DYNAMIC COSTS AND MORAL HAZARD

GUY ARIE - PRELIMINARY VERSION

Abstract. The cost of effort often increases in past effort. In sales, for example, the first few sales of a quarter are easier to make than the last ones – the pool of easy customers is depleted. In an agency setting with unobservable effort, dynamic increase in costs complicates the optimal contract problem. If the agent shirks today, his cost tomorrow will be lower. The main result to date is that the one-shot-deviation principle does not extend to this setting. I show that the *optimal* contract does satisfy a one-shot-deviation condition and characterize the resulting contract as a *dynamic quota* contract. The results are obtained by the construction of a *dynamic dual* representation of dynamic moral hazard problems. The dynamic dual problem has monotonicity and comparative static properties that the standard problem does not, which allows novel results. The findings are consistent with the use of quotas and convex incentive schemes for sales agents.

1. Introduction

Increasing marginal costs is a standard component of economic analysis. In organizational settings, the increase in cost often has a dynamic motivation – the worker gets tired or, as in the example below, the cost of improving outcomes increases. This paper considers optimal long term contracts when effort is unobserved and costs increase with past effort. The main result in the current literature for related settings is negative – Fernandes and Phelan (2000) show that the one shot deviation principle cannot be used. Using a different dynamic representation of the problem, I show that the first deviation is the most profitable in any *optimal* contract. This simplifies the problem and allows a characterization of the optimal contract.

To fix ideas, consider incentives for a sales person. Sales performance is measured over a period, typically quarter or year.[1] As the quarter progresses, the agent depletes all the "easy" sales leads and must exert more effort to generate later sales. Sales effort is inherently hard to monitor and pay is often performance based.[2] If the firm would know the agent's true cost, it may want to increase incentives towards the end of the quarter. However, agents may "game the system" – delay some easy sales leads and use them

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[1]The sales-related statements made here are motivated by the data and analysis in Oyer (1998).

[2]Joseph and Kalwani (1998) provide survey based evidence to the widespread of convex and quota based incentives.

only at the end of the quarter. Indeed, several authors (e.g. Oyer (1998); Larkin (2007); Misra and Nair (2009)) suggest that most incentive schemes reward late effort too much and provide compelling evidence that agents indeed "game the system" in their timing of sales.[3] The authors observe that the main features of the reward scheme are inefficient (i.e. too early) stochastic termination and excessive rewards. If the agent is unlucky in the first part of the quarter, he is very unlikely to make enough sales to obtain a high commission. In response, he stops exerting effort for the remainder of the quarter. In contrast, an agent that obtains a 20% commission on several million dollars deals in a quarter is probably being compensated for more than his efforts on the deal.

I study a simple model to capture the effect of increasing costs. Every day a risk neutral agent either exerts costly effort or not. The probability of success (a sale) increases with effort. The cost of effort today is a convex function of past effort. Effort is unobserved and the principal can commit to a contract at the outset.

To see why increasing costs provide stronger incentives to shirk, suppose that the probability of a sale each day is $\frac{1}{2}$ if the agent exerts effort on a customer and zero otherwise, and that the agent's cost for making the n-th effort is n. If both the principal and the agent consider only current day incentives, a contract paying the agent $2n$ for a sale in day n is incentive compatible and provides the agent zero expected utility – clearly first best. However, if the agent considers future payoffs, this contract is no longer incentive compatible. Shirking in the first period and then working whenever asked obtains the agent an expected utility of 1 *each* period. By shirking today the agent increases his rents from any future contract. The optimal contract must account for this additional incentive to shirk: the agent's utility difference between success and failure must increase.

The optimal contract is characterized by the two features cited from the sales literature – very high volatility of the work length and seemingly excessive rewards for succesfull agents. The optimal contract can be informally described as a *dyanmic quota:* the agent starts in an evaluation stage and eventually moves to a compensation stage. In the compensation stage the agent is rewarded a fixed piece-rate for each sale and works for an additional fixed time that is independent of any new outcomes. In the evaluation stage the agent is rewarded *only* by changes to the expected fixed piece-rate, the length of the compensation stage and the conditions for entering the compensation stage. If the agent had enough early successes, his compensation per sale later in the quarter will be high. If the agent did not accumulate enough early successes, the contract encourages the agent to stop working. Formally, during the evaluation stage, the contract is a list of contingent compensations. For example: "succeed in the next period and move to a four period compensation stage", "fail next period and then succeed – move to a two period compensation stage", "fail in the next two periods and you're fired".

The analysis departs from the existing literature by formulating the original problem as a linear program, deriving its dual and then obtaining a recursive dynamic representation of the dual. The dual analysis transforms the multiple deviation incentive constraints to a single constraint for different agent "types". This allows using standard mechanism

[3] In a recent study of a software maker, Larkin (2007) documents that a sales-person obtains a 2% commission on the first sale of the quarter but a 20% commission for any revenue above $6M.

design techniques to prove that the worst type of agent for any optimal contract in any history is the agent that never deviated in the past. Thus, local deviations are sufficient.

The dynamic dual formulation of the moral hazard problem is the main methodological contribution of the paper. The value of the dynamic dual problem at each history is the increase in expected profit at *time zero* from comitting to the history's contract terms. In contrast, the primal dynamic value in a history is the expected continuation value (see e.g. Spear and Srivastava (1987)). The difference between the two is most striking in periods in which the agent is "given the firm". The principal's continuation profit – the primal value – may be *negative* in such periods. However, the utility obtained by the agent in such periods was used to motivate past work. The dual analysis adds this utility to the history's value. As a result, the dual value of a work period is never negative.

The control variable for the period is the shadow price for the IC. This is the "credit in the firm" that is at risk in the period. If the agent succeeds, he may use the credit to provide liability – buy the firm. If the agent fails, he will have to use future credit to offset his losses.

The dual state variables measure the *degree of agency frictions*. The model has two agency frictions - limited liability and information asymmetry. Each defines a variable that is updated as a function of the credit at risk in the period and the period's outcome. For example, the limited liability control decreases whenever the agent succeeds and increases whenever the agent fails.

There are several advantages to thinking about the dynamic relationship through the progression of agency frictions. The main advantage is that the principal's value is now monotonic in each state variable – a lower limit on liability is better, less information asymmetry is better. This is not typically the case in the standard recursive formulation.[4] The agent's expected continuation utility is obtained as the derivative of the dual value function with respect to the liability friction, and is also monotonic in the state. Thus, the dual analysis recovers the monotonic relation between the value of the period and the agent's utility.

The dynamic dual analysis provides natural interpretation of the optimal contract. As in the standard dynamic models with a limited liability agent, the optimal contract uses future credit to motivate effort. If the agent obtains enough credit to buy the firm, it is no longer possible to use any future credit. Instead, the contract "gives the firm to the agent" in exchange for his credit. However, the outcome of each period affects both the agent's liability limit and the information asymmetry. If the agent shirked in the past, costs are lower and the firm is worth more to the agent than the principal believes. If the principal ignores this when giving the firm to the agent, the agent has an incentive to shirk early on and increase his future gains. The contract controls for this incentive by comitting to reduce the value of the firm at the end of each period by an amount exactly sufficient to counter this extra shirking incentive. When the firm is "given" to the agent, it is given under terms that lower surplus as required.

[4]Following Spear and Srivastava (1987), dynamic moral hazard analysis uses the agent's continuation utility as the state. If the agent's continuation utility is exactly his outside option, the contract must typically terminate and the principal obtains his outside option. If the agent's continuation utlity is very high, the principal must either give away the firm (in limited liability settings, see e.g. Clementi and Hopenhayn (2006)), or provide the agent costly insurance. In all cases, the principal's expected continuation value is highest for some expected agent's continuation utility between the two extremes.

The dynamic dual representation allowed extending moral hazard theory to a new setting – increasing marginal cost. However, dynamic dual analysis may also be useful for the more standard dynamic moral hazard models as well. While the development (but not the economic intuitions or analysis) relies heavily on the duality property of linear programming, the basic duality arguments generalize to concave programs as well (see Rockafellar (1997)). Moreover, as the current model, some existing dynamic moral hazard models may be transformed into a linear program.

Following a short literature review, section 2 lays out the dynamic production model, formulates it as a linear problem and identifies a condition for the sufficiency of local deviations. Section 3 develops and analyzes the dual problem and its dynamic formulation considering only local deviations. Section 4 extends the dynamic dual to multiple deviations and proves the sufficiency of local deviations. Section 5 concludes.

1.1. **Relation To Existing Literature.**

Of the models that consider history *independent* technology, the agency setting and result characterization here is closest to Clementi and Hopenhayn (2006) and DeMarzo and Fishman (2007). Both consider a risk neutral agent subject to limited liability. The two papers emphasize that long term contracts increase surplus by allowing the principal to reward outcomes today with future rents instead of current utility. This allows the agent to provide "de-facto" liability - his unpaid effort - in return for future equity in the firm. Once the agent fully paid for the firm through effort, it is effectively sold to the agent for free - the agent is promised the full fruits of all future labor.[5] In these models, the contract is always in one of three states, either (i) the agent is given the firm or (ii) the contract is terminated or (iii) the contract is in the process of resulting in one of the two other states. Positive period outcomes move the contract towards the first state (giving the firm) and negative period results move the contract towards the second (termination).

The current analysis shows the effects of private-history dependence of technology on this intuition. The optimal contract destroys some surplus after each period to prevent shirking, with the destruction much larger after a failure. While I have not attempted to provide additional insights from using duality to the history independent production setting, one would expect it is possible to do so.[6]

This paper contributes to recent progress in dynamic agency theory with private history dependent technology. Fernandes and Phelan (2000) consider a agency settings in which today's information or effort affects tomorrow's productivity but limit the history dependence to one last period via a Markov assumption. Nevertheless, the result in Fernandes and Phelan (2000) for moral hazard settings is negative - whenever today's effort affects tomorrows productivity, the one-shot-deviation principle does not apply.

Several recent papers follow the modeling approach of Fernandes and Phelan (2000) to analyze dynamic moral hazard settings with payoff relevant private histories. In a recent working paper, DeMarzo and Sannikov (2008) extend the aggregation problem of

[5]Earlier models with similar intuitions for a risk averse agent with unlimited liability are Rogerson (1985) and Spear and Srivastava (1987). There, however, the risk aversion of the agent introduces other considerations.

[6]Adding a discount factor β to the model and setting $c(n) = c$ generates a binary action choice history independent model.

Holmstrom and Milgrom (1987) to settings in which where the agent also obtains private shocks to his productivity that are correlated with past effort. While DeMarzo and Sannikov (2008) is closest to the setting studied here, the additional aggregation problem restricts the results. Appendix C outlines an implementation of the duality method with the aggregation problem in discrete time. Another working paper, Tchistyi (2006), maintains the Markov structure in Fernandes and Phelan (2000) and devises a transformation of the agent continuation payoffs under deviations to obtain sufficiency of local deviations in the presence of unobservable utility shocks to the agent. Williams (2011) focuses on dynamic adverse selection – the agent only reports his income. Bergemann and Hege (2005) and Bonatti and Hörner (2011) consider the case that surplus is private history dependent but the principal only cares about the first success. This simplifies the agency problem, and restores the one-shot-deviation principle as there is only a single instance in which rewards need to be provided. As a result, more involved questions – alternative contracting frameworks and collaboration between multiple agents can be studied.

The paper's contribution to this literature is twofold. First, it provides new and positive results for an important setting. Second, it introduces an approach that separates the different agency frictions and may be useful to other applications.

Related dual approaches to dynamic problems have been suggested before in other contexts. Benveniste and Scheinkman (1982) consider a macro-economic equilibrium model in continuous time and the classic text by Rockafellar (1997) considers several examples that suggest the methods introduced here should apply to the standard concave (non-linear) moral hazard models. Both of these texts predate the dynamic moral hazard analysis. More recently, Vohra (2011) extends the analysis of static adverse selection models by analyzing the dual of the classic adverse selection problem. Among the many new results provided by Vohra (2011), the closest to ours regard the sufficiency of local deviations with respect to the agent's type. Vohra's analysis considers a much richer type and allocation space than ours, but does not consider moral hazard or long term contracts.

2. Model

2.1. **The Setting.**

There is a principal and an agent, both risk neutral. Both have an outside option set to zero. The agent has limited liability – i.e. money can only be transferred to the agent. Time is discrete. In each period the agent either works or not. The agent's work is costly to the agent and unobservable to the principal. The cost of effort in a period is c_n for a commonly known function $c : N \to R_+$ where n denotes the number of periods in which the agent works in the past. The analysis will focus on the case that c_n is an increasing and convex function. However, most parts of the analysis, and specifically the derivation and analysis of the dynamic problem, apply to any general c_n. In particular, the methodology developed can be used to characterize the optimal contract if c_n is fixed or decreasing.

Assumption 1. *c_n is increasing and convex*

A period's production outcome is either success or failure, denoted by $y \in Y = \{s, f\}$. The principal earns a revenue of v from each success and zero from a failure. The probability of the outcome s (resp. f) in a period in which the agent works is $p \in (0, 1)$ (resp.

$1-p$). If the agent does not work, p is replaced with $p_0 \in [0,p)$. To prevent the principal from making free profits, assume the principal incurs a cost of $v \cdot p_0$ for every period in which the contract is still active.[7]

As costs are increasing, the surplus from working becomes negative after enough effort was exerted. Let N^{FB} denote the maximum number of periods in which consecutive work increases surplus:

$$N^{FB} = \max n : c_n \leq v\,(p - p_0)\ .$$

The increase in costs is sufficient to prevent an infinite contract from being optimal. Therefore, the analysis is simplified without loss by assuming that the agent and principal do not discount the future. Section 3.4 adds a common discount factor and shows the effect amounts to a simple accounting exercise.

By the revelation principle, consider contracts that specify for each period a work decision and a wage based on the period's history. Before defining the contract, the following result simplifies the exposition and notation. Given the risk-neutrality and no-discounting assumptions, the proof is straightforward and omitted.

Lemma 1. (1) *The optimal contract provides incentives for the agent to work for at most N^{FB} periods*

(2) *In the optimal contract, the required work decision is a stopping decision: if the agent is ever asked not to work, the contract terminates.*

(3) *The optimal contract never pays the agent in a period without work or with outcome f.*

The dynamic dual formulation that will be developed is simplified by the finiteness of the contract but can be easily extended to accommodate infinite horizon models with discounting.

Given lemma 1, the space of (payoff or contract relevant) pubic histories H is the space of past outcomes:

$$H = \bigcup_{n=0}^{N^{FB}} Y^n \quad .$$

A public history $h \in H$ denotes a sequence of outcomes. Let n_h denote the length of h (i.e. the number of past periods.) The agent's private information is, for every past period, whether he did actually work. The agent *deviated* in a period if he did not work.

As the cost to the agent of working in a period is a function of the *number* of periods in which the agent actually worked in the past, the only information in the agent's private history that is payoff relevant is the number of past deviations:

Definition 1. The agent's *private* history (h,d) is the public history h and the number of past deviations d.

Cost depends on the *private* history. With a slight abuse of notation let $c_{h-d} \equiv c_{n_h-d}$ denote the cost for any history h with past deviations d and $c_h \equiv c_{h-0}$. As the difference in cost between two work periods will play an important role, let $\delta_{h-d} \equiv c_{n_h-d} - c_{n_h-d-1}$ denote the cost difference if the agent shirked d times in the past and $\delta_h \equiv \delta_{h-0}$. To simplify the notation later on, set $\delta_0 = c_0$.

[7]This assumption only simplifies the exposition and is without loss of generality.

The analysis makes extensive use of histories *following* and *preceding* other histories. Let $h = \left\langle h^1, h^2 \right\rangle$ denote the history h^1 followed by the history h^2. That is, the sequence of outcomes h^1 happened and then the sequence h^2 happened. For example, if the current history is h, then the next history will be either $\langle h, s\rangle$ or $\langle h, f\rangle$. Say that the history $\left\langle h^1, h^2 \right\rangle$ *follows* history h^1 and denote the "follows" relation by $\succeq$. That is

$$\tilde{h} \succeq h \quad \Longleftrightarrow \quad \exists \hat{h} \in H : \tilde{h} = \left\langle h, \hat{h} \right\rangle .$$

The set H also includes the empty set and thus $h \succeq h$.

2.2. **The Contract.**

By the revelation principle, consider contracts that specify in each period whether the agent works and the resulting wage. Lemma 1 allows simplifying further and considering the work decision as a stopping rule. Let $(1 - \alpha_h)$ denote the probability that the contract is terminated in history h, if h is reached. Rather than specifying the contract in terms of the conditional stopping rule α, the contract specifies the *ex-ante* probability that a complying agent is still asked to work in history h, denoted q_h:

$$\begin{aligned} q_{\langle h,s\rangle} &= q_h \cdot p \cdot \alpha_{\langle h,s\rangle} \quad ; \\ q_{\langle h,f\rangle} &= q_h \cdot (1-p) \cdot \alpha_{\langle h,f\rangle} . \end{aligned} \tag{2.1}$$

To define the payment, let ω_h denote the payment to the agent if he is asked to work in history h and succeeds. By lemma 1 the payment to the agent in history h is zero if he is either not asked to work or fails. Thus there is no loss of generality in having the contract specify the ex-ante expected wage for success in the history:

$$w_h = q_h \cdot \omega_h \ . \tag{2.2}$$

Definition 2. A contract is a pair of functions $\langle q, w\rangle$, with $q : H \to [0,1]$ specifying for each history h the ex-ante probability that the agent will be asked to work in the period and $w : H \to R_+$ specifying the ex-ante expected payment to the agent for a success in history h.

A special type of contracts will play an important role in the solution:

Definition 3. A contract is *fixed to N from history h* if once the contract reaches history h the agent works exactly until period $N \leq N^{FB}$ regardless of new outcomes and is paid $\frac{c_N}{p-p_0}$ for success in all remaining periods.

The wage $\frac{c_N}{p-p_0}$ is optimal in a single period game with cost c_N. Thus if the agent faces a fixed contract to N from h, he gets paid in all remaining periods the static optimal for period N. Lemma 12 shows that the optimal contract becomes fixed after the first payment is made.

2.3. **The Optimal Contract Problem.**

If the agent complies with the contract (i.e. works when asked to), the ex-ante expected revenue for the principal from a history h is the probability that the history is reached and the contract did not terminate, q_h, multiplied by the expected revenue from work $v \cdot p$ less the principal's direct costs $v \cdot p_0$. The ex-ante expected payment in a period h for a compliant agent is simply $p \cdot w_h$. Thus, the principal's expected profit from a contract the agent complies with is

$$V(q,w) = \sum_{h\in H} \left[q_h (p-p_0) v - w_h p\right] . \tag{2.3}$$

The agent's strategy specifies, for each private history, the probability that the agent works when asked to. Given lemma 1, there is no loss in considering an agent's *effort plan*, $e \in E$, with a typical element $e_h \in \{0,1\}$ that specifies the agent's pure action at each *public* history h if the agent is asked to work in the history. As the agent is risk neutral, given lemma 1 there is no loss in assuming the agent does not mix. As the agent can fully reconstruct the private history h,d at any period using the past effort plan and public history h, letting e be only a function of the public history is without loss of generality.

Let $U(h,d;q,w,e)$ denote the agent's expected continuation profit from effort plan e, starting from private history h,d , having been asked to work in the history and facing the contract q,w.[8] Let $U^c(h,d;q,w)$ be the agent's expected continuation utility from complying with the contract in all remaining periods, starting at private history h,d. Note that it is not required that history h,d could be reached if the agent follows the plan e from the start of the relationship. Specifically $U^c(\cdot)$ is well defined even if the agent deviated in the past $(d>0)$.

Let $\emptyset$ denote the starting (null) history. The optimal contract problem is:

(2.4)

$$\begin{array}{llll}
V^* = \max\limits_{q\geq 0, w\geq 0} & \sum_{h\in H} \left[q_h (p-p_0) v - w_h p\right] & \\
 & q_\emptyset \leq 1 & \text{(Probability at } h=\emptyset) \\
\forall h & q_{\langle h,s\rangle} \leq q_h p & \text{(Probability after success)} \\
\forall h & q_{\langle h,f\rangle} \leq q_h (1-p) & \text{(Probability after failure)} \\
\forall h,e & U^c(h,0;q,w) \geq U(h,0;q,w,e) & \text{(IC)} \\
\forall h & U^c(h,0;q,w) \geq 0 & \text{(IR)}
\end{array}$$

The first three constraints follow directly from the definition of q in equation 2.1. The analysis will refer to these as the "Probability" constraints as they reflect the upper bound on the probability of work. As any plan e may detail several deviations, there is no loss in restricting the IC to histories in which the agent never shirked.[9] The IR constraint is presented for completeness. However, as the agent always has a possible effort plan – never work – that provides non-negative expected payoff, it is redundant given IC and thus will be subsequently ignored.

It can be shown that by specifying the contract in ex-ante terms using q_h and w_h rather than in conditional terms (the α_h and ω_h used in deriving q_h and w_h) allows writing the IC constraint for every possible alternative effort plan e as a linear inequality. Thus, problem 2.4 is a *linear program.* As in any linear program, the difficulty lies in identifying the binding set of constraints. Specifically, the set of IC is too large and the

[8]If the agent is not asked to work in the history, his continuation profit is exactly zero.

[9]As the agent is risk neutral, the IC and IR constraint may be written for only $h=\emptyset$. However, this does not simplify the problem and writing the IC for all histories simplifies the analogy to the alternative problems developed below.

main problem is to identify the relevant subset to consider. This will be the set of Local Deviation Incentive Constraints (LDIC).

To define the local deviations constraints, let $U^D(h,d;q,w)$ be the agent's expected continuation profit starting at a period with private history h,d if the agent *deviates* in the current period and complies with the contract in all following periods.[10]

Definition 4. The set of Local Deviation Incentive Constraints (LDIC) is

$$\forall h: \quad U^c(h,0;q,w) \quad \geq \quad U^D(h,0;q,w) \quad \text{(LDIC)} \tag{2.5}$$

The set LDIC is a clear relaxation of the IC as it limits the agent to a single deviation. The problem would be simpler if the set LDIC was sufficient to imply IC. However, not all contracts that satisfy LDIC are IC.[11] Instead, I will show that the set of *optimal* solutions to the problem considering only LDIC does satisfy all IC and thus there is no loss in considering the relaxed problem. To show this, I first define a larger set of constraints, the Final Deviations Incentive Constraints (FDIC). These constraints require that there is no private history h,d such that the agent prefers deviating in history h,d and complying with the contract in all remaining periods to complying in history h,d and all remaining periods.

Definition 5. The set of Final Deviations Incentive Constraints IC (FDIC) is

$$\forall h,d: \quad U^c(h,d;q,w) \quad \geq \quad U^D(h,d;q,w) \quad \text{(FDIC)}$$

As a first step, the next lemma shows that FDIC is a stricter set of constraints then IC. Intuitively, any most profitable non-compliant plan must have some period in which it is profitable to make a last deviation and thus violate some FDIC. The proof is standard and relegated to appendix B.1.

Lemma 2. *If a contract is FDIC it is IC*

As FDIC is stricter than IC and LDIC is weaker than IC, the following simple result provides a criterion for the sufficiency of the LDIC.

Corollary 1. *If any optimal contract subject to LDIC satisfies FDIC, then any optimal contract subject to LDIC is optimal subject to IC.*

2.4. **Deriving the FDIC and LDIC.**

This section derives all FDIC as linear inequalities. The LDIC are the subset of FDIC with $d=0$. As the contract terms $\langle q,w\rangle$ are identical for all continuation utilities they are omitted from the parametrization of $U^D(\cdot)$ and $U^C(\cdot)$.

The FDIC in history h,d requires that the agent's expected continuation utility from following the contract at history h,d is at least his expected continuation utility from making a final deviation in the history. If the agent will never be asked to work in history h ($q_h=0$) then his expected continuation utility is zero for all continuation effort plans and the FDIC trivially holds at h for all d.

For $q_h>0$, working has three effects. First, at the current period, work costs the agent c_{h-d} and increases the expected payment from $p\cdot w_h$ to $p_0\cdot w_h$. Second, working affects the transition probabilities into the next public history. If the agent shirks the

[10] Formally, $U^D(h,d;q,w)$ is defined only for $d\leq n_h$.

[11] Appendix A provides an example.

next $'p'$ in the ex-ante probability of arriving to history $\langle h,s\rangle$ is replaced with the lower probability of success p_0 , and similarly, for $\langle h,f\rangle$ $1-p$ is replaced by $1-p_0$. Finally, work today increases the agent's costs in all continuation periods. If the agent would shirk, his cost in all future periods will be lower. The FDIC requires that the total of these effects will be positive. As the agent is risk neutral, the FDIC can be evaluated at the outset of the contract. Accounting for all these, the FDIC for private history h,d is:[12]

$$\begin{aligned}(2.6)\qquad & (p-p_0)\,w_h - q_h c_{h-d} \\ &+ (p-p_0)\left(\tfrac{q_{\langle h,s\rangle}}{p}U^c\left(\langle h,s\rangle,d\right) - \tfrac{q_{\langle h,f\rangle}}{1-p}U^c\left(\langle h,f\rangle,d\right)\right) \\ &+ p_0\tfrac{q_{\langle h,s\rangle}}{p}\left(U^c\left(\langle h,s\rangle,d\right) - U^c\left(\langle h,s\rangle,d+1\right)\right) \\ &+ (1-p_0)\tfrac{q_{\langle h,f\rangle}}{1-p}\left(U^c\left(\langle h,f\rangle,d\right) - U^c\left(\langle h,f\rangle,d+1\right)\right) \qquad \geq 0\end{aligned}$$

The first line is the simple static tradeoff. Note that w_h also accounts for the ex-ante probability that the contract arrives to history h and is not terminated. The second line is the incentive effect of continuation utilities, ignoring the change in costs. I will call this the "naive" dynamic incentive. The third and fourth lines are the expected losses due to higher future costs from working, adjusted for the correct continuation probabilities. I will call this the "dynamic information rent".

All continuation utility terms, $U^c(\cdot)$, in the FDIC (2.6) are the ex-ante continuation utility from *complying* with the contract starting at some history $\langle h,y\rangle$ ($\langle h,s\rangle$ or $\langle h,f\rangle$) with some private number of deviations d. Thus, they may be directly defined using the model's primitives. For any history $\langle h,y\rangle$, the continuation utility is simply the expected wages less costs:

$$(2.7)\qquad q_{\langle h,y\rangle}\cdot U^c\left(\langle h,y\rangle,d\right) = \sum_{\tilde h\succeq\langle h,y\rangle} p w_{\tilde h} - \sum_{\tilde h\succeq\langle h,y\rangle} q_{\tilde h} c_{\tilde h-d} \quad .$$

Note that only the expected costs depend on the private information d, but the expected payment is unaffected by d. This implies that the information rent terms (the last two lines of 2.6), are determined only through the lower costs:

$$\begin{aligned}(2.8)\qquad q_{\langle h,y\rangle}\left(U^c\left(\langle h,y\rangle,d+1\right) - U^c\left(\langle h,y\rangle,d\right)\right) &= \sum_{\tilde h\succeq\langle h,y\rangle} q_{\tilde h}\left(c_{\tilde h-d} - c_{\tilde h-d-1}\right) \\ &= \sum_{\tilde h\succeq\langle h,y\rangle} q_{\tilde h}\delta_{\tilde h-d}\end{aligned}$$

To evaluate separately the impact of the "information rent" dynamic incentive, add to the FDIC an exogenous parameter $\theta\in[0,1]$ that multiplies the "information rent" (lines 3-4) part of (2.6). Using (2.7) and (2.8) in (2.6), adding the parameter θ and changing sides to prepare for the dual obtains the final form of the FDIC:

[12]Appendix B.2 provides a detailed derivation.

(2.9)

$$
\begin{aligned}
FDIC: \quad & -(p-p_0)\,w_h + q_h c_{h-d} \\
& -\tfrac{p-p_0}{p}\textstyle\sum_{\tilde{h}\succeq\langle h,s\rangle} p w_{\tilde{h}} + \tfrac{p-p_0}{p}\sum_{\tilde{h}\succeq\langle h,s\rangle} q_{\tilde{h}}\cdot c_{\tilde{h}-d} \\
& +\tfrac{p-p_0}{1-p}\textstyle\sum_{\tilde{h}\succeq\langle h,f\rangle} p w_{\tilde{h}} - \tfrac{p-p_0}{1-p}\sum_{\tilde{h}\succeq\langle h,f\rangle} q_{\tilde{h}} c_{\tilde{h}-d} \\
& +\theta\cdot\tfrac{p_0}{p}\textstyle\sum_{\tilde{h}\succeq\langle h,s\rangle} q_{\tilde{h}}\cdot\delta_{\tilde{h}-d} \\
& +\theta\cdot\tfrac{1-p_0}{1-p}\textstyle\sum_{\tilde{h}\succeq\langle h,f\rangle} q_{\tilde{h}}\cdot\delta_{\tilde{h}-d} \qquad \leq 0
\end{aligned}
$$

The FDIC is linear in the q and w variables. The complete linear program 2.4 subject to the FDIC (2.9), parametrized by θ , is given by :

(2.10)

$$
\begin{aligned}
V^{FD}(\theta) = \max_{q\geq 0, w\geq 0} \quad & \textstyle\sum_{h\in H}\left[q_h\,(p-p_0)\,v - w_h p\right] \\
s.t. \quad & \\
& q_{h_0} \leq 1 \\
\forall h \quad & q_{\langle h,s\rangle} - q_h p \leq 0 \\
\forall h \quad & q_{\langle h,f\rangle} - q_h\,(1-p) \leq 0 \\
\forall h,d \quad & FDIC \quad (2.9)
\end{aligned}
$$

The Local-Deviations Problem is problem 2.10 with the FDIC limited to $d=0$. Problem (2.10) with $\theta=1$ is the original problem subject to FDIC.

Before deriving the dual, the following lemma establishes a useful result. Recall definition (3) of a fixed contract – the agent works from history h to period N regardless of new outcomes and is paid $\frac{c_N}{p-p_0}$ for success in all remaining periods. A work plan is fixed if it can be part of a fixed contract.

Lemma 3. *If the work plan is fixed to N from history h, the contract that minimizes the agent's continuation utility is fixed to N from history h. The agent is paid $\frac{c_N}{p-p_0}$ for success in all remaining histories. All remaining LDIC bind and all remaining FDIC for $d>0$ are slack.*

Proof. Suppose the work plan is fixed from h to N. As the information rent depends only on the work plan the agent's gain from shirking is given by (see equation 2.8):

$$
\begin{aligned}
& q_{\langle h,y\rangle}U^c\left(\langle h,y\rangle, d+1\right) - q_{\langle h,y\rangle}U^c\left(\langle h,y\rangle, d+1\right) = \\
& \textstyle\sum_{n=n_h+1-d}^{N-d}\left(c_{n-d} - c_{n-d-1}\right) = c_{N-d} - c_{n_h-d}\,.
\end{aligned}
$$

A fixed work plan also implies that any difference in continuation utilities between $\langle h,s\rangle$ and $\langle h,f\rangle$ depends only on wages. Because the agent is risk neutral, any incentive compatible future wage difference can be incorporated into the current period wage. Thus there is an optimal contract such that if the work plan is fixed,

$$U^c\left(\langle h,s\rangle; q,w\right) = U^c\left(\langle h,f\rangle; q,w\right)$$

Finally, if the work plan is fixed and the contract is not terminated at the end of the period, $q_{\langle h,s\rangle} = pq_h$ and $q_{\langle h,f\rangle} = (1-p)\,q_{\langle h,y\rangle}$. The FDIC in period h is thus simply:

$$(p-p_0)\,w_h - q_h c_{h-d} - \theta q_h\left(p_0 + (1-p_0)\right)\left(c_{N-d} - c_{h-d}\right) \quad \geq \quad 0\,.$$

Which simplifies to

$$\frac{w_h}{q_h} \geq \frac{\theta c_{N-d} + (1-\theta)\, c_{h-d}}{p - p_0} \ .$$

If $\theta = 1$ – the agent is fully cognizant of the dynamic effects of shirking – only the cost in the last period of work matters. By definition of w_h, the payment to the agent for success in history h is exactly $\frac{w_h}{q_h}$. Observe that the constraint for w_h *does not* depend on any future wage. As c_{n-d} decreases in d, all but the LDIC (in which $d = 0$) must be slack. The contract that minimizes the agent's continuation utility minimizes the total payments. Thus, all LDIC bind and the payment to the agent for success in history h with $\theta = 1$ is

$$\frac{w_h}{q_h} = \frac{c_N}{p - p_0} \ .$$

□

3. The Dual Dynamic Analysis

Dynamic moral hazard analysis typically proceeds by restating the Optimal Contract Problem 2.4 (or its Local/Finite Deviations alternatives) as a dynamic problem. The agent's continuation utility from complying with the contract is the state for period h and the incentive constraint is written in terms of the difference between the continuation utility after success and after failure. In case of history dependent costs, the agent's utility if he deviated once in the past is added as a control (see e.g. Fernandes and Phelan (2000)).

Instead, I derive a dynamic program for the dual of the LDIC and FDIC optimal contract problems.[13] The dynamic dual program allows a relatively straightforward proof for the sufficiency of local deviations. It is sufficient to show that the agency frictions in any period are largest if the agent never shirked in the past. This is clearly the case for the limited liability friction as the agent's cost this period is largest if he never shirked. The information asymmetry friction is largest when the agent never shirked in the past if the marginal gain for the agent from shirking is largest in the first shirk. This is indeed the case whenever costs are convex – the first shirk generates the largest cost difference in each period. Thus, it will be sufficient to consider only local deviations.

To ease the economic interpretation, the derivation and analysis of the dual is done in two steps. This section derives and analyzes the dual of the local deviations problem. Section 4 adds to the dual all possible final deviations and proves that the local deviations problem is sufficient. Therefore, section 4 proves that the analysis of this section describes the optimal contract.

3.1. **The LDIC Dual – An Intuitive Derivation.**

The derivation of the Dual requires intensive notation. To make the reasoning more explicit, the analysis start with a two period example and then apply a recursive argument to add more periods. A formal derivation of the LDIC and FDIC dynamic dual problems is provided in appendix B.4.

3.1.1. *The Two Period Model.*

In a two period model, the possible public histories are $h \in \{\emptyset, s, f\}$. The primal linear problem subject only to LDIC is

---

[13] See appendix B.3 for a review of the basic duality concepts.

$$\max_{(q_h,w_h)\geq 0} \quad (q_\emptyset + q_s + q_f)\, v\,(p-p_0) - p\,(w_0 + w_f + w_s)$$

$s.t.$

$$q_\emptyset \leq 1$$

$$q_s - p q_\emptyset \leq 0$$

$$q_f - (1-p)\, q_\emptyset \leq 0$$

$$LDIC\ h=\emptyset \qquad \begin{array}{l} -(p-p_0)\, w_\emptyset + q_\emptyset c_0 \\ -\frac{p-p_0}{p} p w_s + \frac{p-p_0}{p} q_s c_1 \\ +\frac{p-p_0}{1-p} p w_f - \frac{p-p_0}{1-p} q_f c_1 \\ +\theta \frac{p_0}{p} q_s \delta_1 + \theta \frac{1-p_0}{1-p} q_f \delta_1 \leq 0 \end{array}$$

$$LDIC\ h=s \qquad -(p-p_0)\, w_s + q_s c_1 \leq 0$$

$$LDIC\ h=f \qquad -(p-p_0)\, w_f + q_f c_1 \leq 0$$

3.1.2. *The Dual Two Period Problem.*

Let μ^h for $h \in \{\emptyset, s, f\}$ be the shadow variable on the probability constraints and λ^h the shadow variable on the LDIC. As only the left-hand side of the probability constraint for $\emptyset$ is not zero, the objective for the dual is

$$\min \mu^\emptyset$$

Each primal variable defines a dual constraint. Consider first the wage constraint for $w_\emptyset$. $w_\emptyset$ appears with a coefficient $-p$ in the objective and with a coefficient $-(p-p_0)$ in the LDIC for $h=\emptyset$. The dual constraint is therefore:

$$-(p-p_0)\, \lambda^\emptyset \geq -p\ .$$

w_s appears in the LDICs for histories $\emptyset$ and s, with a coefficient $-(p-p_0)$ in both. The constraint is thus:

$$-(p-p_0)\left(\lambda^s + \lambda^\emptyset\right) \geq -p\ .$$

w_f appears in the LDICs for histories $\emptyset$ and f . The coefficient in the first is $-(p-p_0)$ and on the second $\frac{p-p_0}{1-p}p$. The constraint is thus

$$-(p-p_0)\left(\lambda^f - \lambda^\emptyset \frac{p}{1-p}\right) \geq -p\ .$$

$q_\emptyset$ appears in all the probability constraints and in the LDIC for $\emptyset$. The constraint for $q_\emptyset$ is:

$$\mu^\emptyset - p\mu^s - (1-p)\,\mu^f + \lambda^\emptyset c_0 \quad \geq \quad v\,(p-p_0)\ .$$

q_s appears in its probability constraint, in the two FDIC for s, and twice in the FDIC for the preceding history $\emptyset$ – once for the continuation utility term and once for the shirking gains term. The constraint for q_s is

$$\mu^s + \frac{p-p_0}{p} c_1 \lambda^\emptyset + \theta \frac{p_0}{p} \delta_1 \lambda^\emptyset + \lambda^s c_1 \geq v\,(p-p_0)\ .$$

The same procedure obtains the probability constraint for q_f:

$$\mu^f - \frac{p-p_0}{1-p}c_1\lambda^\emptyset + \theta\frac{1-p_0}{1-p}\delta_1\lambda^\emptyset + \lambda^f c_1 \geq v\left(p-p_0\right)$$

It will be convinient to divide the wage constraints by $-\left(p-p_0\right)$. Summarizing, the dual is

(3.1)
$$\min_{(\mu,\lambda)\geq 0}\mu^\emptyset$$

$$\begin{array}{lll} s.t. & & \\ \left(q_\emptyset\right): & \mu^\emptyset - p\mu^s - (1-p)\,\mu^f + \lambda^\emptyset c_0 & \geq v\left(p-p_0\right) \\ \left(q_s\right): & \mu^s + \frac{p-p_0}{p}c_1\lambda^\emptyset + \theta\frac{p_0}{p}\delta_1\lambda^\emptyset + \lambda^s c_1 & \geq v\left(p-p_0\right) \\ \left(q_f\right): & \mu^f - \frac{p-p_0}{1-p}c_1\lambda^\emptyset + \theta\frac{1-p_0}{1-p}\delta_1\lambda^\emptyset + \lambda^f c_1 & \geq v\left(p-p_0\right) \\ \left(w_\emptyset\right): & \lambda^\emptyset & \leq \dfrac{p}{p-p_0} \\ \left(w_s\right): & \lambda^s + \lambda^\emptyset & \leq \dfrac{p}{p-p_0} \\ \left(w_f\right): & \lambda^f - \lambda^\emptyset\frac{p}{1-p} & \leq \dfrac{p}{p-p_0} \end{array}$$

3.1.3. *A Recursive Formulation for the Second Period Dual.*

The objective for (3.1) is to minimize $\mu^\emptyset$. The dual constraint for $q_\emptyset$ shows that $\mu^\emptyset$ is minimized if the lowest values for μ^s and μ^f are obtained for any $\lambda^\emptyset$. In turn, the constraints for q_s and q_f show that it is possible to consider the problems of minimizing μ^s and μ^f separately for any $\lambda^\emptyset$. Isolating μ^s in the constraint for q_s :

$$\mu^s\left(\lambda^\emptyset\right) = \max \quad \left[0, \min_{\lambda^s\geq 0} v\left(p-p_0\right) - \tfrac{p-p_0}{p}c_1\lambda^\emptyset - \theta\tfrac{p_0}{p}\delta_1\lambda^\emptyset - \lambda^s c_1\right]$$
$$s.t.$$
$$\lambda^s + \lambda^\emptyset \leq \tfrac{p}{p-p_0}$$

Replacing λ^s with λ^f, the problem for μ^f only differs in the coefficients for $\lambda^\emptyset$:

$$\mu^f\left(\lambda^\emptyset\right) = \max \quad \left[0, \min_{\lambda^f\geq 0} v\left(p-p_0\right) + \tfrac{p-p_0}{1-p}c_1\lambda^\emptyset - \theta\tfrac{1-p_0}{1-p}\delta_1\lambda^\emptyset - \lambda^f c_1\right]$$
$$s.t.$$
$$\lambda^f - \tfrac{p}{1-p}\lambda^\emptyset \leq \tfrac{p}{p-p_0}$$

It would be convinient to solve the same problem for both second period histories. For that, define $r^s\left(\lambda^\emptyset\right) \equiv \theta\frac{p_0}{p}\lambda^\emptyset$, $r^f\left(\lambda^\emptyset\right) \equiv \theta\frac{1-p_0}{1-p}\lambda^\emptyset$, $l^s\left(\lambda^\emptyset\right) \equiv -\frac{p-p_0}{p}\lambda^\emptyset$, and $l^f\left(\lambda^\emptyset\right) \equiv \frac{p-p_0}{1-p}\lambda^\emptyset$. The second period problem becomes

(3.2)
$$\mu\left(n=1,l,r\right) = \max \quad \left[0, \min_{\lambda\geq 0} v\left(p-p_0\right) + lc_1 - r\delta_1 - \lambda c_1\right]$$
$$s.t. \qquad \lambda \leq \tfrac{p}{p-p_0}\left(1+l\right)$$

Observe that $\mu\left(1,l,r\right)$ only depends on the l, r and the period number. The optimal solution will always set λ to the maximum feasible and

$$\mu\left(n=1,l,r\right) = \max\left[0, v\left(p-p_0\right) - \frac{c_1}{p-p_0}\left(p-p_0 l\right) - r\delta_1\right]$$

3.1.4. *The LDIC Recursive Formulation.*

Given the second period problems, the first period problem is defined by the $q_\emptyset$ and $w_\emptyset$ dual constraints:

$$\min_{\lambda\geq 0}\mu^\emptyset$$

$$\begin{array}{lll} s.t. & & \\ (q_\emptyset): & \mu^\emptyset - p\mu^s(\lambda) - (1-p)\mu^f(\lambda) + \lambda c_0 & \geq v(p-p_0) \\ (w_\emptyset): & \lambda & \leq \dfrac{p}{p-p_0} \end{array}$$

The $q_\emptyset$ constraint may be written using the definitions of l^h and r^h used to derive problem 3.2 to construct μ^s and μ^f:

$$\begin{aligned} \mu^\emptyset \geq\ & p\mu\left(1, l-\frac{p-p_0}{p}\lambda, r+\theta\frac{p_0}{p}\lambda\right) + (1-p)\mu\left(1, l+\frac{p-p_0}{1-p}\lambda, r+\theta\frac{1-p_0}{1-p}\lambda\right) \\ & +v(p-p_0) - c_0\lambda \end{aligned}$$

Therefore, one can set

$$\mu^\emptyset = \mu(0,0,0)$$

and suggest the following formulation:

Definition 6. The LDIC Dual is

$$\begin{aligned} \mu(n,l,r) &= \max\left[0, \min_{\lambda\geq 0}\mu(n,l,r,\lambda)\right] \qquad (3.3) \\ & s.t. \\ \mu(n,l,r,\lambda) &= p\mu\left(n+1, l-\frac{p-p_0}{p}\lambda, r+\theta\frac{p_0}{p}\lambda\right) + (1-p)\mu\left(n+1, l+\frac{p-p_0}{1-p}\lambda, r+\theta\frac{1-p_0}{1-p}\lambda\right) \\ & +v(p-p_0) - c_0\lambda + c_0 l - \delta_0 r \\ & \lambda \leq \frac{p}{p-p_0}(1+l) \end{aligned}$$

With $\mu(n,l,r) = 0$ for all $n > N^{FB}$. The law of motion for the state variables is defined implicitly in 3.3 . The following definitions formalize the state variables and suggest their economic content.

The *liability limit* in the public history h is l^h, defined by:

$$l^h = (p-p_0)\left(\sum_{\tilde{h}:h\succeq\langle\tilde{h},f\rangle}\frac{\lambda^{\tilde{h}}}{1-p}\right) - \left(\sum_{\tilde{h}:h\succeq\langle\tilde{h},s\rangle}\frac{\lambda^{\tilde{h}}}{p}\right). \qquad (3.4)$$

The *information rent* in the public history h is r^h, defined by:

$$r^h = \theta\left(\sum_{\tilde{h}:h\succeq\langle\tilde{h},s\rangle}\frac{p_0}{p}\lambda^{\tilde{h}}\right) + \theta\left(\sum_{\tilde{h}:h\succeq\langle\tilde{h},f\rangle}\frac{1-p_0}{1-p}\lambda^{\tilde{h}}\right) \qquad (3.5)$$

The state variables' names are motivated in the next section, together with the interpretation of the other components of the problem. The next lemma completes the recursive formulation.

Lemma 4. *For every h, $\mu^h = \mu(n_h, l^h, r^h)$. Specifically, $\mu(0,0,0)$ and the corresponding optimal λ's are a solution to the dual of the LDIC problem.*

Proof. A formal proof is provided in appendix B.4. An intuitive exposition follows. □

Given the two period example, it is left to show that the formulation of problem 3.3 remains valid when considering a longer horizon than just the first two periods. To see the intuition, consider the effect of extending work to a new history ,$\left\langle \tilde{h}, y \right\rangle$ for $y \in \{s, f\}$, on all the LDIC in the existing work histories. Clearly, the new history only affects the incentives in histories that preceed it. For these histories, the effect amounts to adding two linear terms - the change in expected utility from complying and the gain from arriving to $\left\langle \tilde{h}, y \right\rangle$ having shirked once in the past. The added terms on the preceeding LDICs depend only on whether $\left\langle \tilde{h}, y \right\rangle$ follows a success or a failure in the preceeding history, and on the cost at $\left\langle \tilde{h}, y \right\rangle$. Therefore, the incentive effect of extending work to $\left\langle \tilde{h}, s \right\rangle$ is identical to the incentive effect of extending work to $\left\langle \tilde{h}, f \right\rangle$ on all histories except for $\tilde{h}$. Moreover, adjusting for costs, it is the same as the incentive effect of $\tilde{h}$. Therefore, if the state at $\tilde{h}$ captures the total incentive effects up to it, the analysis can proceed as if $\tilde{h}$ is the first period with state l, r.

3.2. **Analysis of the Optimal LDIC Contract.**

This section analyzes the optimal solution to the LDIC Dual, defined by problem 3.3. The analysis first interprets the components. The analysis refers to the optimal LDIC contract as simply the optimal contract. Theorem 1 in section 4 proves the two are indeed equal. The proof is independent of the analysis in this section but is relegated to the next section as it requires more involved notation.

The following standard simplification of the notation will reduce clutter. Function variables are omitted when these are obvious (i.e. μ refers to $\mu(n, l, r)$). Subscripts (e.g. μ_l) denote partial derivatives. A superscript s or f will denote the optimal continuation values. For example l^s is the optimal continuation value for l after a success in the state (n, l, r). Combining the three, μ_r^f is the partial derivative $\frac{\partial \mu\left(n+1, l^f, r^f\right)}{\partial r}$. Note first the following standard result.

Lemma 5. *$\mu(n, l, r)$ is continuous and convex in (l, r). $\mu(n, l, r, \lambda)$ is continuous and convex in λ for every l, r.*

Proof. The constraints in problem 3.3 are convex, the problem is a recursive minimization problem that stops at some N and the non-recursive part of the objective is linear. Therefore, the result is standard. A detailed proof is provided in appendix B.6 □

3.2.1. *Interpreting the Dual.*

The key element in interpreting $\mu(n, l, r)$ and the optimal solution follows from the Duality Theorem.

Lemma 6. *$\mu(0, 0, 0)$ is the principal's expected profit from the optimal LDIC contract.*

Proof. $\mu(0, 0, 0)$ is the solution to the dual problem of the LDIC. It is immidiate that $\mu(0, 0, 0)$ exists and is bounded. The Duality Theorem (Dantzig (1963)) applies. □

In a general history, μ^h is the multiplier on the feasibility constraint. Technically, μ^h is the marginal value of increasing the ex-ante probability of work in the history. Economically, this is the marginal value, calculated at time zero, of *committing to work* in the history h. Recall that in the standard dynamic formulation, the value at each state (denoted $V(h)$) is the expected *continuation profit.* While $\mu(0, 0, 0) = V(\emptyset)$, the two values differ at all other histories: $\mu(n,l,r)$ accounts for the change in *total* expected profits, starting at the first period, from increasing the probability of work in the history, while $V(h)$ accounts only for the continuation profits starting at history h. The zero lower bound on μ reflects the fact that the contract only asks the agent to work in periods in which working increases the total expected profit. The difference between μ and V is most striking in periods in which the agent is "given the firm". The principal's continuation profit, $V(h)$, may be negative in such periods.[14] However, all the utility obtained by the agent in such periods was used to motivate past work. Therefore, lemma 12 will show that μ in such periods takes into account the agent's continuation utility, and is positive.

The state variable l is derived from the multiplier on the continuation utility effect in the IC (2.9). The agent obtains rents from working. If work today is conditional on past outcomes, today's utility is used as an incentive in the past instead of wages. If work today is allowed after a failure in the past, today's rent reduces past incentives to work. l measures this effect of work in the current history on past incentives. Conditioning work on past outcomes reduces surplus. The optimal contract does this to relax the limited liability constraint. A negative change in continuation utility is equal, for the agent, to a negative wage. If the agent could be forced to pay in case of negative outcomes, manipulating the continuation utility would be sub-optimal. Therefore, I call l the *liability limit*.

The state variable r is derived from the multiplier on the "dynamic information rent" effect in the IC. This is the agent's incentive to shirk in the past in order to obtain the secret cost reduction δ_h from working in the current history h. Extending work to history h requires increasing the agent's incentives in all past histories that may lead to h to balance the shirking incentive. The agent is paid the *information rent* in all histories that preceed h to exactly compensate him for the δ_h extra utility he would obtain this period by shirking in a previous period.

By lemma 5, the value function μ is differentiable w.r.t. l, r almost everywhere and the first order condition is necessary and sufficient. However, due to the zero lower bound, μ has kinks. Let $\lambda(n,l,r)$ be the optimal λ for state (n,l,r) and $\omega(n,l,r)$ the payment for success in the period. The first order effect of increasing λ, $\mu_{\lambda+}$ is given by

$$\mu_\lambda = (p-p_0)\left(\mu^f_{l+}-\mu^s_{l-}\right)+p_0\mu^s_{r+}+(1-p_0)\,\mu^f_{r-}-c_n+\omega(n,l,r)(p-p_0) \tag{3.6}$$

Lemma 5 implies the following:

Lemma 7. *In any state along the optimal contract:*

$$\lambda(n,l,r) \quad = \quad \min\left[\frac{p}{p-p_0}(1+l), \sup\{\lambda:\lambda\geq 0, \mu_\lambda\leq 0\}\right]. \tag{3.7}$$

If $\lambda < \frac{p}{p-p_0}(1+l)$, $\omega = 0$. If $\lambda = \frac{p}{p-p_0}(1+l)$, $\omega(n,l,r)$ is set so the inequality is an equality:

$$\omega = -\frac{(p-p_0)\left(\mu^f_{l+}-\mu^s_{l-}\right)+p_0\mu^s_{r+}+(1-p_0)\,\mu^f_{r-}-c_n}{p-p_0}$$

Proof. The first order condition is necessary and sufficient given convexity. The wage paid for success is the multiplier on the wage constraint. If $\lambda < \frac{p}{p-p_0}(1+l)$ the constraint does not bind and therefore $\omega = 0$. If the constraint does bind, ω is set so that the

[14] See section 3.2.3.

first order condition is exactly zero. The original constraint was divided by $-(p-p_0)$ when constructing the dual and so the standard wage variable is multiplied by the same amount. □

The first order condition 3.6 is suspiciously similar to the LDIC in equation 2.6. To save on notation, let $U^h \equiv U^c(h,0)$ denote the continuation utility from complying starting at history h given no shirks in the past. Let $D^h \equiv U^c(h,1) - U^c(h,0)$ denote the extra gains to the agent starting at history h from having shirked once in the past. The LDIC may be written as

$$(3.8)\qquad (p-p_0)\left(\omega_h + U^{\langle h,s\rangle} - U^{\langle h,f\rangle}\right) \quad -p_0 D^{\langle h,s\rangle} - (1-p_0)\, D^{\langle h,f\rangle} - c_n \geq 0$$

Observe that if we let $\mu_l^y = -U^{\langle h,y\rangle}$ and $\mu_r^y = -D^{\langle h,f\rangle}$, the term in 3.8 is exactly μ_λ whenever all derivatives exist. Thus, the problem is to find in each state the smallest feasible λ that does not violate the IC. λ is the "credit at risk" for the agent in the current period. The agent's credit balance increases or decreases whenever the period outcome is, respectively, a success or a failure. As any credit can be used to provide liability, this balance is the negative of the liability limit l. The credit is eventually used to "sell the firm to the agent". If the agent obtained enough credit to completely compensate for his limited liability, no more credit can be offered. This is captured by the dual wage constraint. The foc 3.6 shows that the optimal contract finds the minimal credit at risk λ in each period that is required to satisfy IC. If stronger incentives are required, the wage constraint kicks in and the agent is paid the difference.

However, as μ may have kinks, the first order condition inequality 3.6 in some states will *not* be an equality and the two values will diverge. Thus, to formally relate the continuation utilities to the dual derivatives, one must either refer to superdifferentials or a smooth approximation for $\mu(n,l,r)$.[15] I show in appendix XX (to add) that a continuous time version of the model in which the termination decision must be smooth results in such a smooth approximation, and retains all the intuitive interpretation of the discrete time model.

Definition 7. The *ε-smooth variation* of the model is a related model that obtains $\tilde{\mu}(n,l,r)$ smooth and for all (n,l,r), $|\tilde{\mu}(n,l,r) - \mu(n,l,r)| \leq \varepsilon$.

Lemma 8. *In the ε-smooth model, as $\varepsilon \to 0$, $\tilde{\mu}_l^h \to -U^h$ and $\tilde{\mu}_r^h \to -D^h$.*

Proof. In any last period the result is immidiate. Appendix B.7 shows that backward induction implies the desired result whenever the envelope theorem applies. □

3.2.2. *Monotonicity.*

A key advantage of the dual dynamic analysis is that it restores monotonicity in the state variables. Each of the state variables l, r reflects the degree of agency frictions – limited liability and information asymmetry. Intuitively, as frictions increase, profits decrease. Thus, profit should decrease in both l and r. I now show this is the case.

For the information asymmetry state r, this is evident by observing problem 3.3: r only decreases the objective and thus decreases profits. For l, one expects that as the agent's liability becomes more limited (l increases), the principal's profit decreases. However, this is not immediately observable as the coefficient on l is positive in the period return.

[15]From here to the end of the next proof – needs polishing.

Nevertheless, profits decrease with l because a low limit on liability reduces the need to provide the agent with an incentive today through λ. The agent expects to "own" the firm soon and so values credit towards a share in the firm more. Technically, every increase in the liability limit l increases period profits by c_n but allows an increase of $\frac{p}{p-p_0} > 1$ in λ that decreases period profits by more than c_n.

Lemma 9. *Period value , $\mu(n,l,r;\theta)$ is strictly decreasing in l, r and θ whenever $\mu > 0$.*

Proof. See above for intution and appendix B.8 for details. □

The agent works only in a period in which $\mu(n,l,r) > 0$. By the law of motion, both state variables are higher after a failure than after success. The previous lemma implies then that $\mu(n+1,\cdot)$ is lower after failure than after success. Therefore, if $\mu(n+1,\cdot) = 0$ after success, the same is also true after failure.

Corollary 2. *If the contract terminates after success, it terminates after failure.*

Monotonicity of μ also implies that whenever feasible, $\lambda > 0$. This is intuitive. If the wage constraint does not bind at $\lambda = 0$, then when $\lambda = 0$ the agent works for free – the continuation contract is not affected by outcomes today and the wage for success is zero. Formally, at $\lambda = 0$, the inequality in the first order condition 3.6 is always strict. As λ is the multiplier on the LDIC, complementary slackness implies that the LDIC binds whenever $l > -1$.

Lemma 10. *Whenever $l > -1$, $\lambda > 0$. Thus, the LDIC binds whenever $l > -1$.*

Proof. Suppose $\lambda = 0$ is optimal and $l > -1$. The f.o.c. 3.6 therefore implies that

$$0 = \arg\inf_{\lambda\geq 0} (p-p_0)\left(\mu^f_{l+} - \mu^s_{l-}\right) + p_0\mu^s_{r+} + (1-p_0)\,\mu^f_{r-} - c_n \geq 0$$

At $\lambda = 0$, $l^f = l^s = l$ and $r^f = r^s = r$. If $\mu(n+1,l,r)$ is smooth w.r.t. l, the proof is done as $\mu^y_r \leq 0$. If $\mu^f_{l+} > \mu^s_{l-}$ is not smooth, by continuity of $\mu(n+1,l,r)$ there is some $\lambda > 0$, such that the continuation values are arbitrarily close to zero while $c_n > 0$. (need to write this nicer). □

Consider now the effect of each friction (l or r) on the other. Suppose the agent's liability limit friction (l) *decreases*. This means the agent's credit is larger. Intuitively, this means the firm is more likely to be sold to the agent. Therefore, the relative profit impact of any past commitments to destroy surplus, which is the value of r, increases. The next lemma establishes this:

Lemma 11. *l, r are substitutes: $\mu(n,l,r)$ is sub-modular in $(-l,r)$.*

Proof. In any last period, $\lambda = \frac{p}{p-p_0}(1+l)$ and

$$\mu(n,l,r) = \min\left[0, v(p-p_0) - \frac{p}{p-p_0}c_n - \frac{p_0}{p}c_n \cdot l - \delta_n r\right]$$

The marginal effect of l and r is either fixed and strictly negative or, if $\mu(n,l,r) = 0$, it is zero.

(1) As l (resp. r) increases, the marginal effect of r (resp. l) *increases* from $-\delta_n$ (resp. $-\frac{p_0}{p}c_n$) to zero. Therefore, in any last period, $\mu(n,l,r)$ is sub-modular in $(-l,r)$.

(2) Now suppose that for $\tilde{n} > n$, $\mu(\tilde{n}, l, r)$ is sub-modular in $(-l, r)$. Then both continuation values are sub-modular in $(-l, r)$ and so is the period return. The positive weights sum of sub-modular functions are sub-modular and therefore $\mu(n, l, r, \lambda)$ is sub-modular in $(-l, r)$ for any λ. As the feasible set defines a lattice, the sub-modularity is preserved under minimization w.r.t. λ (Theorem 2.7.6 of Topkis (1998)).

□

To see the implication of lemma 11, suppose $\mu(n, l, r)$ is smooth at (n, l, r) and consider μ_l. Lemma 11 implies that μ_l increases in r. As $\mu(n, l, r)$ is convex, μ_l also increases in l. As $\mu_l < 0$, this implies that the magnitude of μ_l decreases as frictions increase. Exactly the intuition provided above. Recall that in the smooth variation of the model (see definition 7), $-\mu_l$ and $-\mu_r$ are, respecitvely the agent's expected continuation utility $(U(n, l, r))$ and gains from shirking $(D(n, l, r))$. Thus, both U and D are decreasing in the state variables, just as the period value μ.

As a higher state implies more frictions, it is natural that the expected efficiency of the contract decreases with the state. A simple case in which this result is direct, is if c_n is linear. In this case δ_n is fixed at δ and the agent's expected gains from a past shirk, D, are δ times the expected number of work periods. As D decreases in the state variables, it follows that the expected number of work periods decreases.

Corollary 3. *In the smooth variation, a change in l, r has the same qualitative effect on the period's value to the principal,μ, the agent's expected payoff starting from the period, U, and the agent's expected gains from one past shirk, D. If c_n is linear, a higher state implies a weakly lower expected continued efficiency.*

** A stronger result is pending (no need for smoothness or linear costs) **

** Results on $\lambda(n, l, r)$ are pending. Conjecture is that λ increases with r and θ **

3.2.3. *Selling the Firm and Dynamic Quotas.*

Suppose that the wage constraint binds – the agent is paid for success in this period. As both the principal and agent are risk neutral, this implies the principal could not offer the agent any more credit. The agent has earned enough credit to "buy the firm". The value of the firm depends on past effort. If the agent worked less than the principal believes he did, future costs are lower and the value of the firm is higher. As a result, if the agent expects to be rewarded in the future by "getting the firm" he has an incentive to shirk. To counter this incentive, the principal limits the agent's work after selling the firm, so that any possible gains from shirking are countered by the reduction in surplus. This need to reduce surplus is only a function of the information asymmetry r, but not the limited liability, l. Therefore, the terms of the contract after giving the firm to the agent will be determined only by r.

Formally, if the agent is paid for success in history h, by complementary slackness, the wage constraint binds. Thus $\lambda = \frac{p}{p-p_0}(1+l)$. The law of motion for l ($l^s = l - \frac{p-p_0}{p}\lambda$), now implies that in history $\langle h, s\rangle$, $l = -1$. The only feasible value for λ in $\langle h, s\rangle$ is zero – no more credit under risk. Intuitively, if the agent earned enough credit to be paid, it is impossible to offer any more credit as future incentives. As a result, in all remaining periods l and r remain unchanged *regardless of new outcomes*. The period profit is

$$\pi(n,r) = \quad v(p-p_0) - c_n - r\delta_n \tag{3.9}$$

l and λ are fixed and therefore do not affect the period profit as variables.

As both c_n and δ_n are positive and increasing, eventually period profit will become negative and production will stop. It follows that the continuation work plan is fixed and, as there is no reason to provide the agent more utility than necessary, lemma 3 implies that the contract is fixed.

Lemma 12. *In the optimal contract, if the agent is paid for success in history h, then starting in history $\langle h,s\rangle$ the contract is fixed to $N(r)$, given by*

$$N(r) = \max n : \quad v(p-p_0) - c_n - r\cdot\delta_n \geq 0 \quad .$$

All remaining LDIC bind and the agent wage in any following history $\tilde{h} \succeq \langle h,s\rangle$ is

$$\omega_{\tilde{h}} = \frac{(1-\theta)\,c_{\tilde{h}} + \theta c_{N(r)}}{p-p_0} \quad .$$

Proof. The discussion above shows that the work plan after payment is fixed and continues to $N(r)$. Lemma 3 implies that the contract after a payment that provides the lowest utility to the agent is as stated here. As the agent is paid for success in the current period and is risk neutral, any additional utility can be provided as payment in the current period and the optimal continuation contract is fixed. □

The number of periods is determined by considering the information asymmetry cost $r\cdot\delta_h$ as an additional cost to the current period. The contract is implemented by paying the agent a wage as if production cost is at the highest value for which the agent should still work. This guarantees the agent will only work the desired number of remaining periods.

Lemma 12 implies the dynamic quota interpretation. The contract can be divided to two stages. If the agent was not yet paid, he is in the evaluation stage. Outcomes affect the state variables and through them the probability that the wage constraint would soon bind. Once the agent is paid, the contract moves to the compensation stage. The agent is paid in all remaining periods as if his cost is the highest cost for which he is still expected to work. This last cost is determined in the evaluation stage.

This implies an important separation result. In the evaluation stage, incentives are provided *only* through changes to the ensuing work plan. In the compensation stage, incentives are provided only through wages.

Corollary 4. *The period outcome affects the future work plan iff the agent was never paid so far.*

An important implication of the information assymetry is that the ex-ante variance of the expected work plan is larger when the cost increase is private and not public information. The possibility to obtain lower cost requires a stronger incentive for the agent to work. As incentives in the evaluation stage are provided only by means of changes to the ensuing work plan, the variance must be larger in the private cost setting.

Corollary 5. *In the evaluation stage the difference between the work plan after failure and after success is larger when if $\theta = 1$ (the true cost is private information) than if $\theta = 0$ (the true cost is public information).*

The implication of private information on the compensation stage is twofold. If there was no information asymmetry ($\theta = 0$ and so $r = 0$), the standard solution applies – the agent works while efficient and is paid the true cost each period (c_n). In contrast, private information both implies concerns about past shirking (through r) and a stronger incentive for future shirking. The additional incentive to shirk in the past to lower costs this period, $r \cdot \delta_n$, is simply considered an additional cost by the contract. Thus, as r increases, the continuation contract after selling the firm to the agent is less efficient and provides the agent less utility. However, even if $r = 0$, the agent can still increase his utility by shirking in the future and lowering costs. As a result, the agent is paid as if the cost each period is the cost in the last period of work $c_{N(r)}$and not the current period c_n. Thus, the forward looking effect of private information increases the agent's utility from the contract after payment.

3.2.4. *Termination Without Payment.*

If the agent's liability is sufficiently unlimited, the agent is "given" the firm, albeit for less periods than is efficient. Suppose the agent was not paid so far, and so $l > -1$. Let $\overline{\lambda} \equiv \frac{p}{p-p_0}(1+l) > 0$ denote the remaining credit the agent needs to "get" the firm. As the information rent increases by at least $\frac{p_0}{p}\lambda$ each period, it will increase by at least $\frac{p_0}{p}\overline{\lambda}$ before the agent will get the firm. Thus, when the agent will get the firm, the information rent will be at least $r + \theta\frac{p_0}{p}\overline{\lambda} = r + \theta\frac{p_0}{p-p_0}(1+l)$. For future promises to have any value to the agent it must be that production is not shut down when the agent gets the firm:

$$v(p-p_0) - c_{n+1} - \left(r + \theta\frac{p_0}{p-p_0}(1+l)\right)\delta_{n+1} \geq 0 \tag{3.10}$$

Suppose now that l and r are sufficiently large so that 3.10 is violated:

$$\left(\frac{p_0}{p-p_0}l\theta + r\right)\delta_{n+1} \geq v(p-p_0) - c_{n+1} - \theta\frac{p_0}{p-p_0}\delta_{n+1}\ . \tag{3.11}$$

Then the information rent in any period after giving the firm to the agent is too large and the contract will stop immidiately after a payment to the agent. Any promise of "future credit in the firm" is worthless. If he works, the agent must be paid for success in this period and then the contract stops. As the work plan starting in this period is fixed – work only in the current period – lemma 12 can be applied to determine that the wage is $\frac{c_n}{p-p_0}$. The wage constraint binds and so $\lambda = \overline{\lambda}$. To determine whether the agent works in this period, evaluate the μ knowing that the agent cannot work in any future periods:

$$\begin{aligned}\mu(n,l,r) &= v(p-p_0) - \frac{p}{p-p_0}(1+l)c_n + lc_n - r\delta_n \\ &= v(p-p_0) - \frac{p}{p-p_0}c_n - \frac{p_0}{p-p_0}lc_n - r\delta_n\end{aligned}$$

The agent will work in this period if and only if $\mu(n,l,r) \geq 0$. Thus, the agent will not work whenever

$$\frac{p_0}{p}lc_n + r\delta_n > v(p-p_0) - \frac{p}{p-p_0}c_n \tag{3.12}$$

Lemma 13. *For any state (n,l,r):*

1. *If (3.11) but not (3.12), the contract terminates at the end of the period regardless of outcomes. The agent works in the period and is paid the spot wage for success.*
2. *If (3.11) and (3.12) the contract terminates at the start of the period.*

Proof. Suppose (3.11) in state (n,l,r). Setting λ so that the wage constraint binds this period minimizes the period return.

(1) The continuation values are zero:
   (a) $\mu(n+1,l^s,r^s)\leq 0:$ this is exactly the right hand side of 3.10. By condition 3.11 the inequality 3.10 is violated and so $\mu(n+1,l^s,r^s)=0$
   (b) $\mu\left(n+1,l^f,r^f\right)\leq 0:$ l,r are both are larger after failure and μ is decreasing in both.

(2) If 3.12 is violated, the period return is still negative and so the agent works in the period. The wage is determined from the first order condition for $\mu(n,l,r)$:

$$c_n-\omega(p-p_0)=0$$

(3) If also (3.12) then $\max_\lambda V(n,l,r,\lambda)>0$ and so the optimal is to not work – $V(n,l,r)=0$

□

*** The lemma above stated sufficient conditions for termination. Necessary conditions are pending. ***

3.3. **More Information May be Less Efficient - Two Period Example.**

This section solves the optimal contract for any last two periods. This allows us to evaluate the effect of private information in more detail. The first subsection defines and solves a complete information benchmark. The second subsection solves the original problem and compares the solutions. The main result is that under some sensible parametrizations the agent is expected to work *less* under complete information – private information increases efficiency.

This happens when the second period work is only profitable as an incentive for first period effort. In such cases, the optimal contract under complete information may randomize the second period work after success so that the agent is exactly rewarded for his first period effort by the expected utility in the second period. The agent's expected utility in this contract is exactly zero. Therefore, if shirking affects second period costs, the agent has nothing to lose by shirking. This causes the optimal contract to pay the agent some wage in the setting with private information in the last *two* periods. The wage cost in the first period decreases with the probability of work in the second period more than the expected second period loss. Therefore, the private information contract never randomizes second period work after first period success.

3.3.1. *Two Periods – The Complete Information Benchmark.*

Suppose that the cost of effort c_n depends on the number of past *periods* rather than past effort. Thus, the second period cost is c_1 regardless of histories and the first period cost is c_0. To rule out trivial solutions, assume that work in the second period increases surplus: $v(p-p_0)>c_1$.

Let $\hat{\mu}(1,l)$ denote the second period value in this case. $\hat{\mu}(1,l)$ may be obtained by placing $r=0$ in 3.2 and observing that the optimal λ is the highest possible:

$$\hat{\mu}(1,l)=\max\left[0,v(p-p_0)-\frac{p}{p-p_0}c_1-lc_1\frac{p_0}{p-p_0}\right]$$

The first period problem is given by adapting problem (3.3) to the case that $r^y = 0$:

$$\begin{aligned}
\hat{\mu}(0,l) &= \max\left[0, \min_{\lambda\geq 0}\hat{\mu}(0,l,\lambda)\right] \\
s.t. & \\
\hat{\mu}(0,l,\lambda) &= p\hat{\mu}\left(1, l - \frac{p-p_0}{p}\lambda\right) + (1-p)\hat{\mu}\left(1, l + \frac{p-p_0}{1-p}\lambda\right) \\
& +v(p-p_0) - c_0\lambda + lc_0 \\
& \\
& \lambda \leq \frac{p}{p-p_0}(1+l)
\end{aligned}$$

Letting w denote the multiplier on the constraint, the f.o.c. for $\hat{\mu}(0,l,\lambda)$ is

$$\lambda^0(l) = \arg\inf_{\lambda\in\left[0,\frac{p}{p-p_0}(1+l)\right]}(p-p_0)\left(\hat{\mu}_{l^+}\left(1, l+\frac{p-p_0}{1-p}\lambda\right) - \hat{\mu}_{l^-}\left(1, l-\frac{p-p_0}{p}\lambda\right)\right) - c_0 + w(p-p_0) \geq 0$$

$\hat{\mu}(1,l)$ is piecewise linear with one kink (when $\hat{\mu}(1,l)$ becomes zero) and is decreasing in l. If both continuation values are non-zero, the partial derivatives cancel each other. If both continuation values are zero, both partial derivatives are also zero. As $\hat{\mu}(1,l)$ is decreasing in l , the only other possibility is $\hat{\mu}\left(1,l^f\right) = 0$ and $\hat{\mu}(1,l^s) > 0$ and so

$$(p-p_0)\left(\hat{\mu}_{l^+}\left(1, l+\frac{p-p_0}{1-p}\lambda\right) - \hat{\mu}_{l^-}\left(1, l-\frac{p-p_0}{p}\lambda\right)\right) = p_0 c_1$$

Suppose that there is some λ such that$\hat{\mu}\left(1,l^f\right) = 0$ and $\hat{\mu}(1,l^s) > 0$. Then if $p_0c_1 \geq c_0$, the optimal contract is defined by the lowest λ such that $\hat{\mu}\left(1,l^f\right) = 0$. If $p_0c_1 < c_0$, then, as increasing λ only increases μ^s the wage constraint must bind and $\mu^s > 0$. The next items summarize the optimal contract, letting l^0 denote the state at the start of the first period.

(1) If $\hat{\mu}\left(1, \frac{l^0}{1-p} + \frac{p}{1-p}\right) \geq 0$, the agent works in both periods regardless of outcomes ($\mu^f > 0$ at $\lambda = \frac{p}{p-p_0}(1+l)$.) The agent is paid $\frac{c_n}{p-p_0}$ for success in period $n \in \{0,1\}$.

(2) If $v(p-p_0) - \frac{pc_1}{p-p_0} \leq 0$ the second period work is profitable only because of incentive effects on the first period. This implies $\hat{\mu}(1,0) \leq 0$. As $\tilde{\mu}(1,l)$ decreases in l, it follows that the agent never works in the second period if he fails in the first. However, by assumption $\hat{\mu}\left(1, -\frac{p}{p-p_0}\right) > 0$ and so the agent will work after success. There are two possible cases:
   (a) If $p_0c_1 < c_0$ working only after success is profitable to the principal even if it requires increasing the agent's first period wage. Then $\lambda = \frac{p}{p-p_0}$, the agent works in the second period if and only if he succeeds in the first period. The agent is paid $w = \frac{c_0 - p_0c_1}{p-p_0}$ for success in the first period and $\frac{c_1}{p-p_0}$ for success in the second period.
   (b) If $p_0c_1 \geq c_0$ working after success increases first period profits only while it decreases the first period wage. The agent is paid nothing for success in the first period and $\frac{c_1}{p-p_0}$ for success in the second period. The probability of work after success α_s is determined by the IC:

$$\begin{aligned}
(p-p_0)\alpha_s\frac{p_0c_1}{p-p_0} &= c_0 \\
\alpha_s &= \frac{c_0}{p_0c_1}.
\end{aligned}$$

(3) If $v(p-p_0)-\frac{pc_1}{p-p_0}\geq 0$ the second period work is profitable in itself. As $\hat{\mu}\left(1,\frac{l^0}{1-p}+\frac{p}{1-p}\right)\leq 0$, (see case #1) there is some λ^* such that second period work after failure becomes too costly. There are two possible cases:

  (a) If $p_0c_1<c_0$ then $\lambda=\frac{p}{p-p_0}$ is optimal. The agent is paid $w=\frac{c_0-p_0c_1}{p-p_0}$ for success in the first period and works in the second period iff he succeeds in the first.

  (b) If $p_0c_1\geq c_0$ then λ^* is optimal. The agent is not paid for success in the first period. The agent is paid $\frac{c_1}{p-p_0}$ for success in the second period. The probability of work after failure α_f is determined by the IC

$$\begin{aligned}(p-p_0)(1-\alpha_f)\frac{p_0c_1}{p-p_0} &= c_0\\ \alpha_f &= 1-\frac{c_0}{p_0c_1}\end{aligned}$$

3.3.2. *Two Periods With Private Histories.*

The second period value is $\mu(1,l,r)$ given in equation (3.2). It is piecewise linear with one kink (when $\mu(1,l,r)$ becomes zero) and is decreasing in l and r. The first period problem is given by problem (3.3), with the first order condition (3.6). As in the common information case, if both continuation values are non-zero – the agent always works in the second period, the partial derivatives in the first order condition cancel each other. If both continuation values are zero, both partial derivatives are also zero. Therefore, the only possibility in which the wage constraint does not bind is that the value of work after failure is negative while the value of work after success is positive. In this case the first order condition is

$$p_0c_1-p_0\delta_1-c_0+w(p-p_0)\geq 0$$

However, by definition $\delta_1=c_1-c_0$ and so

$$p_0c_1-p_0\delta_1-c_0=p_0c_1-p_0c_1+p_0c_0-c_0=-(1-p_0)c_0<0$$

Therefore, the wage constraint must bind. This is an important difference between the common and private information cases. Working after success is always profitable if it increases surplus (subject to the past state variables). Thus, the optimal contract always sets $\lambda=\frac{p}{p-p_0}$. Letting l^0,r^0 denote the starting state, the possible solutions are

(1) If $\mu\left(1,-1,r^0+\frac{p_0}{p-p_0}\left(1+l^0\right)\right)\leq 0$, the optimal contract is the optimal single period contract for the first period. This is the case that second period *total value* is lower than the added first period shirking incentive

$$v(p-p_0)-c_1\leq\delta_1\cdot\left(r^0+\frac{p_0}{p-p_0}\left(1+l^0\right)\right)$$

(2) If $\mu\left(1,l^0+\frac{p}{1-p}\left(1+l^0\right),r^0+\frac{1-p_0}{1-p}\frac{p}{p-p_0}\left(1+l^0\right)\right)\geq 0$ the optimal contract is fixed from the starting history to the second period. This is the case that second period *profit* is higher than the added first period shirking incentive from committing to always work in the second period.

(3) If $\mu\left(1,-1,r^0+\frac{p_0}{p-p_0}\left(1+l^0\right)\right)\geq 0\geq\mu\left(1,l^0+\frac{p}{1-p}\left(1+l^0\right),r^0+\frac{1-p_0}{1-p}\frac{p}{p-p_0}\left(1+l^0\right)\right)$ the agent works in the second period only if he succeeds in the first period. The agent is paid $\frac{c_0(1-p_0)}{p-p_0}$ for success in the first period and $\frac{c_1}{p-p_0}$ for success in the second period.

Comparing the two solutions, there are several cases in which the optimal contract is more efficient for the public case. This is not surprising - more frictions should reduce efficiency.

However, the opposite may also be true – the potential for private information increases surplus. Specifically, let $l^0 = r^0 = 0$, so the first period may be the first period of the contract and suppose that $p_0 c_1 \geq c_0$. Under common information, the second period work after success is only used to decrease first period wages. As a result, the work decision is randomized just enough for the agent to be indifferent between working and shirking for no wages in the first period. Under private information, the additional information rents force the principal to *always* pay the agent in the first period to prevent shirking. As increasing the probability of work after success decreases first period wages, it is never profitable for the principal to randomize the work decision after payment. Therefore,

Lemma 14. *In any two period setting, the optimal contract is more efficient for private case iff* $p_0 c_1 > c_0$

$$v\left(p - p_0\right) \quad \in \left[\tfrac{pc_1 - p_0 c_0}{p - p_0}, \tfrac{pc_1}{p - p_0}\right]$$

Proof. By construction, $l^0 = r^0 = 0$. In the private case, the agent works in the second period after success whenever

$$v\left(p - p_0\right) - c_1 - \frac{p_0}{p - p_0}\delta_1 \geq 0$$

Replacing $\delta_1 = c_1 - c_0$ and collecting terms the condition is

$$v\left(p - p_0\right) \geq \frac{pc_1 - p_0 c_0}{p - p_0}$$

In the common knowledge case, if the second period work is not statically profitable and $p_0 c_1 \leq c_0$, case 3b applies and the work after success is with probability less than one. This is equivalent to

$$v\left(p - p_0\right) \leq \frac{pc_1}{p - p_0} \ .$$

Thus, under the condition of the lemma, work after success is more likely in the private history setting.

By observation of the remaining conditions, in all other cases the public history work plan implies more work than the private history work plan. □

3.4. **Discounting.**

Suppose the principal and agent discount the future by a factor $\beta \in (0, 1)$. Discounting requires some technical adjustments – accounting for the right value of future utility and shirking gains, and allows the principal to punish the agent by delaying work. I first consider the required technical adjustments. Suppose the contract is still limited to stopping contracts – the principal is not allowed to "pause" work. Then we can adjust the definition of q_h from (2.1) to

$$\begin{aligned} q_{\langle h,s\rangle} &= \beta \cdot q_h \cdot p \cdot \alpha_{\langle h,s\rangle} \quad ; \\ q_{\langle h,f\rangle} &= \beta \cdot q_h \cdot (1-p) \cdot \alpha_{\langle h,f\rangle}. \end{aligned} \tag{3.13}$$

The discount factor has two effects on the remaining analysis.

First, in the primal probability constraints, q_h is multiplied by β. The technical implication for the dual is simple: in the objective for $V\left(n\right)$, the continuation values $V\left(n+1,\cdot\right)$

are multiplied by the discount factor. As $V(n,l,r)$ is concave, it is clear that the optimal contract never delays work (asks the agent not to work for a period and then work in the next period). As both the agent and principal are risk neutral and have the same discount factor, this is equivalent to increasing the probability of termination in a period. As the latter is never optimal without discounting, so is the former.

Corollary 6. *If the principal and the agent have a common discount factor $\beta \in (0,1)$, the optimal contract is still a finite stopping contract.*

Second, the derivation of the wage contract for a fixed plan in lemma 3 used the relevant part of the primal problem assuming $q_h = 1$. This must be updated to account for the discounting as $q_h = \beta^{n_h}$. The resulting problem is

$$\max_{w_n} \sum_{n=n_0}^{N(r)} -\beta^{n-n_0} p w_n$$

$$s.t.$$

$$(p - p_0)\, w_{N(r)} \geq c_{N(r)}$$

$$\forall n_1 \in \{n_0, ..., N(r) - 1\} \quad (p - p_0)\, n \geq c_n + \sum_{n=n_1+1}^{N(r)} \beta^{n-n_1} (c_n - c_{n-1})$$

It is immediate that all constraints still bind. Standard calculations obtain the wage for a fixed contract with discount factor β.

Lemma 15. *If the principal and agent have a common discount factor $\beta \in (0,1)$, then the optimal contract after payment is fixed through $N(r)$, where $N(r)$ is determined as in lemma 12 and the wage is given by:*

$$\begin{aligned} w_{N(r)} &= \frac{c_{N(r)}}{p - p_0} \\ w_n &= \beta w_{n+1} + (1-\beta) \frac{c_n}{p - p_0} \end{aligned}$$

4. Sufficiency of Local Deviations

This section establishes that it is sufficient to consider only local deviations. Intuitively, the construction of the dual for the FDIC problem can be thought of as considering, in each public history, which private history requires the strongest incentives, keeping the state variables fixed. Local deviations are sufficient if the strongest incentives to shirk are when the agent never shirked in the past. Intuitively, if the agent did shirk in the past, his costs this period are lower (c_n is increasing) and the effect of his shirking on future costs is lower (δ_n is increasing). Therefore, if the agent did shirk in the past lower incentives are sufficient. This section formalizes this intuition. I first identify the relation between the dual variables and the sufficiency of local deviations. Then, a "summation by parts" exercise allows qualitative comparison and makes the sufficiency argument transparent.

4.1. The Dual Variables and Sufficiency of Local Deviations.

Recall that the dual problem assigns a non-negative variable (the shadow price or multiplier) to each constraint, and a constraint to each of the original variables. Let μ^h be the dual variable associated with the probability constraint in problem 2.10 in which q_h appears with coefficient one. Let λ_d^h be the dual variable associated with the FDIC for h, d.

I first establish the formal relation between the λ variables and the sufficiency of local deviations. For this, let D-FDP denote the dual of problem 2.10. Observe that the dual of problem 2.10 subject only to LDIC is equivalent to D-FDP with the additional constraint

$$\forall h, \forall d > 0: \ \lambda_d^h = 0 \quad . \tag{4.1}$$

Lemma 16. *If every solution to D-FDP satisfies also the constraint 4.1, then any optimal solution subject to LDIC is an optimal contract in the original problem.*

Proof. If every solution to D-FDP satisfies also the constraint 4.1, then every optimal solution to D-LDP is optimal to D-FDP. By the Duality theorem of Linear Programming it follows that any optimal contract subject to LDIC is feasible subject to FDIC. Thus, the condition of Corollary 1 holds. □

4.2. **From λ to Λ – Private Information and Summation By Parts.**

Recall that the different λ in the same period reflect different possible *private histories*. Equivalently, one may think of each private history as an agent *type*. The analysis tries to determine which agent type requires the strongest incentives – for which type the incentive constraint binds first. It is known from adverse-selection models that integration by parts over the agent's types is useful in answering such questions. The same will apply here. To see why summation by parts is required, consider the two period problem given in section 3.1, but add the two FDIC for the second period histories s and f, with $d = 1$. These are simply

$$\forall y \in \{s, f\}: \qquad -(p - p_0)\, w_y + q_y c_0 \leq 0$$

Letting λ_d denote the multiplier on the FDIC for $d \in \{0, 1\}$, the resulting second period dual problem is

(4.2)

$$\begin{aligned} \mu(n = 1, l, r) = \max \quad & \left[0, \min_{(\lambda_0, \lambda_1) \geq 0} v(p - p_0) + l c_1 - \delta_1 r - \lambda_0 c_1 - \lambda_1 c_0\right] \\ s.t. \quad & \lambda_0 + \lambda_1 \leq \tfrac{p}{p - p_0}(1 + l) \end{aligned}$$

As $c_1 > c_0$, it is immidiate that $\lambda_1 = 0$. However, there are two drawbacks to this formulation. First, the feasible space of $\lambda \times l$ is *not* a sub-lattice.[16] This will create difficulties when conducting monotone comparative statics analysis. Second, λ_0 and λ_1 have the same sign in the objective. This would complicate determining that $\lambda_0 > 0$ while $\lambda_1 = 0$: if the problem is super-modular in λ_0, it is super-modular in λ_1.

These problems are averted by redefining the period control variables and the resulting state variables using partial sums. Let Λ^h be a vector of length $n_h + 1$ such that Λ_m^h for $m = \{0, ..., n_h\}$ is the sum of the last $n_h - m$ elements of λ^h. That is:

$$\Lambda_m^h \equiv \sum_{d=m}^{n} \lambda_d^h \, . \tag{4.3}$$

Note that in the local-deviations problem both λ and Λ have just one element ($d = 0$) and are equal to each other. For the FDIC problem, the following additional constraints

[16]That is, it may very well be that

$$\lambda_0 + \lambda_1 \leq \frac{p}{p - p_0}(1 + l)$$

and

$$\hat{\lambda}_1 + \hat{\lambda}_1 \leq \frac{p}{p - p_0}\left(1 + \hat{l}\right) \quad ,$$

but

$$\max\left[\hat{\lambda}_0, \lambda_0\right] + \max\left[\hat{\lambda}_1, \lambda_1\right] > \frac{p}{p - p_0}\left(1 + \max\left[\hat{l}, l\right]\right) \quad .$$

verify that $\lambda_d^h \geq 0$:

$$\begin{aligned} \Lambda_m^h - \Lambda_{m+1}^h &\geq 0 \\ \Lambda_m^h &\geq 0\,. \end{aligned} \tag{4.4}$$

The condition (4.1) is equivalent to the condition

$$\forall h,\ \Lambda_1^h = 0 \quad . \tag{4.5}$$

Corollary 7. *If every solution to D-FDP satisfies also the constraint 4.5, then any optimal contract subject to LDIC is optimal in the original problem.*

To see the effect on the recursive formulation, in the two period setting, the second period problem is now

(4.6)

$$\begin{aligned} \mu(n=1,l,r) = \max \quad & \left[0, \min_{(\Lambda_0,\Lambda_1)\geq 0} v(p-p_0) + lc_1 - \delta_1 r - \Lambda_0 c_1 + \Lambda_1 (c_1 - c_0)\right] \\ s.t. \quad & \Lambda_0 \leq \tfrac{p}{p-p_0}(1+l) \\ & \Lambda_0 - \Lambda_1 \geq 0 \end{aligned}$$

The sufficiency of local deviations is now simpler than it was in problem 3.2. Λ_0 decreases the objective while Λ_1 increases it, and setting $\Lambda_1 = 0$ does not impose a limit on Λ_0 . Note also that the space of $\Lambda \times l$ that are feasible is a lattice.

4.3. The Multiple Periods FDIC Dual.

To make the difference between the LDIC and FDIC duals explict, denote the FDIC recursive dual by $F(\cdot)$. Deriving a recursive formulation of the multi-period FDIC dual $F(\cdot)$ for a general history h is mostly a technical extension of previous steps. I first extend the LDIC dual to account for all the FDIC, and then apply the summation by parts exercise described above. This section states the resulting problem and outlines the intuition. The technical steps are provided in appendix B.4.

To derive the multiple periods Dual of the FDIC problem, observe that the only difference from the LDIC is the introduction of the new λ_d^h variables. The variables in the primal are identical for the FDIC and LDIC problems. The dual problem for the FDIC has exactly the same constraints as the LDIC, with the additional variables λ_d^h. For every history h, observe that the FDIC for $d > 0$ is exactly the same as of $d = 0$, with only the subscripts on c_n and δ_n changing to $n - d$. Letting λ be a vector of length $n+1$ and l, r be vectors of length n defined using the same law of motion as in the LDIC formulation, the dual q_h constaint is

$$\mu^h \geq p\mu^{\langle h,s\rangle} + (1-p)^{\langle h,f\rangle} + v(p-p_0) - \sum_{d=0}^{n+1} \lambda_d c_{n-d} + \sum_{d=0}^{n} (l_d c_{n-d} - r_d \delta_{n-d}) \ .$$

Now apply the summation by parts exercise from above on λ as well as l and r. That is, using Λ as defined above we let

$$L_m \equiv \sum_{d=m}^{n} l_d \quad ; \quad \text{and } R_m = \sum_{d=m}^{n} r_d \ .$$

As summation by parts is perserved under summation, it may be verified that the laws of motion of L and R w.r.t. Λ are the same as those of l and r w.r.t. λ. The constraint is rewritten as

$$\begin{aligned}\mu^h &\geq p\mu^{\langle h,s\rangle}+(1-p)\,\mu^{\langle h,f\rangle}+v\,(p-p_0)-\Lambda_0 c_n+\sum_{m=1}^{n+1}\Lambda_m\,(c_{n-m+1}-c_{n-m})\\ &\quad +L_0c_n-R_0\delta_n-\sum_{m=1}^{n}\left(L_m\,(c_{n-m+1}-c_{n-m})-R_m\,(\delta_{n-m+1}-\delta_{n-m})\right)\ .\end{aligned}$$

Therefore, define the FDIC period return, $f\,(n,L,R,\Lambda)$:

(4.7)

$$\begin{aligned}f\,(n,L,R,\Lambda) &\equiv v\,(p-p_0)-c_n\,(\Lambda_0-L_0)-\delta_n R_0\\ &\quad+\sum_{m=1}^{n_h}\left[\delta_{n-m}\,(\Lambda_m-L_m)+(\delta_{n-m+1}-\delta_{n-m})\,R_m\right]\quad .\end{aligned}$$

And the FDIC dual is

$$\begin{aligned}F\,(n,L,R) &= \max\left[0,\min_{\Lambda\geq 0}F\,(n,L,R,\Lambda)\right] \qquad (4.8)\\ &s.t.\\ F\,(n,L,R,\Lambda) &= pF\left(n+1,L-\frac{p-p_0}{p}\Lambda,R+\theta\frac{p_0}{p}\Lambda\right)\\ &\quad+(1-p)\,F\left(n+1,L+\frac{p-p_0}{1-p}\Lambda,R+\theta\frac{1-p_0}{1-p}\Lambda\right)\\ &\quad+f\,(n,L,R,\Lambda)\\ &s.t.\\ &\Lambda_0\leq\frac{p}{p-p_0}\,(1+L_0)\\ &\Lambda_m-\Lambda_{m+1}\geq 0\end{aligned}$$

Lemma 17. *For every h, $\mu^h=F\left(n_h,L^h,R^h\right)$. Specifically,$F\,(0,0,0)$ and the corresponding optimal Λ's define a solution to the dual of problem (2.10).*

Proof. See appendix B.4. □

4.4. **Sufficiency Of Local Deviations.**

This section proves that in the solution for the problem $F\,(0,0,0)$, $\Lambda_m=0$ for all $m>0$. Thus, corollary 7 applies and local deviations are sufficient. Again note the convexity result:

Lemma 18. *$F\,(n,L,R)$ is convex in (L,R). $F\,(n,L,R,\Lambda)$ is convex in Λ for every L,R.*

Proof. The proof is identical to the proof of lemma 5 □

Next, note the following, which may be verified directly and is also an application of part (d) of example 2.6.2 in Topkis (1998).

Lemma 19. *For any $x,x^{'}\in R^n$, let $x\wedge x^{'}$ denote the meet (pairwise minimum) of x and $x^{'}$and $\vee$ denote the join (pairwise maximum). For every two pairs L,Λ and $L^{'},\Lambda^{'}$ such that Λ is feasible for L and $\Lambda^{'}$ is feasible for $L^{'}$, it must be that $\Lambda\wedge\Lambda^{'}$ is feasible for $L\wedge L^{'}$ and $\Lambda\vee\Lambda^{'}$ is feasible for $L\vee L^{'}$.*

Proof. See part (d) of example 2.6.2 in Topkis (1998). □

Suppose first that $\theta = 0$. Intuitively, one can imagine that while shirking affects future costs, the agent only realizes this "after the fact" and not while considering his actions. The next lemma proves that in this problem, the optimal LDIC contract is FDIC. This is not surprising – the strongest incentives are required to prevent shirking with the highest cost.[17]

Lemma 20. *If* $\theta = 0$ *then* $\Lambda_m = 0$ *for all* $m > 0$.

Proof. As $\theta = 0$, in all periods $R = 0$. As the proof will apply monotone comparative static results, it will be easier to consider $G(n, L) \equiv -F(n, L, 0)$, which is defined by:

$$\begin{aligned} G(n, L) &= \min\left[0, \max_{\Lambda} G(n, L, \Lambda)\right] \\ s.t. & \\ G(n, L, \Lambda) &= pG\left(n+1, L - \frac{p-p_0}{p}\Lambda\right) \\ &\quad + (1-p)\, G\left(n+1, L + \frac{p-p_0}{1-p}\Lambda\right) \\ &\quad + g(n, L, \Lambda) \end{aligned}$$

$$\Lambda_0 \leq \frac{p}{p-p_0}(1 + L_0)$$
$$\Lambda_m - \Lambda_{m+1} \geq 0$$

With

$$g(n, L, \Lambda) = -v(p - p_0) + c_n(\Lambda_0 - L_0) - \sum_{m=1}^{n_h} \delta_{n-m}(\Lambda_m - L_m)$$

As $F(n, L, R)$ is convex in (L, R) and $F(n, L, R, \Lambda)$ is convex in Λ for every L, R, $G(n, L)$ is concave in L and $G(n, L, \Lambda)$ is concave in Λ for any L.

(1) For any $m > 0$, $G(n, L)$ is super-modular in $(-L_0, L_m)$:
   (a) L_0 and L_m have opposite signs in $g(n, L, \Lambda)$ and the feasible set is a sub-lattice. Thus, by Theorem 2.7.6 in Topkis (1998) $g(n, L, \Lambda)$ is super-modular in $(-L_0, L_m)$ (corollary 2.7.3 in Topkis (1998) is also sufficient).
   (b) By backward induction, $G(n, L)$ is the max of a positive weights sum of super-modular functions in $(-L_0, L_m)$: $G\left(n+1, L - \frac{p-p_0}{p}\Lambda\right)$, $G\left(n+1, L + \frac{p-p_0}{1-p}\Lambda\right)$ and $g(n, L, \Lambda)$ and is thus super-modular in $(-L_0, L_m)$ (see corollary 2.7.3 in Topkis (1998)).
(2) For any m, let $G_{L_m^+}(n+1, L)$ and $G_{L_m^-}(n+1, L)$ denote the directional partial derivatives. By concavity, both exist everywhere, are equal almost everywhere and
$$G_{L_m^+}(n+1, L) \leq G_{L_m^-}(n+1, L)$$
(3) Suppose $\Lambda = 0$. If the wage constraint binds, $\Lambda = 0$ is the only solution and the proof is done. So, suppose the wage constraint does not bind.

[17]However, this is not trivial either. See the second example in appendix A.

(a) For any $m>0$, the effect of a marginal increase for Λ_m at $\Lambda=0$ on $G(n,L,\Lambda)$is

(4.9)

$$\begin{aligned} G_{\Lambda_{m>0}}(n,L,\Lambda=0) = & \quad (p-p_0)\left(-G_{L_m^-}(n+1,L)+G_{L_m^+}(n+1,L)\right)-\delta_{n-m} \\ \leq & \quad \delta_{n-m}<0 \end{aligned}$$

(b) For $m=0$, the effect of a marginal increase for Λ_0 is either positive or negative, and is positive whenever G is smooth at L_0:

$$G_{\Lambda_0}(n,L,\Lambda=0)=(p-p_0)\left(-G_{L_0^-}(n+1,L)+G_{L_0^+}(n+1,L)\right)+c_n$$

(4) By convexity, if $G_{\Lambda_0}(n,L,\Lambda=0)\leq 0$, the optimal solution is $\Lambda=0$. So it remains to prove that if $G_{\Lambda_0}(n,L,\Lambda=0)>0$, $\Lambda_{m>0}=0$

(5) If $\Lambda_0>0$ and $\Lambda_m=0$ for all $m>0$, then the marginal effect of an increase for Λ_m at Λ is negative. Therefore, by $G(n,L,\Lambda)$ concave in Λ, the solution sets $\Lambda_m=0$ for all $m>0$:

(a) By the law of motion for L, as $\Lambda\geq 0$, L_0 is lower after success (the first derivative) than after failure (the second derivative).

(b) By $G(n,L)$ sub-modular in $(-L_0,L_m)$ as L_0 decreases the marginal effect of L_m increases: $G_{L_m}\left(n+1,L-\frac{p-p_0}{p}\Lambda\right)\geq G_{L_m^-}(n+1,L)\geq G_{L_m^+}(n+1,L)\geq G_{L_m}\left(n+1,L+\frac{p-p_0}{1-p}\Lambda\right)$. Therefore

$$-G_{L_m}\left(n+1,L-\frac{p-p_0}{p}\Lambda\right)+G_{L_m}\left(n+1,L+\frac{p-p_0}{1-p}\Lambda\right)\leq -G_{L_m^-}(n+1,L)+G_{L_m^+}(n+1,L)$$

(c) For any $m>0$, the effect of a marginal increase for Λ_m at Λ on $G(n,L,\Lambda)$is

$$\begin{aligned} G_{\Lambda_{m>0}}(n,L,\Lambda) &= (p-p_0)\left(-G_{L_m}\left(n+1,L-\frac{p-p_0}{p}\Lambda\right)+G_{L_m}\left(n+1,L+\frac{p-p_0}{1-p}\Lambda\right)\right)-\delta_{n-m} \\ &\leq (p-p_0)\left(-G_{L_m^-}(n+1,L)+G_{L_m^+}(n+1,L)\right)-\delta_{n-m}<0 \end{aligned}$$

□

Consider now the effect of information asymmetry by increasing θ. As the agent considers the effect of shirking on his future costs, the value of the first shirk relative to any following shirks increases. The marginal effect of the first shirk is an extra δ_h in all future periods while the marginal effect of the second shirk is only an extra $\delta_{h-1}<\delta_h$. Thus it is always the case that the private history with no past shirking is the one that requires strongest incentives, which implies the sufficiency of local deviations. Formally, the problem is monotonic increasing in R_0 and monotonic decreasing in $R_{m\neq 0}$. As Λ_m increases r_m, the marginal effect of $\Lambda_{m\neq 0}$ decreases in θ. As this value is negative at $\theta=0$, it remains negative at $\theta=1$.

Lemma 21. *For any $\theta\geq 0$, $\Lambda_m=0$ for all $m>0$ and $\Lambda_0>0$ whenever feasible.*

Proof. The claim is true for $\theta=0$ by the previous lemma.

(1) Whenever $\theta>0$, $F(\cdot)$ is decreasing in R_0 and increasing in R_m for $m>0$.

(a) Neither R nor θ affect any constraint.

(b) By the period return function $f(\cdot)$ (equation 4.7) the period effect is negative for R_0 as $\delta_n > 0$ and positive for $R_{m\neq 0}$ as $\delta_n \geq \delta_{n-1}$ by $c(n)$ convex.

(c) The same proof as for the LDIC case for r now applies (see appendix B.8).

(2) For any $m > 0$, $\Lambda_m = 0$ whenever $\theta > 0$:

(a) This is true for $\theta = 0$ from the previous lemma.

(b) In any last period, the marginal effects of Λ are unaffected by θ.

(c) In any period before last, for any Λ such that $\Lambda_0 \geq 0$ and $\Lambda_{m>0} = 0$, the marginal effect of increasing Λ_m is negative:

(i) If $\theta = 0$ the previous lemma applies.

(ii) If $\theta > 0$ the marginal effect of increasing Λ_m for $m > 0$ also includes the effect of increasing $R_{m>0}$ in both continuations. As $F(n+1, L, R)$ decreases in $R_{m>0}$, this effect is negative.

□

Concluding:

Theorem 1. *Any optimal contract subject to LDIC is an optimal contract.*

Proof. From lemma 21, at $\theta = 1$, $\Lambda_m = 0$ for all $m > 0$ and $\Lambda_0 > 0$ whenever feasible. Corollary 7 now implies the result. Specifically, $\Lambda_m = 0$ for all $m > 0$ implies that any solution to the dual considering only local deviations is feasible in when considering all final deviations and thus IC. □

5. Conclusion

This paper evaluated the optimal contract for an agent exerting unobservable effort that affects future costs. The optimal contract responded to two agency frictions – limited liability and asymmetric information.

The contract handled the limited liability friction as usual in dynamic moral hazard settings. The agent first works for credit in the firm. When the credit is too negative the agent is fired. When the credit is larger than the forward looking value of the firm, the agent is given the firm in return for his credit.

The asymmetric information friction forced the contract to artificially reduce the value of the firm after each period, depending on the period's outcome. This guaranteed that the agent never gains by creating a discrepancy between his private information and the public information.

The optimal contract explains features of real world contracts that puzzled economic observers. The variance in the expected total effort is larger with private cost information than without. Such large variation in ex-post incentives and effort across agents is inefficient and led several authors (see e.g. Oyer (1998); Larkin (2007); Misra and Nair (2009)) to suggest that there is significant room for improvement in either the design of real world incentives or models of the moral hazard setting. The model shows that this variance allows the firm to provide sufficient incentives for effort when it is relatively cheap and to provide high powered incentives when those are required *without fear that agents misrepresent their effort* (delay "easy sales" to the end of the period). The optimal contract must balance between efficiency (having the agents work longer) and profitability. While a high linear commission would guarantee all agents make the efficient level of effort, the firm's profits would all be provided as rents. Consistent with the model, in the firm documented by Larkin (2007) the top end of the reward scale provides the salesperson a 25% commission on *revenues,* a figure very close to the industry's accounting profit margins.

The analysis used duality based arguments to design a dynamic program that separates the two frictions. There is reason to hope that a duality based approach to dynamic moral hazard should prove fruitful in many additional settings as well.

The model abstracted from several important aspects of dynamic agency settings. I briefly review the main abstractions and consider their implications.

It was assumed there is no information aggregation problem, as in Holmstrom and Milgrom (1987) and DeMarzo and Sannikov (2008). The aggregation problem occurs if the principal only learns of the aggregate outcome at some final period $\overline{N} \leq N^{FB}$. Critically, the actual order of successes becomes private information. The aggregation problem in the current setting is thus equivalent to constraining the wage to depend only on the *number* of successes. I show in appendix C that the problem may still be written as a linear problem and form the resulting dynamic dual problem. Interestingly, preliminary analysis suggests that the optimal contract in such settings is exactly the convex reward scheme.

The possible action and outcome space per period was constructed to be the simplest possible – binary effort and binary outcomes. It may be possible to extend the model in these dimensions without breaking any of the assumptions, as in Rogerson (1985). However, the resulting model is very involved notationally. A more promising approach may be to use duality theory of convex programming (see e.g. Rockafellar (1997)) to apply similar methodology using the more convenient first order approach.

It was assumed the principal could commit to a long term contract. Renegotiation with private histories creates issues on many levels that merit further research.

The structure imposed also implied that the optimal contract is finite. However, all the methodology used in creating the dynamic dual problem can be equally applied to infinite horizon models with discounting. Finally, the agent is assumed to be risk neutral, an assumption that clearly simplifies the analysis and was critical to writing the IC as linear inequalities.

The interaction between private history and current technology has several organizational implications for future research. Perhaps the main one is the increased gains from sequential division of labor. In moral hazard settings in which only the *public* history affects current technology, the space of incentive compatible contracts increases when considering the agency relationship as a long term one. It is always possible to offer the agent the optimal short-term contract each period, and the agent will comply. However, when the *private* history affects current technology, as it does here, contracts that are incentive compatible in the short term may not be incentive compatible in the long term – the agent may have a strict incentive to shirk *only* to improve his future utility. Thus, the set of long term incentive compatible contracts no longer includes the set of short term incentive compatible contracts. If private history has a significant effect, it may be profitable to ex-ante limit the duration of the contract, perhaps shifting production to another agent. Such sequential division of labor – an incentive based hierarchy – can allow the principal to extract the agent's private information without paying the high information rent. I explore these ideas in related work.

Finally, conceptually, the principal in this paper attempts to base incentives on histories – pay more when costs *should* be higher. This is closely related to the “ratcheting” approach, that has been traditionally considered problematic (Berliner (1957) introduced the famous “ratchet effect”). Theoretic studies of ratcheting however tended to assume a specific mechanism and study its shortcomings. In the closest related paper, Weitzman (1980) identifies the effort distortion in an intuitive sub-optimal ratcheted contract for a production setting in which the agent privately observes a production shock each period. In contrast, the current paper identifies what can be interpreted as the optimal ratcheting mechanism and shows that it improves on the optimal no-ratcheting – fixed – alternative.

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Appendix A. Example – LDIC does not imply IC

A.1. **Full Model.** Suppose $c_n = n$, $p = \frac{1}{2}$ and $p_0 = 0$ and consider the following contract. The agent stops if he succeeds in any of the first two periods. If the agent fails in the first two periods, he is asked to work for eight more periods regardless of new outcomes and is paid 20 for each success. The agent is paid 48 for success in any of the first two periods.

To verify that the contract is LDIC, note that in the last eight periods, the contract is fixed and the agent is paid as if his cost is 10. Lemma 3 establishes the optimality of this wage plan. Letting $U^n(d)$ denote the agent's continuation utility after being asked to work in period n having shirked d times in the past:

$$U^3(d) = \sum_{n=3}^{10} \frac{1}{2} \cdot 20 - \sum_{n=3}^{10} (n-d) = 80 - 52 + 8 \cdot d = 28 + 8d$$

Now consider the agent's problem if he is asked to work in the second period.

(1) If the agent did not shirk in the first period ($d = 0$), he works whenever

$$\begin{aligned} \frac{1}{2} \cdot 48 + \frac{1}{2} \cdot U^3(0) - 2 &\geq U^3(1) \\ 24 + \frac{28}{2} - 2 &\geq 28 + 8 \end{aligned}$$

Which holds as an equality

(2) If the agent did shirk in the first period ($d = 1$), he works whenever

$$\begin{aligned} \frac{1}{2} \cdot 48 + \frac{1}{2} \cdot U^3(1) - 1 &\geq U^3(2) \\ 24 + \frac{28}{2} + \frac{8}{2} - 1 &\geq 28 + 8 + 8 \end{aligned}$$

As the agent works more after failing than after succeeding, the previous shirk is worth more after failing than after succeeding. As a result, the left hand side increased less than the right hand side and the IC is violated.

It is easy to verify that the first period IC just binds:

$$\begin{aligned} U^2(0) &= 24 + \frac{28}{2} - 2 = 36 \\ U^2(1) &= 24 + \frac{28}{2} + \frac{8}{2} - 1 = 41 \end{aligned}$$

$$\frac{1}{2} \cdot 48 + \frac{1}{2} \cdot U^2(0) - 1 = 24 + 18 - 1 = U^2(1).$$

A.2. **"Naive" Agent.** Suppose now that $\theta = 0$ – the agent is "naive" about the effect of his shirking. The next example shows that LDIC still does not imply FDIC. For this, consider the contract that has the same work plan as in the previous sub-section but pays the agent $2n$ in period n. The LDIC in this setting may be written as

$$\frac{1}{2}\cdot 2n+\frac{1}{2}U^{c}\left(\left\langle h,s\right\rangle ,d\right)+\frac{1}{2}U^{c}\left(\left\langle h,f\right\rangle ,d\right)-\left(n-d\right)\geq U^{c}\left(\left\langle h,f\right\rangle ,d\right)$$

It is immediate that $U^{c}\left(h,0\right)=0$ for all h and so the contract is LDIC. The agent expects zero rents in all future periods and so is willing to work for zero rents today.

To see why this contract is not FDIC, first observe that the agent's continuation utility after being asked to work in the third period having shirked d times in the past is $U^{3}\left(d\right)=8d$. Now suppose the agent shirked in the first period and consider the second period FDIC for $d=1$. If he succeeds, the agent is paid 4. If he fails, he obtains a future utility of 8 – double his pay for success. Thus, the IC is violated:

$$\frac{1}{2}\cdot 4+\frac{1}{2}\cdot 0+\frac{1}{2}\cdot 8-1=5<8$$

Appendix B. Detailed Proofs and Derivations

B.1. **Proof that FDIC implies IC.**

Lemma 22. *If a contract is FDIC it is IC*

Proof. Suppose the contract q,w is not IC. Then it violates FDIC:

1. As the set of possible work plans for the agent is finite and the agent's expected profit is well defined and bounded for each work plan given q,w, there is a set $\hat{E}\left(q,w\right)$ of most profitable work plans given q,w.
2. Take $\hat{e}\in\hat{E}$, $\hat{e}\neq e^{c}$ be a most profitable deviating work plan.
3. Consider the set of histories $\hat{H}$ in which the agent makes a "final deviation" according to $\hat{e}$. That is, $\hat{h}\in\hat{H}$ if $\hat{e}_{\hat{h}}=0$ and for every $h\succeq\hat{h}$, $h\neq\hat{h}$, either $q_{h}=0$ or $\hat{e}_{h}=1$. Let $\hat{d}$ be the number of past deviations at $\hat{h}$ according to $\hat{e}$. Clearly, if the agent profits from making this final deviation, the FDIC for $\hat{h},\hat{d}$ is violated and the proof is complete.
4. If the agent does not profit from making this final deviation then the effort plan that complies in this last period provides at least the same expected profit to the agent. Thus, the effort plan with $\hat{e}_{\hat{h}}=1$ provides at least the same expected profit for the agent. We can now repeat the process of searching for a profitable final deviation after setting $\hat{e}_{\hat{h}}=1$. As H is a finite set, the process ends either in finding a history in which FDIC is violated or if we change all periods in which $\hat{e}_{h}=0$ to $\hat{e}_{h}=1$ while weakly increasing the agent's expected profit, implying that $\hat{e}$ was not more profitable than e^{c}.

□

B.2. **Longer Derivation of the FDIC.**

As the contract is fixed, I omit $\left\langle q,w\right\rangle$ from the definition of $U^{D}\left(\cdot\right)$ and $U^{C}\left(\cdot\right)$. The FDIC in history h,d requires that the agent's expected utility from following the contract at history h,d is at least his expected utility from making a final deviation in the history. Write this in the form convenient for taking the dual later:

$$U^{D}\left(h,d\right)-U^{c}\left(h,d\right)\leq 0\ . \tag{B.1}$$

If the agent will never be asked to work in history h ($q_{h}=0$) then $U^{c}\left(\cdot\right)=U^{D}\left(\cdot\right)=0$ and (B.1) trivially holds at h for all d. For $q_{h}>0$, it will be convenient to write $U^{c}\left(h,d\right)$

and $U^D(h,d)$ in recursive form. The agent's expected utility from complying with the contract when asked to work in history h,d is his expected payment in the period less the cost, plus the expected continuation utility from complying after success and after failure:

(B.2)

$$U^c(h,d) = \frac{1}{q_h}\left[pw_h - q_h c_{h-d} + q_{\langle h,s\rangle}U^c(\langle h,s\rangle, d) + q_{\langle h,f\rangle}U^c(\langle h,f\rangle, d)\right].$$

To construct $U^D(h,d)$, observe that shirking has three effects. First, it saves the agent the cost c_{h-d} in the current period. Second, it replaces the success probability p with p_0. This changes the term $p\cdot w_h$ to $p_0\cdot w_h$. It also changes the conditional probability of arriving to the public history $\langle h,s\rangle$ from p to p_0 and the probability of moving to the history $\langle h,f\rangle$ from $1-p$ to $1-p_0$. As the term $q_{\langle h,s\rangle}$ (resp. $q_{\langle h,f\rangle}$) assumes the correct probability p (resp. $1-p$), it must be multiplied by $\frac{p_0}{p}$ (resp. $\frac{1-p_0}{1-p}$). Lastly, in the continuation utilities, the total deviations increase by one and so d is replaced with $d+1$:

(B.3)

$$U^D(h,d) = \frac{1}{q_h}\left[p_0 w_h + \frac{p_0}{p}q_{\langle h,s\rangle}U^c(\langle h,s\rangle, d+1) + \frac{1-p_0}{1-p}q_{\langle h,f\rangle}U^c(\langle h,f\rangle, d+1)\right].$$

To facilitate comparison between $U^c(h,d)$ and $U^D(h,d)$, note that for any history, the continuation utility after shirking $d+1$ times in the past equals the continuation utility after shirking d times in the past, plus the utility increase from the last shirk:

$$U^c(\langle h,y\rangle, d+1) = U^c(\langle h,y\rangle, d) + \left(U^c(\langle h,y\rangle, d+1) - U^c(\langle h,y\rangle, d)\right).$$

Thus, we may write

(B.4)

$$\begin{aligned} U^D(h,d) &= \frac{1}{q_h}\left[p_0 w_h + \frac{p_0}{p}q_{\langle h,s\rangle}U^c(\langle h,s\rangle, d) + \frac{1-p_0}{1-p}q_{\langle h,f\rangle}U^c(\langle h,f\rangle, d)\right] \\ &\quad + \frac{1}{q_h}\left[\frac{p_0}{p}q_{\langle h,s\rangle}\left(U^c(\langle h,s\rangle, d+1) - U^c(\langle h,s\rangle, d)\right)\right] \\ &\quad + \frac{1}{q_h}\left[\frac{1-p_0}{1-p}q_{\langle h,f\rangle}\left(U^c(\langle h,f\rangle, d+1) - U^c(\langle h,f\rangle, d)\right)\right] \end{aligned}$$

Placing (B.2) and (B.3) in the FDIC B.1 and multiplying by $q_h > 0$ obtains the linear inequality

$$\begin{aligned} & p_0 w_h + \tfrac{p_0}{p}q_{\langle h,s\rangle}U^c(\langle h,s\rangle, d) + \tfrac{1-p_0}{1-p}q_{\langle h,f\rangle}U^c(\langle h,f\rangle, d) \\ & + \tfrac{p_0}{p}q_{\langle h,s\rangle}\left(U^c(\langle h,s\rangle, d+1) - U^c(\langle h,s\rangle, d)\right) \\ & + \tfrac{1-p_0}{1-p}q_{\langle h,f\rangle}\left(U^c(\langle h,f\rangle, d+1) - U^c(\langle h,f\rangle, d)\right) \\ & - \left(pw_h - q_h c_h + q_{\langle h,s\rangle}U^c(\langle h,s\rangle, d) + q_{\langle h,f\rangle}U^c(\langle h,s\rangle, d)\right) \\ & \leq 0 \end{aligned}.$$

As the first and last rows have the same terms, collect terms to obtain:

$$
\begin{aligned}
&-\left(p-p_0\right) w_h + q_h c_{h-d} + \\
&-\tfrac{p-p_0}{p} q_{\langle h,s\rangle} U^c\left(\langle h,s\rangle, d\right) \\
&+\tfrac{p-p_0}{1-p} q_{\langle h,f\rangle} U^c\left(\langle h,f\rangle, d\right) \\
&+\tfrac{p_0}{p} q_{\langle h,s\rangle}\left(U^c\left(\langle h,s\rangle, d+1\right)-U^c\left(\langle h,s\rangle, d\right)\right) \\
&+\tfrac{1-p_0}{1-p} q_{\langle h,f\rangle}\left(U^c\left(\langle h,f\rangle, d+1\right)-U^c\left(\langle h,f\rangle, d\right)\right) \\
&\leq 0 \quad .
\end{aligned} \tag{B.5}
$$

Which is identical to the FDIC (2.6)

B.3. **Duality - Main Theorems.** The classic reference is Dantzig (1963). The results are given in current textbooks on static optimization (see e.g. Vohra (2005)). Any linear problem may be written as

$$\max_{x\geq 0} c\cdot x \qquad s.t. \quad Ax \leq b \quad . \tag{B.6}$$

With c a vector of coefficients and A a matrix that holds in each row the coefficients on a constraint. The dual of the problem is

$$\min_{y\geq 0} y\cdot b \qquad s.t. \quad yA \geq c$$

The main results of interest are:

(1) The Duality Theorem: If x^* and y^* are optimal, $y^*\cdot b = c\cdot x^*$ whenever both exist and are finite; and
(2) Complementary Slackness: If $y_i^* = 0$ then the constraint associated with the i-th row in A never binds when solving the primal.

The linearity of the objective then implies:

(1) If $y_i^* = 0$ then the constraint associated with the i-th row in A can be ignored when solving the primal.

While this last result may not have a formal name, it is a combination of the Complementary Slackness result and the Fundamental Theorem of Linear Programming. See e.g. the discussion in Vohra (2005) preceding theorem 4.10 (Complementary Slackness).

B.4. **The Dual LDIC Dynamic Program - Detailed Derivation.**

This section proves that the dual of the optimal contract problem subject to LDIC is given by the LDIC dynamic dual (problem 3.3).

B.4.1. *Prelimiary Definitions.*

For convinience, the relevant definitions that will be used in the proof are reproduced here. The optimal contract problem subject only to LDIC is given in problem 2.10, with the FDIC replaced by the relaxed LDIC:

$$\begin{array}{rcc} V^{LD}(\theta) = \max_{q\geq 0, w\geq 0} & \sum_{h\in H}\left[q_h\left(p-p_0\right)v - w_h p\right] & \\ s.t. & & \\ & q_{h_0}\leq 1 & \\ \forall h & q_{\langle h,s\rangle} - q_h p \leq 0 & \\ \forall h & q_{\langle h,f\rangle} - q_h\left(1-p\right)\leq 0 & \\ \forall h & LDIC & . \end{array}$$

The LDIC for history h is given by setting $d = 0$ in 2.9:

$$\begin{array}{c} -\left(p-p_0\right)w_h + q_h c_h \\ -\frac{p-p_0}{p}\sum_{\tilde h\succeq\langle h,s\rangle} p w_{\tilde h} + \frac{p-p_0}{p}\sum_{\tilde h\succeq\langle h,s\rangle} q_{\tilde h}\cdot c_{\tilde h} \\ +\frac{p-p_0}{1-p}\sum_{\tilde h\succeq\langle h,f\rangle} p w_{\tilde h} - \frac{p-p_0}{1-p}\sum_{\tilde h\succeq\langle h,f\rangle} q_{\tilde h} c_{\tilde h} \\ +\theta\cdot\frac{p_0}{p}\sum_{\tilde h\succeq\langle h,s\rangle} q_{\tilde h}\cdot\delta_{\tilde h} \\ +\theta\cdot\frac{1-p_0}{1-p}\sum_{\tilde h\succeq\langle h,f\rangle} q_{\tilde h}\cdot\delta_{\tilde h} \qquad \leq 0 \quad . \end{array} \tag{B.7}$$

The dynamic representation 3.3 of the dual for the LDIC problem V^{LD} is

$$\begin{array}{lll} \mu\left(n,l,r\right) & = \max & \left[0, \min_{\lambda\geq 0}\mu\left(n,l,r,\lambda\right)\right] \\ & s.t. & \\ & \mu(n,l,r,\lambda) & = p\mu\left(n+1, l-\frac{p-p_0}{p}\lambda, r+\theta\frac{p_0}{p}\lambda\right) + \left(1-p\right)\mu\left(n+1, l+\frac{p-p_0}{1-p}\lambda, r+\theta\frac{1-p_0}{1-p}\lambda\right) \\ & & +v\left(p-p_0\right) - c_0\lambda + c_0 l - \delta_0 r \\ & & \lambda\leq\frac{p}{p-p_0}\left(1+l\right) \quad . \end{array}$$

The dynamic problem implicitly defines the state variables l and r. It will facilitate the proof to recall the explicit definition of the state variables (equations 3.4 and 3.5):

$$l^h = \left(p-p_0\right)\left(\sum_{\tilde h: h\succeq\left\langle\tilde h,f\right\rangle}\frac{\lambda^{\tilde h}}{1-p}\right) - \left(\sum_{\tilde h: h\succeq\left\langle\tilde h,s\right\rangle}\frac{\lambda^{\tilde h}}{p}\right) .$$

and

$$r^h = \theta\left(\sum_{\tilde h: h\succeq\left\langle\tilde h,s\right\rangle}\frac{p_0}{p}\lambda^{\tilde h}\right) + \theta\left(\sum_{\tilde h: h\succeq\left\langle\tilde h,f\right\rangle}\frac{1-p_0}{1-p}\lambda^{\tilde h}\right) .$$

B.4.2. *Derivation of the w_h constraint.*

Lemma 23. *The dual constraint associated with w_h in the LDIC is:*

$$\forall h: \quad \lambda^h \leq \frac{p}{p-p_0}\left(1+l^h\right) \tag{B.8}$$

Proof. For any h, the right hand side of the w_h constraint is the coefficient on w_h in the objective: $-p$. To construct the left hand side, observe that w_h appears with a coefficient $-\left(p-p_0\right)$ in each LDIC (B.7) for h. The variable w_h also appears with a coefficient $\frac{p-p_0}{p}p = \left(p-p_0\right)$ in all FDIC for $\tilde h$ such that $h\succeq\left\langle\tilde h,s\right\rangle$ and with a coefficient $\frac{p-p_0}{1-p}p$ in all FDIC for $\tilde{h}$ such that $h \succeq \left\langle \tilde{h}, f \right\rangle$. Summing up, the constraint is:

$$\forall h \quad - (p - p_0)\,\lambda^h - (p - p_0)\left(\sum_{\tilde{h}:h \succeq \langle \tilde{h}, s\rangle} \lambda^{\tilde{h}}\right) + (p - p_0)\,\frac{p}{1-p}\left(\sum_{\tilde{h}:h \succeq \langle \tilde{h}, f\rangle} \lambda^{\tilde{h}}\right) \quad \geq \quad -p\;.$$

Collecting terms:

$$\forall h \quad - (p - p_0)\,\lambda^h + p \cdot (p - p_0)\left(\left(\sum_{\tilde{h}:h \succeq \langle \tilde{h}, f\rangle} \frac{\lambda^{\tilde{h}}}{1-p}\right) - \left(\sum_{\tilde{h}:h \succeq \langle \tilde{h}, s\rangle} \frac{\lambda^{\tilde{h}}}{p}\right)\right) \geq -p\;.$$

Observe that the p in the second term is exactly multiplying l^h (given in equation 3.4), and so:

$$- (p - p_0)\,\lambda^h + p l^h \quad \geq \quad -p\;.$$

Dividing by $-(p - p_0)$ obtains B.8 □

B.4.3. *Derivation of the q_h constraint.*

Lemma 24. *The dual constraint associated with q_h in the LDIC is:*

$$\mu^h - (1-p)\,\mu^{\langle h, f\rangle} - p\mu^{\langle h, s\rangle} + c_h\lambda^h - c_h l^h + \delta_h r^h \quad \geq v\,(p - p_0)\;. \tag{B.9}$$

Proof. The variable q_h appears in the objective with coefficient $v\,(p - p_0)$. In the primal constraints, q_h appears in the three probability constraints and in all the LDIC for histories that h follows. The probability constraints generate the continuation value terms:

$$\mu^h - (1-p)\,\mu^{\langle h, f\rangle} - p\mu^{\langle h, s\rangle} \tag{B.10}$$

To simplify the rest of the derivation, I consider each case that q_h appears in an LDIC separately.

(1) In the LDIC for history h, q_h appears with a coefficient c_h . This generates the term

$$c_h\lambda^h \tag{B.11}$$

(2) The variable q_h also appears twice in each of the LDIC for $\tilde{h}$ such that $h \succeq \tilde{h}$, once as part of the continuation utility term (in either the second or third row of B.7) and once as part of the future gains from shirking term (in either the fourth or fifth row of B.7). The continuation utility term will determine the coefficient on l. The shirking gains term will determine the coefficient on r.
   (a) In the continuation utility term in the LDIC for all histories that h follows, the coefficient for $\lambda^{\tilde{h}}$ is the cost at history h multiplied by the same coefficient as the wage: $\frac{p-p_0}{p} \cdot c_h$ if $h \succeq \left\langle \tilde{h}, s \right\rangle$ and $-\frac{p-p_0}{1-p} c_h$ if $h \succeq \left\langle \tilde{h}, f \right\rangle$. Summarizing these terms obtains the sum:

$$c_h\,(p - p_0)\left[\left(\sum_{\tilde{h}:h \succeq \langle \tilde{h}, s\rangle} \frac{\lambda^{\tilde{h}}}{p}\right) - \left(\sum_{\tilde{h}:h \succeq \langle \tilde{h}, f\rangle} \frac{\lambda^{\tilde{h}}}{1-p}\right)\right].$$

   By the definition of l^h, this is simply

$$c_h\left(-l^h\right)\;. \tag{B.12}$$

(b) In the shirking gains term in the LDIC for all histories that h follows, the coefficient for $\lambda^{\tilde{h}}$ is $\theta\frac{p_0}{p}\delta_{h-d}$ if $h \succeq \left\langle \tilde{h}, s\right\rangle$ and $\theta\frac{1-p_0}{1-p}\delta_{h-d}$ if $h \succeq \left\langle \tilde{h}, f\right\rangle$. Again the coefficients depend only on whether h follows a success or failure in $\tilde{h}$. The sum is:

$$\delta_h \cdot \left(\frac{p_0}{p}\left(\sum_{\tilde{h}:h\succeq\left\langle\tilde{h},s\right\rangle}\lambda_{\tilde{h},d}\right) + \frac{1-p_0}{1-p}\left(\sum_{\tilde{h}:h\succeq\left\langle\tilde{h},f\right\rangle}\lambda_{\tilde{h},d}\right)\right) .$$

By the definition of r^h, this is simply

$$\delta_h r^h \ . \tag{B.13}$$

Combining B.10,B.11,B.12 and B.13 obtains the left hand side of B.9. □

B.4.4. *The LDIC Dual.*

Lemma 25. *The LDIC dual is*

$$\min_{(\mu,\lambda)\geq 0} \mu^{\emptyset} \tag{B.14}$$

s.t.

$\forall h:$ $\quad \lambda^h \leq \frac{p}{p-p_0}\left(1+l^h\right)$ *(wage constraint)*

$\forall h:$ $\quad \mu^h \geq v\left(p-p_0\right) + (1-p)\,\mu^{\langle h,f\rangle} + p\mu^{\langle h,s\rangle} - c_h\lambda^h + c_h l^h - \delta_h r^h$ *(q constraint)*

subject to the definitions of l^h and r^h given above.

Proof. The only primal constraint with a non negative right hand side is $q_{\emptyset} \leq 1$. This establishes the objective. The constraints and state variables were established in the previous two lemmas. □

B.4.5. *Recursive LDIC and Principle of Optimality.*

Lemma 26. *For every h, $\mu^h = \mu\left(n_h, l^h, r^h\right)$. Specifically,$\mu\left(0,0,0\right)$ and the corresponding optimal λ's are a solution to the dual of problem (B.14).*

Proof. In steps:

(1) By the definition of $l^h$and r^h (equations 3.4 and 3.5, reproduced at the beginning of this section): If $l^h = l$ $r^h = r$ and $\lambda^h = \lambda$ then $l^{\langle h,s\rangle} = l - \frac{p-p_0}{p}\lambda$, $l^{\langle h,f\rangle} = \frac{p-p_0}{1-p}\lambda$, $r^{\langle h,s\rangle} = \theta\frac{p_0}{p}\lambda$, $r^{\langle h,f\rangle} = \theta\frac{1-p_0}{1-p}\lambda$.

(2) The q constraint in the dual B.14 binds whenever $\mu^h > 0$.
   (a) Suppose not. Let h be the first history in which the q constraint does not bind.
   (b) If $h = \emptyset$ then decrease $\mu^{\emptyset}$. This decreases the objective. Therefore, the constraint binds.
   (c) For any other h , decrease μ^h by ε. As the constraint in the previous period binds, this allows decreasing the previous history's μ by either $\varepsilon{\cdot}p$ or $\varepsilon\left(1-p\right)$. Continuing backwards, this will decrease $\mu^{\emptyset}$. This decreases the objective. Therefore, the constraint binds.

(3) Let $\mu^*\left(n,l,r\right)$ be the optimal solution for the problem 3.3 starting from $\mu\left(0,0,0\right)$. By the principle of optimality, $\mu^*\left(n,l,r\right)$ exists and is unique. Let μ^{h*} be an optimal solution to problem B.14 . It remains to show that for every h, $\mu^*\left(n_h,l^h,r^h\right) = \mu^{h*}$

(4) Suppose that for some h this is not true. As the contract is finite, choose h so that there is no future $\tilde{h} \succeq h$ such that $\mu^*\left(n_{\tilde{h}}, l^{\tilde{h}}, r^{\tilde{h}}\right) \neq \mu^{\tilde{h}*}$. Let $\lambda^*\left(n_h, l^h, r^h\right)$ be the recursive problem's optimal solution at $\left(n_h, l^h, r^h\right)$ and λ^{h*} be given by the optimal solution to problem B.14.
   (a) If $\mu^*\left(n_h, l^h, r^h\right) > \mu^{h*}$ replace $\lambda^*\left(n_h, l^h, r^h\right)$ with λ^{h*}:
      (i) The dual wage constraint is equivalent to the constraint in problem 3.3 and thus λ^{h*} is feasible at $\left(n_h, l^h, r^h\right)$.
      (ii) By construction, $\mu^*\left(n_h, l^h, r^h\right)$ is now exactly μ^{h*}
      (iii) As $\mu^*\left(n_h, l^h, r^h\right) > \mu^{h*}$ and λ^{h*} is feasible at $\left(n_h, l^h, r^h\right)$, it must be that $\mu^*\left(n_h, l^h, r^h\right)$ was not optimal.
   (b) If $\mu^*\left(n_h, l^h, r^h\right) < \mu^{h*}$then replace λ^{h*} with $\lambda^*\left(n_h, l^h, r^h\right)$:
      (i) The dual wage constraint is equivalent to the constraint in problem 3.3 and thus $\lambda^*\left(n_h, l^h, r^h\right)$ does not violate the wage constraint.
      (ii) The change does not affect any previous periods, except through μ^h .
      (iii) For the future periods, by constructiont $\mu^{\langle h,y\rangle^*} = \mu^*\left(n_{\langle h,y\rangle}, l^{\langle h,y\rangle}, r^{\langle h,y\rangle}\right)$
      (iv) Thus, μ^{h*} is now exactly $\mu^*\left(n_h, l^h, r^h\right)$
      (v) As $\mu^*\left(n_h, l^h, r^h\right) < \mu^{h*}$ the decrease in μ^{h*} allows a decrease in the previous period's μ and so on by backward induction up to $\mu^{\emptyset}$. Therefore, setting $\mu^{h*} > \mu^*\left(n_h, l^h, r^h\right)$ cannot be optimal.

□

B.5. **Derivation of the FDIC Dual.** This section derives the recursive dual of the FDIC problem. Formally, it proves that This section proves that $V^{FD}(\theta) = F(0,0,0;\theta)$ where $V^{FD}(\theta)$ is given in problem 2.10 and $F(n, L, R; \theta)$ is given in problem 4.8.

B.5.1. *Preliminary Definitions.*

For convinience, the relevant definitions that will be used in the proof are reproduced here. The primal $V^{FD}(\theta)$ is :

$$\begin{aligned}
V^{FD}(\theta) = \max_{q\geq 0, w\geq 0} \quad & \textstyle\sum_{h\in H}\left[q_h\left(p-p_0\right)v - w_h p\right] \\
s.t. \quad & \\
& q_{h_0} \leq 1 \\
\forall h \quad & q_{\langle h,s\rangle} - q_h p \leq 0 \\
\forall h \quad & q_{\langle h,f\rangle} - q_h\left(1-p\right) \leq 0 \\
\forall h, d \quad & FDIC \quad (2.9)
\end{aligned}$$

The FDIC is (2.9) is:

$$\begin{aligned}
FDIC \quad (2.9): \quad & -\left(p-p_0\right)w_h + q_h c_{h-d} \\
& -\tfrac{p-p_0}{p}\textstyle\sum_{\tilde{h}\succeq\langle h,s\rangle} p w_{\tilde{h}} + \tfrac{p-p_0}{p}\sum_{\tilde{h}\succeq\langle h,s\rangle} q_{\tilde{h}}\cdot c_{\tilde{h}-d} \\
& +\tfrac{p-p_0}{1-p}\textstyle\sum_{\tilde{h}\succeq\langle h,f\rangle} p w_{\tilde{h}} - \tfrac{p-p_0}{1-p}\sum_{\tilde{h}\succeq\langle h,f\rangle} q_{\tilde{h}} c_{\tilde{h}-d} \\
& +\theta\cdot\tfrac{p_0}{p}\textstyle\sum_{\tilde{h}\succeq\langle h,s\rangle} q_{\tilde{h}}\cdot\delta_{\tilde{h}-d} \\
& +\theta\cdot\tfrac{1-p_0}{1-p}\textstyle\sum_{\tilde{h}\succeq\langle h,f\rangle} q_{\tilde{h}}\cdot\delta_{\tilde{h}-d} \qquad \leq 0 \quad .
\end{aligned}$$

The recursive dual is given in problem 4.8:

$$
\begin{aligned}
&4.8:\\
F(n,L,R) &= \max\left[0, \min_{\Lambda\geq 0} F(n,L,R,\Lambda)\right]\\
&s.t.\\
F(n,L,R,\Lambda) &= pF\left(n+1, L-\frac{p-p_0}{p}\Lambda, R+\theta\frac{p_0}{p}\Lambda\right)\\
&\quad + (1-p)F\left(n+1, L+\frac{p-p_0}{1-p}\Lambda, R+\theta\frac{1-p_0}{1-p}\Lambda\right)\\
&\quad + f(n,L,R,\Lambda)\\
&s.t.\\
&\quad \Lambda_0 \leq \frac{p}{p-p_0}(1+L_0)\\
&\quad \Lambda_m - \Lambda_{m+1} \geq 0
\end{aligned}
$$

with $f(n,L,R,\Lambda)$ given in 4.7:

$$
\begin{aligned}
f(n,L,R,\Lambda) &\equiv v(p-p_0) - c_n(\Lambda_0 - L_0) - \delta_n R_0\\
&\quad + \sum_{m=1}^{n_h}\left[\delta_{n-m}(\Lambda_m - L_m) + (\delta_{n-m+1} - \delta_{n-m}) R_m\right] \quad .
\end{aligned}
$$

Problem $F(n,L,R)$ is a recursive formulation of the dual of problem V^{FD}.

Λ, L, R are vectors for each history h. The control variables, Λ are the partial sums of the dual variables for the FDIC, λ_d^h. For a detalied exposition see section 4.3. Λ^h is defined in equation 4.3:

$$
(4.3): \qquad \Lambda_m^h \equiv \sum_{d=m}^{n} \lambda_d^h
$$

The state variables L, R are implictly defined in problem 4.8. The proof is clearer if these are given in explicit form:

$$
L_m^h = (p-p_0)\left(\sum_{\tilde{h}:h\succeq\langle\tilde{h},f\rangle} \frac{\Lambda_m^{\tilde{h}}}{1-p} - \sum_{\tilde{h}:h\succeq\langle\tilde{h},s\rangle} \frac{\Lambda_m^{\tilde{h}}}{p}\right) \tag{B.15}
$$

$$
R_m^h = \theta p_0\left(\sum_{\tilde{h}:h\succeq\langle\tilde{h},s\rangle} \frac{\Lambda_m^{\tilde{h}}}{p}\right) + \theta(1-p_0)\left(\sum_{\tilde{h}:h\succeq\langle\tilde{h},f\rangle} \frac{\Lambda_m^{\tilde{h}}}{1-p}\right) \tag{B.16}
$$

B.5.2. *Derivation of the w_h constraint.*

Lemma. *The constraint associated with w_h in the FDIC dual is*

$$
\forall h \quad \Lambda_0^h \leq \frac{p}{p-p_0}\left(1+L_0^h\right) \tag{B.17}
$$

Proof. The right hand side of each constraint is the coefficient on w_h in the objective of V^{FD}: $-p$. To construct the left hand side, observe that w_h appears with a coefficient $-(p-p_0)$ in all the FDIC (2.9) for h. The variable w_h also appears with a coefficient $\frac{p-p_0}{p}p = (p-p_0)$ in all FDIC for $\tilde{h}$ such that $h \succeq \left\langle \tilde{h}, s\right\rangle$ and with a coefficient $\frac{p-p_0}{1-p}p$ in all FDIC for $\tilde{h}$ such that $h \succeq \left\langle \tilde{h}, f \right\rangle$. Summing up:

(B.18)

$$\forall h \qquad \sum_{d=0}^{n_h} \left[-(p-p_0)\lambda_d^h - (p-p_0)\left(\sum_{\tilde{h}:h\succeq\langle\tilde{h},s\rangle} \lambda_d^{\tilde{h}}\right) + (p-p_0)\frac{p}{1-p}\left(\sum_{\tilde{h}:h\succeq\langle\tilde{h},f\rangle} \lambda_d^{\tilde{h}}\right)\right] \geq -p \, .$$

Insert the summand into the right hand side of B.18 and factor out $-p$ as in the LDIC to get

$$-(p-p_0)\sum_{d=0}^{n_h}\lambda_d^h + p(p-p_0)\left[\left(\sum_{\tilde{h}:h\succeq\langle\tilde{h},f\rangle}\sum_{d=0}^{n_h}\frac{\lambda_d^{\tilde{h}}}{1-p}\right) - \left(\sum_{\tilde{h}:h\succeq\langle\tilde{h},s\rangle}\sum_{d=0}^{n_h}\frac{\lambda_d^{\tilde{h}}}{p}\right)\right] \geq -p \, .$$

Using the definition of Λ (4.3) for the summation over λ_d^h obtains

$$-(p-p_0)\Lambda_0^h + p(p-p_0)\left[\left(\sum_{\tilde{h}:h\succeq\langle\tilde{h},f\rangle}\frac{\Lambda_0^{\tilde{h}}}{1-p}\right) - \left(\sum_{\tilde{h}:h\succeq\langle\tilde{h},s\rangle}\frac{\Lambda_0^{\tilde{h}}}{p}\right)\right] \geq -p \, .$$

The definition of L (B.15) can now be applied as was l in the LDIC dual to obtain B.17 □

B.5.3. *Derivation of the q_h constraint.*

Lemma 27. *The constraint associated with q_h in the FDIC dual is*

(B.19)

$$\begin{aligned} &\mu^h - (1-p)\mu^{\langle h,f\rangle} - p\mu^{\langle h,s\rangle} + c_h\Lambda_0^h - c_h L_0^h + \delta_h R_0^h \\ &-\sum_{m=1}^{n_h}\left[\delta_{h-m}\left(\Lambda_m^h - L_m^h\right) + \left(\delta_{h-m+1}-\delta_{h-m}\right)R_m^h\right] \quad \geq \quad v(p-p_0) \end{aligned}$$

Proof. The variable q_h appears in the objective with coefficient $v(p-p_0)$. In the primal constraints, q_h appears in the three probability constraints and in all the FDIC for histories that h follows. The probability constraints generate the continuation value terms:

$$\mu^h - (1-p)\mu^{\langle h,f\rangle} - p\mu^{\langle h,s\rangle} \tag{B.20}$$

To simplify the rest of the derivation, I consider each case that q_h appears in an FDIC separately.

(1) In all the FDIC for history h (i.e., for each d), q_h appears with a coefficient c_{h-d} . This generates the term

$$\sum_{d=0}^{n_h} c_{h-d}\lambda_d^h \, .$$

To obtain the term in Λ, apply summation by parts:

$$\begin{aligned} \sum_{d=0}^{n_h} c_{h-d}\lambda_d^h &= \\ c_h\Lambda_0^h - \sum_{m=1}^{n_h}\left(c_{h-m+1}-c_{h-m}\right)\Lambda_m^h &= \\ c_h\Lambda_0^h - \sum_{m=1}^{n_h}\delta_{h-m+1}\Lambda_m^h & . \end{aligned} \tag{B.21}$$

(2) The variable q_h also appears twice in each of the FDIC for $\tilde{h}, d$ such that $h \succeq \tilde{h}$, once as part of the continuation utility term (in either the second or third row of 2.9) and once as part of the future gains from shirking term (in either the fourth or fifth row of 2.9). The continuation utility term will determine the coefficients on L. The shirking gains term will determine the coefficients on R.

(a) In the continuation utility term in the FDIC for all histories that h follows, the coefficient for $\lambda_d^{\tilde{h}}$ is the current cost multiplied by $\frac{p-p_0}{p}$ if $h \succeq \left\langle \tilde{h}, s \right\rangle$ and by$-\frac{p-p_0}{1-p}$ if $h \succeq \left\langle \tilde{h}, f \right\rangle$. Summarizing these terms obtains the sum:

$$\sum_{d=0}^{n_h} \frac{c_{h-d}}{p} (p-p_0) \left[\left(\sum_{\tilde{h}:h\succeq\langle\tilde{h},s\rangle} \lambda_d^{\tilde{h}} \right) - \frac{p}{1-p} \left(\sum_{\tilde{h}:h\succeq\langle\tilde{h},f\rangle} \lambda_d^{\tilde{h}} \right) \right] \quad . \tag{B.22}$$

For the FDIC, use by "summation by parts":[18]

$$\begin{aligned} \sum_{d=0}^{n_h} c_{h-d} \cdot \lambda_d^{\tilde{h}} &= c_h \cdot \sum_{d=0}^{n_h} \lambda_d^{\tilde{h}} - \sum_{d=1}^{n_h} (c_{h-d+1} - c_{h-d}) \cdot \left(\sum_{m=d}^{n_h} \lambda_d^{\tilde{h}} \right) \\ &= c_h \Lambda_0^{\tilde{h}} - \sum_{d=1}^{n_h} \delta_{h+1-d} \Lambda_d^{\tilde{h}} \end{aligned}$$

Thus, rewrite B.22

$$(p-p_0) \cdot c_h \left[\left(\sum_{\tilde{h}:h\succeq\langle\tilde{h},s\rangle} \frac{\Lambda_0^{\tilde{h}}}{p} \right) - \left(\sum_{\tilde{h}:h\succeq\langle\tilde{h},f\rangle} \frac{\Lambda_0^{\tilde{h}}}{1-p} \right) \right]$$
$$-(p-p_0) \sum_{m=1}^{n_h} \delta_{h+1-m} \left[\left(\sum_{\tilde{h}:h\succeq\langle\tilde{h},s\rangle} \frac{\Lambda_m^{\tilde{h}}}{p} \right) - \left(\sum_{\tilde{h}:h\succeq\langle\tilde{h},f\rangle} \frac{\Lambda_m^{\tilde{h}}}{1-p} \right) \right].$$

As should be expected, the terms in the square brackets are the agent's liability limit L_m^h. Using the definition of L_m^h in (B.15) to replace the internal summations, the continuation utility term becomes

$$-c_h L_0^h + \sum_{m=1}^{n_h} \delta_{h-m+1} L_m^h \tag{B.23}$$

(b) Finally, consider the terms reflecting the information rents in each FDIC constraint (the last two rows in 2.9). The coefficient for $\lambda_d^{\tilde{h}}$ is $\theta \frac{p_0}{p} \delta_{h-d}$ if $h \succeq \left\langle \tilde{h}, s \right\rangle$ and $\theta \frac{1-p_0}{1-p} \delta_{h-d}$ if $h \succeq \left\langle \tilde{h}, f \right\rangle$. Again the coefficients depend only on whether h follows a success or failure in $\tilde{h}$. Combine these to obtain

$$\sum_{d=0}^{n_h} \delta_{h-d} \cdot \theta \left(\frac{p_0}{p} \left(\sum_{\tilde{h}:h\succeq\langle\tilde{h},s\rangle} \lambda_d^{\tilde{h}} \right) + \frac{1-p_0}{1-p} \left(\sum_{\tilde{h}:h\succeq\langle\tilde{h},f\rangle} \lambda_d^{\tilde{h}} \right) \right) \quad . \tag{B.24}$$

Summation by parts can be used to obtain

$$\begin{aligned} \sum_{d=0}^{n_h} \delta_{h-d} \cdot \lambda_d^{\tilde{h}} &= \delta_h \sum_{d=0}^{n_h} \lambda_d^{\tilde{h}} - \sum_{d=1}^{n_h} (\delta_{h-d+1} - \delta_{h-d}) \cdot \left(\sum_{m=d}^{n_h} \lambda_d^{\tilde{h}} \right) \quad . \\ &= \delta_h \Lambda_0^{\tilde{h}} - \sum_{d=1}^{n_h} (\delta_{h-d+1} - \delta_{h-d}) \cdot \Lambda_d^h \end{aligned}$$

Thus, rewrite B.24 as:

$$\theta \delta_h \left(\frac{p_0}{p} \left(\sum_{\tilde{h}:h\succeq\langle\tilde{h},s\rangle} \Lambda_0^{\tilde{h}} \right) + \frac{1-p_0}{1-p} \left(\sum_{\tilde{h}:h\succeq\langle\tilde{h},f\rangle} \Lambda_0^{\tilde{h}} \right) \right)$$
$$\theta \sum_{d=1}^{n_h} (\delta_{h-d+1} - \delta_{h-d}) \left(\frac{p_0}{p} \left(\sum_{\tilde{h}:h\succeq\langle\tilde{h},s\rangle} \Lambda_d^{\tilde{h}} \right) + \frac{1-p_0}{1-p} \left(\sum_{\tilde{h}:h\succeq\langle\tilde{h},f\rangle} \Lambda_d^{\tilde{h}} \right) \right)$$

[18] This may be verified directly. Intuitively, this is the same as letting λ be the derivative of Λ and δ the derivative of c. Then

$$\int_0^{n_h} c_t \lambda_t^h dt = \Lambda^h c_{n_h} - \int_0^{n_h} \delta_t \Lambda_t dt$$

Applying the definition of R in B.16 obtains the "information rent" term:

$$\delta_h R_0^h - \sum_{m=1}^{n_h} \left(\delta_{h-m+1} - \delta_{h-m}\right) R_m^h \tag{B.25}$$

Combine B.10 , B.21, B.23 and B.25 to obtain the dual constraint for q_h B.19. □

B.5.4. *The FDIC Dual.*

Lemma 28. *The dual problem for $V^{FD}(\theta)$ is*

$$\min_{\Lambda^h \geq 0} \quad \mu^{\emptyset} \tag{B.26}$$

$$\begin{aligned}
&\textit{s.t., for every} \quad h \in H && \\
&\textit{The state variables} && B.15, B.16 \\
&\lambda_d^h \geq 0: && \Lambda_m^h - \Lambda_{m+1}^h \geq 0 \\
&\textit{The } q_h \textit{ dual constraint} && B.19 \\
&\textit{The } w_h \textit{ dual constraint} && \textbf{??}
\end{aligned}$$

Proof. The only primal constraint with a non negative right hand side is $q_{\emptyset} \leq 1$. This establishes the objective. The constraints and state variables were established in the previous two lemmas. □

B.5.5. *Recursive Derivation.*

Lemma 29. *For every h, $\mu^h = F\left(n_h, L^h, R^h\right)$. Specifically,$F(0,0,0)$ and the corresponding optimal Λ's are a solution to problem (B.26).*

Proof. The proof is a step by step repitition of the proof in section B.4.5, with the variables renamed. For clarity, I repeat the steps with the adjusted notation. The same arguments as in B.4.5 apply. □

(1) The definition of $L^h$and R^h (equations B.15 and B.16) are consistent with the law of motion for the state variables given in the definition of $F(n, L, R)$
(2) The q constraint in the dual B.26 binds whenever $\mu^h > 0$.
(3) Let $F^*(n, L, R)$ be the optimal solution for the recursive dual problem 4.8 starting from $F(0,0,0)$. By the principle of optimality, $F^*(n, L, R)$ exists and is unique. Let μ^{h*} be an optimal solution to problem B.26 . It remains to show that for every h, $F^*\left(n_h, L^h, R^h\right) = \mu^{h*}$
(4) Suppose that for some h this is not true. As the contract is finite, choose h so that there is no future $\tilde{h} \succeq h$ such that $F^*\left(n_{\tilde{h}}, L^{\tilde{h}}, R^{\tilde{h}}\right) \neq \mu^{\tilde{h}*}$. Let $\Lambda^*\left(n_h, L^h, R^h\right)$ be the recursive problem's optimal solution at $\left(n_h, L^h, R^h\right)$ and λ^{h*} be given by the optimal solution to problem B.14.
    (a) If $F^*\left(n_h, L^h, R^h\right) > \mu^{h*}$ replace $\Lambda^*\left(n_h, L^h, R^h\right)$ with those given by λ^{h*}. Obtain that $F^*\left(n_h, L^h, R^h\right)$ is not optimal.
    (b) If $F^*\left(n_h, L^h, R^h\right) < \mu^{h*}$, replace λ^{h*} with those given by $\Lambda^*\left(n_h, L^h, R^h\right)$. Obtain that μ^{h^*} cannot be optimal.

B.6. **Proof for Lemma 5.**

Lemma 30. *$\mu(n, l, r)$ is continuous and convex in (l, r). $\mu(n, l, r, \lambda)$ is continuous and convex in λ for every l, r. The optimal λ is continuous in l and r.*

Proof. Separately for each claim.

(1) For $\mu(n,l,r)$:
   (a) In any last period, $\mu(n,l,r)$ is linear and thus continuous and convex.
   (b) Assume that $\mu(n+1,l,r)$ is continuous and convex. As the positive sum of three continuous and convex functions is continuous and convex, for every λ, $V(n,l,r,\lambda)$ is convex in (l,r). As the feasible set is convex and the objective is to minimize a convex function, $\mu(n,l,r)$ is continuous and convex.

(2) For $\mu(n,l,r,\lambda)$, the period return π is linear in λ and so it is sufficient to show that the continuation is convex in λ. I show this for the continuation after success $-\mu\left(n+1,l-\frac{p-p_0}{p}\lambda,r+\frac{p_0}{p}\lambda\right)$. The same proof applies for the continuation after failure. Let λ^1 and λ^2 be feasible solutions. Then by convexity of $\mu(n+1,l,r)$ for any $\alpha\in(0,1)$:

$$\alpha\mu\left(n+1,l-\tfrac{p-p_0}{p}\lambda^1,r+\tfrac{p_0}{p}\Lambda^1\right)+(1-\alpha)\mu\left(n+1,l-\tfrac{p-p_0}{p}\lambda^2,r+\tfrac{p_0}{p}\Lambda^2\right)\leq$$
$$\mu\left(n+1,\alpha\left(l-\tfrac{p-p_0}{p}\lambda^1\right)+(1-\alpha)\left(l-\tfrac{p-p_0}{p}\lambda^2\right),\alpha\left(r+\tfrac{p_0}{p}\lambda^1\right)+(1-\alpha)\left(r+\tfrac{p_0}{p}\lambda^2\right)\right)=$$
$$\mu\left(n+1,l-(p-p_0)\left(\alpha\mu^1+(1-\alpha)\mu^2\right),r+\alpha\tfrac{p_0}{p}\mu^1+(1-\alpha)\tfrac{p_0}{p}\mu^2\right)$$

(3) Given the previous result, the optimal λ is either unique or an interval. Continuity in both cases is standard.

□

B.7. **Proof for Lemma 8.**

Lemma. *8 In the ε-smooth model, as $\varepsilon\to 0$, $\tilde{\mu}_l^h\to -U^h$ and $\tilde{\mu}_r^h\to -D^h$.*

Proof. I prove that if μ is everywhere differentiable, $\mu_l=-U$ and $\mu_r=-D$. The proof applies backward induction.

(1) If the agent does not work in a period, $\mu=\mu_l=\mu_r=-U=-D=0$.

(2) For every last period in which the agent works, all the continuation terms in the first order condition are zero. Therefore,

$$\mu_\lambda=-c_n+\omega\cdot(p-p_0)$$

As $c_n>0$, it is always the case that $\omega>0$ and so by complementary slackness, $\lambda(n,l,r)=\frac{p}{p-p_0}(1+l)$ and the agent is paid $\frac{c_n}{p-p_0}$ for success. Therefore, the agent's utility is

$$\begin{aligned}U&=\frac{pc_n}{p-p_0}-c_n=\frac{p_0c_n}{p-p_0}\quad\text{and}\\ D&=\delta_n\end{aligned}$$

Placing λ in μ in 6 obtains

$$\begin{aligned}\mu(n,l,r)&=\min\left[0,v(p-p_0)-c_n\frac{p}{p-p_0}-c_nl\left(\frac{p}{p-p_0}-1\right)-r\delta_n\right]\\ &=\min\left[0,v(p-p_0)-c_n\frac{p}{p-p_0}-c_nl\left(\frac{p_0}{p-p_0}\right)-r\delta_n\right]\end{aligned}$$

   (a) If $\mu>0$

$$\begin{aligned}\mu_l&=-c_n\frac{p_0}{p-p_0}=-U^h\\ \mu_r&=-\delta_n=-D^h\end{aligned}$$

(b) If $v(p-p_0) - c_n\frac{p}{p-p_0} - c_n l\left(\frac{p_0}{p-p_0}\right) - r\delta_n = 0$ then let $\partial\mu_l$ and $\partial\mu_r$ denote the sub-differentials. Then $-U^h \in \partial\mu_l$ and $-D^h \in \partial\mu_r$

(3) For every non last period, suppose the relation holds for all future periods. If $\mu$is smooth, the envelope theorem applies and

$$\begin{aligned}\mu_l &= p\mu_l^s + (1-p)\mu_l^f + c_n - \frac{p}{p-p_0}\omega \cdot (p-p_0) \\ &= p\left(-U^{\langle h,s\rangle}\right) + (1-p)\left(-U^f\right) + c_n - p\omega \\ &= -U^h\end{aligned}$$

and

$$\begin{aligned}\mu_r &= p\mu_r^s + (1-p)\mu_r^f - \delta_n \\ &= p\left(-D^{\langle h,s\rangle}\right) + (1-p)\left(-D^{\langle h,f\rangle}\right) - \delta_n \\ &= -D^h\end{aligned}$$

□

B.8. **Proof for Lemma 9:** $\mu(n,l,r;\theta)$ **decreases in** l, r, θ**.**

Proof. I first show the result for r and then for l. The proof for θ is identical. To simplify the notation, assume $\theta = 1$.

- For r:
  - In any last period if $\mu(n,l,r) > 0$, $\mu_r = -\delta_n < 0$.
  - Suppose $\mu(n+1,l,r)$ weakly decreases in r . Then for any $\varepsilon > 0$, it must be that $\mu(n,l,r+\varepsilon) < \mu(n,l,r)$:
    Let λ^* be optimal at the state (n,l,r). As the constraint is not affected by r, λ^* is feasible for $(n,l,r+\varepsilon)$. Therefore

$$\begin{aligned}\mu(n,l,r+\varepsilon) &\le p\mu\left(n+1, l - \frac{p-p_0}{p}\lambda^*, r + \varepsilon\frac{p_0}{p}\lambda^*\right) \\ &\quad + (1-p)\mu\left(n+1, l - \frac{p-p_0}{1-p}\lambda^*, r + \varepsilon\frac{1-p_0}{1-p}\lambda^*\right) \\ &\quad + v(p-p_0) - \lambda^* c_n + l c_n - r\delta_n - \varepsilon\delta_n \\ &< \mu(n,l,r)\end{aligned}$$

    Therefore, for whenever $\mu(n,l,r) > 0$, μ strictly decreases in r .
- For l:
  - In any last period, if $\mu(n,l,r) > 0$, $\mu_l = -\frac{p_0}{p-p_0}c_n < 0$
  - Suppose $\mu(n+1,l,r)$ weakly decreases in l . Then for any $\varepsilon > 0$, it must be that $\mu(n,l+\varepsilon,r) < \mu(n,l,r)$:
    Let λ^* be optimal at the state (n,l,r). Then $\lambda^* + \frac{p}{p-p_0}\varepsilon$ is feasible for $(n,l+\varepsilon,r)$. Therefore

$$\begin{aligned}
\mu(n, l+\varepsilon, r) \leq & \quad p \cdot \mu\left(n+1, l+\varepsilon-\tfrac{p-p_0}{p}\lambda^*-\varepsilon, r+\tfrac{p_0}{p}\lambda^*+\varepsilon\tfrac{p_0}{p-p_0}\right) \\
& +(1-p)\cdot\mu\left(n+1, l+\varepsilon+\tfrac{p-p_0}{1-p}\lambda^*+\tfrac{p}{1-p}\varepsilon, r+\tfrac{1-p_0}{1-p}\lambda^*+\varepsilon\tfrac{1-p_0}{1-p}\cdot\tfrac{p}{p-p_0}\right) \\
& \quad v(p-p_0)-c_n\left(\lambda^*+\varepsilon\tfrac{p}{p-p_0}-l-\varepsilon\right)-r\delta_n \\
\leq & \quad p\cdot\mu\left(n+1, l-\tfrac{p-p_0}{p}\lambda^*, r+\tfrac{p_0}{p}\lambda^*\right) \\
& +(1-p)\cdot\mu\left(n+1, l+\tfrac{p-p_0}{1-p}\lambda^*, r+\tfrac{1-p_0}{1-p}\lambda^*\right) \\
& \quad v(p-p_0)-c_n\left(\lambda'-l'+\tfrac{p}{p-p_0}\varepsilon\right)+r\delta_n \\
= & \quad \mu(n,l,r)-c_n\varepsilon\tfrac{p}{p-p_0}
\end{aligned}$$

The second inequality removes the added ε terms on r^y and l^y as they are positive and so increase the function by the result on r and the induction assumption. □

B.9. **Contract After Payment.**

Lemma. *12 In the optimal contract, if the agent is paid for success in history h, then starting in history $\langle h,s\rangle$ the contract is fixed to $N(r)$. All remaining LDIC bind and if $\theta=1$, the agent is paid $\frac{c_{N(r)}}{p-p_0}$ for each additional success.*

$$N(r)=\max n: \quad v(p-p_0)-c_n-r\cdot\delta_n\geq 0$$

Proof. The discussion in the text show that after a payment, $\lambda=0$ and the period profit is $\pi(n,r)$ as in equation (3.9). As both c_n and δ_n are increasing, eventually period profit will become negative and production will stop. It therefore remains to show that the principal does not gain by randomizing in any last period. As $\pi(n,r)$ is independent of any new outcomes, this is always true. To see this, write the dual for the last two periods in explicit form:

$$\begin{aligned}
\min_{\mu_h,\mu_{\langle h,s\rangle},\mu_{\langle h,f\rangle}} & \quad \mu_h \\
s.t. & \\
& \mu_h-p\mu_{\langle h,s\rangle}-\mu_{\langle h,f\rangle} \geq \pi(n,r) \\
& \mu_{\langle h,s\rangle} \geq \pi(n+1,r) \\
& \mu_{\langle h,f\rangle} \geq \pi(n+1,r)
\end{aligned}$$

With $\pi(n,r)\geq 0$ and $\pi(n,r+1)<0$. The primal is simply

$$\begin{aligned}
\max_{q\geq 0} & \quad q_h\pi(n,r)+\left(q_{\langle h,s\rangle}+q_{\langle h,f\rangle}\right)\pi(n,r+1) \\
s.t. & \\
& q_h\leq 1 \\
& q_{\langle h,s\rangle}\leq pq_h \\
& q_{\langle h,f\rangle}\leq(1-p)\,q_h
\end{aligned}$$

It is immediate that if $\pi(n,r+1)<0$ the optimal solution sets $q_{\langle h,y\rangle}=0$ and as $\pi(n,r)\geq 0$, an optimal solution is to set $q_h=1$. □

Appendix C. Aggregation Model

Suppose the principal only observes the aggregate outcome after N^{FB} periods. This section outlines the implications for the model. It is clear that the wage scheme can only depend on the number of outcomes. I state without proof that this constrains the contract enough to imply that LDIC are sufficient for IC.

Let ω_h denote the extra wage for a success in a period h. Then if the agent is supposed to work also at $\langle h, f\rangle$, $\omega_h = \omega_{\langle h,f\rangle}$. If the agent will not be asked to work at $\langle h, f\rangle$, $\omega_{\langle h,f\rangle} = 0$ and requiring $\omega_h \geq \omega_{\langle h,f\rangle}$ does not constrain the contract. As the optimal contract minimizes wages, I can relax the constraint $\omega_h = \omega_{\langle h,f\rangle}$ to[19]

$$\omega_h \geq \omega_{\langle h,f\rangle} \ .$$

The model uses $w_h = \omega_h q_h$. We assume that $q_{\langle h,f\rangle} \in \{(1-p)\, q_h, 0\}$ and so whenever $q_{\langle h,f\rangle} \neq 0$, the constraint may be written as

$$\frac{w_h}{q_h} \geq \frac{w_{\langle h,f\rangle}}{q_{\langle h,f\rangle}} = \frac{w_{\langle h,f\rangle}}{q_h\,(1-p)}$$

Multiplying by q_h obtains the following additional set of constraints to the linear problem:

$$\forall h \quad w_h\,(1-p) \geq w_{\langle h,f\rangle}$$

Call this the aggregation constraint. To obtain the constraint in the most similar form to the wage constraints, multiply it by $(p-p_0)$ and collect sides:

$$\forall h \quad (p-p_0)\left(w_{\langle h,f\rangle} - w_h\,(1-p)\right) \leq 0 \ .$$

Let ϕ^h be the multiplier on the aggregation constraint in which w_h appears with coefficient $(p-p_0)\,(1-p)$.

For any history h, let ϕ^- denote the value of ϕ in the previous period. Recall that the dual wage constraint is multiplied by (-1). The aggregation constraint adds to the wage constraint of every period that follows a failure the term :

$$(p-p_0)\left(-\phi^- + (1-p)\,\phi^h\right) \ ,$$

and to every period that follows a success the term

$$(p-p_0)\,(1-p)\,\phi^h \ .$$

It can be shown that in this setting any LDIC contract is IC – the payment for success cannot depend on the period in which the effort was exerted and so the agent has no informational gain from shirking. Then ammend the LDIC problem 3.3 to account for the added constraint by adding the state variable f (for “fixed” wage):

$$\mu\,(n,f,l,r) = \max\left[0, \min_{\lambda,\phi\geq 0} \mu\,(n,f,l,r,\lambda)\right]$$

$$s.t.$$

$$\begin{aligned}\mu\,(n,f,l,r,\lambda) = \ & pV\left(n+1, 0, l - \frac{p-p_0}{p}\lambda, r + \theta\frac{p_0}{p}\lambda\right) \\ & + (1-p)\,V\left(n+1, \phi, l + \frac{p-p_0}{1-p}\lambda, r + \theta\frac{1-p_0}{1-p}\lambda\right) \\ & v\,(p-p_0) - \lambda c_n + l\cdot c_n - r\delta_n\end{aligned}$$

$$\lambda + (1-p)\,\phi \leq \frac{p}{p-p_0} + l + f$$

[19]It is possible to verify later that indeed in the optimal solution this will bind when $q_{\langle h,f\rangle} = 1$.

Paying a higher wage today increases the feasible wage tomorrow. As a result, the only effect of ϕ is an increase in the f after failure. As f only relaxes the wage constraint, it is immidiate that $V_f \geq 0$. Thus ϕ is set to the highest possible value and the wage constraint binds in all periods – every success earns the agent a direct increase in pay.