Matching in Networks with Bilateral Contracts

John William Hatfield
Graduate School of Business
Stanford University

Scott Duke Kominers*
Harvard Business School
Harvard University

April 2010

Abstract

We introduce a model in which firms trade goods via bilateral contracts which specify a buyer, a seller, and the terms of the exchange. This setting subsumes (many-to-many) matching with contracts, as well as supply chain matching. When firms' relationships do not exhibit a supply chain structure, stable allocations need not exist. By contrast, in the presence of supply chain structure, a natural substitutability condition characterizes the maximal domain of firm preferences for which stable allocations always exist. Furthermore, the classical lattice structure, rural hospitals theorem, and one-sided strategy-proofness results all generalize to this setting.

JEL Classification: C78, D4, L14

Keywords: Matching, Networks, Stability, Quasisubmodularity, Substitutes

*The authors are grateful for the helpful comments of Drew Fudenberg, Jason D. Hartline, Sonia Jaffe, Fuhito Kojima, Paul Milgrom, Muriel Niederle, Alvin E. Roth, Alexander Westkamp, E. Glen Weyl, and seminar audiences at Chicago, Harvard, Northwestern, and Stanford. Additionally, Kominers gratefully acknowledges the support of a National Science Foundation Graduate Research Fellowship. Please e-mail comments or suggestions to hatfield@stanford.edu or skominers@hbs.edu.

1 Introduction

The theoretical literature on two-sided matching began with the simple one-to-one (marriage) model of Gale and Shapley (1962), in which agents on opposite sides of a market (men and women) seek to match into pairs. The central solution concept in this literature is *stability*, the requirement that for no unmatched pair does each agent prefer the other to their current assigned partner. Gale and Shapley (1962) showed that stable one-to-one matches exist in general, and obtained conditions under which this existence result is preserved even if agents on one side of the market are allowed to match to multiple partners, that is, when the matching is many-to-one (as in college admissions and doctor-hospital matching). Following high-profile applications of matching in labor markets and school choice programs,[1] the foundational work on matching has been extensively generalized.[2] Recently, Ostrovsky (2008) illustrated that matching markets need not be two-sided—they may instead consist of a market of firms organized into supply chains. Earlier matching models easily embed into the supply chain framework: for example, the many-to-one matching market between doctors and hospitals may be thought of as a "one-step supply-chain" in which doctors sell their services to hospitals.

Although the expanding work on matching has eliminated nearly all the theoretical restrictions imposed in the early literature, two assumptions have been maintained throughout, either implicitly or explicitly:

[1]Roth and Sotomayor (1990) provide a survey of the pre-1990 theory of matching. Roth (2008) gives an updated account, as well as references for historical and recent applications of matching. For examples of specific applications, see Roth (2002) (National Resident Matching Program) and Abdulkadìroğlu et al. (2009) (school choice).

[2]Kelso and Crawford (1982) extended many-to-one matching to a setting in which matches are supplemented by wage negotiations, as well as allowing for more general preferences than responsiveness for those agents desiring multiple contracts; Hatfield and Milgrom (2005) generalized this framework still further, by allowing agents to negotiate contracts which fully specify both a matching and the conditions of the match.

Meanwhile, a host of work has studied the existence of stable matchings in many-to-many matching settings, two-sided markets in which all agents may match to multiple partners (as in the matching of consultants to firms). Many-to-many matching has been studied, for example, in the work of Sotomayor (1999a, 2004), Echenique and Oviedo (2006), and Konishi and Ünver (2006). Recently, Hatfield and Kominers (2010) merged this line of research with that of Hatfield and Milgrom (2005), introducing a theory of many-to-many matching with contracts.

- *acyclicity* – no agent may both buy from and sell to another agent, even through intermediaries, and

- *full substitutability* – upon being endowed with an additional item, an agent's demand for other items is lower, both in the sense of a reduced desire to buy additional items and an increased desire to sell items he currently owns.[3]

In this paper, we introduce a new matching model in which firms trade via bilateral contracts which specify a buyer, a seller, and the terms of the exchange. This model subsumes all classical matching models, and hence can be used to understand many-to-many matching markets (such as the matching of auto-parts suppliers and assemblers (Fox, 2008)), as well as more complex markets where agents are both buyers and sellers (such as in multi-firm supply chains (Ostrovsky, 2008)).

The generality of our framework allows us to make two novel theoretical contributions.

First, we show that both acyclicity and full substitutability are necessary for classical matching theory. If either condition is violated, then stable allocations cannot be guaranteed.[4] Intuitively, if a contracting relationship contains a cycle, and if a firm in the cycle has an outside option which he prefers to one contract in the cycle, then both the outside option and the complete trading cycle are unstable; the necessity of acyclicity follows. The necessity of full substitutability is more technical to illustrate, but follows closely upon prior results of Hatfield and Kominers (2010).

Second, in the presence of acyclicity and fully substitutable preferences, we fully generalize the key results of classical matching theory. We prove that under these conditions, stable allocations correspond bijectively to fixed points of an isotone operator; Tarski's fixed

[3]Full substitutability is a condition on firms' preferences familiar from auction theory. Indeed, full substitutability is an ordinal analogue of the conventional notion of substitutes from auction theory (see, for instance, Milgrom (2004)) and we prove more formally that it is equivalent to quasisubmodularity of the associated indirect utility function.

[4]Note that we use a notion of stability which is distinct from *chain stability* introduced by Ostrovsky (2008). Our stability concept is more stringent than chain stability, although these two notions coincide on acyclic contract domains over which firm preferences are fully substitutable. As we detail in Section 3.3, for domains where these conditions do not hold, chain stability has some unappealing properties.

point theorem then guarantees the existence of a lattice of stable allocations. We also prove a generalization of the classical rural hospitals theorem of Roth (1986) and the strategy-proofness results of Hatfield and Milgrom (2005) and Hatfield and Kojima (2009).[5] These latter results display a surprising structure which can only be elicited within a framework as general as ours: in particular, we show that the difference between the numbers of buy- and sell-contracts held, rather than the absolute number of contracts held, is invariant across stable allocations for each agent.[6]

In light of our necessity results, our work establishes a frontier of matching theory. Without acyclicity and fully substitutable preferences, stable allocations are not guaranteed to exist in general, and hence the results of classical matching theory fail. Up to the failure of these conditions, however, all of the results of classical matching theory hold.

The remainder of this paper is organized as follows. We formalize our model in Section 2 and discuss the various restrictions on preferences, proving our characterization of full substitutability. In Section 3, we prove the sufficiency and necessity of fully substitutable preferences for the existence of stable contract allocations. In Section 4, we discuss the structure of the set of stable allocations, proving our rural hospitals and strategy-proofness results. We conclude in Section 5.

---

[5]As in other work in the theory of generalized matching our rural hospitals theorem provides an invariant on agents' net position in contracts. It thus only partially generalizes the original rural hospitals theorem of Roth (1986). However, as we discuss in Footnote 26, this generalization is maximal.

[6]The Roth (1986) rural hospitals theorem and its subsequent generalizations by Hatfield and Milgrom (2005) and Hatfield and Kominers (2010) all showed that, under certain conditions, the *number* of contracts signed by each firm is invariant across stable allocations. The natural conjecture that this exact result would extend to our setting is false, as we demonstrate in Section 4.1. We instead find that the proper invariant for each firm in our framework is the difference between the numbers of buy- and sell-contracts held by that firm. This result implies the previous rural hospitals results because, in a two-sided market, no firm can be both a buyer and a seller.

2 Model

2.1 Framework and Solution Concepts

2.1.1 Agents and Contracts

There is finite set F of of firms, and a finite set X of contracts. Each contract $x \in X$ is associated with both a buyer x_B and a seller x_S; several contracts in X may have the same buyer and seller.[7]

For concreteness, one may suppose each contract $x \in X$ denotes the exchange of a single unit of a good from x_S to x_B.[8] However, contracts need not use a constant unit. For example, labor markets might allow both full- and part-time job contracts.[9]

Let $x_F \equiv \{x_B, x_S\}$ be the set of the firms associated with contract x. For a set of contracts Y, we denote

$$Y_B \equiv \bigcup_{y \in Y} \{y_B\}, \quad Y_S \equiv \bigcup_{y \in Y} \{y_S\}, \quad Y_F \equiv Y_B \cup Y_S.$$

The contract set X is **acyclic** if there does not exist a **cycle**, i.e. a set of contracts

$$\left\{x^1, \ldots, x^N\right\} \subseteq X$$

such that $x^1_B = x^2_S, x^2_B = x^3_S, \ldots, x^{N-1}_B = x^N_S, x^N_B = x^1_S$ (as pictured in Figure 1).[10] This condition is equivalent to the condition that there is an ordering $\triangleright$ on F such that for all $x \in X$, $x_S \triangleleft x_B$; for an acyclic contract set X, if $f \triangleleft f'$, we will say that f is **upstream** of f' and that f' is **downstream** of f. We say that X is **exhaustive** if there is a contract

[7] Note that since X is finite, we may interpret X as being a subset of the set $F \times F \times T$, for some finite set T of possible contract terms. With this notation, a contract $x \in X$ is a 3-tuple: $x = (x_B, x_S, t)$ with $x_B, x_S \in F$ and $t \in T$.

[8] In this case, an exchange of 17 units from x_S to x_B would technically occur through 17 different contracts. While in practice the actual sale would not transact in this fashion—a single contractual document would cover the sale of all 17 units—use of primitive contract units lose no generality and will help us interpret the numerical implications of our results. For example, when primitive units are used, our Theorem 8 characterizes the excess stock of goods held by each firm at every stable allocation.

[9] As we point out in Section 4.1, the practical implications of our results which involve numerical contract counts are unclear if contracts are not denoted in a fixed unit. An in-depth discussion of these issues is presented by Hatfield and Kominers (2010).

[10] In our diagrams, an arrow $h_1 \xrightarrow{z} h_2$ between two firms denotes a contract z with seller $z_S = h_1$ and buyer $z_B = h_2$.

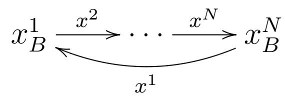

Figure 1: A contract cycle.

between any two firms, that is, if for all $f \neq f'$, $f, f' \in F$ there exists a contract x such that $x_F = \{f, f'\}$.

Each $f \in F$ has a strict preference relation P^f over sets of contracts involving f. Let $Y|_f \equiv \{y \in Y : f \in y_F\}$, the set of contracts in Y associated with firm f.

For any $Y \subseteq X$, we first define the **choice set** of f as the set of contracts he chooses from Y. Define

$$C^f(Y) \equiv \max_{P^f} \{Z \subseteq Y : x \in Z \Rightarrow f \in x_F\}.$$

It will also be convenient to define the choice function for f as a buyer when f has access to the set of contracts Y as a buyer and the set of contracts Z as a seller. Hence we define

$$C_B^f(Y|Z) \equiv \left\{x \in C^f(\{y \in Y : y_B = f\} \cup \{z \in Z : z_S = f\}) : x_B = f\right\}.$$

Analogously, we define

$$C_S^f(Z|Y) \equiv \left\{x \in C^f(\{y \in Y : y_B = f\} \cup \{z \in Z : z_S = f\}) : x_S = f\right\}.$$

We also define the **rejected set** of contracts when acting as a buyer or as a seller as

$$R_B^f(Y|Z) \equiv Y - C_B^f(Y|Z),$$
$$R_S^f(Z|Y) \equiv Z - C_S^f(Z|Y).$$

Let $C_B(Y|Z) \equiv \bigcup_{f \in F} C_B^f(Y|Z)$ be the set of contracts chosen from Y by some firm as a buyer, and $C_S(Z|Y) \equiv \bigcup_{f \in F} C_S^f(Z|Y)$ be the set of contracts chosen from Z by some firm as a seller. Let $R_B(Y|Z) \equiv Y - C_B(Z|Y)$ and $R_S(Z|Y) \equiv Z - C_S(Z|Y)$.

An **allocation** is a set of contracts $A \subseteq X$. Preference relations are extended to allocations in a natural way: for two allocations $W, V \subseteq X$, write $W \succ_f V$ to mean $W|_f \succ_f V|_f$.

2.1.2 Stability

The key question in matching theory is whether or not an allocation A is stable, that is, whether or not there exists a blocking set of contracts Z such that all firms in Z_F will choose their contracts in Z from $Z \cup A$ (and possibly drop contracts in A).

Definition 1. An allocation A is **stable** if it is

1. **Individually rational**: for all $f \in F$, $C^f(A) = A|_f$.
2. **Unblocked**: There does not exist a nonempty **blocking set** $Z \subseteq X$ such that $Z \cap A = \varnothing$ and for all $f \in F$, $Z|_f \subseteq C^f(A \cup Z)$.

This notion is the natural generalization of the prior notions of stability in the one-to-one and many-to-one literature.[11,12] Stability is inherently a price-theoretic notion. For an allocation A to be stable, we must be able to find offer sets (that is, sets of contracts offered to each firm) such that

- each contract in X is offered to some firm and
- each contract not part of the allocation A is rejected by every firm to which it is offered.[13]

This is similar to the idea of competitive equilibrium in price theory, where the demand for each item, given the prices, exactly equals the supply.

Stronger notions of stability allow for firms to play "strategically", that is, to take on a set contracts Z from which they obtain a better overall allocation, even though a particular

[11] See Roth and Sotomayor (1990) and Hatfield and Milgrom (2005) for definitions of stability in the one-to-one and many-to-one contexts. See Hatfield and Kominers (2010) for a discussion of the relationship between these concepts.

[12] This definition of stability is also strictly stronger than the "chain stability" and "tree stability" concepts introduced by Ostrovsky (2008); we discuss further the relationship between these concepts in Section 3.3.

[13] Here, we use the terminology "offer set" instead of "choice set" since firms are typically allowed to choose only one option (or point, or bundle) from a choice set; here they may choose any subset of contracts offered. (See the definition given on the first page of Chapter 1 of Mas-Colell et al. (1995) for instance.) Formally the choice set is the power set of the offer set.

contract $z \in Z$ may not be part of $C^f(Z \cup A)$ for some f. However, allocations satisfying these stronger notions of stability, such as *setwise stability* (Roth (1984); Sotomayor (1999b)) or *strong pairwise stability* (Echenique and Oviedo (2006)) often do not exist even for reasonable preferences.

2.1.3 Strategy-proofness

We also consider strategic properties of matching mechanisms. A **matching mechanism** ψ is a mapping from the set of preference profiles to the set of allocations. In particular, we examine whether certain matching mechanisms are strategy-proof for some firms; that is, whether or not it is a weakly dominant strategy for firms to truthfully reveal their preferences.

Definition 2. A matching mechanism ψ is **strategy-proof** for $G \subseteq F$ if, for all $g \in G$, for any preference profile P, $\psi(P) \succsim_g \psi\left(\hat{P}^g, P^{-g}\right)$ for all $\hat{P}^g$.

Similarly, we can consider the incentives of groups of firms under a given matching mechanism.

Definition 3. A matching mechanism ψ is **group strategy-proof** for $G \subseteq F$ if, for any preference profile P, for at least one $g \in G$, $\psi(P) \succsim_g \psi\left(\hat{P}^G, P^{-G}\right)$ for all $\hat{P}^G$.

As is standard in the matching literature, for a matching mechanism to not be group strategy-proof, the deviation from truth-telling must make all firms in the coalition strictly better off.[14]

2.2 Conditions on Preferences

2.2.1 Full Substitutability

We now proceed to introduce the two conditions on preferences of Ostrovsky (2008). The primary condition on preferences studied in matching theory is *substitutability*. Intuitively,

[14]See Hatfield and Kojima (2009) for a discussion of the motivation behind this definition.

contracts x and y are substitutes for f if they are the same type of contract for f and they are not complements. For example, if $x_B = y_B = f$, and f rejects the contract y from Y as a buyer while having access to Z as a seller, f will not choose y from the larger set $\{x\} \cup Y$ while still having access to Z as a seller. The formal definition of substitutes is given below:

Definition 4. Preferences are **same-side substitutable** for $f \in F$ if for all $Y' \subseteq Y \subseteq X$ and $Z' \subseteq Z \subseteq X$,

1. $R_B^f(Y'|Z) \subseteq R_B^f(Y|Z)$ and
2. $R_S^f(Z'|Y) \subseteq R_S^f(Z|Y)$.

Note that this condition is over offer sets; it states that any contract that is rejected from a smaller offer set is also rejected from a larger one.

However, for models where firms can be both buyers and sellers, we must consider how additional offers on one side of the market changes firms' choices on the other side of the market. The key condition here is *cross-side complementarity*. Intuitively, contracts y and z are cross-side complements for f where $y_B = f = z_S$ whenever f chooses y from Y as a buyer when the set Z of contracts is available to f as a seller, f still chooses y from Y when $\{z\} \cup Z$ is available to f as a seller.

Definition 5. Preferences are **cross-side complementary** for $f \in F$ if for all $Y' \subseteq Y \subseteq X$ and $Z' \subseteq Z \subseteq X$,

1. $R_B^f(Y|Z) \subseteq R_B^f(Y|Z')$ and
2. $R_S^f(Z|Y) \subseteq R_S^f(Z|Y')$.

Same-side substitutability and cross-side complementarity are closely linked. For illustration, suppose that each contract delineates the transfer of an object: that is, the transfer of an object from the seller to the buyer. If a firm's preferences satisfy both same-side substitutes and cross-side complements, then the firm has "substitutable" preferences over objects: that is, the firm is more willing to buy an object if either there are less other objects available to buy (same-side substitutes), or there are more opportunities for the firm to sell objects he already possesses (cross-side complements). Similarly, the firm is more willing to sell an object if either there are fewer other opportunities to sell objects the firm already possesses (same-side substitutes), or more opportunities to buy an object to replace the one the firm is losing (cross-side complements). In other words, the more objects the firm currently holds, the less willing the firm is to buy/keep new objects. Hence we shall call **fully substitutable** any preference relation that satisfies both same-side substitutes and cross-side complements.

We can characterize the set of preferences which are fully substitutable. We describe the set of contracts that f may choose to sign by the **offer vector** $\vec{q}^f = (\vec{q}^f_x)_{x \in X|_f}$ defined by

$$\vec{q}^f_x(Y) = \begin{cases} 0 & x_B = f \text{ and } x \in Y \\ -1 & x_B = f \text{ and } x \notin Y \\ 0 & x_S = f \text{ and } x \in Y \\ 1 & x_S = f \text{ and } x \notin Y. \end{cases}$$

Intuitivively, $\vec{q}^f_x(Y)$ refers to the "object" associated with contract x. A value of 1 is given if the firm currently owns the object but can not sell it. A value of 0 is given if either the firm does not currently own the object but may choose to buy it, or the firm currently owns the object but may choose to sell it. Finally, a value of -1 is given if the firm does not currently own the object and can not buy it.

Using the above notation, we can represent preferences over offer sets with an indirect utility function u over offer vectors. An indirect utility function u **represents** a preference relation P^f if

$$u\left(\vec{q}^f(Y)\right) > u\left(\vec{q}^f(Y')\right) \Leftrightarrow C^f(Y) \succ_f C^f(Y') \text{ for all } Y, Y' \subseteq X.$$

Of particular interest are preferences that induce a quasisubmodular indirect utility function.[15,16]

Theorem 1. *The preferences of $f \in F$ are fully substitutable if and only if every indirect utility function representing these preferences is quasisubmodular.*

Submodularity of the indirect utility function is the key condition in demand theory for preferences to be demand-theory substitutes; see, e.g., Milgrom (2009). However, in the absence of transferable utility it is impossible to quantify the increase in utility from a newly available contract and therefore we can only characterize the utility functions in terms of the ordinal notion of quasisubmodularity.[17]

2.2.2 The Laws of Aggregate Supply and Demand

A number of important results in two-sided matching theory rely on the law of aggregate demand, which was first introduced by Hatfield and Milgrom (2005).[18] We generalize this concept to the matching in networks framework with the following definitions:

Definition 6. Preferences of f satisfy the **law of aggregate demand** if for all $Y, Z \subseteq X$ and $Y' \subseteq Y$,

$$\left|C_B^f(Y|Z)\right| - \left|C_B^f(Y'|Z)\right| \geq \left|C_S^f(Z|Y)\right| - \left|C_S^f(Z|Y')\right|,$$

and satisfy the **law of aggregate supply** if for all $Y, Z \subseteq X$ and $Z' \subseteq Z$,

$$\left|C_S^f(Z|Y)\right| - \left|C_S^f(Z'|Y)\right| \geq \left|C_B^f(Y|Z)\right| - \left|C_B^f(Y|Z')\right|.$$

[15]Recall that $u(\cdot)$ is quasisubmodular if for all $\vec{q} \leq \vec{r}$ and $\vec{s} \geq 0$ we have that

$$u(\vec{r}+\vec{s}) - u(\vec{r}) > 0 \Rightarrow u(\vec{q}+\vec{s}) - u(\vec{q}) > 0,$$
$$u(\vec{q}+\vec{s}) - u(\vec{q}) < 0 \Rightarrow u(\vec{r}+\vec{s}) - u(\vec{r}) < 0.$$

[16]The utility function is indirect as it is a function of what are contracts are available to the firm, as opposed to what contracts the firm actually chooses.

[17]For firms who are either only buyers or only sellers, Hatfield and Kominers (2010) show that if preferences satisfy (same-side) substitutes, then not only is every function that represents these preferences quasisubmodular, but one can always find a submodular function that represents these preferences. This second result relies on a technique introduced by Chambers and Echenique (2009), who show in general that there exists a monotonic transformation for any quasisubmodular utility function that transforms it to a submodular utility function. A similar technique can not be applied to the current setting, as utility is not monotonic in the offer vector.

[18]Alkan and Gale (2003) introduced a related condition called "size monotonocity".

Note that the laws of aggregate demand and supply generalize the analogous definitions from two-sided matching. In two-sided matching, if a firm is a buyer, then that firm does not choose any contracts as a seller, and so the right-hand side of the law of aggregate demand vanishes, and hence the law of aggregate demand reduces to its usual statement, that a firm, upon recieving additional offers, chooses at least as many offers as it did before. However, when a firm can be both a buyer and a seller of contracts, the condition is more subtle. The law of aggregate demand now imposes that when an firm obtains additional offers as a buyer, that firm takes on at least as many new contracts as a buyer as he takes on as a seller. Intuitively, the condition states that when a firm has a new offer where he is a buyer that he accepts, while holding onto his other offers, that firm will then sell at most one new item. Similarly, the law of aggregate supply can be interpeted to say that when the firm receives a new offer to sell that he accepts while holding onto his other offers, he will choose to buy at most one new item.

3 Existence of Stable Allocations

3.1 Sufficiency of Substitutable Preferences

To prove the existence of a stable allocation, we introduce the operator

$$\Phi_S\left(X^B, X^S\right) \equiv X - R_B\left(X^B | X^S\right)$$
$$\Phi_B\left(X^B, X^S\right) \equiv X - R_S\left(X^S | X^B\right)$$
$$\Phi\left(X^B, X^S\right) = \left(\Phi_B\left(X^B, X^S\right), \Phi_S\left(X^B, X^S\right)\right),$$

a generalized version of the deferred acceptance algorithm of Gale and Shapley (1962). The inputs X^B and X^S are sets of contracts which, respectively, contain the options available to the firms as buyers, and the options available to the firms as sellers. At each iteration of Φ, we obtain a new set of seller options which includes all of X except $R_B\left(X^B | X^S\right)$, the set of contracts currently available to firms as buyers that they are rejecting. Similarly, the new set of buyer options is all of X except $R_S\left(X^S | X^B\right)$.

We first consider fixed points of the operator, and show that these fixed points correspond to stable allocations. Intuitively, at a fixed point we have that every contract is either being accepted (and hence is in both X^B and X^S), is being rejected by a buyer (and hence is in X^B but not X^S) or is being rejected by a seller (and hence is in X^S but not X^B). Since every contract not in $X^B \cap X^S$ is being rejected by some firm, there does not exist a blocking set of contracts Z such that each firm will desire all of those contracts, so long as preferences are fully substitutable.

Theorem 2. *Suppose that the set of contracts X is acyclic and that preferences are fully substitutable. Then if $\Phi\left(X^B, X^S\right) = \left(X^B, X^S\right)$, the allocation $X^B \cap X^S$ is stable. Conversely, if A is a stable allocation, there exist $X^B, X^S \subseteq X$ such that $\Phi\left(X^B, X^S\right) = \left(X^B, X^S\right)$ and $X^B \cap X^S = A$.*

Full substitutability is necessary for both directions of the above result. To see that full substitutablity is necessary for stable allocations to generate fixed points, consider an example where $F = \{f_1, f_2\}$ and $X = \{x, y\}$, where $x_S = y_S = f_1$ and $x_B = y_B = f_2$, as shown in Figure 2.

Now, consider the following preferences:

$$P^{f_1} : \{x, y\} \succ \varnothing,$$
$$P^{f_2} : \{x\} \succ \{y\} \succ \varnothing.$$

Note that the preferences of f_1 are not substitutable. Then $\varnothing$ is the unique stable match, and yet there do not exist $\left(X^B, X^S\right)$ such that $X^B \cap X^S = \varnothing$ and $\Phi\left(X^B, X^S\right) = \left(X^B, X^S\right)$. If either x or y is in X^B, then one of these contracts is not rejected by the buyer f_2, and hence this contract is in $X^S = \Phi_S\left(X^B, X^S\right) = X - R_B\left(X^B | X^S\right)$, contradicting the fact that $X^B \cap X^S = \varnothing$. If both x and y are in X^S, then neither is rejected by the seller f_1, and so both are also in $X^B = \Phi_B\left(X^B, X^S\right) = X - R_S\left(X^S | X^B\right)$, again contradicting the fact that $X^B \cap X^S = \varnothing$.

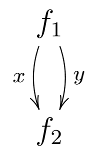

Figure 2: A simple contract structure where the fixed-point characterization fails if preferences are not fully substitutable.

To see that full substitutablity is necessary for fixed points to imply stability, consider altering the example above so that preferences are given by

$$P^{f_1}: \{x, y\} \succ \varnothing,$$

$$P^{f_2}: \{x, y\} \succ \varnothing.$$

In this case, $\{x, y\}$ is the unique stable match, but $(\{x\}, \{y\})$ is a fixed point and corresponds to the match $\varnothing$.

Furthermore, acyclicity is necessary for fixed points to imply stable matches. Consider the case where there are three firms, f_1, f_2, g, and the contract structure is as shown in Figure 3. When firms' preferences are given by

$$P^{f_1}: \left\{y, x^2\right\} \succ \left\{x^1, x^2\right\} \succ \varnothing, \quad P^{f_2}: \left\{x^2, x^1\right\} \succ \varnothing,$$

$$P^g: \{y\} \succ \varnothing,$$

no stable allocation exists. However, $\left(X^B, X^S\right) = (\varnothing, \{y, x^1, x^2\})$ is a fixed point of the operator.[19]

Theorem 2 shows that, when preferences are fully substitutable, and the contract set is acyclic, there is a bijective correspondence between the set of fixed points of Φ and the set

[19] It is not necessary that no stable match exist for fixed points to not correspond to stable matches. Consider the case where $F = \{f_1, f_2\}$ and $X = \{x, y\}$, where $x_S = y_B = f_1$ and $x_S = y_B = f_2$, and preferences are given by:

$$P^{f_1}: \{x, y\} \succ \varnothing,$$
$$P^{f_2}: \{x, y\} \succ \varnothing,$$

which satisfy full substitutability. In this example, $\{x, y\}$ is the unique stable match, but $(\{x, y\}, \varnothing)$ is a fixed point of $\Phi(\cdot)$ even though $\varnothing$ is not a stable match.

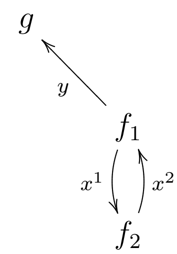

Figure 3: A contract cycle with an outside option.

of stable allocations. We now define the order $\sqsubseteq$ on $X \times X$ as

$$\left(X^B, X^S\right) \sqsubseteq \left(\hat{X}^B, \hat{X}^S\right) \text{ if } X^B \subseteq \hat{X}^B \text{ and } X^S \supseteq \hat{X}^S.$$

It is clear that Φ is isotone with respect to this order if preferences are fully substitutable.[20] Hence, by Tarski's theorem, there exists a lattice of fixed points of this operator. Furthermore, if the contract set is acyclic, these fixed points correspond to stable allocations.

Theorem 3. *Suppose that the set of contracts X is acyclic and that preferences are fully substitutable. Then there exists a nonempty finite lattice of fixed points $\left(X^B, X^S\right)$ of Φ which correspond to stable allocations $A = X^B \cap X^S$.*

Furthermore, this lattice of fixed points has the same structure as in standard bilateral matching contexts. An allocation $\hat{A}$ is the **buyer-optimal stable allocation** if it is preferred to all other stable allocations A by all firms who are exclusively buyers, that is $\hat{A} \succsim_g A$ for all stable allocations A and for all $g \in \{f \in F : \nexists x \in X \text{ such that } x_S = g\}$. Since the set of the fixed points is a lattice, there exists a highest fixed point with respect to the order $\sqsubseteq$, which we denote by $\left(\hat{X}^B, \hat{X}^S\right)$. Since this fixed point has the largest set of contracts for the buyers to choose from, it is the unanimously most preferred stable allocation for all firms who are buyers but not sellers. Similarly, the lowest fixed point with respect to $\sqsubseteq$,

[20]This means that, when preferences are fully substitutable,

$$\left(X^B, X^S\right) \sqsubseteq \left(\hat{X}^B, \hat{X}^S\right) \Rightarrow \Phi\left(X^B, X^S\right) \sqsubseteq \Phi\left(\hat{X}^B, \hat{X}^S\right).$$

denoted $\left(\check{X}^B, \check{X}^S\right)$, is the unanimously most preferred stable allocation for all firms who are sellers but not buyers.

Theorem 4. *Suppose that the set of contracts X is acyclic and that preferences are fully substitutable. Then the highest fixed point $\left(\hat{X}^B, \hat{X}^S\right)$ of Φ corresponds to the buyer-optimal stable allocation $\hat{X}^B \cap \hat{X}^S$, and the lowest fixed-point $\left(\check{X}^B, \check{X}^S\right)$ of Φ corresponds to the seller-optimal stable allocation $\check{X}^B \cap \check{X}^S$.*

Our proof of the existence of a stable allocation naturally generalizes the deferred acceptance approach of Gale and Shapley (1962) to the context of matching with networks. Consequently, our existence (Theorem 3) and opposition of interest (Theorem 4) results naturally and directly generalize those of Gale and Shapley (1962) (for one-to-one and many-to-one matching), Hatfield and Milgrom (2005) (for many-to-one matching with contracts), and Hatfield and Kominers (2010) (for many-to-many matching with contracts). Additionally, these results generalize the analogous results due to Echenique and Oviedo (2006) (for many-to-many matching) and Ostrovsky (2008) (for supply chain matching).

3.2 Necessary Conditions for Stability

From the preceding analysis, it is clear that, in order to ensure the existence of a stable allocation, it is sufficient that preferences are fully substitutable and that the set of contracts is acyclic. In many-to-one matching with contracts, however, there exist weaker conditions on preferences that guarantee the existence of a stable allocation.[21] However, for the more general model of matching in networks, both conditions are necessary: if either of the conditions fails, then there exist preferences for the (other) firms satisfying these conditions such that no stable allocation exists.

[21] Hatfield and Kojima (2008) show that substitutes is not necessary to guarantee existence of a stable allocation for many-to-one matching with contracts. Hatfield and Kojima (Forthcoming) and Hatfield and Kominers (2010) have found in that setting weaker conditions on preferences than substitutability that guarantee the existence of a stable allocation.

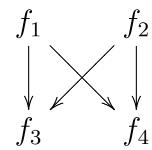

Figure 4: A contract structure in which stable allocations may not exist when preferences are not fully substitutable.

Theorem 5. *If the set of contracts X admits a cycle $L = \left\{x^1, \ldots, x^N\right\}$ and there exists a firm $f \notin L_F$ and a contract between f and some firm in the cycle, then there exist fully substitutable preferences such that no stable allocation exists.*

An example of fully substitutable preferences where no stable match exists is given in the discussion of acyclicity in Section 3.1. Consider again the contract structure given in Figure 3. Intuitively, if contracts form a cycle (such as that between f_1 and f_2), then it is easy to construct preferences satisfying full substitutability such that each firm wants either both contracts it is associated with in the cycle or no contracts at all. With these preferences, the empty allocation is not stable, as all members of the cycle agree that the contract cycle is better than nothing. However, if one member of the cycle (f_1) has an outside option as a buyer (y) then the cycle itself may not be stable as that one member may most prefer to choose this outside option as a buyer while keeping the contract where he is a seller (x^2) from the cycle. This implies that the cycle itself is unstable. However, any other allocation (such as $\{y, x^2\}$) is also unstable, as it is not individually rational for some agent. Hence, we see that even when preferences are reasonable, contract sets containing cycles may not admit stable allocations.

Furthermore, it is also necessary that the preferences of each firm be fully substitutable. As an example, consider the following, where $F = \{f_1, f_2, f_3, f_4\}$ and $X = \{(i, j) : f_i, f_j \in F \text{ and } i \leq 2 < j\}$, as pictured in Figure 4. Here, firms f_1 and f_2 are sellers, and firms f_3 and f_4 are buyers. Suppose that preferences are given by:

$$P^{f_1} : \{(1,3)\} \succ \{(1,4)\} \succ \varnothing,$$

$$P^{f_2} : \{(2,4)\} \succ \{(2,3)\} \succ \varnothing,$$

$$P^{f_3} : \{(1,3), (2,3)\} \succ \varnothing,$$

$$P^{f_4} : \{(1,4)\} \succ \{(2,4)\} \succ \varnothing.$$

It is not stable for both sellers f_1 and f_2 to sell to f_3, as then f_2 would like to deviate and sell to f_4, and f_4 would like to buy from f_2. That, however, is also not stable, as then f_3 would buy nothing, and so f_1 would then sell to f_4. This is also not stable, as then both sellers would prefer to sell to f_3. Hence, no allocation is individually rational and unblocked. Generalizing this example, we obtain the following theorem.

Theorem 6. *Suppose X is exhaustive, there exists a firm f whose preferences are not fully substitutable, and there exist at least two firms upstream and two firms downstream of f. Then, there exist fully substitutable preferences for the firms other than f such that no stable allocation exists.*[22]

Theorem 6 is a generalization of the previous necessity results in the matching literature (see Hatfield and Kominers (2010)). Unlike the previous necessity results, the generality of our framework allows us to demonstrate through Theorem 6 that we are generally unable to find stable assignments in the roommate problem of Gale and Shapley (1962).[23]

3.3 Chain Stability

We now compare our notion of stability to **chain stability**, the notion of stability considered by Ostrovsky (2008). Although chain stability is a less stringent solution concept than the

[22] Slightly weaker conditions on the number of firms other than f can be stated. However, these conditions are exceedingly technical; see the proof of Theorem 6 for details.

[23] Our model does not directly subsume the roommate problem, since contracts in our setting are directed, while in a roommate problem, a match is comprised of undirected links. However, it is possible to embed the roommate problem in our framework by imposing a partial order $\gg$ on agents and expressing a match between agents $f_1 \gg f_2$ by a contract x with $x_S = f_1$ and $x_B = f_2$. (The order structure dictates for bookkeeping purposes that a match between two agents is encoded by a contract in which the higher agent sells to the lower.) With this structure preferences do not generally satisfy cross-side complementarity, hence Theorem 6 applies.

notion of stability used in this paper, these concepts coincide on domains where X is acyclic and preferences are fully substitutable. However, as we illustrate below, chain stability has some unappealing properties on domains where these conditions do not hold. To address these issues formally, we first define a chain of contracts.

Definition 7. A set of contracts $\left\{x^1, \ldots, x^N\right\}$ is a **chain** if

1. $x_B^n = x_S^{n+1}$ for all $n = 1, \ldots, N-1$.

2. $x_S^n = x_S^m$ implies that $n = m$.

3. $x_B^N \neq x_S^1$.

Intuitively, contracts form a chain if each firm holding a contract in the chain sells to the next firm in the chain (condition 1), the chain never doubles back on itself (condition 2), and the chain is not a loop, i.e. the buyer of the last contract is not the seller of the first (condition 3). Naturally, then, an allocation is chain stable if it is not blocked by chains of contracts.

Definition 8. An allocation A is **chain stable** if it is individually rational and there is no chain that is a blocking set.

It is immediate that any stable allocation, regardless of restrictions on preferences or the contract set, is chain stable.[24] However, for fully substitutable preferences over acyclic contract sets, these notions are equivalent.

Theorem 7. *Suppose that the set of contracts X is acyclic and that preferences are fully substitutable. Then an allocation A is stable if and only if it is chain stable.*

Unfortunately, the equivalence of Theorem 7 only holds when the contract set is acyclic and preferences are fully substitutable. If either condition fails, then the set of stable allocations may be a strict subset of the set of chain stable allocations, and furthermore the set of

[24]This follows as chain stability requires that blocking sets be chains, whereas our notion of stability puts no restrictions on the structure of the blocking set.

chain stable allocations may be an intuitively unappealing solution concept. For instance, consider the following example, where $x_S = y_B = f$, and $x_B = y_S = g$:

$$P^f : \{x, y\} \succ \varnothing,$$

$$P^g : \{x, y\} \succ \varnothing.$$

These preferences are fully substitutable; however, the contract set X is not acyclic, as it admits the cycle $\{x, y\}$. For this example, the set $\{x, y\}$ is the only stable allocation, while both $\{x, y\}$ and $\varnothing$ are chain stable. The $\varnothing$ allocation is an unappealing solution to this problem on both normative and positive grounds; $\{x, y\}$ is Pareto preferred to $\varnothing$, and it also seems unreasonable to consider a solution concept which does not allow f and g to take part in a joint deviation from $\varnothing$ to $\{x, y\}$.

If preferences do not satisfy full substitutability, then chain stability is again a strictly weaker concept than stability. Furthermore, chain stability is not equivalent to the standard notions of stability used in the many-to-many matching literature, which are special cases of our stability notion (see Echenique and Oviedo (2006); Klaus and Walzl (2009); Hatfield and Kominers (2010)). Rather, chain stability is equivalent to *pairwise stability*, a much weaker concept. Consider again the (many-to-many matching) example of Section 3.2, where $F = \{f_1, f_2, f_3, f_4\}$ and $X = \{(i, j) : f_i, f_j \in F \text{ and } i \leq 2 < j\}$ (pictured in Figure 4):

$$P^{f_1} : \{(1,3)\} \succ \{(1,4)\} \succ \varnothing,$$

$$P^{f_2} : \{(2,4)\} \succ \{(2,3)\} \succ \varnothing,$$

$$P^{f_3} : \{(1,3), (2,3)\} \succ \varnothing,$$

$$P^{f_4} : \{(1,4)\} \succ \{(2,4)\} \succ \varnothing.$$

Note that the preferences of firm f_3 do not satisfy same-side substitutes. There does not exist a stable allocation; however, $\{(1,4)\}$ is chain stable, as the only blocking set is $\{(1,3), (2,3)\}$, which is not a chain. Furthermore, for preferences such as those of firm

f_3, a chain stable allocation exists for any fully substitutable preferences for other firms so long as the contract set X is acyclic. These chain stable allocations can be found by using the isotone operator Φ introduced above on the set $X - \{x \in X : f_3 \in x_F\}$, that is on the set of contracts in X that do not involve firm f_3; these allocations will be chain stable as there are clearly (by Theorem 2) no blocking sets not involving firm f_3, and the only blocking set involving firm f_3 could be $\{(1,3),(2,3)\}$, which is not a chain. However, for such problems, it is not clear that $\{(1,4)\}$ is the expected outcome; the blocking set $Z = \{(1,3),(2,3)\}$ seems a very natural deviation that makes all members of Z_F strictly better-off.

4 The Structure of the Set of Stable Allocations

4.1 The Rural Hospitals Theorem

In the celebrated *rural hospitals theorem*, Roth (1986) proved that in a many-to-one (doctor-hospital) matching market with responsive preferences, any hospital that has unfilled positions at some stable matching is assigned exactly the same set of doctors at every stable matching.[25] In the context of many-to-one (Hatfield and Milgrom (2005); Hatfield and Kojima (Forthcoming)) and many-to-many matching with contracts (Hatfield and Kominers (2010)), the rural hospitals theorem has been (partially) generalized to a statement regarding the number of contracts: in a many-to-many matching market with contracts, if preferences are substitutable and satisfy the law of aggregate demand, then every firm holds the same number of contracts in each stable allocation.[26]

[25]Recall that hospital preferences are called **responsive** if they are consistent with a complete strict order over individual doctors.

[26]Generalizations of the Roth (1986) rural hospitals theorem to the theory of matching with contracts have focused only upon the total number of contracts signed by each hospital h, since in general both the set of doctors with whom h contracts and the contract terms to which h agrees may vary across stable allocations. Indeed, Hatfield and Kojima (Forthcoming) give (in their Footnote 21) a many-to-one matching with contracts example in which there is a hospital h with unfilled positions such that both

- the set of doctors h contracts with, and
- the contract terms h receives from doctors

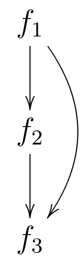

Figure 5: A contract structure in which the most immediate (putative) generalization of the rural hospitals theorem need not hold.

However, this statement is false in the context of matching in networks. For instance, consider the following simple set of preferences, where $F = \{f_1, f_2, f_3\}$ and $X = \{(i, j) : f_i, f_j \in F \text{ and } i < j\}$, as pictured in Figure 5:

$$P^{f_1} : \{(1, 2)\} \succ \{(1, 3)\} \succ \varnothing,$$
$$P^{f_2} : \{(1, 2), (2, 3)\} \succ \varnothing,$$
$$P^{f_3} : \{(1, 3)\} \succ \{(2, 3)\} \succ \varnothing.$$

Note that firm f_2 has the simplest possible fully substitutable preferences where the set of contracts that firm f_2 signs as a buyer truly depends on the set of contracts she has access to as a seller. Furthermore, these preferences satisfy the laws of aggregate demand and supply. However, both $\{(1, 3)\}$ and $\{(1, 2), (2, 3)\}$ are stable allocations, and so we see that the number of contracts in different stable allocations may vary across stable allocations even when preferences satisfy the laws of aggregate supply and demand. Rather, the difference for each firm between the number of contracts where he is a buyer and the number where he is a seller is constant across the two stable allocations; it turns out this "balancing" property holds in general.

Theorem 8. *Suppose that the set of contracts X is acyclic and that preferences are fully substitutable and satisfy the laws of aggregate supply and demand. Then, for each firm, the*

vary across stable allocations. Hence, the invariance of the number of contracts per firm across stable allocations seems to be the sharpest possible matching with contracts generalization of the Roth (1986) theorem.

difference between the number of contracts the firm buys and the number of contracts the firm sells is invariant across stable allocations.

When all contracts are denoted in a fixed unit (as in exchange economies), Theorem 8 implies that each agent holds the same excess stock at every stable allocation. When contracts do not use a constant unit (as, for example, in labor markets in which both full- and part-time job contracts are available), the exact numerical implications of Theorem 8 are less clear, as Hatfield and Kominers (2010) discuss. However, even in this case, the rural hospitals result is crucial for the strategy-proofness results of Section 4.2.

Note that Theorem 8 does generalize the prior rural hospitals theorems: in previous two-sided matching models, a firm f either only buys or only sells, hence Theorem 8 implies that f signs exactly the same number of contracts in every stable allocation. We have that the net position in contracts of a firm which both buys and sells is invariant across stable allocations, but the total number of contracts such a firm signs can vary across stable allocations. Furthermore, the laws of aggregate supply and demand are the weakest possible conditions that ensure this additional structure on the set of stable allocations.

Theorem 9. *Suppose that the set of contracts X is acyclic and exhaustive. Then if the preferences of some firm f do not satisfy the law of aggregate supply or the law of aggregate demand but are fully substitutable, then there exist preferences for the other firms satisfying full substitutability and the laws of aggregate supply and demand such that there exist two stable allocations such that the difference between the number of contracts f buys and the number of contracts f sells is different.*

4.2 Strategy-Proofness

In many-to-one matching with contracts, substitutes and the law of aggregate demand are enough to ensure that the buyer-optimal stable mechanism is strategy-proof for buyers when buyers demand at most one contract. Unfortunately, in many-to-many contexts, it is not strategy-proof for either side of the market to reveal its preferences truthfully to any mechanism which chooses a stable allocation. However, even in the matching in networks framework, if some subset of firms acts only as buyers, and each of these buyers demands at most one contract, then the mechanism which chooses the buyer-optimal stable allocation will be strategy-proof for these buyers.[27]

Theorem 10. *Suppose that the set of contracts X is acyclic and that preferences are fully substitutable and satisfy the laws of aggregate supply and demand. If additionally, for all $g \in G \subseteq F$, the preferences of g exhibit unit demand, then any mechanism that selects the buyer-optimal stable allocation is (group) strategy-proof for G.*

Theorem 10 generalizes results of Dubins and Freedman (1981), Abdulkadìroğlu (2005), Hatfield and Milgrom (2005), and Hatfield and Kojima (2009); its proof follows exactly as in Hatfield and Kojima (2009). This result can be used to show a common corollary that the set of unit-demand buyers weakly prefers the buyer-optimal stable allocation to all other individually rational allocations.

Corollary 1. *Suppose that the set of contracts X is acyclic and that preferences are fully substitutable and satisfy the laws of aggregate supply and demand. If additionally, for all $g \in G \subseteq F$, the preferences of g exhibit unit demand, then there does not exist an individually rational allocation that every member of G strictly prefers to the buyer-optimal stable allocation.*[28]

This result is commonly called "weak Pareto optimality" and was first shown by Roth (1982) for one-to-one matching; the most result general result to date is for many-to-one matching with contracts and is shown by Hatfield and Kojima (2009) and Kojima (2007).

5 Conclusion

In this paper we have extended the model of classical matching theory to consider networks of contracts. We have shown that, on the one hand, if the set of contracts is acyclic,

[27] By symmetry, an analogous result applies to sellers.

[28] As with Theorem 10, the proof of Corollary 1 follows as in Hatfield and Kojima (2009).

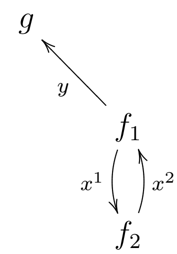

Figure 3: A contract cycle with an outside option.

and preferences are fully substitutable, not only do stable allocations exist, but they form a lattice. Moreover, classical results of matching theory, such as the rural hospitals and strategy-proofness theorems, generalize to this setting. On the other hand, Theorem 5 shows that, in the presence of fully substitutable preferences over contracts, stability cannot be guaranteed if there is a single cycle in the set of contracts; furthermore, Theorem 6 shows that if even one firm does not have fully substitutable preferences, stability again cannot be guaranteed. Hence our work delineates a strict frontier for matching, in the sense that both acyclity and full substitutability are both necessary and sufficient for classical matching theory.

However, supplementing the set of contracts with a numeraire (over which utility is quasilinear) may allow us to go further. Consider the simplest example where a cyclic contract set confounds stability, drawn from the proof of Theorem 5. There are three firms, f_1, f_2, g, and the contract structure is as shown in Figure 3, which we reproduce here. When firms' preferences are given by

$$P^{f_1} : \left\{y, x^2\right\} \succ \left\{x^1, x^2\right\} \succ \varnothing, \quad P^{f_2} : \left\{x^2, x^1\right\} \succ \varnothing,$$
$$P^g : \{y\} \succ \varnothing,$$

there exists no stable allocation. Indeed, firm f_1 prefers its outside option y over x^1, and hence the trade cycle $\{x^1, x^2\}$ always breaks down.

Problems of this form often arise in contracting relationships, and they have a well-known solution, albeit one outside the scope of classical matching theory. One resolution to this dilemma dates back to ideas of Vickrey (1961) and Pigou: firm f_1 should pay a transfer to f_2 equal to the value of the externality f_1 causes by dropping contract x^1 in favor of y. Generalizing this intuition, Hatfield et al. (2010) find that transferable utility promotes stability in some new settings.

Even with transferable utility, full substitutability is necessary in order to guarantee the existence of stable allocations.[29] However, many problems naturally generate complementarities; a hospital may open a new wing only if it acquires doctors of multiple specialities, or a firm may be able to operate more efficiently with more units.[30] Much work remains to be done to understand the dynamics and equilibria of matching markets with complementarities; we leave this topic for future research.

[29]It is easy to construct examples based, for instance, on examples such as those constructed by Milgrom (2007), in which competitive equilibria do not exist.

[30]Milgrom (2007) and Day and Milgrom (2008) discuss some of the issues that arise with complementary preferences in package auctions. Klaus and Klijn (2005) discuss the "couples problem,"a type of complementary preferences that commonly arises in bilateral matching contexts.

References

Abdulkadìroğlu, Atila, "College Admissions with Affirmative Action," *International Journal of Game Theory*, 2005, *33*, 535–549.

_ **, Parag Pathak, and Alvin E. Roth**, "Strategyproofness versus Efficiency in Matching with Indifferences: Redesigning the NYC High School Match," *American Economic Review*, 2009, pp. 1954–1978.

Alkan, Ahmet and David Gale, "Stable Schedule Matching uner Revealed Preference," *Journal of Economic Theory*, 2003, *112*, 289–306.

Chambers, Christopher P. and Federico Echenique, "Supermodularity and preferences," *Journal of Economic Theory*, 2009, *144*, 1004–1014.

Day, Robert and Paul Milgrom, "Core-selecting auctions," *International Journal of Game Theory*, 2008, *36*, 393–407.

Dubins, L. E. and D. A. Freedman, "Machiavelli and the Gale-Shapley Algorithm," *American Mathematical Monthly*, 1981, *88*, 485–494.

Echenique, Federico and Jorge Oviedo, "A theory of stability in many-to-many matching markets," *Theoretical Economics*, 2006, *1*, 233–273.

Fox, Jeremy T., "Estimating matching games with transfers," *NBER Working Paper 14382*, 2008.

Gale, David and Lloyd S. Shapley, "College admissions and the stability of marriage," *American Mathematical Monthly*, 1962, *69*, 9–15.

Hatfield, John William and Fuhito Kojima, "Matching with Contracts: Comment," *American Economic Review*, 2008, *98*, 1189–1194.

— **and** — , "Group Incentive Compatibility for Matching with Contracts," *Games and Economic Behavior*, 2009, *67*, 745–749.

— **and** — , "Substitutes and Stability for Matching with Contracts," *Journal of Economic Theory*, Forthcoming.

— **and Paul R. Milgrom**, "Matching with Contracts," *American Economic Review*, 2005, *95*, 913–935.

— **and Scott Duke Kominers**, "Contract Design and Stability in Matching Markets," 2010. Mimeo, Stanford Graduate School of Business.

— **,** — **, Alexandru Nichifor, Michael Ostrovsky, and Alexander Westkamp**, "Matching with Money," 2010. In preparation.

Kelso, Alexander S. and Vincent P. Crawford, "Job Matching, Coalition Formation, and Gross Substitutes," *Econometrica*, 1982, *50*, 1483–1504.

Klaus, Bettina and Flip Klijn, "Stable matchings and preferences of couples," *Journal of Economic Theory*, 2005, *121*, 75–106.

— **and Markus Walzl**, "Stable Many-to-Many Matchings with Contracts," *Journal of Mathematical Economics*, 2009, *45*, 422–434.

Kojima, Fuhito, "The law of aggregate demand and welfare in the two-sided matching market," *Economics Letters*, 2007, *99*, 581–584.

Konishi, Hideo and M. Utku Ünver, "Credible group-stability in many-to-many matching problems," *Journal of Economic Theory*, 2006, *129*, 57–80.

Mas-Colell, Andreu, Michael D. Whinston, and Jerry R. Green, *Microeconomic theory*, Oxford University Press New York, 1995.

Milgrom, Paul, *Putting Auction Theory to Work* Churchill Lectures in Economics, Cambridge University Press, 2004.

_ , "Package auctions and exchanges," *Econometrica*, 2007, *75*, 935–965.

_ , "Assignment Messages and Exchanges," *American Economic Journal: Microeconomics*, 2009, *1*, 95–113.

Ostrovsky, Michael, "Stability in supply chain networks," *American Economic Review*, 2008, *98*, 897–923.

Roth, Alvin E., "The economics of matching: Stability and incentives," *Mathematics of Operations Research*, 1982, *7*, 617–628.

_ , "Stability and polarization of interests in job matching," *Econometrica*, 1984, *52*, 47–57.

_ , "On the Allocation of Residents to Rural Hospitals: A General Property of Two-Sided Matching Markets," *Econometrica*, 1986, *54*, 425–427.

_ , "The Economist as Engineer: Game Theory, Experimentation, and Computation as Tools for Design Economics," 2002, *70*, 1341–1378.

_ , "Deferred acceptance algorithms: history, theory, practice, and open questions," *International Journal of Game Theory*, 2008, *36*, 537–569.

_ **and Marilda A. O. Sotomayor**, *Two-sided matching: a study in game-theoretic modeling and analysis* Econometric Society Monographs, Cambridge University Press, 1990.

Sotomayor, Marilda A. O., "Three remarks on the many-to-many stable matching problem," *Mathematical Social Sciences*, 1999, *38*, 55–70.

_ , "Three remarks on the many-to-many stable matching problem," *Mathematical Social Sciences*, 1999, *38*, 55–70.

_ , "Implementation in the many-to-many matching market," *Games and Economic Behavior*, 2004, *46*, 199–212.

Vickrey, William, "Counterspeculation, Auctions and Competitive Sealed Tenders," *Journal of Finance*, 1961, *16*, 8–37.

Appendix

Proof of Theorem 1

Proof of the "only if" Direction

Let the preferences of f be given by

$$X^N \succ X^{N-1} \succ \cdots \succ X^2 \succ X^1 \succ \varnothing$$

and let

Consider any two offer vectors $\vec{q}^f(Y^q)$ and $\vec{r}^f(Y^r)$ such that $\vec{q} \leq \vec{r}$ and suppose that $\vec{q}^f_w = \vec{r}^f_w = 0$. Hence $Y^q \subseteq Y^r$ and $Z^q \supseteq Z^r$. Suppose that

$$u\left(\vec{q}^f\left(\{w\} \cup Y^r\right)\right) - u\left(\vec{q}^f\left(Y^r\right)\right) > 0$$

then $w \in C^f_B(\{w\} \cup Y^r)$, so $w \in C^f_B(\{w\} \cup Y^q)$ by same-side substitutes, and so $w \in C^f_B(\{w\} \cup Y^q)$ by cross-side complements. Hence,

$$u\left(\vec{q}^f\left(\{w\} \cup Y^q\right)\right) - u\left(\vec{q}^f\left(Y^q\right)\right) > 0$$

and quasisubmodularity is satisfied when we add any contract to the offer set as a buyer. (Clearly, when we add a contract to the offer set, the utility of the firm can not fall.)

If

$$u\left(\vec{q}^f\left(Y^q\right)\right) - u\left(\vec{q}^f\left(Y^q\right)\right) < 0$$

then $w \in C^f_S(Z^q)$, so $w \in C^f_S(Z^r)$ by same-side substitutes, and so $w \in C^f_S(Z^r)$ by cross-side complements. Hence,

$$u\left(\vec{q}^f\left(Y^r\right)\right) - u\left(\vec{q}^f\left(Y^r\right)\right) < 0$$

and quasisubmodularity is satisfied when we remove any contract to the offer set as a seller. (Clearly, when we remove a contract from the offer set, the utility of the firm can not rise.)

Proof of the "if" Direction

Suppose that preferences violate condition #1 of same-side substitutability. Then there exists contracts $x, w \in X$ and $Y, Z \subseteq X$ such that

$$w \notin C_B^f(Y \cup \{w\}) \text{ and } w \in C_B^f(\{x\} \cup Y \cup \{w\})$$

Now consider any indirect utility function u which represents these preferences. Hence we have that

$$u\left(\vec{q}^f(Y \cup \{w\})\right) - u\left(\vec{q}^f(Y)\right) = 0 < u\left(\vec{q}^f(\{x\} \cup Y \cup \{w\})\right) - u\left(\vec{q}^f(\{x\} \cup Y)\right)$$

so u is not quasisubmodular.

Similarly, suppose that preferences violate condition #2 of same-side substitutability. Then there exists contracts $x, w \in X$ and $Y, Z \subseteq X$ such that

$$w \notin C_S^f(Z \cup \{w\}) \text{ and } w \in C_S^f(\{x\} \cup Z \cup \{w\})$$

Now consider any indirect utility function u which represents these preferences. Hence we have that

$$u\left(\vec{q}^f(Y)\right) - u\left(\vec{q}^f(Y \cup \{w\})\right) = 0 > u\left(\vec{q}^f(Y)\right) - u\left(\vec{q}(Y)\right)$$

so u is not quasisubmodular.

Now suppose that preferences violate condition #1 of cross-side complementarity. Then there exists contracts $x, w \in X$ and $Y, Z \subseteq X$ such that

$$w \in C_B^f(Y \cup \{w\}) \text{ and } w \notin C_B^f(Y \cup \{w\})$$

Now consider any indirect utility function u which represents these preferences. Hence we have that

$$u\left(\vec{q}^f(Y \cup \{w\})\right) - u\left(\vec{q}^f(Y)\right) > 0 = u\left(\vec{q}^f(Y \cup \{w\})\right) - u\left(\vec{q}^f(Y \cup \{z\})\right)$$

so u is not quasisubmodular.

Similarly, suppose that preferences violate condition #2 of cross-side complementarity. Then there exists contracts $x, w \in X$ and $Y, Z \subseteq X$ such that

$$w \in C_S^f (Z \cup \{w\}) \text{ and } w \notin C_S^f (Z \cup \{w\} \ \cup Y)$$

Now consider any indirect utility function u which represents these preferences. Hence we have that

$$u\left(\vec{q}^f (Y)\right) - u\left(\vec{q}^f (Y)\right) > 0 = u\left(\vec{q}^f (\{x\} \cup Y)\right) - u\left(\vec{q}^f (\{x\} \cup Y)\right)$$

so u is not quasisubmodular.

Proof of Theorem 2

First Part

Suppose that $X^B \cap X^S \equiv A$ is a fixed point, but that A is either not individually rational or admits a blocking set Z. If it is not individually rational, there must exist $x \in A$ such that $x \in R^f (A)$ for some $f \in F$. Then either $x \in R_B^f (A|A)$ and $x_B = f$ or $x \in R_S^f (A|A)$ and $x_S = f$. Assume the former. (The latter case is symmetric.) Then $x \in R_B^f \left(X^B|A\right)$ by same-side substitutes. However, every contract in the set $X^S - A$ is rejected by some firm as a seller, and so $R_B^f \left(X^B|A\right) = R_B^f \left(X^B|X^S\right)$ by individual rationality. Hence $x \in R_B^f \left(X^B|X^S\right)$, and hence $x \notin X^S$, and hence $x \notin X^B \cap X^S = A$, a contradiction.

If there exists a blocking set Z, consider a contract z such that $z_S \trianglelefteq y_S$ for all other $y \in Z$.[31] By same-side substitutes, since $z \in C_S^{z_S} (Z \cup A|Z \cup A)$, we have that $z \in C_S^{z_S} (\{z\} \cup A|Z \cup A) = C_S^{z_S} (\{z\} \cup A|A)$ as there are no contracts in Z such that z_S is a buyer by assumption. Hence, by individual rationality, $z \in C_S^{z_S} \left(\{z\} \cup X^S|A\right)$ and then by cross-side complements, $z \in C_S^{z_S} \left(\{z\} \cup X^S|X^B\right)$. Hence, if $z \in X^S$, then it would also be in $X^B = X - R_S \left(X^S|X^B\right)$. But $z \notin A = X^B \cap X^S$ by assumption, and $X^B \cup X^S = X$, and so $z \in X^B$. Now consider an arbitrary contract $w \in Z$, and suppose that for all contracts $y \in Z$ such that $y_S \triangleleft w_S$, $y \in X^B$. By same-side substitutes, since

[31] Recall that acyclicity guarantees there exists an order $\triangleleft$ on F such that $x_S \triangleleft x_B$ for all $x \in X$.

$w \in C_S^{ws}(Z \cup A|Z \cup A)$, we have that $w \in C_S^{ws}(\{w\} \cup A|Z \cup A)$. Now, by induction, for any contract $y \in Z$ such that $y_B = w_S$, $y \in X^B$. Hence, $\{y \in Z : y_B = w_S\} \subseteq X^B$, and $A \subseteq X^B$, hence $\{y \in Z : y_B = w_S\} \cup A \subseteq X^B$, and so by cross-side complemenarity, $w \in C_S^{ws}\left(\{w\} \cup A|X^B\right)$. Hence, by individual rationality, $w \in C_S^{ws}\left(\{w\} \cup X^S|X^B\right)$. Hence, if $w \in X^S$, then it would also be in $X^B = X - R_S\left(X^S|X^B\right)$. But $w \notin A = X^B \cap X^S$ by assumption, and $X^B \cup X^S = X$, and so $w \in X^B$. Using induction then, we have that $Z \subseteq X^B$. Working symmetrically for buyers, we have that $Z \subseteq X^S$. Hence, $Z \subseteq X^S \cap X^B = A$ and hence Z is not a blocking set, a contradiction.

Second Part

Suppose A is a stable allocation. We construct X^B and X^S iteratively over firms; since contracts are acyclic, we order the firms $f_1, \ldots, f_N$ so that $f_n \lhd f_{n+1}$ for all $n = 1, \ldots, N-1$. Let $X^B(0) = X^S(0) = A$. Let

$$X^B(n) = \left\{x \in \left(X - X^S(n-1)\right) : x_B = f_n\right\} \cup X^B(n-1)$$
$$X^S(n) = \left\{x \in X : x \in R_S^{f_n}\left(\{x\} \cup A|X^B(n)\right)\right\} \cup X^S(n-1)$$

Finally, let $X^B = X^B(N)$ and $X^S = X^S(N)$.

We now show that $\left(X^B, X^S\right)$ is a fixed point. We have

$$\begin{aligned}
X - R_S\left(X^S|X^B\right) &= X - \textstyle\bigcup_{n=1}^N R_S^{f_n}\left(X^S|X^B\right) \\
&= X - \textstyle\bigcup_{n=1}^N R_S^{f_n}\left(X^S|X^B(n)\right) \text{ as } X^B(n) = \left\{x \in X^B : x_B = f_n\right\} \\
&= X - \textstyle\bigcup_{n=1}^N R_S^{f_n}\left(X^S(n)|X^B(n)\right) \text{ as } X^S(n) = \left\{x \in X^S : x_S = f\right\} \\
&= X - \textstyle\bigcup_{n=1}^N R_S^{f_n}\left(X_S(n) - A|_{f_n}\right) \text{ as } f_n \text{ has fully substitutable preferences} \\
&= X - \left(X^S - A\right) \text{ by the definition of } X^S \\
&= X^B \text{ by the definition of } X^B
\end{aligned}$$

Similarly,

$$
\begin{aligned}
X - R_B\left(X^B|X^S\right) &= X - \bigcup_{n=1}^N R_B^f\left(X^B|X^S\right) \\
&= X - \bigcup_{n=1}^N R_B^f\left(X^B|X^S(n)\right) \text{ as } X^S(n) = \left\{x \in X^S : x_S = f_n\right\} \\
&= X - \bigcup_{n=1}^N R_B^f\left(X^B(n)|X^S(n)\right) \text{ as } X^B(f) = \left\{x \in X^B : x_B = n\right\} \\
&= X - \bigcup_{n=1}^N \left(X^B(n) - A|_{f_n}\right) \text{ as shown below} \\
&= X - \left(X^B - A\right) \text{ by the definition of } X^B \\
&= X^S \text{ by the definition of } X^S
\end{aligned}
$$

To see the fourth equality, suppose that there exists a nonempty set of contracts

$$Y \equiv \left\{y^1, \ldots, y^k\right\} \in \bigcup_{n=1}^N \left(X^B(n) - A|_{f_n}\right) - \bigcup_{n=1}^N R_B^f\left(X^B(n)|X^S(n)\right)$$

We also know that no contract Y is rejected by a seller (assuming they have access to X^B as buyers) as these are contracts X^B. Hence, Y is a blocking set and hence A is not stable, a contradiction. Finally, $A|_{f_n} \cap R_B^f\left(X^B(n)|X^S(n)\right) = \varnothing$ as A is individually rational.

Finally, we need to show that $X^B \cap X^S = A$. First, since $X^B(0) \cap X^S(0) = A$ and $X^B(n-1) \subseteq X^B(n)$ and $X^S(n-1) \subseteq X^S(n)$, $A \subseteq X^B(0) \cap X^S(0)$. Suppose that $z \in X^S - A$. Then $z \notin X^B$, as it could only be added in the z_B-th step and since $z \in X^S$, $a \in X^S(f_{z_B} - 1)$.

Proof of Theorem 4

For any stable allocation $\left(X^B, X^S\right)$, we have that

$$
\begin{aligned}
X^B \cap X^S &= X^B \cap \left(X - R_B\left(X^B|X^S\right)\right) \\
&= X^B \cap \left(X - \left(X^B - C_B\left(X^B|X^S\right)\right)\right) \\
&= X^B - \left(X^B - C_B\left(X^B|X^S\right)\right) \\
&= C_B\left(X^B|X^S\right)
\end{aligned}
$$

Since for each firm f who is only a buyer, $C_B^f\left(X^B|X^S\right) = C^f\left(X^B\right)$, the firm f has a strictly larger choice set under $\left(\hat{X}^B, \hat{X}^S\right)$ than under any other stable allocation, and hence (weakly) prefers $\left(\hat{X}^B, \hat{X}^S\right)$. The proof that $\left(\check{X}^B, \check{X}^S\right)$ is the seller-optimal stable allocation is symmetric.

Proof of Theorem 5

Let y be the contract with f and some member of the cycle. Define $f_n \equiv x^n_S$. Suppose that $y_S = f_1$ and $y_B = g$. (The other case is symmetric.) Then for $n = 2, \ldots, N$, let the preferences of firm f_n be

$$P^{f_n} : \left\{x^n, x^{n+1}\right\} \succ \varnothing$$

which satisfies same-side substitutes and cross-side complements. Let

$$P^g : \{y\} \succ \varnothing$$
$$P^{f_1} : \left\{y, x^N\right\} \succ \left\{x^1, x^N\right\} \succ \varnothing$$

which satisfies full substitutability; let all other firms desire no contracts. Any set $Y \nsubseteq L \cup \{y\}$ is not stable, as it is not individually rational. Any $Z \subsetneq L \cup \{y\}$ is not stable, as it not individually rational unless $Z = \varnothing$, in which case L is a blocking set, or $Z = L$, in which case $\{y\}$ is a blocking set.

Proof of Theorem 6

If the preferences of a firm f do not satisfy same-side substitutes, then there exists contracts x, y and sets of contracts Y, Z such that

$$y \notin C^f_B(Y|Z) \text{ but } y \in C^f_B(\{x\} \cup Y|Z) \text{ or}$$
$$y \notin C^f_S(Y|Z) \text{ but } y \in C^f_S(Y|\{x\} \cup Z)$$

Assume the former; the latter case is symmetric. There are two cases.

Case 1: $x_S \neq y_S$. Let the firms in Y_S and Z_B (except x_S and y_S) have preferences such that they are willing to accept any all contracts with f that they are associated with. Let y_S have preferences such that he would be willing to accept any and all of the contracts he is associated with in Y, except that he wants exactly one of w and y, and would prefer y, where $w_S = y_S$ and $w_B = x_B$. Let x_S have preferences such that he would be willing to accept any and all of the contracts he is associated with in Y, except that he also desires w and only desires x if he obtains w. Finally, let all other firms not want any contracts. Now suppose that there is a stable allocation A.

1. Suppose $A|_f \prec_f C^f(Y \cup Z)$. If A is individually rational, all other firms want their contracts in $C^f(Y \cup Z)$ irrespective of their other contracts, so $C^f(Y \cup Z)$ is a blocking set.
2. Suppose $A|_f = C^f(Y \cup Z)$. Then both x_S and y_S desire the contract w; hence $w \in A$. But then $C^f(\{x\} \cup Y \cup Z)$ is a blocking set.
3. Suppose $C^f(\{x\} \cup Y \cup Z) \succ_f A|_f \succ_f C^f(Y \cup Z)$. In this case, $x \in A$ as otherwise $A|_f$ is available to f from $Y \cup Z$, so we could not have $A|_f \succ_f C^f(Y \cup Z)$. But then, $C^f(\{x\} \cup Y \cup Z)$ is a blocking set.
4. Suppose $C^f(\{x\} \cup Y \cup Z) = A|_f$. Then if $w \in A$, A is not individually rational for y_S; if $w \notin A$, A is not individually rational for x_S.

Case 2: $x_S = y_S \equiv d$. Let the firms in Y_S and Z_B (except y_S) have preferences such that they are willing to accept any all contracts with f that they are associated with. By assumption, there are two firms, g and h, downstream of f, and one firm, e upstream of f, and hence upstream of g and h. Now consider the contracts v, v', w and w' such that $v_S = w_S = d$, $v'_S = w'_S = e$, $v_B = v'_B = g$ and $w_B = w'_B = h$ (which exist as X is exhaustive). g and h are willing to accept any and all contracts in Y, and outside of that, have preferences given by

$$P^g : \{v\} \succ \{v'\} \succ \varnothing$$

$$P^h : \{w'\} \succ \{w\} \succ \varnothing$$

and e is always willing to accept any and all contracts in Y, and outside of that, has preferences

$$P^e : \{v'\} \succ \{w'\} \succ \varnothing$$

Finally, d is always willing to accept any and all contracts in Y except y; his preferences over x, y, w, v are responsive, where he wishes to obtain two contracts in accord with the preferences over singleton contracts given by $w \succ z \succ x \succ v$. Now suppose there exists a stable allocation A.

1. Suppose $A|_f \prec_f C^f(Y \cup Z)$. Then $C^f(Y \cup Z)$ constitutes a blocking set, as all the firms want their contracts in $C^f(Y)$, irrespective of other contracts.

2. Suppose $A|_f = C^f(Y \cup Z)$. Since d does not obtain x or y, he desires both v and w. For A to be stable then, d obtains v. Furthermore, since e does not obtain v', for A to be stable, e must obtain w'. Hence $\{w', v\} \subseteq A$ and $w \notin A$. Hence $C^f(\{x\} \cup Y \cup Z)$ is a blocking set.

3. Suppose $C^f(Y \cup Z) \prec_f A|_f \prec C^f(\{x\} \cup Y \cup Z)$. Then $x \in A$, so $C^f(\{x\} \cup Y \cup Z) - \{x\}$ is a blocking set, as d will always take y and the other firms in Y and Z will always accept offers of any and all contracts in Y.

4. Suppose $C^f(\{x\} \cup Y \cup Z) = A|_f$. Then $v' \in A$, as otherwise $\{v'\}$ is a blocking set. (Note that in this case $v \notin A$, as then $\{x, y, v\} \subseteq A$, and so A is not individually rational for d.) But $v' \in S$ implies that $w' \notin S$, by the individual rationality of e. Hence w is a blocking set. (Note that $w \notin A$, as then $\{w, y, w\} \subseteq A$, and so A is not individually rational for d.)

If the preferences of a firm f do not satisfy cross-side complements, then there exists contracts y, z and sets of contracts Y, Z such that

$$y \in C_B^f(\{y\} \cup Y | Z) \text{ but } y \notin C_B^f(\{y\} \cup Y | \{z\} \cup Z) \text{ or}$$

$$z \in C_S^f(\{z\} \cup Z | Y) \text{ but } z \notin C_S^f(\{z\} \cup Z | \{y\} \cup Y)$$

Assume the latter; the former case is symmetric.

Let the firms in Y_S and Z_B (except y_S and z_B) have preferences such that they are willing to accept any all contracts with f that they are associated with. Let y_S have preferences such that he would be willing to accept any and all of the contracts he is associated with in Y, except that he wants exactly one of w and y, and would prefer w, where $w_S = y_S$ and $w_B = z_B$. Let z_B have preferences such that he would be willing to accept any and all of the contracts he is associated with in Z, except that he wants exactly one of w and z, and would prefer z. Finally, let all other firms not want any contracts. Now suppose that there is a stable allocation A.

1. Suppose $A|_f \prec_f C^f(Y \cup \{z\} \cup Z)$. If A is individually rational, all other firms want their contracts in $C^f(Y \cup \{z\} \cup Z)$ irrespective of their other contracts, so $C^f(Y \cup \{z\} \cup Z)$ is a blocking set.

2. Suppose $A|_f = C^f(Y \cup Z)$. Then both y_S and z_B desire the contract w; hence $w \in A$. But then $A \supseteq \{w\} \cup C^f(Y \cup \{z\} \cup Z)$ is not individually rational for y_S.

3. Suppose $C^f(\{y\} \cup Y \cup \{z\} \cup Z) \succ_f A|_f \succ_f C^f(Y \cup \{z\} \cup Z)$. In this case, $y \in A$ as otherwise $A|_f$ is available to f from $Y \cup \{z\} \cup Z$, so we could not have $A|_f \succ_f C^f(Y \cup \{z\} \cup Z)$. But then, $C^f(\{x\} \cup Y \{z\} \cup \cup Z)$ is a blocking set.

4. Suppose $C^f(\{y\} \cup Y \cup \{z\} \cup Z) = A|_f$. Then if $w \in A$, A is not individually rational for z_B; if $w \notin A$, $\{w\}$ is a blocking set.

Proof of Theorem 7

Consider an allocation A that is not stable. If it is not individually rational, then it is not chain stable by definition. Hence, suppose there is a blocking set Z. Since X is acyclic, there is an ordering of firms in Z_F such that $x_B \rhd x_S$. Find the firm f for which $f \rhd g$ for all $g \in Z_F$, and consider one contract $y^1 \in Z$ such that $y^1_B = f$. Now consider y^1_S. By same-side substitutes, $y^1 \in C^{y^1_B}_B(\{y^1\} \cup A|A)$ and $y^1 \in C^{y^1_S}_S(\{y^1\} \cup A|A \cup Z)$.

If $y^1 \in C_S^{y_S^1}(\{y^1\} \cup A|A)$, then $\{y^1\}$ is a chain and a blocking set and we are done, if not there exists a contract $y^2 \in Z$ such that $y^1 \in C_S^{y_S^1}(\{y^1\} \cup A|A \cup \{y^2\})$ and $y^2 \in C_B^{y_S^1}(\{y^2\} \cup A|A \cup \{y^1\})$, by same-side substitutes. (If no such y^2 exists, then preferences do not satisfy same-side substitutes, as there exists $y^2 \notin C_B^{y_S^1}(\{y^2\} \cup A|A \cup \{y^2\})$, but $y^2 \in C_B^{y_S^1}(Z \cup A|A \cup \{y^2\})$.) Using the same arguments for the contract y^2, either $\{y^1, y^2\}$ is a chain and a blocking set, or there exists $y^3 \in Z$ such that $y^2, y^3 \in C^{y_S^2}(\{y^1, y^2, y^3\} \cup A)$. More generally, $\{y^1, y^2, \ldots, y^n\}$ is a chain and a blocking set, or there exists $y^{n+1} \in Z$ such that $y^n, y^{n+1} \in C^{y_S^n}(\{y^1, \ldots, y^{n+1}\} \cup A)$. Since X is finite, Z is finite, and hence there exists a chain $\left\{y^1, \ldots, y^N\right\}$ that is a blocking set.

5.1 Proof of Theorem 8

Consider any stable allocation A associated with the fixed point $\left(X^S, X^B\right)$ and the seller-optimal stable allocation $\hat{A}$ associated with the fixed point $\left(\hat{X}^S, \hat{X}^B\right)$. Consider an arbitrary firm f. We have that

$$\begin{aligned}\left|C_f^S\left(\hat{X}^S|\hat{X}^B\right)\right| - \left|C_f^B\left(\hat{X}^B|\hat{X}^S\right)\right| &\geq \left|C_f^S\left(X^S|\hat{X}^B\right)\right| - \left|C_f^B\left(\hat{X}^B|X^S\right)\right| \\ &\geq \left|C_f^S\left(X^S|X^B\right)\right| - \left|C_f^B\left(X^B|X^S\right)\right|,\end{aligned}$$

where the first inequality follows by the LoAS, as $\hat{X}^S \supseteq X^S$, and the second follows by the LoAD, as $\hat{X}^B \subseteq X^B$. Hence, the difference between the number of contracts f sells and the number f buys is weakly greater under $\left(\hat{X}^S, \hat{X}^B\right)$ than $\left(X^S, X^B\right)$. However, summing over all firms, the difference between the number of contracts bought and the number of contracts sold is zero. Hence, for each firm the change in the difference between the number of sold and the number bought is zero.

Proof of Theorem 9

Let the ordering of firms given X such that no firm sells to a lower-orded firm by $\lhd$. There are three possibilities:

1. There exists a contract $z \in X$ such that

$$x, y \in C^f_S(Z|Y) \text{ and } x, y \notin C^f_S(\{z\} \cup Z|Y)$$

(The case where f is a buyer is symmetric.) For any firm $g \in (Y \cup Z) - \{x_B, y_B, z_B\}$ let them desire any and all contracts in $Y \cup Z$ they are associated with. Furthermore, let $\{x_B, y_B, z_B\}$ all desire any and all contracts they are associated with in Z regardless of their other contracts.

There are two subcases:

(a) We have either $f \lhd x_B \lhd z_B$ or $f \lhd y_B \lhd z_B$; assume the former (the latter is symmetric). Then let $\{x_B, y_B, z_B\}$ have these additional preferences over contracts besides $Z - \{x, y, z\}$:

$$P^{x_B} : \{x, \hat{y}\} \succ \{x, \hat{z}\} \succ \{x\} \succ \{\hat{y}\} \succ \{\hat{z}\} \succ \varnothing$$
$$P^{y_B} : \{y\} \succ \{\hat{y}\} \succ \varnothing$$
$$P^{z_B} : \{\hat{z}\} \succ \{z\} \succ \varnothing$$

There are two stable allocations: $C^f(Z \cup Y) \cup \{\hat{z}\}$ and $C^f(\{z\} \cup Z \cup Y) \cup \{\hat{y}\}$, and hence the conclusion of the rural hospitals theorem fails for f.

(b) We have $f \lhd z_B \lhd x_B, y_B$. Then let $\{x_B, y_B, z_B\}$ have these additional preferences over contracts besides $Z - \{x, y, z\}$:

$$P^{z_B} : \{z, \hat{x}\} \succ \varnothing$$
$$P^{y_B} : \{y\} \succ \varnothing$$
$$P^{x_B} : \{x\} \succ \{\hat{x}\} \succ \varnothing$$

There are two stable allocations: $C^f(Z \cup Y)$ and $C^f(\{z\} \cup Z \cup Y) \cup \{\hat{x}\}$, and hence the conclusion of the rural hospitals theorem fails for f.

2. There exists a contract $z \in X$ such that

$$x \in C_S^f(Z|Y), y \notin C_B^f(Y|Z) \text{ and } x \notin C_S^f(\{z\} \cup Z|Y), y \in C_B^f(Y|\{z\} \cup Z)$$

(The case where f is a buyer is symmetric.) For any firm $g \in (Y \cup Z) - \{x_B, y_B, z_B\}$ let them desire any and all contracts in $Y \cup Z$ they are associated with. Furthermore, let $\{x_B, y_B, z_B\}$ all desire any and all contracts they are associated with in Z regardless of their other contracts. Then let $\{x_B, y_B, z_B\}$ have these additional preferences over contracts besides $Z \cup Y - \{x, y, z\}$:

$$P^{x_B} : \{x\} \succ \varnothing$$

$$P^{y_S} : \{y\} \succ \{\hat{y}\} \succ \varnothing$$

$$P^{z_B} : \{\hat{y}\} \succ \{z\} \succ \varnothing$$

There are two stable allocations: $C^f(Z \cup Y) \cup \{\hat{y}\}$ and $C^f(\{z\} \cup Z \cup Y)$, and hence the conclusion of the rural hospitals theorem fails for f.

3. There exists a contract $z \in X$ such that

$$x, y \notin C_B^f(Y|Z) \text{ and } x, y \in C_B^f(Y|\{z\} \cup Z)$$

(The case where f is a buyer is symmetric.) For any firm $g \in (Y \cup Z) - \{x_B, y_B, z_B\}$ let them desire any and all contracts in $Y \cup Z$ they are associated with. Furthermore, let $\{x_B, y_B, z_B\}$ all desire any and all contracts they are associated with in Z regardless of their other contracts. Then let $\{x_B, y_B, z_B\}$ have these additional preferences over contracts besides $Z \cup Y - \{x, y, z\}$:

$$P^{x_S} : \{x\} \succ \varnothing$$

$$P^{y_S} : \{y\} \succ \{\hat{y}\} \succ \varnothing$$

$$P^{z_B} : \{\hat{y}\} \succ \{z\} \succ \varnothing$$

There are two stable allocations: $C^f(Z \cup Y) \cup \{\hat{y}\}$ and $C^f(\{z\} \cup Z \cup Y)$, and hence the conclusion of the rural hospitals theorem fails for f.