G} \leq G_0$  for all incentive-compatible contract. On the other hand, under the optimal contract  $\Phi^*$  the investors' payoff  $\tilde{G}(\Phi^*)$  achieves  $G_0$  because the above weak inequality holds in equality when  $t \rightarrow \infty$ .

Finally, we require that the agent's shirking benefit  $\phi \equiv \lambda\mu$  be sufficiently small to ensure the optimality of  $a = 1$  all the time. Similar to DeMarzo and Sannikov (2007) and He (2007), there is a sufficient condition for the optimality of  $a = \{\mu\}$  against  $a_t = 0$  for some  $t$ . Let  $\hat{w} = \arg \max_w p(w)$ , and we require that

$$\frac{(p(w) - wp'(w) - 1)^2}{2\theta} \leq (r + \delta)p(w) - p'(w)[(\gamma + \delta)w - \phi] \text{ for all } w$$

Since the left side is increasing in  $w$ , and right side dominates  $p\left(\frac{\phi}{\gamma + \delta}\right) - \frac{\gamma - r}{r + \delta}\left(p(\hat{w}) - p\left(\frac{\phi}{\gamma + \delta}\right)\right)$  (see the proof in DeMarzo and Sannikov (2006)), a sufficient condition is

$$\frac{(p(\bar{w}) + \bar{w} - 1)^2}{2\theta} \leq p\left(\frac{\phi}{\gamma + \delta}\right) - \frac{\gamma - r}{r + \delta}\left(p(\hat{w}) - p\left(\frac{\phi}{\gamma + \delta}\right)\right).$$

## B Proof for Proposition 2

We need to verify that under the proposed scheme, the agent's value function given the relevant state  $(M, K)$  is  $V(M, K) = \lambda M$ , and his optimal policy is the one we obtained in Proposition 1. We take the "guess and verify" approach. Under the proposed implementation, the evolution of the cash reserve  $M$  is

$$\begin{aligned}
dM_t &= rM_t dt + dY_t - dD_t - dX_t \\
&= rM_t dt + K_t dA_t - K_t c(i_t) dt - [K_t \mu dt - K_t c(i_t) dt - (\gamma - r)M_t dt] - dX_t \\
&= \gamma M_t dt + K_t (dA_t - \mu dt) - dX_t,
\end{aligned}$$

and  $dK_t = I_t dt - \delta K_t dt$ . Given the agent's value function  $V(M, K) = \lambda M$ , we have his HJB equation as

$$\gamma \lambda M dt = \sup_{a_t \in [0,1], dX_t \geq 0} \lambda (\gamma M_t dt + K_t (a_t \mu dt - \mu dt) - dX_t) + \lambda K_t (1 - a_t) \mu + \lambda dX_t$$

where the first term is  $\mathbb{E}_t [V_M(M, K) dM]$  under the policy  $a_t$  and  $dX_t$ , the second term is the agent's private benefit by exerting  $a_t$ , and the third term is his portion of special dividend. Then

$$\gamma \lambda M dt = \sup_{a_t \in [0,1], dX_t} \lambda (\gamma M dt + K (a_t \mu dt - \mu dt) - dX) + \lambda K (1 - a_t) \mu + \lambda dX = \gamma \lambda M dt$$

where the action choice  $a_t$  drops out because of the binding incentive-compatibility constraint, and  $\lambda dX$  cancels the agent is indifferent between keeping the cash inside the firm so grows at a rate of  $\gamma$ , or paying it out for consumption. This proves that the agent's value function and his optimal policy under the proposed implementation coincide with those in Proposition 1. In particular, as the agent is indifferent from any investment policy, he will follow the optimal investment policy we derived in Proposition 1.

## C Renegotiation-proof Contract

As we have indicated in Section 4.1, our contract may not be renegotiation-proof. Intuitively, whenever  $p'(w) > 0$  (or,  $P_W(W, K) > 0$ ) both parties may achieve an *ex post* Pareto-improving allocation by renegotiating the contract. Therefore, the value function  $p(w)$  that is renegotiation-proof must be weakly decreasing in the agent's scaled continuation payoff  $w$ . Moreover,  $p^{RP}(w)$  has an (endogenous) renegotiation boundary  $\underline{w}^{RP}$ , where the scaled investors' value function  $p^{RP}(w)$  has the following boundary conditions:

$$p^{RP}(\underline{w}^{RP}) = l, \tag{C.1}$$

$$p^{RP'}(\underline{w}^{RP}) = 0. \tag{C.2}$$

Specifically,  $\underline{w}^{RP}$  (rather than  $w = 0$  in the baseline model) becomes the lower bound for the agent's scaled continuation payoff  $w$  during the equilibrium employment path. The scaled investors' value function  $p^{RP}(w)$  solves the ODE (18) for  $w \in [\underline{w}^{RP}, \bar{w}^{RP}]$ , with two sets of free-boundary conditions: one is the boundary conditions (16) and (20) at the payout boundary  $\bar{w}^{RP}$ , and the other is the boundary conditions (C.1) and (C.2) at the renegotiation boundary  $\underline{w}^{RP}$ .

The dynamics of the scaled agent's payoff  $w$  takes the following form:

$$dw_t = (\gamma + \delta - i(w_t)) w_t dt + \lambda \sigma dZ_t - du_t + (dv_t - \underline{w}^{RP} dM_t), \tag{C.3}$$

where the first (drift) term implies that the expected rate of change for the agent's scaled continuation payoff  $w$  is  $(\gamma + \delta - i(w))$ , the second (diffusion) term captures incentive provisions in the continuation-payoff region (away from the boundaries), and the (third) nondecreasing process  $u$  captures the reflection of the process  $w$  at the upper payment boundary  $\bar{w}^{RP}$ . Unlike the dynamics (17) for the agent's scaled payoff process  $w$  without renegotiation, the last term  $dv_t - \underline{w}^{RP} dM_t$  in dynamics (C.3) captures the effect at the renegotiation boundary. The nondecreasing process  $v$  reflects  $w$  at the renegotiation boundary  $\underline{w}^{RP}$ . The intensity of the counting process  $dQ$  is  $dv_t / \underline{w}^{RP}$ ; and once  $dQ = 1$ ,  $w$  becomes 0, and the

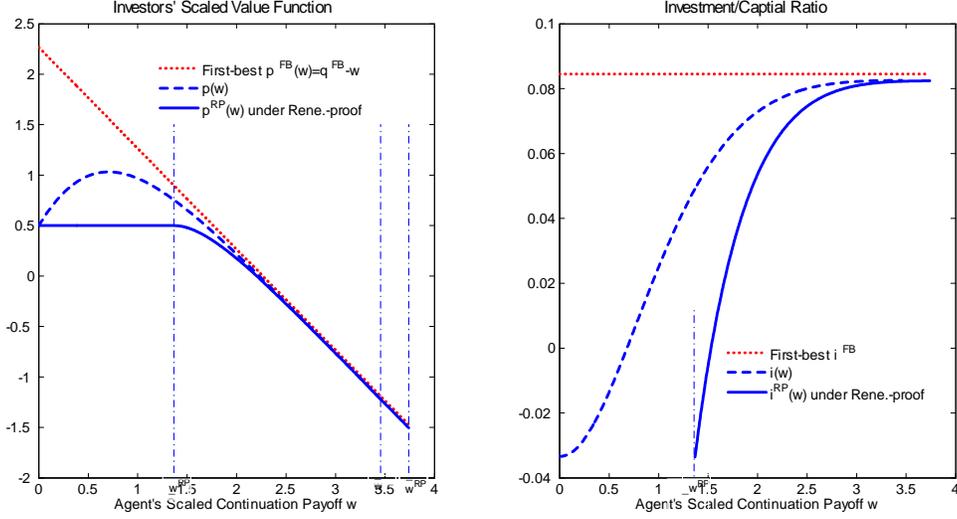


Figure 8: **Renegotiation proofness.** The original scaled value function  $p(w)$  is not renegotiation-proof, because  $p'(0) > 0$ . For the renegotiation-proof contract,  $\underline{w}^{RP}$  is the lower bound for the agent's scaled continuation payoff  $w$ , with the following properties:  $p(\underline{w}^{RP}) = p(0) = l$ , and  $p'(\underline{w}^{RP}) = 0$ . The value function  $p^{RP}(w)$  solves the ODE (18) subject to the boundary conditions (16)-(20) and the above stated conditions at  $\underline{w}^{RP}$ .

firm is liquidated.<sup>19</sup> Note that the additional term  $dv_t - \underline{w}^{RP} dQ_t$  is a compensated Poisson process, and hence a martingale increment.

We illustrate the contracting behavior at the renegotiation boundary through the following intuitive way. When the agent's poor performance drives  $w$  down to  $\underline{w}^{RP}$ , the two parties run a lottery. With a probability of  $dv_t/\underline{w}^{RP}$ , the firm is liquidated. If the firm is not liquidated, the agent stays at the renegotiation boundary  $\underline{w}^{RP}$ . Here, the stochastic termination is to achieve the "promise-keeping" constraint so that  $w$  is indeed the scaled continuation payoff with expected growth rate  $\gamma + \delta - i(w)$  as specified in Proposition 1. To see this, by running this lottery, the agent could potentially lose  $(dv_t/\underline{w}^{RP}) \cdot \underline{w}^{RP} = dv_t$ , which just compensates the reflection gain  $dv_t$  if the firm is not liquidated.

Since renegotiation further worsens the agency conflict, we expect not only a greater value reduction for investors, but also a stronger underinvestment distortion. Figure 8 shows this result.

<sup>19</sup>Technically speaking, the counting process has a survival probability  $\Pr(Q_t = 0) = \exp(-v_t/\underline{w}^{RP})$ .

## D Appendix for Section 6

### D.1 Characterization of Optimal Contracting with Stochastic Prices

Fix regime  $L$  as the current regime (similar results hold for regime  $H$  upon necessary relabelling.) Based on (31) and (34) in Section 6, the following Bellman equation holds for  $P(K, W, \pi^L)$ :

$$\begin{aligned} rP(K, W, \pi^L) = & \sup_{I, \Psi} (\mu\pi^L K - I - G(I, K)) + (I - \delta K) P_K + (\gamma W - \Psi_{LH}(K, W)\xi^L) P_W \\ & + \frac{\lambda^2 \sigma^2 K^2}{2} P_{WW} + \xi^L (P(K, W + \Psi_{LH}(K, W), \pi^H) - P(K, W, \pi^L)), \end{aligned} \quad (\text{D.1})$$

where the investment  $I$  and compensation adjustment  $\Psi_{LH}(K, W)$  (both are unscaled) are state-dependent controls.

The investment policy  $I(K, W, n)$ , by taking a FOC condition, is similar to the baseline case. The first-order condition (FOC) for optimal  $\Psi_{LH}(K, W)$ , given that the solution takes an interior solution, yields that

$$P_W(K, W, \pi^L) = P_W(K, W + \Psi_{LH}(K, W), \pi^H), \quad (\text{D.2})$$

As discussed in the main text, the optimal contract equates the marginal cost of delivering compensation, i.e.,  $-P_W$ , across different Markov states at any time. However, in general, the solution of  $\Psi_{nm}(K, W)$  might be binding (corner solution), as the agent's continuation payoff after the regime change has to be positive. Therefore, along the equilibrium path the optimal  $\Psi_{nm}(K, W)$  might bind, i.e.,  $\Psi_{nm}(K, W) + W \geq 0$  holds with equality.

The scale invariance remains: with  $w = W/K$ , we let  $p_n(w) = P(K, W, \pi^n)/K$ ,  $i_n(w) = I(K, W, \pi^n)/K$ ,  $\psi_{nm}(w) = \Psi_{nm}(K, W, n)/K$ , and upper payment boundary  $\bar{w}_n = \bar{W}(K, \pi^n)/K$ . Similar to equation (19),

$$i_n(w) = \frac{P_K(K, W, n) - 1}{\theta} = \frac{p_n(w) - wp'_n(w) - 1}{\theta}. \quad (\text{D.3})$$

Combining this result with the above analysis regarding  $\psi_{nm}(w)$ 's (notice that  $P_W(K, W, \pi^n) = p'_n(w)$ ), the following proposition characterizes the ODE system  $\{p_n\}$  when the output price is stochastic.

**Proposition 4** *For  $0 \leq w \leq \bar{w}_n$  (the continuation-payoff region for regime  $n$ ), the scaled investor's value function  $p_n(w)$  and the optimal payment threshold  $\bar{w}_n$  solve the following coupled ODEs:*

$$\begin{aligned} (r + \delta) p_L(w) = & \mu\pi^L + \frac{(p_L(w) - wp'_L(w) - 1)^2}{2\theta} + p'_L(w) [(\gamma + \delta)w - \xi^L \psi_{LH}(w)] + \frac{\lambda^2 \sigma^2}{2} p''_L(w) \\ & + \xi^L (p_H(w + \psi_{LH}(w)) - p_L(w)), \quad 0 \leq w \leq \bar{w}_L, \\ (r + \delta) p_H(w) = & \mu\pi^H + \frac{(p_H(w) - wp'_H(w) - 1)^2}{2\theta} + p'_H(w) [(\gamma + \delta)w - \xi^H \psi_{HL}(w)] + \frac{\lambda^2 \sigma^2}{2} p''_H(w) \\ & + \xi^H (p_L(w + \psi_{HL}(w)) - p_H(w)), \quad 0 \leq w \leq \bar{w}_H, \end{aligned} \quad (\text{D.4})$$

subject to the following boundary conditions at the upper boundary  $\bar{w}_n$ :

$$p'_n(\bar{w}_n) = -1, \quad (\text{D.5})$$

$$p''_n(\bar{w}_n) = 0, \quad (\text{D.6})$$

and the left boundary conditions at liquidation:

$$p_n(0) = l_n, \quad n = 1, 2.$$

The scaled endogenous compensation adjustment functions  $\psi_{nm}(w)$  satisfy:

$$p'_n(w) = p'_m(w + \psi_{nm}(w))$$

if  $w + \psi_{nm}(w) > 0$  (interior solution); otherwise  $\psi_{nm}(w) = -w$ . For  $w > \bar{w}_n$  (cash-payment regions),  $p_n(w) = p_n(\bar{w}_n) - (w - \bar{w}_n)$ .

**Proof.** See Section D.2. ■

In solving the coupled ODEs in Proposition 4 with the optimal payment threshold  $\bar{w}_n$  and the endogenous compensation adjustment functions  $\psi_{nm}(w)$ , we take the following numerical iteration steps:

1. Solve for  $p_L^{(0)}$  and  $p_H^{(0)}$  without price changes (i.e., setting  $\xi$ 's to zero.) This is essentially solving a nonlinear ODE with a free-boundary condition about  $\bar{w}_n^{(0)}$ .
2. Given  $p_L^{(0)}$  and  $p_H^{(0)}$ , obtain the compensation adjustment function  $\psi_{HL}^{(0)}(w)$  based on Proposition 4.
3. Given  $\psi_{HL}^{(0)}(w)$  and  $p_L^{(0)}$ , calculate  $p_H^{(1)}$  by solving the high-state one-dimensional nonlinear ODE (i.e., the second equation) in Eq. (D.4). The resulting solution is  $p_H^{(1)}$  with the free boundary  $\bar{w}_H^{(1)}$ .
4. Obtain  $\psi_{LH}^{(0)}(w)$  based on  $p_H^{(1)}$  and  $p_L^{(0)}$ , and then calculate  $p_L^{(1)}$  by solving the low-state one-dimensional nonlinear ODE (i.e., the first equation) in (D.4). The resulting solution is  $p_L^{(1)}$  with the free boundary  $\bar{w}_L^{(1)}$ .
5. Given  $p_L^{(1)}$  and  $p_H^{(1)}$ , update  $\psi_{HL}^{(1)}(w)$ .
6. Then repeat the procedures after step 3, until converge. The convergence is criteria is

$$\max \left[ \sup_w \left( p_L^{(j+1)} - p_L^{(j)} \right), \sup_w \left( p_H^{(j+1)} - p_H^{(j)} \right) \right] < 10^{-5}.$$

## D.2 Proof for Proposition 3

**Lemma 3** Both  $p_n$ 's are strictly concave for  $0 \leq w < \bar{w}_n$ .

**Proof.** Denote two states as  $n, m$ . By differentiating (D.4) we obtain

$$\begin{aligned} (r + \delta) p'_n &= -\frac{(p_n - wp'_n - 1) wp''_n}{\theta} + p''_n \cdot [(\gamma + \delta) w - \xi^n \psi_{nm}(w)] + p'_n (\gamma + \delta - \xi^n \psi'_{nm}(w)) + \frac{\lambda^2 \sigma^2}{2} p'''_n \\ &\quad + \xi^n (p'_m(w + \psi_{nm}(w)) (1 + \psi'_{nm}(w)) - p'_n). \end{aligned}$$

Notice that when  $\psi_{nm}(w)$  takes an interior solution,  $p'_m(w + \psi_{nm}(w)) = p'_n(w)$ ; and otherwise  $\psi'_{nm}(w) = -1$ . Either condition implies that

$$(r + \delta) p'_n = -\frac{(p_n - wp'_n - 1) wp''_n}{\theta} + p''_n \cdot [(\gamma + \delta) w - \xi^n \psi_{nm}(w)] + p'_n (\gamma + \delta) + \frac{\lambda^2 \sigma^2}{2} p'''_n, \quad (\text{D.7})$$

which takes a similar form as in (A.6). Similar to the argument in the proof of Proposition 2 we can show that  $p_n''(\bar{w}_n - \epsilon) < 0$  and  $p_m''(\bar{w}_m - \epsilon) < 0$ .

Denote  $\mu_n = \mu\pi^n$ . Let  $q_n(w) = p_n(w) - wp_n'(w)$ , i.e., the marginal  $q$  that captures the investment benefit. We have

$$\begin{aligned} (r + \delta + \xi^n) q_n(w) &= \mu_n + \frac{(q_n(w) - 1)^2}{2\theta} + \xi^m q_m(w + \psi_{nm}(w)) + (\gamma - r) wp_n'(w) + \frac{\lambda^2 \sigma^2}{2} p_n'' \\ (r + \delta + \xi^m) q_m(w + \psi_{nm}(w)) &= \mu_m + \frac{(q_m(w + \psi_{nm}(w)) - 1)^2}{2\theta} + \xi^n q_n(w) \\ &\quad + (\gamma - r)(w + \psi_{nm}(w)) p_m'(w + \psi_{nm}(w)) + \frac{\lambda^2 \sigma^2}{2} p_m''(w + \psi_{nm}(w)) \end{aligned} \quad (\text{D.8})$$

Recall that the first-best pair  $(q_n^{FB}, q_m^{FB})$  solves the system

$$\begin{cases} (r + \delta + \xi^n) q_n^{FB} = \mu_n + \frac{(q_n^{FB} - 1)^2}{2\theta} + \xi^m q_m^{FB} \\ (r + \delta + \xi^m) q_m^{FB} = \mu_m + \frac{(q_m^{FB} - 1)^2}{2\theta} + \xi^n q_n^{FB} \end{cases} .$$

It is understood throughout the proof that as in the Hayashi (1982) with quadratic adjustment cost,  $(q_n, q_m)$  in Eq. (D.8) and  $(q_n^{FB}, q_m^{FB})$  all take the smaller root in solving the quadratic equations.

Suppose that there exists some points so that  $p_n$  is convex. Pick the largest  $\tilde{w}$  such that  $p_n''(\tilde{w}) = 0$  but  $p_n''(\tilde{w} + \epsilon) < 0$ , and  $p_n''(w) \leq 0$  for  $w \in (\tilde{w}, \bar{w}_n)$ . If  $\psi_{nm}(\tilde{w})$  is interior, then let

$$k = p_n'(\tilde{w}) = p_m'(\tilde{w} + \psi_{nm}(\tilde{w})), p_n''(\tilde{w}) = p_m''(\tilde{w} + \psi_{nm}(\tilde{w}))(1 + \psi_{nm}'(\tilde{w})) = 0,$$

Clearly, if  $p_m''(\tilde{w} + \psi_{nm}(\tilde{w})) = 0$ , then

$$\begin{aligned} (r + \delta + \xi^n) q_n(\tilde{w}) &= \mu_n + \frac{(q_n(\tilde{w}) - 1)^2}{2\theta} + \xi^m q_m(\tilde{w} + \psi_{nm}(\tilde{w})) + (\gamma - r) \tilde{w} k \\ (r + \delta + \xi^m) q_m(\tilde{w} + \psi_{nm}(\tilde{w})) &= \mu_m + \frac{(q_m(\tilde{w} + \psi_{nm}(\tilde{w})) - 1)^2}{2\theta} + \xi^n q_n(\tilde{w}) + (\gamma - r)(\tilde{w} + \psi_{nm}(\tilde{w})) k \end{aligned}$$

Since a positive  $k$  in the above system will imply that  $q_n > q_n^{FB}$  and  $q_m > q_m^{FB}$ , we must have  $k < 0$ . Then evaluating (A.6) at the point  $\tilde{w}$ , we obtain

$$\frac{\lambda^2 \sigma^2}{2} p_n'''(\tilde{w}) = (r - \gamma) p_n'(\tilde{w}) = (r - \gamma) k > 0.$$

This is inconsistent with the choice of  $\tilde{w}$  where  $p_n''(\tilde{w}) = 0$  but  $p_n''(\tilde{w} + \epsilon) < 0$ . Notice that the above argument applies to the case  $p_m''(\tilde{w} + \psi_{nm}(\tilde{w})) > 0$ .

Now we consider the case  $1 + \psi_{nm}'(\tilde{w}) = 0$  but  $p_m''(w + \psi_{nm}(w)) < 0$ . We must have  $p_n'(\tilde{w}) > 0$  and  $p_n'''(\tilde{w}) < 0$  according to the above argument. In the neighborhood of  $\tilde{w}$  find two points  $\tilde{w} - \epsilon < \tilde{w} < \tilde{w} + \eta\epsilon$  where  $\epsilon, \eta$  are positive, such that  $p_n''(\tilde{w} - \epsilon) > p_n''(\tilde{w}) = 0 > p_n''(\tilde{w} + \eta\epsilon)$ , but  $(\tilde{w} - \epsilon) p_n'(\tilde{w} - \epsilon) = (\tilde{w} + \eta\epsilon) p_n'(\tilde{w} + \eta\epsilon) = k > 0$ . Therefore,

$$\begin{aligned} (r + \delta + \xi^n) q_n(\tilde{w} - \epsilon) &= \mu_n + \frac{(q_n(\tilde{w} - \epsilon) - 1)^2}{2\theta} + \xi^m q_m(\tilde{w} - \epsilon + \psi_{nm}(\tilde{w} - \epsilon)) + (\gamma - r) k + \frac{\lambda^2 \sigma^2}{2} p_n''(\tilde{w} - \epsilon) \\ (r + \delta + \xi^n) q_n(\tilde{w} + \eta\epsilon) &= \mu_n + \frac{(q_n(\tilde{w} + \eta\epsilon) - 1)^2}{2\theta} + \xi^m q_m(\tilde{w} + \eta\epsilon + \psi_{nm}(\tilde{w} + \eta\epsilon)) + (\gamma - r) k + \frac{\lambda^2 \sigma^2}{2} p_n''(\tilde{w} + \eta\epsilon) \end{aligned}$$

Because  $1 + \psi'_{nm}(\tilde{w}) = 0$ , the difference in  $q_m$  will be dominated (since it is in a lower order) by the difference in  $p'_n$ 's. Now since  $p''_n(\tilde{w} - \epsilon) > p''_n(\tilde{w} + \eta\epsilon)$ , it implies that  $q_n(\tilde{w} - \epsilon) > q_n(\tilde{w} + \eta\epsilon)$ . But because  $q_n(\tilde{w} - \epsilon) - q_n(\tilde{w} + \eta\epsilon) = p_n(\tilde{w} - \epsilon) - p_n(\tilde{w} + \eta\epsilon) < 0$  according to  $(\tilde{w} - \epsilon)p'_n(\tilde{w} - \epsilon) = (\tilde{w} + \eta\epsilon)p'_n(\tilde{w} + \eta\epsilon)$ , contradiction.

Finally consider the case where  $\psi_{nm}(\tilde{w})$  is binding at  $-w$ . Take the same approach; notice that in this case the points after regime switching are exactly 0. Therefore the same argument applies, and  $p_n(\cdot)$  is strictly concave over the whole domain  $[0, \bar{w}_n)$ . Q.E.D. ■

Once the concavity of  $p_n$ 's is established, the verification argument is similar to the baseline case in Appendix A.

Now we show the properties of compensation adjustment functions. For simplicity take  $\xi^n = \xi^m = \xi$ . We focus on  $\psi_{HL}(w)$ . Once  $\psi_{HL}(w) < 0$  is shown, it immediately follows that  $\psi_{LH}(w) > 0$ .

Because  $p_H(w) > p_L(w)$  while  $p_L(0) = p_H(0) = l$ , we know  $p'_H(0) > p'_L(0)$ , and  $\psi_{HL}(w) = -w$  when  $w$  is small. Then the claim that  $\psi_{HL}(w) = -w$  for  $w$  lower than a threshold  $w^c$  such that  $p'_H(w^c) = p'_L(0)$  follows from the concavity of both  $p$ 's. From then on, we focus on the area where the jump functions take interior solutions.

Consider  $\Delta q(w) = q_H(w) - q_L(w + \psi_{HL}(w))$ , which is the difference between marginal  $q$ 's across the two points before and after jump. It starts at zero (both  $q = l$ ), and must be positive in the payment boundary, as eventually  $q_L = p_L(w) + w < p_H(w) + w = q_H$ . Notice that

$$\Delta q'(w) = -wp''_H(w) + (w + \psi_{HL}(w))(1 + \psi'_{HL}(w))p''_L(w + \psi_{HL}(w)),$$

and the slope becomes zero on the upper boundary  $\bar{w}_H$ . Focus on  $w$ 's that are below the payment region. When  $\psi$  takes interior solutions,  $p'_H(w) = p'_L(w + \psi_{HL}(w))$ , and  $p''_H(w) = p''_L(w + \psi_{HL}(w))(1 + \psi'_{HL}(w))$ . Therefore  $\Delta q'(w) = p''_H(w)\psi_{HL}(w)$ , and  $\psi_{HL}(w) = 0$  if and only if  $\Delta q'(w) = 0$ . Moreover,  $\Delta q(w)$  is decreasing if and only if  $\psi_{HL}(w)$  is positive.

The following lemma shows that  $\psi_{HL}(\bar{w}_H) = \bar{w}_L - \bar{w}_H \leq 0$  on the upper payment boundary; later on we will show  $\psi_{HL}(\bar{w}_H) < 0$  strictly.

**Lemma 4** *The upper payment boundary  $\bar{w}_H \geq \bar{w}_L$  so that  $\psi_{HL}(\bar{w}_H) = \bar{w}_L - \bar{w}_H \leq 0$ .*

**Proof.** We prove by contradiction. Suppose that  $\psi_{HL}(\bar{w}_H) = \bar{w}_L - \bar{w}_H > 0$ . According to (D.8), at the upper boundary, we have  $(r + \delta + \xi)\bar{q}_n = \mu_n + \frac{(\bar{q}_n - 1)^2}{2\theta} + \xi\bar{q}_m - (\gamma - r)\bar{w}_n$ , therefore

$$(r + \delta + 2\xi)(\bar{q}_H - \bar{q}_L) = \mu_H - \mu_L + \frac{(\bar{q}_H - \bar{q}_L)(\bar{q}_H + \bar{q}_L + 2)}{2\theta} + (\gamma - r)(\bar{w}_L - \bar{w}_H). \quad (\text{D.9})$$

Moreover,  $\psi_{HL}(\bar{w}_H) > 0$  implies that  $\Delta q(w)$  is decreasing for  $w < \bar{w}_H$  around its vicinity. Take  $\hat{w}$ , which is the largest  $w$  such that  $\psi_{HL}(\hat{w}) = 0$  and  $\Delta q'(\hat{w}) = 0$ . (The existence of such  $\hat{w}$  follows from the fact  $\psi_{HL}(0+) < 0$ .) This implies that  $\Delta q(\hat{w})$  reaches its local maximum and  $\Delta q(\hat{w}) > \bar{q}_H - \bar{q}_L$ ; and  $\psi'_{HL}(\hat{w}) \geq 0$ . However, denoting  $q_L(\hat{w}) = \hat{q}_L$  and  $q_H(\hat{w}) = \hat{q}_H$ , (D.8) implies that  $(r + \delta + \xi)\hat{q}_n = \mu_n + \frac{(\hat{q}_n - 1)^2}{2\theta} + \xi\hat{q}_m + (\gamma - r)\hat{w}p'_n(\hat{w}) + \frac{\lambda^2\sigma^2}{2}p''_n(\hat{w})$ , which says that

$$(r + \delta + 2\xi)\Delta q(\hat{w}) = \mu_H - \mu_L + \frac{\Delta q(\hat{w})(\hat{q}_L + \hat{q}_H + 2)}{2\theta} + \frac{\lambda^2\sigma^2}{2}[p''_H(\hat{w}) - p''_L(\hat{w})]. \quad (\text{D.10})$$

Comparing to (D.9), since  $\hat{q}_L + \hat{q}_H < \bar{q}_L + \bar{q}_H$  as  $q$  is increasing with  $w$ , in order for  $\Delta q(\hat{w}) > \bar{q}_H - \bar{q}_L$  we must have the last term in (D.10) to be strictly positive, or  $p''_H(\hat{w}) > p''_L(\hat{w})$ . But recall that  $p''_H(\hat{w}) = p''_L(\hat{w})(1 + \psi'_{HL}(\hat{w}))$ ; concavity of  $p_n$ 's and  $\psi'_{HL}(\hat{w}) \geq 0$  yields a contradiction. ■

Now we proceed to show that  $\psi_{HL}(w) \leq 0$  always. Since  $\psi_{HL}(\bar{w}_H) \leq 0$  and  $\psi_{HL}(0+) < 0$ , for any  $w$  such that  $\psi_{HL}(w) > 0$ , we always can find two points  $\hat{w} < \tilde{w}$  closest to  $w$  such that  $\psi_{HL}(\hat{w}) = \psi_{HL}(\tilde{w}) = 0$ ,  $\Delta q'(\hat{w}) = \Delta q'(\tilde{w}) = 0$ ,  $\Delta q(\hat{w}) > \Delta q(\tilde{w})$ , and  $\psi'_{HL}(\hat{w}) > 0$ ,  $\psi'_{HL}(\tilde{w}) < 0$ . We have

$$\begin{aligned} (r + \delta + 2\xi) \Delta q(\hat{w}) &= \mu_H - \mu_L + \frac{\Delta q(\hat{w})(\hat{q}_L + \hat{q}_H + 2)}{2\theta} + \frac{\lambda^2 \sigma^2}{2} p_L''(\hat{w}) \psi'_{HL}(\hat{w}) \\ &< \mu_H - \mu_L + \frac{\Delta q(\hat{w})(\hat{q}_L + \hat{q}_H + 2)}{2\theta} \end{aligned}$$

and  $(r + \delta + 2\xi) \Delta q(\tilde{w}) = \mu_H - \mu_L + \frac{\Delta q(\tilde{w})(\tilde{q}_L + \tilde{q}_H + 2)}{2\theta} + \frac{\lambda^2 \sigma^2}{2} p_L''(\tilde{w}) \psi'_{HL}(\tilde{w}) > \mu_H - \mu_L + \frac{\Delta q(\tilde{w})(\tilde{q}_L + \tilde{q}_H + 2)}{2\theta}$ . Finally, because  $\hat{q}_L + \hat{q}_H < \tilde{q}_L + \tilde{q}_H$ ,  $\Delta q(\hat{w}) < \Delta q(\tilde{w})$ , contradiction.

A similar argument rules out the case that there  $\exists \hat{w}$ , s.t.  $\psi_{HL}(\hat{w}) = \psi'_{HL}(\hat{w}) = 0$ , while  $\psi_{HL}(w) \leq 0$  for all  $w$ . Suppose not, then  $\Delta q'(\hat{w}) = 0$ , and

$$(r + \delta + 2\xi) \Delta q(\hat{w}) = \mu_H - \mu_L + \frac{\Delta q(\hat{w})(\hat{q}_L + \hat{q}_H + 2)}{2\theta}.$$

For  $w = \hat{w} + \epsilon$ , after neglecting some terms with an order higher than  $\epsilon^2$  (note that  $\psi_{HL}(\hat{w} + \epsilon)$  is in the order of  $\epsilon^2$ )

$$\begin{aligned} (r + \delta + 2\xi) \Delta q(\hat{w} + \epsilon) &= \mu_H - \mu_L + \frac{\Delta q(\hat{w} + \epsilon)(q_L(\hat{w} + \epsilon) + \hat{q}_H(\hat{w} + \epsilon) + 2)}{2\theta} \\ &\quad + (\gamma - r) \psi_{HL}(\hat{w} + \epsilon) p_H'(\hat{w} + \epsilon) + \frac{\lambda^2 \sigma^2}{2} p_L''(\hat{w} + \epsilon) \psi'_{HL}(\hat{w} + \epsilon) \end{aligned}$$

The first term in the second line is in a lower order than  $\epsilon$ , as  $\psi_{HL}(\hat{w}) = \psi'_{HL}(\hat{w}) = 0$ . The second term in fact is positive. Because  $q_L(\hat{w} + \epsilon) + \hat{q}_H(\hat{w} + \epsilon) - (\hat{q}_L + \hat{q}_H)$  is in the order of  $\epsilon$ , this contradicts with  $\Delta q'(\hat{w}) = 0$ .

Finally we rule out the case of  $\bar{w}_H = \bar{w}_L$  so  $\psi_{HL}(\bar{w}) = 0$ . At  $\bar{w}$ , we have  $\psi'_{HL}(\bar{w}) = 0$ . It is because  $p_H'''(w) = p_L'''(w)(1 + \psi'_{HL}(\bar{w}))$ , and  $p_L'''(w) = p_H'''(w) = \frac{2(\gamma-r)}{\lambda^2 \sigma^2}$ . Now consider the point  $\bar{w} - \epsilon$ :

$$\begin{aligned} (r + \delta + 2\xi) \Delta \bar{q}(\bar{w} - \epsilon) &= \mu_H - \mu_L + \frac{\Delta \bar{q}(\bar{w} - \epsilon)(\bar{q}_L(\bar{w} - \epsilon) + \bar{q}_H(\bar{w} - \epsilon) + 2)}{2\theta} + \\ &\quad (\gamma - r) \psi_{HL}(\bar{w} - \epsilon) p_H'(\bar{w} - \epsilon) + \frac{\lambda^2 \sigma^2}{2} p_L''(\bar{w} - \epsilon) \psi'_{HL}(\bar{w} - \epsilon). \end{aligned}$$

One can show that, compared to the value at  $\bar{w}$ ,  $\bar{q}_L(\bar{w} - \epsilon) + \bar{q}_H(\bar{w} - \epsilon)$  is  $\epsilon^2$  order smaller. Again, the first term in the second line is in lower order, and the second term is negative. Therefore,  $\Delta \bar{q}(\bar{w} - \epsilon)$  should be in  $\epsilon^2$  order smaller. However,

$$\Delta \bar{q}''(\bar{w}) = p_H'''(\bar{w}) \psi_{HL}(\bar{w}) + p_H''(\bar{w}) \psi'_{HL}(\bar{w}) = 0,$$

contradiction. Q.E.D.

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