Dynamic Variational Preferences*

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July, 2005

Abstract

We introduce and axiomatize dynamic variational preferences, the dynamic version of the variational preferences we axiomatized in [14], which include the Multiplier Preferences inspired by robust control and used in macroeconomics, as in Hansen and Sargent ([9]), as well as Mean Variance Preferences of Markovitz and Tobin, used in finance. We provide a condition that makes dynamic variational preferences time consistent, and their representation recursive. This gives them the analytical tractability needed in finance and macroeconomic applications. A corollary of our results is that Multiplier Preferences are time consistent, but Mean Variance Preferences are not.

1 Introduction

In the Multiple Priors (MP) model agents rank acts h using the criterion

$$V(h) = \inf_{p \in C} \mathrm{E}^p [u(h)], \tag{1}$$

where C is a closed and convex subset of the set Δ of all probabilities on states. This model has been axiomatized by Gilboa and Schmeidler [7] with the goal of modeling ambiguity averse agents, who exhibit the Ellsberg-type behavior first observed in the seminal paper of Ellsberg [4].

The nonsingleton nature of C reflects the limited information that MP agents may have, which may not be enough to quantify their beliefs with a single probability, and it is, instead, compatible with a nonsingleton set C of probabilities.

On the other hand, the cautious attitude featured by MP agents can also be viewed as the result of the effect that an adversarial influence, which we may call "Nature," has on the realizations of the state. Under this view, Nature chooses a probability p over states with the objective of minimizing agents' utility, conditional on their choice of an act and under the constraint that the probability p has to be chosen in a fixed set C. This interpretation of the MP model provides an intuitive notion of ambiguity aversion, which can be regarded as the agents' diffidence for any lack of precise

*We thank Larry Epstein and Gino Favero for helpful suggestions. Part of this research was done while the first two authors were visiting the Department of Economics of Boston University and CERMSEM (Université Paris 1), which they thank for their hospitality. They also gratefully acknowledge the financial support of the Ministero dell'Istruzione, dell'Università e della Ricerca. Rustichini gratefully acknowledges the financial support of the National Science Foundation (grant # 0136556).

definition of the uncertainty involved in a choice, something that provides room for the malevolent influence of Nature.[1]

In a recent paper, [14], we extended the MP representation by generalizing Nature's constraint. Specifically, in our extension the constraint on Nature is given by a cost $c(p)$ associated with the choice of probability, and agents rank acts according to the criterion:

$$V(h) = \inf_{p \in \Delta} \left(\mathrm{E}^p [u(h)] + c(p) \right), \tag{2}$$

where c is a closed and convex function on Δ. Preferences represented by (2) are called *variational preferences* (VP), and the function c is their *ambiguity index*. In [14] we axiomatize the representation (2) and we discuss in detail its ambiguity interpretation, as well as the malevolent Nature interpretation we emphasize here in view of possible macroeconomic applications.

The VP representation generalizes the MP representation, which is the special case where there is an infinite cost for choosing outside the set C, with the cost being constant (and hence, without loss of generality, zero) inside that set. In other words, the cost for Nature in the MP model is given by the indicator function $\delta_C : \Delta \to [0, \infty]$ of C, defined as

$$\delta_C(p) = \begin{cases} 0 & \text{if } p \in C, \\ \infty & \text{if } p \notin C, \end{cases} \tag{3}$$

and it is immediate to see that

$$\inf_{p \in \Delta} \left(\mathrm{E}^p [u(h)] + \delta_C(p) \right) = \inf_{p \in C} \mathrm{E}^p [u(h)].$$

The notion of ambiguity aversion has found an important application in the last years in the literature, pioneered by Hansen and Sargent (see, e.g., [9]), that applies the idea of robust control to the choice of agents in macroeconomic models. While the initial definition of robust control was different from that of ambiguity aversion, the intuition is closely related: an agent prefers a robust control if he is not confident that his (probabilistic) model of the uncertainty is correct, and so he wants to avoid the possibility that a small error in the formulation of the stochastic environment produces a large loss. Ambiguity aversion comes up because the agents' information is too limited to be represented by a single probabilistic model.

In the *multiplier preferences* model, the most important choice model used in this macroeconomic literature (see [9]), the constraint on Nature is represented by a cost c based on a reference probability $q \in \Delta$: Nature can deviate away from q, but the larger the deviation, the larger the cost. In particular, this cost is assumed to be proportional to the relative entropy $R(p\|q)$ between the chosen probability p and reference probability q; that is,

$$c(p) = \theta R(p\|q),$$

where $\theta > 0$. Multiplier preferences are, therefore, the special case of variational preferences given by

$$\inf_{p \in \Delta} \left(\mathrm{E}^p [u(h)] + \theta R(p\|q) \right),$$

and their analytical tractability is important in deriving optimal policies.

Even though the motivation behind multiplier preferences was similar to that used for MP preferences, formally multiplier preferences are not MP preferences. In fact, in [14] we show that

[1] As Hart, Modica , and Schmeidler [10, p. 352] write "In Gilboa and Schmeidler [7] it is shown that preferences ... are represented by functionals of the form $f \longmapsto \min_{q \in Q} \sum_s u(f(s))\, q(s)$, for some closed convex set $Q \subset \Delta(S)$. So the ambiguity averse decision maker behaves 'as if' there were an opponent who could partially influence occurence of states to his disadvantage (i.e., think of the opponent as choosing $q \in Q$)."

they are examples of divergence preferences, a special class of variational preferences featuring tractable cost functions, but which are not MP preferences. Variational preferences are, therefore, the generalization needed in order to encompass both MP and multiplier preferences, as discussed at length in [14].

In view of applications, however, the static analysis of [14] is insufficient and a dynamic extension is required. This is the purpose of the present paper, in which we introduce and axiomatize dynamic variational preferences.

The first observation to make is that, while in a static environment acts are functions from states to consequences, in a dynamic environment they are functions from times and states to consequences. We impose on acts the usual measurability conditions ensuring that agents' choices are consistent with the information they have. As a result, agents' evaluations are conditional to time and state, and they are modelled by a family of (conditional) preferences $\succsim_{t,\omega}$ indexed by time and state pairs (t,ω). In the main results of the paper we provide necessary and sufficient conditions guaranteeing that agents' preferences $\succsim_{t,\omega}$ are represented by the preference functional $V_t(h) : \Omega \to \mathbb{R}$ given by

$$V_t(h) = \inf_{p \in \Delta} \left(\mathrm{E}^p \left[\sum_{\tau \geq t} \beta^{\tau - t} u(h_\tau) \,|\mathcal{G}_t \right] + c_t(p\,|\mathcal{G}_t) \right), \tag{4}$$

and we show what restrictions on c_t guarantee time consistency.[2] Under time consistency the representation (4) becomes recursive, and so it has the analytical tractability required in applications.

Besides tractability, time consistency has also an intuitive appeal. In fact, suppose that two acts are the same in every contingency up to the present period, and the first is preferred to the second according to the conditional preference in the next period in every state. Then time consistency requires that the first act should be preferred to the second in the present period. Equivalently, think of a plan as a sequence of conditional choices, so that the choice of a plan in the current period includes a plan of choices in all future periods, conditional on all future contingencies. Then, an agent is time consistent if he never formulates a plan of future choices that he wants to revise later in some event that is conceivable today.

1.1 The No-Gain Condition and Bayesian Updating

Our work extends to the VP setting the recent dynamic version of the MP model provided by Epstein and Schneider [5]. They give a condition, called rectangularity, that guarantees time consistency of MP preferences. Since rectangularity is a restriction on the sets of probabilities from which Nature can select at every time and state, it is therefore natural that our corresponding condition is formulated as a restriction on cost functions.

Specifically, our condition is given by (11) of Theorem 1. To facilitate the exposition, we present it in a simplified form, dropping the time index (the reader may think of this as the condition for the two-period version of the model). The agent has a partition $\mathcal{G}$ over the set of possible states (see the picture at p. 7). Nature has a cost c_Ω in the first period, so that $c_\Omega(q)$ is Nature's cost of choosing the probability q over the states. To each event G in this partition it is associated a new, second period, cost c_G. The announced condition requires that:

$$c_\Omega(q) = \inf_{\{p : p(G) = q(G)\ \forall G \in \mathcal{G}\}} [\beta \sum_{G \in \mathcal{G}} q(G) c_G(q_G) + c_\Omega(p)], \tag{5}$$

where $q(G) = \sum_{\omega \in G} q(\omega)$, β is the discount factor, and

$$q_G(\omega) = \begin{cases} q(\omega)/q(G) & \text{if } \omega \in G, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

[2] Here β is a discount factor and $\mathcal{G}_t$ represents the information available to the agents at time t.

The condition has a simple interpretation. The choice of probability by Nature over two periods can be thought of as consisting of two steps. The first period choice is a choice of probability over the events that realize in the first period. The second period choice is a choice of probability over states in every event, conditional on that event.

Nature can make this choice in a time consistent way: choose q in the first period, pay the appropriate cost $c_\Omega(q)$, wait for the realization of the second period event G, do nothing, pay nothing, and get the probability q_G on the states in the event G. The total cost of this is the term in the l.h.s. of (5).

Alternatively, Nature can achieve the same result in a time inconsistent way, with total cost given by the r.h.s. of (5). Nature can choose today a probability p that induces the same probability over events in the second period as q does. This constraint is described by the condition $p(G) = q(G)$ for every event G. Nature pays for its choice p the appropriate cost, which is the term $c_\Omega(p)$ in the r.h.s. of (5). After the realization of the event G, the probability over states in that event would be p_G. Nature can now change the conditional probability to q_G, and again pay the appropriate cost, represented by the term $c_G(q_G)$ in the r.h.s. of (5). Overall, in this second more indirect way, Nature achieves the same result as in the first choice: a probability $q(G)$ of every event G in the first period, and a conditional probability q_G if G obtains.

The condition requires that this second, time inconsistent, choice is not less costly for Nature. A simple way of stating our main result is therefore the following: *A decision maker is dynamically consistent if and only if (he thinks that) Nature is dynamically consistent.*

In view of all this, we call (5), and more generally (11) of Theorem 1, a "no-gain condition." We will formally prove that the no-gain condition generalizes rectangularity, and it coincides with it when cost functions are indicators δ_C.

Equation (5) provides a link between cost functions in different periods. One important aspect of this link is that in the second period the probability over states conditional on the event G is the conditional probability q_G as defined by (6), namely according to Bayes' Rule. This link extends to variational preferences the connection between time consistency and Bayes' Rule.

As well known, Subjective Expected Utility preferences are time consistent if and only if their subjective beliefs are updated according to Bayes' Rule. This result is generalized in [5] to MP preferences by showing that they are time consistent if and only if their sets of subjective beliefs are rectangular and updating is done belief by belief (prior by prior in the terminology of the MP model) according to Bayes' Rule. Our Theorem 1, in turn, further generalizes all these results and it shows that variational preferences are time consistent if and only if their cost functions satisfy the no-gain condition and updating is done according to Bayes' Rule.

Moreover, the recursive structure of the no-gain condition makes it possible to construct by backward induction cost functions that satisfy it. This is shown by Theorem 2, which thus provides a way to construct via (4) examples of variational preferences that are time consistent.

Some papers have recently studied related issues, in particular dynamic aspects of the MP model. We already mentioned Epstein and Schneider [5], which is turn closely related to Wang [21]. Some aspects of their work have been extended by Ghirardato, Maccheroni, and Marinacci [6] and Hayashi [11]. More recently, Hanany and Klibanoff [8] proposed a dynamic version of the MP model that it is dynamically consistent but it does not satisfy Consequentilism, while Siniscalchi [19] focused on dynamic MP models that relax Dynamic Consistency. Finally, Ozdenoren and Peck [17] have studied some dynamic games against Nature that lead to ambiguity averse behavior, thus providing a game-theoretic underpinning of the game against Nature interpretation of ambiguity we discussed above and in [14].

The paper is organized as follows. Section 2 introduces the setup and notation, Section 3 presents the axioms needed for our derivation, whereas Section 4 contains the main results of the paper. Section 5 illustrates the main results with two important classes of variational preferences, the multiple priors preferences of Gilboa and Schmeidler [7] and the multiplier preferences of Hansen and Sargent [9]. All proofs are collected in the Appendix.

2 Setup

2.1 Information

Time is discrete and varies over $\mathcal{T} = \{0, 1, ..., T\}$. In our results we model information as an event tree $\{\mathcal{G}_t\}_{t\in\mathcal{T}}$, given and fixed throughout, which is defined on a finite space Ω. The elements of this tree are partitions $\mathcal{G}_t$ of Ω consisting of non-empty sets, with $\mathcal{G}_0 = \{\Omega\}$, $\mathcal{G}_{t+1}$ finer than $\mathcal{G}_t$ for all $t < T$, and $\mathcal{G}_T = \{\{\omega\} : \omega \in \Omega\}$; in particular, $G_t(\omega)$ is the element of $\mathcal{G}_t$ that contains ω.

The main interpretation we have in mind for this standard modelling of information is as follows. Given an underlying (and possibly unverifiable) state space S, endowed with a σ-algebra Σ, observations are generated by a sequence of random variables $\{Z_t\}_{t>0}$ taking values on finite observation spaces Ω_t. Each random variable $Z_t : S \to \Omega_t$ is Σ-measurable and for convenience we assume that they are surjective, so that all elements of Ω_t can be viewed as observations generated by Z_t.

The sample space $\prod_{t=1}^T \Omega_t$ is denoted by Ω, and its points $\omega = (\omega_1, ..., \omega_T)$ are the possible observation paths generated by the sequence $\{Z_t\}$. Given $t \in \mathcal{T}$, denote by $\{\omega_1, ..., \omega_t\}$ the cylinder

$$\{\omega_1\} \times \cdots \times \{\omega_t\} \times \Omega_{t+1} \times \cdots \times \Omega_T.$$

The event tree $\{\mathcal{G}_t\}$ records the building up of observations and it is given by $\mathcal{G}_0 = \{\Omega\}$,

$$\mathcal{G}_t = \{\{\omega_1, ..., \omega_t\} : \omega_\tau \in \Omega_\tau \text{ for each } \tau = 1, ..., t\},$$

and $\mathcal{G}_T = \{\{\omega\} : \omega \in \Omega\}$. In other words, the atoms of the partition $\mathcal{G}_t$ are the observation paths up to time t and they can be viewed as the nodes of the event tree $\{\mathcal{G}_t\}$.

Denote by $\Delta(\Omega)$ the set of all probability distributions $p : 2^\Omega \to [0, 1]$. The elements of $\Delta(\Omega)$ represent the agent's subjective beliefs over the observation paths. Their conditional distributions

$$p(\omega_{t+1}, ..., \omega_T \mid \omega_1, ..., \omega_t) = \frac{p(\omega_1, ..., \omega_T)}{p(\omega_1, ..., \omega_t)}$$

are called *predictive distributions* and they represent the agent's (subjective) probability that $(\omega_{t+1}, ..., \omega_T)$ will be observed after having observed $(\omega_1, ..., \omega_t)$.[3] Using the standard notation for conditional probabilities, the predictive distributions are given by the collection $\{p(\cdot \mid \mathcal{G}_t)\}_{t>0}$.

Observe that in the literature on MP preferences, the probabilities $p : 2^\Omega \to [0, 1]$ are often called priors and the conditional probabilities $p(\cdot \mid \mathcal{G}_t)$ are called the Bayesian updates of the prior. This terminology is, however, a bit confusing as in Statistics priors are often probabilities on parameters (and posteriors are their Bayesian updates given observations). Here no parametric representation is assumed for the probabilities $p : 2^\Omega \to [0, 1]$, and so we prefer not to use the term prior for them.

We now illustrate these notions with few examples.

Example 1 Suppose that observations are given by heads and tails from a given coin. We can set $\Omega_t = \{0, 1\}$ for each $t = 1, ..., T$, so that $\Omega = \{0, 1\}^T$ is the sample space. A possible $p \in \Delta(\Omega)$

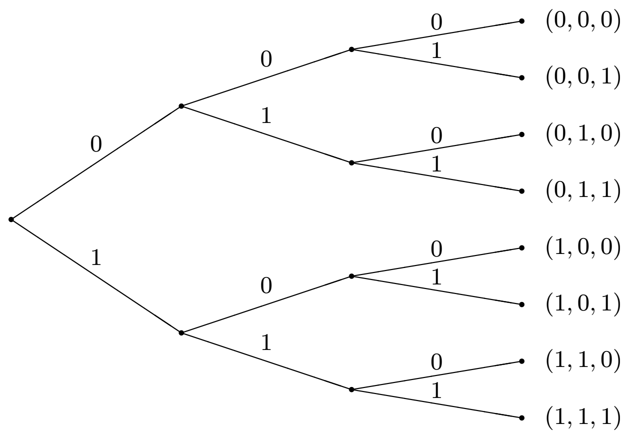
[3] For convenience, we write $p(\omega_1, ..., \omega_t)$ in place of $p(\{\omega_1, ..., \omega_t\})$. Moreover, $(\omega_{t+1}, ..., \omega_T)$ stands for

$$\Omega_1 \times \cdots \times \Omega_t \times \{\omega_{t+1}\} \times \cdots \times \{\omega_T\}.$$

is the one that assigns equal probability to all observation paths ω; that is, $p(\omega) = 2^{-T}$ for each $\omega \in \Omega$. In this case, $p(\omega_1, ..., \omega_t) = 2^{-t}$ and

$$p(\omega_{t+1}, ..., \omega_T \mid \omega_1, ..., \omega_t) = \frac{p(\omega_1, ..., \omega_T)}{p(\omega_1, ..., \omega_t)} = 2^{t-T}.$$

For example, if $T = 3$, we have $\Omega = \{0,1\}^3$ and Ω consists of 2^3 states. This case can be illustrated with a simple binomial tree



and the above probability p is such that $p(\omega) = 1/8$ for all $\omega \in \Omega$, while its predictive distributions are:

$$p(\omega_3 \mid \omega_1, \omega_2) = 1/2 \quad \textit{and} \quad p(\omega_2, \omega_3 \mid \omega_1) = 1/4.$$

▲

In the next examples we assume that $\Omega_t = \mathcal{Z}$ for all t, so that $\Omega = \mathcal{Z}^T$. For instance, in the previous example we had $\mathcal{Z} = \{0,1\}$.

Example 2 Consider a $p \in \Delta(\Omega)$ that makes the sequence $\{Z_t\}$ i.i.d., with common marginal distribution $\pi : 2^{\mathcal{Z}} \to [0,1]$. In this case, p is a product probability on 2^{Ω} uniquely determined by π as follows:

$$p(\omega) = \prod_{i=1}^{T} \pi(\omega_i) \qquad \forall \omega \in \Omega.$$

The predictive distributions are given by:

$$p(\omega_{t+1}, ..., \omega_T \mid \omega_1, ..., \omega_t) = \prod_{i=t+1}^{T} \pi(\omega_i),$$

that is, $p(\omega_{t+1}, ..., \omega_T \mid \omega_1, ..., \omega_t) = p(\omega_{t+1}, ..., \omega_T)$. Hence, information is irrelevant for prediction. ▲

Example 3 Consider a $p \in \Delta(\Omega)$ that makes the sequence $\{Z_t\}$ exchangeable, i.e.,

$$p(\omega_1, ..., \omega_T) = p(\omega_{i_1}, ..., \omega_{i_T}) \tag{7}$$

for all permutations $i_1, ..., i_T$. For simplicity, suppose $\mathcal{Z} = \{0,1\}$ and set

$$v_l^t = \binom{t}{l} p\left(\sum_{i=1}^{t} \omega_i = l\right),$$

where $p\left(\sum_{i=1}^{t}\omega_i = l\right)$ is the probability of having l successes among $t \in \mathcal{T}$ trials. Some algebra shows that here the predictive distributions are given by

$$p(\omega_{t+1},...,\omega_T \mid \omega_1,...,\omega_t) = \frac{\upsilon_{l+k}^T / \binom{T}{l+k}}{\upsilon_l^t / \binom{t}{l}},$$

where $l = \sum_{i=1}^{t}\omega_i$ and $k = \sum_{i=t+1}^{T}\omega_i$. Because of exchangeability, only the quantities l and k matter for the predictive distributions. Here information, as recorded by l and r, is relevant for prediction. ▲

Example 4 Finally, suppose that $p \in \Delta(\Omega)$ makes the sequence $\{Z_t\}$ a homogeneous Markov chain with transition function $\pi : \Omega_{t-1} \times 2^{\mathcal{Z}} \to [0,1]$ for $t \geq 2$, where $\pi(\omega_{t-1}, \cdot) : 2^{\mathcal{Z}} \to [0,1]$ is a probability measure on $\mathcal{Z}$ for each $\omega_{t-1} \in \Omega_{t-1}$. Given an initial probability distribution π^0 on $2^{\mathcal{Z}}$, p is uniquely determined by π as follows:

$$p(\omega) = \pi^0(\omega_1)\prod_{i=2}^{T}\pi(\omega_{i-1},\omega_i) \quad \forall\omega \in \Omega.$$

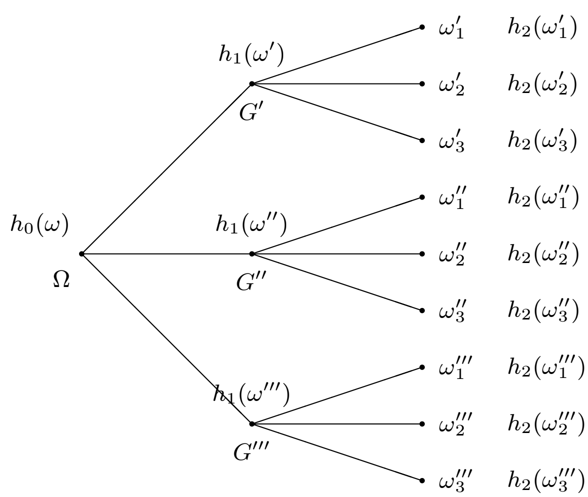
so that,

$$p(\omega_{t+1},...,\omega_T \mid \omega_1,...,\omega_t) = \prod_{i=t+1}^{T}\pi(\omega_{i-1},\omega_i). \tag{8}$$

Also in this Markov example information matters for prediction. In particular, (8) shows that here the relevant information is given by ω_t. ▲

2.2 Consumption Streams

The acts among which agents choose are here given by consumption processes. Formally, *acts* are X-valued adapted processes of the form $h = (h_0, h_1, ..., h_T)$, where each $h_t : \Omega \to X$ is $\mathcal{G}_t$-measurable and takes values on a convex consumption set X (e.g., $X = \mathbb{R}^+$ or $\Delta(\mathbb{R}^+)$).



Denote by $\mathcal{H}$ the set of all acts; we indifferently write $h_t(\omega)$ or $h(t,\omega)$ to denote consumption at time t if ω obtains (and sometimes $h(t,G)$ to denote consumption at time t if $G \in \mathcal{G}_t$ occurs). Notice that in our finite setting acts can be regarded as functions defined on $\bigcup_{t\in\mathcal{T}}\mathcal{G}_t$, that is, on the set of all nodes.

We can identify $\mathcal{H}$ with the set of all maps $h : \Omega \to X^{\mathcal{T}}$ such that $h_\tau(\omega) = h_\tau(\omega')$ if $G_\tau(\omega) = G_\tau(\omega')$; in this perspective $h(\omega)$ is the element $(h_0(\omega), h_1(\omega), ..., h_T(\omega)) \in X^{\mathcal{T}}$ for any given ω. For all $\alpha \in [0,1]$, and all $h, h' \in \mathcal{H}$ we set

$$(\alpha h + (1-\alpha)h')(t,\omega) \equiv \alpha h(t,\omega) + (1-\alpha)h'(t,\omega) \quad \forall(t,\omega) \in \mathcal{T}\times\Omega.$$

If the values of an act $y \in \mathcal{H}$ depend only on time but not on state, that is, for every fixed t

$$y(t,\omega) = y(t,\omega') = y_t \quad \forall \omega, \omega' \in \Omega,$$

with a little abuse we write $y = (y_0, y_1, ..., y_T) \in X^{\mathcal{T}}$. Moreover, if $y_0 = y_1 = ... = y_T = x \in X$, the act is called constant and, with another little abuse, we denote it by x.

Example 5 Suppose as in Example 1 that $\Omega = \{0,1\}^T$. A consumption process $h = (h_0, h_1, ..., h_T)$ is such that:

$$\begin{aligned} h_0(\omega) &= h_0(\omega'), \quad \forall \omega, \omega' \in \Omega, \\ h_1(\omega) &= h_1(\omega'), \quad \forall \omega, \omega' \in \Omega \text{ with } \omega_1 = \omega_1', \\ &\cdots \\ h_t(\omega) &= h_t(\omega'), \quad \forall \omega, \omega' \in \Omega \text{ with } (\omega_1, ..., \omega_t) = (\omega_1', ..., \omega_t'), \\ &\cdots \end{aligned}$$

In other words, h_0 is a constant, h_1 only depends on the first observation, and h_t only depends on the first t observations. ▲

2.3 Notation

We close by introducing some notation, which is usually a bit heavy in dynamic settings. If $p \in \Delta(\Omega)$, we denote by $p_{|\mathcal{G}_t}$ its restriction to the algebra generated by $\mathcal{G}_t$, and by $p(\cdot\,|\mathcal{G}_t)$ the conditional probability given $\mathcal{G}_t$.[4] As we already observed, the conditional probabilities $p(\cdot\,|\mathcal{G}_t)$ are called predictive distributions.

For all $t \in \mathcal{T}$, $\Delta(\Omega, \mathcal{G}_t)$ denotes the set of all probabilities on the algebra $\mathcal{A}(\mathcal{G}_t)$ generated by $\mathcal{G}_t$; i.e., $\Delta(\Omega, \mathcal{G}_t) = \{p_{|\mathcal{G}_t} : p \in \Delta(\Omega)\}$. In particular, $\Delta(\Omega, \mathcal{G}_T) = \Delta(\Omega)$.

For each $E \in \mathcal{A}(\mathcal{G}_t)$, we set

$$\begin{aligned} \Delta(E, \mathcal{G}_t) &\equiv \{p \in \Delta(\Omega, \mathcal{G}_t) \mid p(E) = 1\} \\ \Delta^{++}(E, \mathcal{G}_t) &\equiv \left\{ p \in \Delta(\Omega, \mathcal{G}_t) \;\middle|\; \begin{array}{ll} p(G) > 0 & \forall G \in \mathcal{G}_t : G \subseteq E \\ p(G) = 0 & \forall G \in \mathcal{G}_t : G \nsubseteq E \end{array} \right\}. \end{aligned}$$

Denoting by $\operatorname{supp} p$ the support $\{\omega \in \Omega : p(\omega) > 0\}$ of $p \in \Delta(\Omega)$, for each subset E of Ω we have:

$$\Delta(E) = \{p \in \Delta(\Omega) : \operatorname{supp} p \subseteq E\} \quad \text{and} \quad \Delta^{++}(E) = \{p \in \Delta(\Omega) : \operatorname{supp} p = E\}.$$

In particular, $\Delta(G_t(\omega))$ is the set of all predictive distributions that can be obtained by conditioning on $G_t(\omega)$ from probabilities $p \in \Delta(\Omega)$ such that $p(G_t(\omega)) > 0$, while $\Delta^{++}(G_t(\omega))$ is the subset of $\Delta(G_t(\omega))$ derived under the further condition that $p \in \Delta(\Omega)$ be such that $p(\omega') > 0$ for all $\omega' \in G_t(\omega)$.

Similarly, for each $E \in \mathcal{A}(\mathcal{G}_t)$ we have

$$\Delta(E, \mathcal{G}_t) = \{p_{|\mathcal{G}_t} : p \in \Delta(E)\} \quad \text{and} \quad \Delta^{++}(E, \mathcal{G}_t) = \{p_{|\mathcal{G}_t} : p \in \Delta^{++}(E)\}.$$

If the vector space of all measures on $\mathcal{A}(\mathcal{G}_t) \simeq \mathbb{R}^{\mathcal{G}_t}$ is endowed with the product topology, then $\Delta^{++}(E, \mathcal{G}_t)$ is the relative interior of the convex set $\Delta(E, \mathcal{G}_t)$ (see Rockafellar [18], to which we refer for the Convex Analysis terminology and notation).

[4] Notice that for all $\omega \in \Omega$ with $p(G_t(\omega)) \neq 0$, $p(\cdot\,|\mathcal{G}_t)(\omega) = p_{G_t(\omega)}$.

3 Axioms

Let the binary relations $\succsim_{t,\omega}$ on $\mathcal{H}$ represent the agent's preferences at any time-state node. Next are stated several properties (axioms) of the preference relation, which will be used in the sequel.

Axiom 1 (Conditional preference—CP) *For each* $(t,\omega) \in \mathcal{T} \times \Omega$*:*

(i) $\succsim_{t,\omega}$ *coincides with* $\succsim_{t,\omega'}$ *if* $G_t(\omega) = G_t(\omega')$*.*

(ii) If $h(\tau,\omega') = h'(\tau,\omega')$ *for all* $\tau \geq t$ *and* $\omega' \in G_t(\omega)$*, then* $h \sim_{t,\omega} h'$*.*

(i) says that preferences orderings are "adapted" and allows to write $\succsim_{t,G}$ if $G \in \mathcal{G}_t$. (ii) states that at time t in event G only "continuation acts" matter for choice.

Axiom 2 (Variational preferences—VP) *For each* $(t,\omega) \in \mathcal{T} \times \Omega$*:*

(i) $\succsim_{t,\omega}$ *is complete and transitive.*

(ii) For all $h, h' \in \mathcal{H}$ *and* $y, y' \in X^{\mathcal{T}}$*, and for all* $\alpha \in (0,1)$*, if* $\alpha h + (1-\alpha)y \succsim_{t,\omega} \alpha h' + (1-\alpha)y$ *then* $\alpha h + (1-\alpha)y' \succsim_{t,\omega} \alpha h' + (1-\alpha)y'$*.*

(iii) For all $h, h', h'' \in \mathcal{H}$*, the sets* $\{\alpha \in [0,1] : \alpha h + (1-\alpha)h' \succsim_{t,\omega} h''\}$ *and* $\{\alpha \in [0,1] : h'' \succsim_{t,\omega} \alpha h + (1-\alpha)h'\}$ *are closed.*

(iv) For all $h, h' \in \mathcal{H}$*, if* $(h_0(\omega'), h_1(\omega'), ..., h_T(\omega')) \succsim_{t,\omega} (h'_0(\omega'), h'_1(\omega'), ..., h'_T(\omega'))$ *for all* $\omega' \in \Omega$*, then* $h \succsim_{t,\omega} h'$*.*

(v) For all $h, h' \in \mathcal{H}$*, if* $h \sim_{t,\omega} h'$*, then* $\alpha h + (1-\alpha) h' \succsim_{t,\omega} h$ *for all* $\alpha \in (0,1)$*.*

The requirement here is that at every time-state node the agent has variational preferences, see Maccheroni, Marinacci, and Rustichini [14] for a discussion of (i)-(v).

Axiom 3 (Risk preference—RP) *Let* $y, y' \in X^{\mathcal{T}}$*:*

(i) For each $(t,\omega) \in \mathcal{T} \times \Omega$*, if* $y_\tau \succsim_{t,\omega} y'_\tau$ *for all* $\tau \in \mathcal{T}$*, then* $y \succsim_{t,\omega} y'$*.*

(ii) For all $x, x', x'', x''' \in X$*, if*

$$\left(y_{-\{\tau,\tau+1\}}, x, x'\right) \succsim_{t,\omega} \left(y_{-\{\tau,\tau+1\}}, x'', x'''\right)$$

holds for some $(t,\omega) \in \mathcal{T} \times \Omega$ *and some* $\tau \geq t$*, then it holds for all* $(t,\omega) \in \mathcal{T} \times \Omega$ *and all* $\tau \geq t$*.*[5]

(iii) For each $(t,\omega) \in \mathcal{T} \times \Omega$ *there exist* $x \succ_{t,\omega} x'$ *in* X *such that for all* $\alpha \in (0,1)$ *there is* $x'' \in X$ *satisfying either* $x' \succ_{t,\omega} \alpha x'' + (1-\alpha) x$ *or* $\alpha x'' + (1-\alpha) x' \succ_{t,\omega} x$*.*

(i) and (ii) are standard monotonicity and stationarity axioms, while (iii) requires that the agent's utility over consumption is unbounded (either below, or above, or both).

Axiom 4 (Dynamic consistency—DC) *For each* $(t,\omega) \in \mathcal{T} \times \Omega$ *with* $t < T$*, and all* $h, h' \in \mathcal{H}$*, if* $h_\tau = h'_\tau$ *for all* $\tau \leq t$ *and* $h \succsim_{t+1,\omega'} h'$ *for all* $\omega' \in \Omega$*, then* $h \succsim_{t,\omega} h'$*.*

[5] Notation: $\left(y_{-\{\tau,\tau+1\}}, x, x'\right) \equiv (y_0, ..., y_{\tau-1}, x, x', y_{\tau+2}, ..., y_T)$ if $\tau < T$ and $(y_0, ..., y_{T-1}, x)$ otherwise.

As Epstein and Schneider [5, p. 6] observe "According to the hypothesis, h and h' are identical for times up to t, while h is ranked (weakly) better in every state at $t+1$. 'Therefore', it should be ranked better also at (t,ω). A stronger and more customary version of the axiom would require the same conclusion given the weaker hypothesis that

$$h_t(\omega) = h'_t(\omega) \text{ and } h \succsim_{t+1,\omega'} h' \text{ for all } \omega' \in G_t(\omega).$$

In fact, given CP, the two versions are equivalent." We refer to [5] for a discussion of dynamic consistency.[6]

A state $\omega'' \in \Omega$ is $\succsim_{t,\omega}$-null if

$$h(\tau',\omega') = h'(\tau',\omega') \text{ for all } \tau' \in \mathcal{T} \text{ and all } \omega' \neq \omega'' \text{ implies } h \sim_{t,\omega} h'.$$

Axiom 5 (Full support—FS) *No state in Ω is $\succsim_{0,\Omega}$-null.*

4 The Representation

We first extend to the current dynamic setting the notion of ambiguity index c we used in the static setting of [14]. A *dynamic ambiguity index* is a family $\{c_t\}_{t\in\mathcal{T}}$ of functions $c_t : \Omega \times \Delta(\Omega) \to [0,\infty]$ such that for all $t \in \mathcal{T}$:

(i) $c_t(\cdot,p) : \Omega \to [0,\infty]$ is $\mathcal{G}_t$-measurable for all $p \in \Delta(\Omega)$,[7]

(ii) $c_t(\omega,\cdot) : \Delta(\Omega) \to [0,\infty]$ is grounded, closed and convex, with $\operatorname{dom} c_t(\omega,\cdot) \subseteq \Delta(G_t(\omega))$ and $\operatorname{dom} c_t(\omega,\cdot) \cap \Delta^{++}(G_t(\omega)) \neq \varnothing$, for all $\omega \in \Omega$.

Observe that the effective domains of the $c_t(\omega,\cdot)$ consist of predictive distributions, that is, of the conditional probabilities on the nodes $G_t(\omega)$. In the terminology more used in the MP model, we would call them the Bayesian updates of the original priors $p \in \Delta(\Omega)$.

In our first result we characterize a dynamic version of variational preferences that do not necessarily satisfy dynamic consistency. Notice that in (9) we consider $\Delta^{++}(\Omega)$ in order to have well defined conditional probabilities $p_{G_t(\omega)}$.

Proposition 1 *The following statements are equivalent:*

(a) $\{\succsim_{t,\omega}\}$ satisfy CP, VP, RP, and for each $(t,\omega) \in \mathcal{T}\times\Omega$ no state in $G_t(\omega)$ is $\succsim_{t,\omega}$-null.

(b) There exist a scalar $\beta > 0$, an unbounded affine function $u : X \to \mathbb{R}$, and a dynamic ambiguity index $\{c_t\}$ such that, for each $(t,\omega) \in \mathcal{T}\times\Omega$, $\succsim_{t,\omega}$ is represented by

$$V_t(\omega,h) = \inf_{p\in\Delta^{++}(\Omega)} \left(\int \sum_{\tau\geq t} \beta^{\tau-t} u(h_\tau)\, dp_{G_t(\omega)} + c_t\left(\omega, p_{G_t(\omega)}\right) \right) \quad \forall h \in \mathcal{H}. \tag{9}$$

Moreover, $(\beta',u',\{c'_t\})$ represent $\succsim_{t,\omega}$ in the sense of (9) if and only if $\beta'=\beta$, $u' = au+b$ for some $a>0$ and $b\in\mathbb{R}$, and $\{c'_t\} = \{ac_t\}$.

As a result, for all $t \in \mathcal{T}$ and all $h \in \mathcal{H}$, the preference functional $V_t(\cdot,h)$ is a $\mathcal{G}_t$-measurable random variable

$$V_t(h) = \inf_{p\in\Delta^{++}(\Omega)} \left(\mathrm{E}^p \left(\sum_{\tau\geq t} \beta^{\tau-t} u(h_\tau) \,|\mathcal{G}_t \right) + c_t(p\,|\mathcal{G}_t) \right).$$

[6] Inspection of our proofs shows that the weaker version of DC in which $\succsim$ is replaced by $\sim$ is enough to obtain the results of the following section.

[7] Equivalently, $c_t(\omega,\cdot) = c_t(\omega',\cdot)$ for all $\omega,\omega' \in \Omega$ such that $G_t(\omega) = G_t(\omega')$.

We call *dynamic variational preferences* the (families of) preferences satisfying CP, VP, RP, and such that no state in $G_t(\omega)$ is $\succsim_{t,\omega}$-null. It is natural to wonder what restriction on the dynamic ambiguity index would characterize the dynamic variational preferences that satisfy dynamic consistency. This condition, which we have called the "no-gain condition" in the Introduction, is given in the next theorem, which is the main result of the paper.

Theorem 1 *The following statements are equivalent:*

(a) $\{\succsim_{t,\omega}\}$ *satisfy CP, VP, RP, FS, and DC.*

(b) There exist a scalar $\beta > 0$*, an unbounded affine function* $u : X \to \mathbb{R}$*, and a dynamic ambiguity index* $\{c_t\}$ *such that, for each* $(t,\omega) \in \mathcal{T} \times \Omega$*,* $\succsim_{t,\omega}$ *is represented by*

$$V_t(\omega, h) = \inf_{p\in\Delta^{++}(\Omega)} \left(\int \sum_{\tau\geq t} \beta^{\tau-t} u(h_\tau)\, dp_{G_t(\omega)} + c_t\left(\omega, p_{G_t(\omega)}\right) \right) \quad \forall h \in \mathcal{H}, \tag{10}$$

and

$$c_t(\omega, q) = \beta \sum_{\substack{G\in\mathcal{G}_{t+1}\\ q(G)>0}} q(G) c_{t+1}(G, q_G) + \min_{\left\{p\in\Delta(G_t(\omega)) : p_{|\mathcal{G}_{t+1}} = q_{|\mathcal{G}_{t+1}}\right\}} c_t(\omega, p), \tag{11}$$

for all $q \in \Delta(G_t(\omega))$ *and all* $t < T$.

Moreover, $\left(\beta', u', \{c'_t\}\right)$ *represent* $\succsim_{t,\omega}$ *in the sense of (10) if and only if* $\beta' = \beta$*,* $u' = au + b$ *for some* $a > 0$ *and* $b \in \mathbb{R}$*, and* $\{c'_t\} = \{ac_t\}$*.*

Therefore, dynamic variational preferences satisfy dynamic consistency if and only if their dynamic ambiguity index has the recursive structure (11), that is, if and only if they satisfy the no-gain condition and updating is done according to Bayes' Rule.

In turn, (11) delivers the recursive representation

$$V_t(\omega, h) = u(h_t(\omega)) + \min_{r\in\Delta(\Omega,\mathcal{G}_{t+1})} \left(\beta \int V_{t+1}(h)\, dr + \gamma_t(\omega, r) \right) \tag{12}$$

of the agent's preference functional V_t, where

$$\gamma_t(\omega, r) = \min_{\left\{p\in\Delta(G_t(\omega)) : p_{|\mathcal{G}_{t+1}} = r\right\}} c_t(\omega, p) \quad \forall r \in \Delta(\Omega, \mathcal{G}_{t+1}), \tag{13}$$

(see Lemma 6 in the Appendix).

In view of all this, we call *recursive variational preferences* the dynamic variational preferences satisfying dynamic consistency, and we call *recursive ambiguity indexes* their dynamic ambiguity indexes, that is, the dynamic indexes satisfying the no-gain condition (11).

Recall from the Introduction that the recursive formula (11) has a transparent interpretation under the game against Nature interpretation of our setting, in which $\{c_t\}$ is a dynamic cost for the Nature. In fact, (11) suggests that the cost for Nature of choosing q at time t in state ω can be decomposed as the sum of: the discounted expected cost of choosing q's conditionals at time $t+1$,[8] plus the cost $\gamma_t\left(\omega, q_{|\mathcal{G}_{t+1}}\right)$ of inducing $q_{|\mathcal{G}_{t+1}}$ as one period ahead marginal. By (11) and (12), both Nature's costs and agent's preferences are recursive.

After completion of an earlier version of this paper, we learned of independent work by Detlefsen and Scandolo [1], who arrive at a condition related to (11) in studying conditions for the time consistency of risk measures.

[8] In fact,

$$\sum_{\substack{G\in\mathcal{G}_{t+1}\\ q(G)>0}} q(G) c_{t+1}(G, q_G) = \int c_{t+1}\left(q\,|\mathcal{G}_{t+1}\right) dq.$$

4.1 Going Backward

A main advantage of the recursive structure of the no-gain condition (11) is that it makes it possible to construct by backward induction recursive dynamic indexes, and so recursive variational preferences via (12) and (13).

The next result provides the key ingredient for the desired backward induction construction

Proposition 2 *Let $\{c_t\}$ be a dynamic ambiguity index. For all $t < T$ and $\omega \in \Omega$, set*[9]

$$\gamma_t(\omega, r) \equiv \min_{\left\{p \in \Delta(G_t(\omega)) : p_{|\mathcal{G}_{t+1}} = r\right\}} c_t(\omega, p) \quad \forall r \in \Delta(\Omega, \mathcal{G}_{t+1}).$$

The family $\{\gamma_t\}_{t<T}$ of functions $\gamma_t : \Omega \times \Delta(\Omega, \mathcal{G}_{t+1}) \to [0, \infty]$ is such that for all $t < T$:

(i) $\gamma_t(\cdot, r) : \Omega \to [0, \infty]$ is $\mathcal{G}_t$-measurable for all $r \in \Delta(\Omega, \mathcal{G}_{t+1})$.

(ii) $\gamma_t(\omega, \cdot) : \Delta(\Omega, \mathcal{G}_{t+1}) \to [0, \infty]$ is grounded, closed and convex, with $\operatorname{dom} \gamma_t(\omega, \cdot) \subseteq \Delta(G_t(\omega), \mathcal{G}_{t+1})$ and $\operatorname{dom} \gamma_t(\omega, \cdot) \cap \Delta^{++}(G_t(\omega), \mathcal{G}_{t+1}) \neq \varnothing$, for all $\omega \in \Omega$.

The index $\gamma_t(\omega, r)$ can be interpreted as the cost for Nature of inducing r as one period ahead marginal, as suggested by (12) and (13). Since the properties of $\gamma_t(\omega, \cdot)$ on $\Delta(\Omega, \mathcal{G}_{t+1})$ are analogous to those of a static (or dynamic) ambiguity index on the set of the agent's subjective beliefs, we call *one-period-ahead ambiguity index* a family $\{\tilde{\gamma}_t\}_{t<T}$ of functions that satisfies conditions (i) and (ii) of Proposition 2.

Next we characterize recursive ambiguity indexes by means of one-period-ahead ones, thus giving the desired backward induction construction of recursive ambiguity indexes. Here δ_C is the indicator function defined in (3) and, given $\omega \in \Omega$, d_ω is the Dirac probability assigning mass 1 to ω.

Theorem 2 *The following statements are equivalent.*

(a) $\{c_t\}$ is a recursive ambiguity index.

(b) There exist $\beta > 0$ and a one period ahead ambiguity index $\{\gamma_t\}$ such that, for all $\omega \in \Omega$,

$$\begin{aligned} c_T(\omega, p) &= \delta_{\{d_\omega\}}, \text{ and} \\ c_t(\omega, q) &= \begin{cases} \beta \sum_{\substack{G \in \mathcal{G}_{t+1} \\ q(G) > 0}} q(G) c_{t+1}(G, q_G) + \gamma_t\left(\omega, q_{|\mathcal{G}_{t+1}}\right) & \forall q \in \Delta(G_t(\omega)) \\ \infty & \forall q \in \Delta(\Omega) \setminus \Delta(G_t(\omega)). \end{cases} \end{aligned}$$

The important implication is (b) $\Rightarrow$ (a), which allows to construct any recursive ambiguity index by backward induction: it suffices to specify at any non-terminal node $G = G_t(\omega)$ a grounded, closed and convex function γ_G on the set of all probabilities on the branches springing from G.

This decomposition of cost functions in one-period-ahead components is a key feature of our derivation. The next example illustrates this feature by showing what happens in a binomial tree if we take at each non-terminal node the Gini index $\chi(p\|q)$ (see (18) below) as one-period-ahead ambiguity index.

Example 6 Consider Example 1 with $T = 2$, that is, $\Omega = \{0, 1\}^2$. We have:

$$\mathcal{G}_1 = \{\{0\}, \{1\}\} \quad \text{and} \quad \mathcal{G}_2 = \{\{0, 0\}, \{0, 1\}, \{1, 0\}, \{1, 1\}\},$$

[9] Here we adopt the convention that the minimum over the empty set is ∞.

where $\{0\} = \{(0,0),(0,1)\}$ and $\{1\} = \{(1,0),(1,1)\}$. Hence,

$$\begin{aligned}
\Delta(\Omega, \mathcal{G}_1) &= \{(r, 1-r) : r \in [0,1]\}, \\
\Delta(\{0\}, \mathcal{G}_2) &= \Delta(\{0\}) = \{(r, 1-r) : r \in [0,1]\}, \\
\Delta(\{1\}, \mathcal{G}_2) &= \Delta(\{1\}) = \{(r, 1-r) : r \in [0,1]\},
\end{aligned}$$

and $\Delta(\Omega, \mathcal{G}_2) = \Delta(\Omega)$. Let $q \in \Delta(\Omega)$ be the uniform distribution with $q(\omega) = 1/4$ for all $\omega \in \Omega$, and set $\varphi(\pi) \equiv 2\pi^2 + 2(1-\pi)^2 - 1$ for each $\pi \in [0,1]$. Define

$$\gamma_0(\Omega, p) = \chi\left(p \| q_{|\mathcal{G}_1}\right) = \varphi(p(0)) \quad \forall p \in \Delta(\Omega, \mathcal{G}_1),$$

$$\gamma_1(\{0\}, p) = \chi\left(p \| q_{\{0\}}\right) = \begin{cases} \varphi(p(0,0)) & \text{if } p \in \Delta(\{0\}), \\ \infty & \text{otherwise,} \end{cases}$$

and

$$\gamma_1(\{1\}, p) = \chi\left(p \| q_{\{1\}}\right) = \begin{cases} \varphi(p(1,0)) & \text{if } p \in \Delta(\{1\}), \\ \infty & \text{otherwise,} \end{cases}$$

By Theorem 2, using these one period ahead ambiguity indexes we can construct a recursive dynamic index, given by:

$$c_1(\{0\}, p) \equiv \gamma_1(\{0\}, p),$$

$$c_1(\{1\}, p) \equiv \gamma_1(\{1\}, p),$$

and,

$$\begin{aligned}
c_0(\Omega, p) &\equiv \beta\left[p(\{0\})\, c_1\left(\{0\}, p_{\{0\}}\right) + p(\{1\})\, c_1\left(\{1\}, p_{\{1\}}\right)\right] + \gamma_0\left(\Omega, p_{|\{\{0\},\{1\}\}}\right) \\
&= \beta\left[(p_{00} + p_{01})\,\varphi\left(\frac{p_{00}}{p_{00} + p_{01}}\right) + (p_{10} + p_{11})\,\varphi\left(\frac{p_{10}}{p_{10} + p_{11}}\right)\right] + \varphi(p_{00} + p_{01}),
\end{aligned}$$

where we set $p_{ij} = p(i,j)$ for $i, j \in \{0,1\}$ and we adopt the convention $0\varphi(0/0) = 0$. ▲

5 Special Cases

5.1 Multiple Prior Preferences

We now show that Epstein and Schneider [5]'s characterization of dynamic MP preferences is a special case of ours, modulo some minor differences in assumptions (they do not assume unboundedness and assume a slightly stronger version of dynamic consistency).

MP preferences are the special class of variational preferences satisfying the certainty independence condition of Gilboa and Schmeidler [7]. In the present dynamic setting, this amounts to consider:

MP(ii) For all $h, h' \in \mathcal{H}$, $y \in X^{\mathcal{T}}$, and $\alpha \in (0,1)$, $h \succsim_{t,\omega} h'$ if and only if $\alpha h + (1-\alpha)y \succsim_{t,\omega} \alpha h' + (1-\alpha)y$,

which is a stronger version of VP(ii) (in [14] we discuss the different behavioral implications of these two axioms).

Under the stronger MP(ii), the ambiguity index $c_t(\omega, \cdot)$ becomes an indicator function, and the no-gain condition (11) coincides with rectangularity, which is the condition that [5] have used to characterize recursive MP preferences.

Corollary 1 *Let $\{\succsim_{t,\omega}\}$ be a family of dynamic variational preferences. The following statements are equivalent:*

(a) $\{\succsim_{t,\omega}\}$ *satisfy MP(ii).*

(b) For every t *and* ω*, there exists a closed and convex subset* $C_t(\omega)$ *of* $\Delta(\Omega)$ *such that* $c_t(\omega,\cdot) = \delta_{C_t(\omega)}(\cdot)$*.*

In this case, condition (11) is equivalent to

$$C_t(\omega) = \left\{ \sum_{G\in\mathcal{G}_{t+1}} p^G r(G) : p^G \in C_{t+1}(G) \ \ \forall G \in \mathcal{G}_{t+1} \text{ and } r \in C_t(\omega)_{|\mathcal{G}_{t+1}} \right\}, \tag{14}$$

for all $\omega \in \Omega$ *and* $t < T$*, where* $C_{t+1}(G) = C_{t+1}(\omega')$ *for all* $\omega' \in G$*, and* $C_t(\omega)_{|\mathcal{G}_{t+1}}$ *is the set of restrictions to the algebra generated by* $\mathcal{G}_{t+1}$ *of the probabilities in* $C_t(\omega)$*.*

5.2 Multiplier Preferences

Given $p, q \in \Delta(\Omega)$, the *relative entropy* (or *Kullback-Leibler distance*) of p w.r.t. q is

$$R(p\|q) = \begin{cases} \sum_{\omega\in\Omega} p(\omega)\log\dfrac{p(\omega)}{q(\omega)} & \text{if } p \ll q, \\ \infty & \text{otherwise,} \end{cases}$$

with the convention $0\ln(0/a) = 0$ for all $a \geq 0$. Analogously, if $p, q \in \Delta(\Omega, \mathcal{G})$, where $\mathcal{G}$ is a partition of Ω, the *relative entropy* of p w.r.t. q *on* $\mathcal{G}$ is

$$R_{\mathcal{G}}(p\|q) = \begin{cases} \sum_{G\in\mathcal{G}} p(G)\log\dfrac{p(G)}{q(G)} & \text{if } p \ll q, \\ \infty & \text{otherwise,} \end{cases}$$

again with the convention $0\ln(0/a) = 0$ for all $a \geq 0$.

Given a reference probabilistic model $q \in \Delta^{++}(\Omega)$, we call *dynamic multiplier preferences* the family of preferences on $\mathcal{H}$ represented for every t and ω by

$$V_t(\omega, h) \equiv \inf_{p\in\Delta^{++}(\Omega)} \left(\int \sum_{\tau\geq t} \beta^{\tau-t} u(h_\tau)\, dp_{G_t(\omega)} + \theta\beta^{-t} R\left(p_{G_t(\omega)} \| q_{G_t(\omega)}\right) \right) \quad \forall h \in \mathcal{H}. \tag{15}$$

The name is inspired by the robust control approach of Hansen and Sargent [9].[10] They interpret θ as a coefficient of uncertainty aversion, an interpretation we formalize and discuss in [14]. Observe that by standard results (see [3]) we can equivalently write (15) as:

$$V_t(\omega, h) = -\theta\beta^{-t}\log\left(\int e^{-\frac{1}{\theta}\sum_{\tau\geq t}\beta^\tau u(h_\tau)} dq_{G_t(\omega)} \right),$$

a formula useful in calculations.

Next we show that multiplier preferences are recursive variational preferences and their ambiguity index is

$$c_t(\omega, p) \equiv \theta\beta^{-t} R\left(p_{G_t(\omega)} \| q_{G_t(\omega)}\right) \tag{16}$$

for all $t \in \mathcal{T}$, $\omega \in \Omega$, and $p \in \Delta(\Omega)$.

[10] Clearly, these preferences are represented also by the functional

$$\beta^t V_t(\omega, h) = \inf_{p\in\Delta^{++}(\Omega)} \left(\int \sum_{\tau\geq t} \beta^\tau u(h_\tau)\, dp_{G_t(\omega)} + \theta R\left(p_{G_t(\omega)} \| q_{G_t(\omega)}\right) \right).$$

Theorem 3 *For all $q \in \Delta^{++}(\Omega)$, $\beta > 0$, unbounded affine $u : X \to \mathbb{R}$, and $\theta > 0$, the dynamic multiplier preferences represented by (15) are recursive variational preferences with ambiguity index given by (16). In particular,*

$$V_t(\omega, h) = u(h_t(\omega)) + \min_{r \in \Delta(\Omega, \mathcal{G}_{t+1})} \left(\beta \int V_{t+1}(h)\, dr + \theta \beta^{-t} R_{\mathcal{G}_{t+1}} \left(r \| \left(q_{G_t(\omega)} \right)_{|\mathcal{G}_{t+1}} \right) \right), \tag{17}$$

for each $h \in \mathcal{H}$, $\omega \in \Omega$, and $t < T$.

The recursive formulation (17) is especially important because it makes it possible to use standard dynamic programming tools in studying optimization problems involving dynamic multiplier preferences. This class of dynamic variational preferences is therefore very tractable, something important for applications.

Finally, the recursive structure of another (continuous time) version of a robust control preference functional is studied by Skiadas [20].

5.3 Mean Variance Preferences

We conclude by observing that Theorem 3 does not hold when we replace the relative entropy with a general convex statistical distance (see [13]). For example, consider the relative *Gini index* (often called χ^2-distance)

$$\chi(p \| q) \equiv \begin{cases} \sum_{\omega \in \Omega} \dfrac{(p(\omega))^2}{q(\omega)} - 1 & \text{if } p \ll q, \\ \infty & \text{otherwise.} \end{cases} \tag{18}$$

In [14] and [16] we show that $\theta \chi(p \| q)$ is the ambiguity index associated with the classic mean-variance preferences. For example, on the domain of monotonicity of such preferences we have:

$$\int f dq - \frac{1}{2\theta} \mathrm{Var}(f) = \min_{p \in \Delta(q)} \left(\int f dp + \theta \chi(p \| q) \right),$$

where $q \in \Delta^{++}(\Omega)$ is again a reference probability.

It is easily seen that the dynamic ambiguity index given by

$$c_t(\omega, p) \equiv \theta \beta^{-t} \chi \left(p_{G_t(\omega)} \| q_{G_t(\omega)} \right)$$

is not recursive, and so the dynamic variational preferences represented by

$$V_t(\omega, h) \equiv \inf_{p \in \Delta^{++}(\Omega)} \left(\int \sum_{\tau \geq t} \beta^{\tau - t} u(h_\tau)\, dp_{G_t(\omega)} + \theta \beta^{-t} \chi \left(p_{G_t(\omega)} \| q_{G_t(\omega)} \right) \right)$$

are not dynamically consistent.

6 Conclusions

Ambiguity adverse behavior is pervasive, and the theory of ambiguity aversion has found applications in macroeconomics, finance, even political analysis. The extension of the standard theory to include this phenomenon has made possible a rigorous and convincing analysis.

A widely accepted theory has been so far the theory of multiple priors of [7]. Different approaches, mostly found under the name of robust preferences, have made desirable an extension of this theory to include a larger class of behaviors. The extension, in the static case, has been provided by the theory of variational preferences introduced by [14]. This is, however, a theory of static choice, while most of the applications we have mentioned are in dynamic environments:

hence, a further extension to the intertemporal problem is desirable. This paper has provided such a theory.

The paper has four main results. The first, Proposition 1, characterizes the intertemporal preferences that have a variational representation, the so-called dynamic variational preferences (intuitively, variational decision makers can be viewed as making their choices "as if" they think to face a malevolent opponent, which we call Nature).

The second result, Theorem 1, characterizes the dynamic preferences that are time consistent. In particular, a variational decision maker is dynamically consistent if and only if he thinks that Nature as well is dynamically consistent.

The third result, Theorem 2, provides a decomposition of the cost function into one step ahead costs, paid by Nature in every period. This decomposition makes it possible the use of recursive analysis in studying the dynamic choice problem of a decision maker with variational preferences.

The fourth and final result, Theorem 3, is an application of Theorem 1 and it shows that the widely used multiplier preferences introduced by Hansen and Sargent are dynamically consistent. In contrast, we observe that mean variance preferences are not.

We close by observing that, though in the paper we assumed both Ω and T finite, we expect that the extension to the infinite case can be carried out in standard ways.

A Proofs and Related Material

A.1 Convex Analysis Preliminaries

An important tool for the proofs is Convex Analysis. Here we report some basic definitions and properties, for details we refer the reader to the classic Rockafellar [18] or Hiriart-Urruty and Lemaréchal [12].

Let K be a non-empty subset of a finite dimensional euclidean space E, and f a function $f : K \to (-\infty, \infty]$:

- f_∞ is the extension of f to E defined by $f_\infty(e) \equiv \infty$ if $e \notin K$;
- the *effective domain* of f is the set $\operatorname{dom} f \equiv \{e \in K : f(e) < \infty\}$ ($\operatorname{dom} f = \operatorname{dom} f_\infty$);
- if K is convex, f is *convex* if for all $e, \bar{e} \in E$ and all $\alpha \in (0,1)$, $f(\alpha e + (1-\alpha)\bar{e}) \leq \alpha f(e) + (1-\alpha) f(\bar{e})$ (iff f_∞ is convex);
- f is *proper* if it is not identically ∞ and there is an affine function minorizing f on K (iff f_∞ is proper);[11]
- the set of all proper convex functions on a convex set K is denoted by $\operatorname{Conv} K$;
- f is *grounded* if $\inf_{e \in K} f(e) = 0$ (iff f_∞ is grounded);[12]
- f is *closed* (or *l.s.c.*) if for all $t \in \mathbb{R}$ the (possibly empty) set $\{e \in K : f(e) \leq t\}$ is closed in E for every $t \in \mathbb{R}$ (iff f_∞ is closed);[13]
- the set of all proper, closed and convex functions on K is denoted by $\overline{\operatorname{Conv}}\, K$;
- the *convex conjugate* of a proper f is the function $f^* : E \to (-\infty, \infty]$

  $$f^*(e') = \sup_{e \in K} (\langle e, e' \rangle - f(e))$$

  for all $e' \in E$ ($f^* = (f_\infty)^*$);
- the *closed and convex hull* of a proper f is the function $\overline{\operatorname{co}}\, f : E \to (-\infty, \infty]$ defined for all $e \in E$ by

  $$(\overline{\operatorname{co}}\, f)(e) \equiv \sup \{\langle e, e' \rangle - b : e' \in E, b \in \mathbb{R}, \langle e^\circ, e' \rangle - b \leq f(e^\circ) \quad \forall e^\circ \in K\},$$

  that is, the supremum of all affine functions that minorize f on K ($\overline{\operatorname{co}}\, f = \overline{\operatorname{co}}\,(f_\infty)$);
- if K is convex, the *relative interior* of K, denoted by $\operatorname{ri} K$ is the interior of K in the relative euclidean topology of the set

  $$\operatorname{aff} K \equiv \left\{ \sum_{i=1}^{n} \alpha_i e_i : n \in \mathbb{N}, e_1, ..., e_n \in K, \alpha_1, ..., \alpha_n \in \mathbb{R}, \sum_{i=1}^{n} \alpha_i = 1 \right\}.$$

A function $j : K \to [-\infty, \infty)$ is *concave* if $-j$ is convex. Its *effective domain* is the set $\{e \in K : j(e) > -\infty\}$.

Lemma 1 *Let $f \in \overline{\operatorname{Conv}}\, E$, $K' \subseteq E$ and $g : K' \to [-\infty, \infty]$. The following statements are equivalent:*

[11] If f is bounded below or K and f are convex, the second requirement is automatically satisfied, see [12, IV.1.2.1].

[12] Indeed, $\inf_{e \in K} f(e) = \inf_{e \in E} f_\infty(e)$.

[13] Indeed, $\{e \in K : f(e) \leq t\} = \{e \in E : f_\infty(e) \leq t\}$ for all $t \in \mathbb{R}$. If K is closed in E, it is enough to require that $\{e \in K : f(e) \leq t\}$ is closed in K.

(a) $f(e) = \sup_{e' \in K'} (\langle e, e' \rangle - g(e'))$ *for all* $e \in E$.

(b) $g : K' \to (-\infty, \infty]$ *is proper and* $\overline{\text{co}}\, g = f^*$.

Proof. (a) $\Rightarrow$ (b). If $g(e^\circ) = -\infty$ for some $e^\circ \in K'$, then $f(e) = \infty$ for all $e \in E$, which is absurd. Then $g : K' \to (-\infty, \infty]$. For all $e \in E$, and $e' \in K'$,

$$f(e) \geq \langle e, e' \rangle - g(e') \text{ and } g(e') \geq \langle e, e' \rangle - f(e).$$

Choosing $\bar{e} \in \text{dom}\, f$, the affine function $\langle \bar{e}, \cdot \rangle - f(\bar{e})$ minorizes g on K'. Moreover, if g were identically ∞, f would be identically $-\infty$, which is absurd. We conclude that $g : K' \to (-\infty, \infty]$ is proper. By [12, X.1.3.5], $\overline{\text{co}}(g_\infty) = (g_\infty)^{**}$. Notice that

$$(g_\infty)^*(e) = g^*(e) = f(e) \quad \forall e \in E,$$

hence $f = (g_\infty)^*$, and $f^* = (g_\infty)^{**} = \overline{\text{co}}(g_\infty) = \overline{\text{co}}(g)$.

(b) $\Rightarrow$ (a). $f^* = \overline{\text{co}}\, g = \overline{\text{co}}(g_\infty) = (g_\infty)^{**}$, hence $f = f^{**} = (g_\infty)^{***} = (g_\infty)^* = g^*$. ■

In particular, if $j : E \to (-\infty, \infty)$ is concave (and hence continuous) and

$$j(e) = \inf_{e' \in K'} (\langle e, e' \rangle + g(e')) \quad \forall e \in E,$$

then, setting $f(e) = -j(-e)$ for all $e \in E$, $f \in \overline{\text{Conv}}\, E$ and

$$f(e) = -\inf_{e' \in K'} (\langle -e, e' \rangle + g(e')) = \sup_{e' \in K'} (\langle e, e' \rangle - g(e'))$$

therefore

$$\begin{aligned} \overline{\text{co}}(g)(e') &= f^*(e') = \sup_{e \in E} (\langle e, e' \rangle - f(e)) = \sup_{e \in E} (\langle -e, e' \rangle - f(-e)) \\ &= \sup_{e \in E} (\langle -e, e' \rangle + j(e)) = -\inf_{e \in E} (\langle e, e' \rangle - j(e)) = -j^*(e'), \end{aligned}$$

where j^* is the *concave conjugate* of j.

Lemma 2 *If K is a convex compact subset of E, and $f \in \overline{\text{Conv}}\, K$, then*

$$\inf_{e \in \text{ri}\, K} f(e) = \min_{e \in K} f(e)$$

if and only if $\text{ri}\, K \cap \text{dom}\, f \neq \varnothing$.

Proof. Notice that by the Weierstrass Theorem f attains its finite minimum in K. If there exists $\bar{e} \in \text{ri}\, K \cap \text{dom}\, f$, let $e^\circ \in \arg\min_{e \in K} f(e)$. By [18, Thm. 6.1], $(1-\lambda)\bar{e} + \lambda e^\circ \in \text{ri}\, K$ for all $\lambda \in (0,1)$ and by [18, Cor. 7.5.1]

$$\min_{e \in K} f(e) = f(e^\circ) = \lim_{\lambda \uparrow 1} f((1-\lambda)\bar{e} + \lambda e^\circ) \geq \inf_{e \in \text{ri}\, K} f(e) \geq \min_{e \in K} f(e).$$

Conversely, if $\inf_{e \in \text{ri}\, K} f(e) = \min_{e \in K} f(e)$, it cannot be $f(e) = \infty$ for all $e \in \text{ri}\, K$. It follows that $\text{ri}\, K \cap \text{dom}\, f \neq \varnothing$. ■

Notice that $\text{ri}\, K \cap \text{dom}\, f \neq \varnothing$ if and only if $\inf_{e \in \text{ri}\, K} f(e) < \infty$.

Finally, let $E = \mathbb{R}^\Omega$, K be a non-empty subset of E, and f a function $f : K \to (-\infty, \infty)$:

- for every $A \subseteq \Omega$, 1_A is the vector defined by $1_A(\omega) \equiv 1$ if $\omega \in A$, $1_A(\omega) \equiv 0$ if $\omega \notin A$;[14]

[14] With a little abuse, we sometimes write b instead of $b1_\Omega$ if $b \in \mathbb{R}$.

- f is *normalized* if $f(b1_\Omega) = b$ for all $b \in \mathbb{R}$ such that $b1_\Omega \in K$;
- f is a *niveloid* if $f(e) - f(\bar{e}) \leq \sup_{\omega \in \Omega}(e(\omega) - \bar{e}(\omega))$ for all $e, \bar{e} \in K$.

Niveloids are comprehensively studied in Dolecki and Greco [2] and Maccheroni, Marinacci, and Rustichini [15]. When $H \in \{\mathbb{R}, \mathbb{R}^+, \mathbb{R}^{++}, \mathbb{R}^-, \mathbb{R}^{--}\}$ and $K = H^\Omega$, f is a niveloid if and only if

- f is *monotonic*, that is $f(e) \geq f(\bar{e})$ for all $e, \bar{e} \in H^\Omega$ such that $e \geq \bar{e}$, and
- f is *vertically invariant*, that is $f(e + b) = f(e) + b$ for all $e \in H^\Omega$ and $b \in H$.

A.2 Some Important Lemmas

Lemma 3 *The following statements are equivalent:*

(a) $\{\succsim_{t,\omega}\}$ *satisfy CP, VP, and RP.*

(b) There exists a family $\{c_t(\omega, \cdot) : (t, \omega) \in \mathcal{T} \times \Omega\}$ *of grounded, closed and convex functions* $c_t(\omega, \cdot) : \Delta(\Omega) \to [0, \infty]$, *such that* $\operatorname{dom} c_t(\omega, \cdot) \subseteq \Delta(G_t(\omega))$ *and* $c_t(\omega, \cdot) = c_t(\omega', \cdot)$ *if* $G_t(\omega) = G_t(\omega')$, $\beta > 0$, *and an unbounded affine* $u : X \to \mathbb{R}$ *such that: for every* t *and* ω, $\succsim_{t,\omega}$ *is represented by* $V_t(\omega, \cdot)$, *where*

$$V_t(\omega, h) = \min_{p \in \Delta(\Omega)} \left(\int \sum_{\tau \geq t} \beta^{\tau - t} u(h_\tau)\, dp + c_t(\omega, p) \right) \quad \forall h \in \mathcal{H}. \tag{19}$$

Moreover, $(\bar{\beta}, \bar{u}, \{\bar{c}_t(\omega, \cdot)\})$ *represent* $\succsim_{t,\omega}$ *in the sense of Eq. (19) iff* $\bar{\beta} = \beta$, $\bar{u} = au + b$ *for some* $a > 0$ *and* $b \in \mathbb{R}$ *and* $\{\bar{c}_t(\omega, \cdot)\} = \{ac_t(\omega, \cdot)\}$.

Finally, if $|G_t(\omega)| > 1$, *the following facts are equivalent:*

(i) for all $h \in \mathcal{H}$

$$V_t(\omega, h) = \inf_{p \in \operatorname{ri} \Delta(G_t(\omega))} \left(\int \sum_{\tau \geq t} \beta^{\tau - t} u(h_\tau)\, dp + c_t(\omega, p) \right); \tag{20}$$

(ii) no state in $G_t(\omega)$ *is* $\succsim_{t,\omega}$*-null;*

(iii) $\operatorname{dom} c_t(\omega, \cdot) \cap \operatorname{ri} \Delta(G_t(\omega)) \neq \varnothing$.

Notice that:

- if $|G_t(\omega)| = 1$, $G_t(\omega)$ is a singleton $\{\omega\}$ and both Eq. (19) and Eq. (20) collapse to

  $$V_t(\omega, h) = \sum_{\tau \geq t} \beta^{\tau - t} u(h_\tau(\omega)),$$

  and (iii) is automatically satisfied.
- Eq. (19) can be rewritten as

  $$V_t(\omega, h) = \min_{p \in \Delta(G_t(\omega))} \left(\int \sum_{\tau \geq t} \beta^{\tau - t} u(h_\tau)\, dp + c_t(\omega, p) \right). \tag{21}$$

Proof. We indifferently write $c_t(\omega,\cdot)$ or $c_{t,\omega}$ and $V_t(\omega,\cdot)$ or $V_{t,\omega}$.

(a) $\Rightarrow$ (b). Assume that $\{\succsim_{t,\omega}\}$ satisfy CP, VP, and RP.

Step 1. There exist $\beta>0$ and an unbounded affine $u: X\to\mathbb{R}$ such that, for all $(t,\omega)\in\mathcal{T}\times\Omega$, $\succsim_{t,\omega}$ on $X^{\mathcal{T}}$ is represented by

$$U_t(y)\equiv\sum_{\tau\geq t}\beta^{\tau-t}u(y_\tau)\quad\forall y\in X^{\mathcal{T}}.$$

Details. Arbitrarily choose $(t,\omega)\in\mathcal{T}\times\Omega$. Notice that, $X^{\mathcal{T}}$ can be regarded as a set of acts $y:\mathcal{T}\to X$, and – by VP and RP – $\succsim_{t,\omega}$ on $X^{\mathcal{T}}$ satisfies the SEU axioms of Anscombe and Aumann.[15] Therefore, there exists an affine function $u^{(t,\omega)}: X\to\mathbb{R}$ and $\lambda^{(t,\omega)}=\left(\lambda_0^{(t,\omega)},...,\lambda_T^{(t,\omega)}\right)\in\Delta(\mathcal{T})$, such that $\succsim_{t,\omega}$ on $X^{\mathcal{T}}$ is represented by

$$\tilde{U}_{t,\omega}(y)\equiv\sum_{\tau=0}^{T}\lambda_\tau^{(t,\omega)}u^{(t,\omega)}(y_\tau)\quad\forall y\in X^{\mathcal{T}};\tag{22}$$

$u^{(t,\omega)}$ is unbounded and it represents $\succsim_{t,\omega}$ on X, $\lambda^{(t,\omega)}$ is unique. If $\lambda_{\tau'}^{(t,\omega)}\neq 0$ for some $\tau'<t$, let $x\succ_{t,\omega}x'$ and arbitrarily choose $y\in X^{\mathcal{T}}$. By CP(ii)

$$(y_{-\tau'},x)\sim_{t,\omega}(y_{-\tau'},x'),$$

hence

$$\sum_{\tau\neq\tau'}\lambda_\tau^{(t,\omega)}u^{(t,\omega)}(y_\tau)+\lambda_{\tau'}^{(t,\omega)}u^{(t,\omega)}(x)=\sum_{\tau\neq\tau'}\lambda_\tau^{(t,\omega)}u^{(t,\omega)}(y_\tau)+\lambda_{\tau'}^{(t,\omega)}u^{(t,\omega)}(x'),$$

this is absurd since $u^{(t,\omega)}(x)>u^{(t,\omega)}(x')$. Eq. (22) then becomes

$$\tilde{U}_{t,\omega}(y)=\sum_{\tau\geq t}\lambda_\tau^{(t,\omega)}u^{(t,\omega)}(y_\tau)\quad\forall y\in X^{\mathcal{T}}.\tag{23}$$

Again RP, (see [5, p. 23]) implies that there exists $\beta>0$ and an affine $u: X\to R$ such that for all (t,ω)

$$U_{t,\omega}(y)\equiv\sum_{\tau\geq t}\beta^{\tau-t}u(y_\tau)\quad\forall y\in X^{\mathcal{T}}\tag{24}$$

is a positive affine transformation of $\tilde{U}_{t,\omega}$. Since the above functional does not depend on ω we can just write U_t. Unboundedness of u descends again by RP(iii). □

Step 2. W.l.o.g. $u(X)\in\{\mathbb{R},\mathbb{R}^+,\mathbb{R}^{++},\mathbb{R}^-,\mathbb{R}^{--}\}$.

Details. Since u is unbounded, there exists $b\in\mathbb{R}$ that, setting $u^b\equiv u+b$, delivers $u^b(X)=u(X)+b\in\{\mathbb{R},\mathbb{R}^+,\mathbb{R}^{++},\mathbb{R}^-,\mathbb{R}^{--}\}$. Then for each t and $y,y'\in X^{\mathcal{T}}$,

$$\begin{aligned}y\succsim_t y' &\Leftrightarrow \sum_{\tau\geq t}\beta^{\tau-t}u(y_\tau)\geq\sum_{\tau\geq t}\beta^{\tau-t}u(y_\tau)\\ &\Leftrightarrow \sum_{\tau\geq t}\beta^{\tau-t}u(y_\tau)+\sum_{\tau\geq t}\beta^{\tau-t}b\geq\sum_{\tau\geq t}\beta^{\tau-t}u(y_\tau)+\sum_{\tau\geq t}\beta^{\tau-t}b\\ &\Leftrightarrow \sum_{\tau\geq t}\beta^{\tau-t}u^b(y_\tau)\geq\sum_{\tau\geq t}\beta^{\tau-t}u^b(y_\tau).\end{aligned}$$

□

[15] More precisely, axioms VP(i), VP(ii), VP(iii), RP(i), and RP(iii), when restricted to $X^{\mathcal{T}}$, deliver SEU with unbounded utility.

Step 3. For all $(t,\omega) \in \mathcal{T} \times \Omega$, and all $h \in \mathcal{H}$ there exists $y = y(t,\omega,h) \in X^{\mathcal{T}}$ (indeed constant) such that $y \sim_{t,\omega} h$.

Details. For all $h \in \mathcal{H}$, let $x, x' \in X$ be such that $x \succsim_{t,\omega} h(\tau,\omega') \succsim_{t,\omega} x'$ for all $(\tau,\omega') \in \mathcal{T} \times \Omega$, then, by RP(i)

$$(x,...,x) \succsim_{t,\omega} (h(0,\omega'),...,h(T,\omega')) \succsim_{t,\omega} (x',...,x') \quad \forall \omega' \in \Omega,$$

by VP(iv)

$$x \succsim_{t,\omega} h \succsim_{t,\omega} x',$$

by VP(iii) the sets $\{\alpha \in [0,1] : \alpha x + (1-\alpha)x' \succsim_{t,\omega} h\}$ and $\{\alpha \in [0,1] : h \succsim_{t,\omega} \alpha x + (1-\alpha)x'\}$ are closed; they are nonempty since 1 belongs to the first and 0 to the second; their union is the whole $[0,1]$. Since $[0,1]$ is connected, their intersection is not empty, hence there exists $\bar{\alpha} \in [0,1]$ such that $\bar{\alpha}x + (1-\bar{\alpha})x' \sim_{t,\omega} h$. □

Step 4. For all $(t,\omega) \in \mathcal{T} \times \Omega$, and $h \in \mathcal{H}$ set

$$V_{t,\omega}(h) \equiv U_t(y) \quad \text{if } h \sim_{t,\omega} y \in X^{\mathcal{T}}.$$

$V_{t,\omega}$ is well defined and it represents $\succsim_{t,\omega}$ on $\mathcal{H}$.

Details. Choose $(t,\omega) \in \mathcal{T} \times \Omega$. For all $h \in \mathcal{H}$, $h \sim_{t,\omega} y$ and $h \sim_{t,\omega} y^\circ$ with $y, y^\circ \in X^{\mathcal{T}}$ implies $y \sim_{t,\omega} y^\circ$ and $U_t(y) = U_t(y^\circ)$, then $V_{t,\omega}$ is well defined. Let $h \sim_{t,\omega} y$ and $h' \sim_{t,\omega} y'$, then $h \succsim_{t,\omega} h'$ iff $y \succsim_{t,\omega} y'$ iff $U_t(y) \geq U_t(y')$ iff $V_{t,\omega}(h) \geq V_{t,\omega}(h')$. Therefore $V_{t,\omega}$ represents $\succsim_{t,\omega}$ on $\mathcal{H}$. □

Step 5. Each $h \in \mathcal{H}$ can be regarded as a function $h : \Omega \to X^{\mathcal{T}}$, and $U_t : X^{\mathcal{T}} \to \mathbb{R}$ is an affine function for every $t \in \mathcal{T}$. Define $U_t(h) : \Omega \to \mathbb{R}$ by

$$U_t(h)(\omega') \equiv U_t(h(\omega')) = \sum_{\tau \geq t} \beta^{\tau-t} u(h_\tau(\omega')) \quad \forall \omega' \in \Omega,$$

that is, $U_t(h) = U_t \circ h$. Notice that, $U_t(\alpha h + (1-\alpha)h') = \alpha U_t(h) + (1-\alpha)U_t(h')$ for all $h, h' \in \mathcal{H}$ and $\alpha \in [0,1]$.

Details. For all $\omega' \in \Omega$,

$$\begin{aligned}
U_t((\alpha h + (1-\alpha)h'))(\omega') &= \sum_{\tau \geq t} \beta^{\tau-t} u(\alpha h_\tau(\omega') + (1-\alpha)h'_\tau(\omega')) \\
&= \sum_{\tau \geq t} \beta^{\tau-t} [\alpha u(h_\tau(\omega')) + (1-\alpha)u(h'_\tau(\omega'))] \\
&= \sum_{\tau \geq t} \left(\beta^{\tau-t} \alpha u(h_\tau(\omega')) + \beta^{\tau-t}(1-\alpha)u(h'_\tau(\omega'))\right) \\
&= \alpha \sum_{\tau \geq t} \beta^{\tau-t} u(h_\tau(\omega')) + (1-\alpha) \sum_{\tau \geq t} \beta^{\tau-t} u(h'_\tau(\omega')) \\
&= \alpha U_t(h)(\omega') + (1-\alpha)U_t(h')(\omega').
\end{aligned}$$

Also notice that, if $y(\tau,\omega') = y_\tau$ for all $\tau \in T$ and all $\omega' \in \Omega$, then

$$U_t(y)(\omega') = \sum_{\tau \geq t} \beta^{\tau-t} u(y_\tau) = U_t(y_0,...,y_T) \quad \forall \omega' \in \Omega,$$

that is, the identification between acts with consequences depending only on time (and not on state) and elements of $X^{\mathcal{T}}$ corresponds here to the equivalence between constant functions on Ω and real numbers. □

Step 6. For all $t \in \mathcal{T}$, $\{U_t(h) : h \in \mathcal{H}\} = u(X)^{\Omega}$.

Details. $\subseteq$ is trivial.

- If $u(X)=\mathbb{R}^{+}$. Let $x^0 \in X$ be an element such that $u\left(x^0\right)=0$. If $\psi \in u(X)^{\Omega}$, for all $\omega' \in \Omega$ there exists $x^{\psi(\omega')} \in X$ such that $u\left(x^{\psi(\omega')}\right)=\beta^{t-T}\psi(\omega')$. Set

$$h(\tau,\omega')=\begin{cases} x^0 & \text{if } \tau<T \\ x^{\psi(\omega')} & \text{if } \tau=T. \end{cases}$$

  This delivers, for all $\omega' \in \Omega$:

$$U_t(h(\omega'))=\sum_{\tau\geq t}\beta^{\tau-t}u(h_\tau(\omega'))=\beta^{T-t}u\left(x^{\psi(\omega')}\right)=\psi(\omega').$$

- If $u(X)=\mathbb{R}^{++}$. If $t=T$ take any $x^0 \in X$ and apply the technique just introduced. Otherwise, notice that, if $\psi \in u(X)^{\Omega}$, there exists $\varepsilon>0$ such that $\psi-\varepsilon \in u(X)^{\Omega}$, choose $x^{\varepsilon} \in X$ such that $u(x^{\varepsilon})=\left(\sum_{T>\tau\geq t}\beta^{\tau-t}\right)^{-1}\varepsilon$. For all $\omega' \in \Omega$, there exists $x^{\psi(\omega')} \in X$ such that $u\left(x^{\psi(\omega')}\right)=\beta^{t-T}(\psi(\omega')-\varepsilon)$. Set

$$h(\tau,\omega')=\begin{cases} x^{\varepsilon} & \text{if } \tau<T \\ x^{\psi(\omega')} & \text{if } \tau=T. \end{cases}$$

  This delivers, for all $\omega' \in \Omega$:

$$\begin{aligned} U_t(h(\omega')) &= \sum_{\tau\geq t}\beta^{\tau-t}u(h_\tau(\omega'))=\sum_{T>\tau\geq t}\beta^{\tau-t}u(x^{\varepsilon})+\beta^{T-t}u\left(x^{\psi(\omega')}\right) \\ &= u(x^{\varepsilon})\left(\sum_{T>\tau\geq t}\beta^{\tau-t}\right)+\psi(\omega')-\varepsilon=\psi(\omega'). \end{aligned}$$

The cases $u(X)=\mathbb{R},\mathbb{R}^{-},\mathbb{R}^{--}$ are very similar. □

Step 7. Let $(t,\omega) \in \mathcal{T}\times\Omega$. For all $\psi \in u(X)^{\Omega}$, set

$$I_{t,\omega}(\psi)\equiv V_{t,\omega}(h) \quad \text{if } \psi=U_t(h) \text{ for some } h\in\mathcal{H}.$$

$I_{t,\omega}: u(X)^{\Omega}\to\mathbb{R}$ is well defined, monotonic, and normalized.

Details. Assume that $\psi=U_t(h)=U_t(h')$ for $h,h' \in \mathcal{H}$. This means that, for all $\omega' \in \Omega$,

$$\sum_{\tau\geq t}\beta^{\tau-t}u(h_\tau(\omega'))=\sum_{\tau\geq t}\beta^{\tau-t}u(h'_\tau(\omega')),$$

then, for all $\omega' \in \Omega$,

$$(h_0(\omega'),h_1(\omega'),...,h_T(\omega'))\sim_{t,\omega}(h'_0(\omega'),h'_1(\omega'),...,h'_T(\omega')).$$

By VP(iv), $h\sim_{t,\omega}h'$ and $V_{t,\omega}(h)=V_{t,\omega}(h')$. Then $I_{t,\omega}$ is well defined since, by Step 5, for every $\psi \in u(X)^{\Omega}$ there is $h\in\mathcal{H}$ such that $\psi=U_t(h)$.

Assume $\psi,\psi' \in u(X)^{\Omega}$, $\psi\geq\psi'$, $\psi=U_t(h)$, $\psi'=U_t(h')$. Then, for all $\omega' \in \Omega$,

$$\sum_{\tau\geq t}\beta^{\tau-t}u(h_\tau(\omega'))\geq\sum_{\tau\geq t}\beta^{\tau-t}u(h'_\tau(\omega')),$$

and, for all $\omega' \in \Omega$,

$$(h_0(\omega'),h_1(\omega'),...,h_T(\omega'))\succsim_{t,\omega}(h'_0(\omega'),h'_1(\omega'),...,h'_T(\omega')),$$

by VP(iv), $h\succsim_{t,\omega}h'$ and $V_{t,\omega}(h)\geq V_{t,\omega}(h')$. That is, $I_{t,\omega}$ is monotonic.

For all $b \in u(X)$, take $x^b \in X$ such that $u(x^b) = \left(\sum_{\tau \geq t} \beta^{\tau-t}\right)^{-1} b$, and the constant act x^b to obtain

$$U_t(x^b)(\omega') = \sum_{\tau \geq t} \beta^{\tau-t} u(x^b) = b \quad \forall \omega' \in \Omega,$$

then $b1_\Omega = U_t(x^b)$ (where x^b is regarded as a constant act) and

$$I_{t,\omega}(b1_\Omega) = V_{t,\omega}(x^b) = U_t(x^b, x^b, ..., x^b) = b.$$

This implies that $I_{t,\omega}$ is normalized. □

Step 8. Let $(t,\omega) \in \mathcal{T} \times \Omega$. For every $\psi \in u(X)^\Omega$ and for every $b \in \mathbb{R}$ such that $\psi + b \in u(X)^\Omega$,

$$I_{t,\omega}(\psi + b) = I_{t,\omega}(\psi) + b.$$

Details. Let $\psi' = U_t(h'), \psi'' = U_t(h'') \in u(X)^\Omega$, $b' = U_t(x'), b'' = U_t(x'') \in u(X)$, VP(ii) guarantees that for all $\alpha \in (0,1)$

$$\alpha h' + (1-\alpha)x' \sim_{t,\omega} \alpha h'' + (1-\alpha)x' \Rightarrow \alpha h' + (1-\alpha)x'' \sim_{t,\omega} \alpha h'' + (1-\alpha)x''.$$

That is

$$\begin{aligned} & V_{t,\omega}(\alpha h' + (1-\alpha)x') = V_{t,\omega}(\alpha h'' + (1-\alpha)x') \\ \Rightarrow \quad & V_{t,\omega}(\alpha h' + (1-\alpha)x'') = V_{t,\omega}(\alpha h'' + (1-\alpha)x'') \end{aligned}$$

hence

$$\begin{aligned} & I_{t,\omega}(U_t(\alpha h' + (1-\alpha)x')) = I_{t,\omega}(U_t(\alpha h'' + (1-\alpha)x')) \\ \Rightarrow \quad & I_{t,\omega}(U_t(\alpha h' + (1-\alpha)x'')) = I_{t,\omega}(U_t(\alpha h'' + (1-\alpha)x'')). \end{aligned}$$

Therefore

$$\begin{aligned} & I_{t,\omega}\left(\alpha\psi' + (1-\alpha)b'\right) = I_{t,\omega}\left(\alpha\psi'' + (1-\alpha)b'\right) \\ \Rightarrow \quad & I_{t,\omega}\left(\alpha\psi' + (1-\alpha)b''\right) = I_{t,\omega}\left(\alpha\psi'' + (1-\alpha)b''\right), \end{aligned}$$

for all $\psi', \psi'' \in u(X)^\Omega$, $b', b'' \in u(X)$, and $\alpha \in (0,1)$. Replacing ψ' with $\frac{\psi'}{\alpha} \in u(X)^\Omega$, ψ'' with $\frac{\psi''}{\alpha} \in u(X)^\Omega$, b' with $\frac{b'}{1-\alpha} \in u(X)$, b'' with $\frac{b''}{1-\alpha} \in u(X)$, we obtain

$$I_{t,\omega}\left(\psi' + b'\right) = I_{t,\omega}\left(\psi'' + b'\right) \Rightarrow I_{t,\omega}\left(\psi' + b''\right) = I_{t,\omega}\left(\psi'' + b''\right), \tag{25}$$

for all $\psi', \psi'' \in u(X)^\Omega$, $b', b'' \in u(X)$.

Notice that for all $\psi \in u(X)^\Omega$, $\min_{\omega'} \psi(\omega'), \max_{\omega'} \psi(\omega') \in u(X)$, since $I_{t,\omega}$ is monotonic and normalized,

$$\begin{aligned} I_{t,\omega}(\psi) &\geq I_{t,\omega}\left(\min_{\omega'} \psi(\omega')\right) = \min_{\omega'} \psi(\omega') \in u(X) \text{ and} \\ I_{t,\omega}(\psi) &\leq I_{t,\omega}\left(\max_{\omega'} \psi(\omega')\right) = \max_{\omega'} \psi(\omega') \in u(X), \text{ then} \\ I_{t,\omega}(\psi) &\in u(X). \end{aligned}$$

Assume $u(X) \supseteq \mathbb{R}^{++}$, $\psi \in u(X)^\Omega$ and $b > 0$, normalization and Eq. (25) guarantee

$$\begin{aligned} & I_{t,\omega}(\psi) = I_{t,\omega}(I_{t,\omega}(\psi)) \\ \Rightarrow \quad & I_{t,\omega}(\psi + b) = I_{t,\omega}(I_{t,\omega}(\psi) + b) = I_{t,\omega}(\psi) + b. \end{aligned}$$

If $b < 0$, then $I_{t,\omega}(\psi) = I_{t,\omega}((\psi + b) - b) = I_{t,\omega}(\psi + b) - b$, as wanted. The case in which $u(X) \supseteq \mathbb{R}^{--}$ is analogous. □

Step 9. Let $(t, \omega) \in \mathcal{T} \times \Omega$. For every $\psi, \psi' \in u(X)^{\Omega}$ such that $I_{t,\omega}(\psi) = I_{t,\omega}(\psi')$, and every $\alpha \in (0,1)$, $I_{t,\omega}(\alpha\psi + (1-\alpha)\psi') \geq I_{t,\omega}(\psi)$.

Details. Let $\psi = U_t(h), \psi' = U_t(h') \in u(X)^{\Omega}$. Since $I_{t,\omega}(\psi) = I_{t,\omega}(\psi')$, then $V_{t,\omega}(h) = V_{t,\omega}(h')$ and, by VP(v), $\alpha h + (1-\alpha) h' \succsim_{t,\omega} h$, that is

$$\begin{aligned} I_{t,\omega}(\alpha\psi + (1-\alpha)\psi') &= I_{t,\omega}(\alpha U_t(h) + (1-\alpha) U_t(h')) = I_{t,\omega}(U_t(\alpha h + (1-\alpha) h')) \\ &= V_{t,\omega}(\alpha h + (1-\alpha) h') \geq V_{t,\omega}(h) = I_{t,\omega}(\psi). \end{aligned}$$

□

Steps 6–8, and the results of Maccheroni, Marinacci, and Rustichini [15], imply that: For all $(t, \omega) \in \mathcal{T} \times \Omega$, $I_{t,\omega}$ is a *concave and normalized niveloid* on $u(X)^{\Omega}$. The restriction of its concave conjugate to $\Delta(\Omega)$,

$$I^*_{t,\omega}(p) \equiv \inf_{\psi \in u(X)^{\Omega}} (\langle \psi, p \rangle - I_{t,\omega}(\psi)) \quad \forall p \in \Delta(\Omega),$$

is the unique concave and upper semicontinuous function $I^{\#}_{t,\omega}$ on $\Delta(\Omega)$ such that

$$I_{t,\omega}(\psi) = \min_{p \in \Delta(\Omega)} \left(\langle \psi, p \rangle - I^{\#}_{t,\omega}(p) \right) \quad \forall \psi \in u(X)^{\Omega}.$$

Moreover, there exists a unique niveloid $J_{t,\omega} : \mathbb{R}^{\Omega} \to \mathbb{R}$ extending $I_{t,\omega}$. It is defined by

$$J_{t,\omega}(\varphi) \equiv I_{t,\omega}(\varphi + b) - b$$

if $\varphi \in \mathbb{R}^{\Omega}$ and $b \in \mathbb{R}$ is such that $\varphi + b \in u(X)^{\Omega}$. $J_{t,\omega}$ is a normalized, concave niveloid and its concave conjugate $J^*_{t,\omega}$ coincides with $I^*_{t,\omega}$ on $\Delta(\Omega)$ and takes value $-\infty$ on $\mathbb{R}^{\Omega} \backslash \Delta(\Omega)$, in particular

$$J_{t,\omega}(\varphi) = \min_{p \in \Delta(\Omega)} (\langle \varphi, p \rangle - I^*_{t,\omega}(p)) \quad \forall \varphi \in \mathbb{R}^{\Omega}.$$

Again, see [15] for details.

Clearly, by CP(i), we can assume $I_{t,\omega} = I_{t,\omega'}$ if $G_t(\omega) = G_t(\omega')$ and set

$$c_{t,\omega}(p) \equiv -I^*_{t,\omega}(p) = -J^*_{t,\omega}(p)$$

for all $p \in \Delta(\Omega)$ and all $(t, \omega) \in \mathcal{T} \times \Omega$. Then

- $c_{t,\omega} : \Delta(\Omega) \to [0, \infty]$ is grounded, closed, and convex for all $(t, \omega) \in \mathcal{T} \times \Omega$;
- $c_{t,\omega'} = c_{t,\omega}$ if $G_t(\omega) = G_t(\omega')$;
- for all $(t, \omega) \in \mathcal{T} \times \Omega$, $\succsim_{t,\omega}$ is represented by

  $$V_{t,\omega}(h) = I_{t,\omega}(U_t(h)) = J_{t,\omega}(U_t(h)) = \min_{p \in \Delta(\Omega)} (\langle U_t(h), p \rangle - I^*_{t,\omega}(p)).$$

  that is

  $$V_t(\omega, h) = \min_{p \in \Delta(\Omega)} \left(\sum_{\omega' \in \Omega} p(\omega') \sum_{\tau \geq t} \beta^{\tau - t} u(h_\tau(\omega')) + c_t(\omega, p) \right).$$

Step 10. Let $(t, \omega) \in \mathcal{T} \times \Omega$. If $\varphi^1, \varphi^2 \in \mathbb{R}^{\Omega}$ are such that $\varphi^1_{|G_t(\omega)} = \varphi^2_{|G_t(\omega)}$, then $J_{t,\omega}(\varphi^1) = J_{t,\omega}(\varphi^2)$.

Details. Assume $\psi^1, \psi^2 \in u(X)^{\Omega}$ are such that $\psi^1_{|G_t(\omega)} = \psi^2_{|G_t(\omega)}$. We show that $I_{t,\omega}(\psi^1) = I_{t,\omega}(\psi^2)$.

- If $u(X)=\mathbb{R}^{+}$. Let $x^{0}\in X$ be an element such that $u\left(x^{0}\right)=0$. For all $\omega'\in\Omega$ and $i=1,2$, there exists $x^{\psi^{i}(\omega')}\in X$ such that $u\left(x^{\psi^{i}(\omega')}\right)=\beta^{t-T}\psi^{i}(\omega')$; also notice that if $\omega'\in G_{t}(\omega)$, since $\psi^{1}(\omega')=\psi^{2}(\omega')$, we can choose $x^{\psi^{1}(\omega')}=x^{\psi^{2}(\omega')}$. Therefore the acts h^{1},h^{2} defined by
$$h^{i}(\tau,\omega')=\begin{cases} x^{0} & \text{if } \tau<T \\ x^{\psi^{i}(\omega')} & \text{if } \tau=T \end{cases}$$
are such that $h^{1}(\tau,\omega')=h^{2}(\tau,\omega')$ for all $\tau\geq t$ and $\omega'\in G_{t}(\omega)$, CP(ii) implies $h^{1}\sim_{t,\omega}h^{2}$ and
$$I_{t,\omega}\left(\psi^{1}\right)=I_{t,\omega}\left(U_{t}\left(h^{1}\right)\right)=V_{t,\omega}\left(h^{1}\right)=V_{t,\omega}\left(h^{2}\right)=I_{t,\omega}\left(\psi^{2}\right).$$
- If $u(X)=\mathbb{R}^{++}$. If $t=T$ take any $x^{0}\in X$ and apply the technique just introduced. Otherwise notice that there exists $\varepsilon>0$ such that $\psi^{i}-\varepsilon\in u(X)^{\Omega}$ (for $i=1,2$), choose $x^{\varepsilon}\in X$ such that $u(x^{\varepsilon})=\left(\sum_{T>\tau\geq t}\beta^{\tau-t}\right)^{-1}\varepsilon$. For all $\omega'\in\Omega$ and $i=1,2$, there exists $x^{\psi^{i}(\omega')}\in X$ such that $u\left(x^{\psi^{i}(\omega')}\right)=\beta^{t-T}\left(\psi^{i}(\omega')-\varepsilon\right)$; again notice that if $\omega'\in G_{t}(\omega)$, since $\psi^{1}(\omega')=\psi^{2}(\omega')$, we can choose $x^{\psi^{1}(\omega')}=x^{\psi^{2}(\omega')}$. Therefore the acts h^{1},h^{2} defined by
$$h^{i}(\tau,\omega')=\begin{cases} x^{\varepsilon} & \text{if } \tau<T \\ x^{\psi^{i}(\omega')} & \text{if } \tau=T \end{cases}$$
are such that $h^{1}(\tau,\omega')=h^{2}(\tau,\omega')$ for all $\tau\geq t$ and $\omega'\in G_{t}(\omega)$, again CP(ii) implies $h^{1}\sim_{t,\omega}h^{2}$ and
$$I_{t,\omega}\left(\psi^{1}\right)=I_{t,\omega}\left(U_{t}\left(h^{1}\right)\right)=V_{t,\omega}\left(h^{1}\right)=V_{t,\omega}\left(h^{2}\right)=I_{t,\omega}\left(\psi^{2}\right).$$
- The cases $u(X)=\mathbb{R},\mathbb{R}^{-},\mathbb{R}^{--}$ are very similar.

If $\varphi^{1},\varphi^{2}\in\mathbb{R}^{\Omega}$ are such that $\varphi^{1}_{|G_{t}(\omega)}=\varphi^{2}_{|G_{t}(\omega)}$, take $b\in\mathbb{R}$ such that $\varphi^{1}+b,\varphi^{2}+b\in u(X)^{\Omega}$, then $\left(\varphi^{1}+b\right)_{|G_{t}(\omega)}=\left(\varphi^{2}+b\right)_{|G_{t}(\omega)}$ and
$$J_{t,\omega}\left(\varphi^{1}\right)=J_{t,\omega}\left(\varphi^{1}+b-b\right)=I_{t,\omega}\left(\varphi^{1}+b\right)-b=I_{t,\omega}\left(\varphi^{2}+b\right)-b=J_{t,\omega}\left(\varphi^{2}\right).$$

□

By Lemma 4, $\operatorname{dom}c_{t,\omega}=\operatorname{dom}J^{*}_{t,\omega}\subseteq\Delta\left(G_{t}(\omega)\right)$. This concludes the proof of (a) ⇒ (b).

(b) ⇒ (a) is straightforward.

Step 11. $\left(\bar{\beta},\bar{u},\{\bar{c}_{t,\omega}\}\right)$ represent $\succsim_{t,\omega}$ in the sense of Eq. (19) iff $\bar{\beta}=\beta$, $\bar{u}=au+b$ for some $a>0$ and $b\in\mathbb{R}$ and $\{\bar{c}_{t,\omega}\}=\{ac_{t,\omega}\}$.

Details. Let $\bar{u}\equiv au+b$ for some $a>0$ and $b\in\mathbb{R}$ and $\bar{c}\equiv ac$, then
$$\begin{aligned}\bar{V}_{t}(\omega,h) &\equiv \min_{p\in\Delta(\Omega)}\left(\int\sum_{\tau\geq t}\beta^{\tau-t}\bar{u}(h_{\tau})\,dp+\bar{c}_{t,\omega}(p)\right)\\ &= \min_{p\in\Delta(\Omega)}\left(\int\sum_{\tau\geq t}\beta^{\tau-t}\left(au(h_{\tau})+b\right)dp+ac_{t,\omega}(p)\right)\\ &= a\min_{p\in\Delta(\Omega)}\left(\int\sum_{\tau\geq t}\beta^{\tau-t}u(h_{\tau})\,dp+c_{t,\omega}(p)\right)+b\sum_{\tau\geq t}\beta^{\tau-t}\\ &= aV_{t}(\omega,h)+b\sum_{\tau\geq t}\beta^{\tau-t}\end{aligned}$$

represents $\succsim_{t,\omega}$ for all $(t,\omega) \in \mathcal{T} \times \Omega$. Clearly, $\bar{u}$ and $\bar{c}$ have the required structural properties.

Conversely, assume there is another family of grounded, closed and convex functions $\bar{c}_{t,\omega}$: $\Delta(\Omega) \to [0,\infty]$, such that $\operatorname{dom} \bar{c}_{t,\omega} \subseteq \Delta(G_t(\omega))$ and $\bar{c}_{t,\omega} = \bar{c}_{t,\omega'}$ if $G_t(\omega) = G_t(\omega')$, $\bar{\beta} > 0$, and an unbounded affine $\bar{u} : X \to \mathbb{R}$ such that: for all $(t,\omega) \in \mathcal{T} \times \Omega$, $\succsim_{t,\omega}$ is represented by $\bar{V}_t(\omega,\cdot)$, where

$$\bar{V}_t(\omega,h) \equiv \min_{p \in \Delta(\Omega)} \left(\int \sum_{\tau \geq t} \bar{\beta}^{\tau-t} \bar{u}(h_\tau)\, dp + \bar{c}_{t,\omega}(p) \right) \quad \forall h \in \mathcal{H}.$$

The function $\bar{U}_t : X^{\mathcal{T}} \to \mathbb{R}$ defined for all $y \in X^{\mathcal{T}}$ by

$$\bar{U}_t(y) \equiv \bar{V}_t(\omega,y) = \sum_{\tau \geq t} \bar{\beta}^{\tau-t} \bar{u}(y_\tau)$$

represents $\succsim_{t,\omega}$ on $X^{\mathcal{T}}$ for all $(t,\omega) \in \mathcal{T} \times \Omega$. In particular, $\bar{U}_0(x) = \bar{u}(x) \sum_{\tau \geq 0} \bar{\beta}^\tau$ represents $\succsim_0$ on X, since the same is true for $U_0(x) = u(x) \sum_{\tau \geq 0} \beta^\tau$, we conclude that there are $a > 0$ and $b \in \mathbb{R}$ such that

$$\bar{u} = au + b.$$

Therefore,

$$\bar{U}_0(y) = a \sum_{\tau \geq 0} \bar{\beta}^\tau u(y_\tau) + b \sum_{\tau \geq 0} \bar{\beta}^\tau$$

and

$$W_0(y) \equiv \sum_{\tau \geq 0} \bar{\beta}^\tau u(y_\tau)$$

also represents $\succsim_0$ on $X^{\mathcal{T}}$. Since U_0 and W_0 are affine on $X^{\mathcal{T}}$, there are $\bar{a} > 0$ and $\bar{b} \in \mathbb{R}$ such that

$$W_0(y) = \bar{a} U_0(y) + \bar{b} \tag{26}$$

for all $y \in X^{\mathcal{T}}$. We restrict our attention to the case in which $u(X) \ni 0, 1$, the remaining cases being analogous. Let $x^0, x^1 \in X$ be elements such that $u(x^0) = 0$, $u(x^1) = 1$. Denote by y^{00} the constant element of $X^{\mathcal{T}}$ defined by $y^{00}_\tau = x^0$ for all $\tau \in \mathcal{T}$, and

$$\begin{aligned} y^{10} &= \left(y^{00}_{-\{0\}}, x^1 \right), \\ y^{01} &= \left(y^{00}_{-\{1\}}, x^1 \right). \end{aligned}$$

Eq. (26) applied to y^{00}, y^{10}, y^{01} delivers

$$0 = \bar{b},\ 1 = \bar{a},\ \bar{\beta} = \beta.$$

Therefore, for all $t \in \mathcal{T}$,

$$\begin{aligned} \bar{U}_t(y) &= a \sum_{\tau \geq t} \beta^{\tau-t} u(y_\tau) + b \sum_{\tau \geq t} \beta^{\tau-t}, \\ \bar{U}_t(y) &= a U_t(y) + b \sum_{\tau \geq t} \beta^{\tau-t}, \\ U_t(y) &= \frac{\bar{U}_t(y)}{a} - \frac{b}{a} \sum_{\tau \geq t} \beta^{\tau-t}, \end{aligned}$$

for all $y \in X^{\mathcal{T}}$. Setting $k \equiv \frac{1}{a}$ and $l \equiv -\frac{b}{a} \sum_{\tau \geq t} \beta^{\tau-t}$, $k\bar{U}_t(y) + l = U_t(y)$.

Let $(t,\omega) \in \mathcal{T} \times \Omega$. For all $\varphi \in u(X)^{\Omega}$ and $h \in \mathcal{H}$ such that $U_t(h) = \varphi$, let $y \sim_{t,\omega} h$, and conclude

$$\begin{aligned} I_{t,\omega}(\varphi) &= V_{t,\omega}(h) = U_t(y) = k\bar{U}_t(y) + l = k\bar{V}_{t,\omega}(y) + l \\ &= k\bar{V}_{t,\omega}(h) + l = k \min_{p \in \Delta(\Omega)} \left(\int \bar{U}_t(h)\, dp + \bar{c}_{t,\omega}(p) \right) + l \\ &= \min_{p \in \Delta(\Omega)} \left(\int \left(k\bar{U}_t(h) + l \right) dp + k\bar{c}_{t,\omega}(p) \right) \\ &= \min_{p \in \Delta(\Omega)} \left(\int U_t(h)\, dp + k\bar{c}_{t,\omega}(p) \right) \\ &= \min_{p \in \Delta(\Omega)} \left(\int \varphi dp + \frac{1}{a}\bar{c}_{t,\omega}(p) \right), \end{aligned}$$

whence, see [15],

$$\bar{c}_{t,\omega} = ac_{t,\omega}.$$

□

Step 12. Let $(t,\omega) \in \mathcal{T} \times \Omega$ be such that $|G_t(\omega)| > 1$. A state $\omega'' \in G_t(\omega)$ is $\succsim_{t,\omega}$-null if and only if $\operatorname{dom} c_{t,\omega} \subseteq \Delta(G_t(\omega) \setminus \{\omega''\})$.

Details. We show that if $\omega'' \in G_t(\omega)$ is $\succsim_{t,\omega}$-null, then $J_{t,\omega}(\varphi^1) = J_{t,\omega}(\varphi^2)$ for every $\varphi^1, \varphi^2 \in \mathbb{R}^{\Omega}$ such that $\varphi^1_{|G_t(\omega)\setminus\{\omega''\}} = \varphi^2_{|G_t(\omega)\setminus\{\omega''\}}$. By Lemma 4, $\operatorname{dom} c_{t,\omega} = \operatorname{dom} J^*_{t,\omega} \subseteq \Delta(G_t(\omega) \setminus \{\omega''\})$. Clearly, it is sufficient to show that for every $\psi^1, \psi^2 \in u(X)^{\Omega}$ such that $\psi^1_{|G_t(\omega)\setminus\{\omega''\}} = \psi^2_{|G_t(\omega)\setminus\{\omega''\}}$, then $I_{t,\omega}(\psi^1) = I_{t,\omega}(\psi^2)$. In fact, take $b \in \mathbb{R}$ such that $\varphi^1 + b, \varphi^2 + b \in u(X)^{\Omega}$, then $\varphi^1_{|G_t(\omega)\setminus\{\omega''\}} = \varphi^2_{|G_t(\omega)\setminus\{\omega''\}}$ implies $(\varphi^1 + b)_{|G_t(\omega)\setminus\{\omega''\}} = (\varphi^2 + b)_{|G_t(\omega)\setminus\{\omega''\}}$ and

$$J_{t,\omega}(\varphi^1) = J_{t,\omega}(\varphi^1 + b - b) = I_{t,\omega}(\varphi^1 + b) - b = I_{t,\omega}(\varphi^2 + b) - b = J_{t,\omega}(\varphi^2).$$

Let $u(X) = \mathbb{R}^+$ and $x^0 \in X$ be an element such that $u(x^0) = 0$. For all $\omega' \in \Omega$ and $i = 1, 2$, there exists $x^{\psi^i(\omega')} \in X$ such that $u\left(x^{\psi^i(\omega')}\right) = \beta^{t-T}\psi^i(\omega')$; also notice that if $\omega' \in G_t(\omega) \setminus \{\omega''\}$, then, since $\psi^1(\omega') = \psi^2(\omega')$, we can choose $x^{\psi^1(\omega')} = x^{\psi^2(\omega')}$. For $i = 1, 2$ and $(\tau, \omega') \in \mathcal{T} \times \Omega$ set

$$h^i(\tau,\omega') = \begin{cases} x^0 & \text{if } \tau < T \\ x^{\psi^i(\omega')} & \text{if } \tau = T \end{cases} \text{ and } g^i(\tau,\omega') = \begin{cases} x^0 & \text{if } \tau < T \\ x^{\psi^i(\omega')} & \text{if } \tau = T \text{ and } \omega' \neq \omega'' \\ x^0 & \text{if } \tau = T \text{ and } \omega' = \omega'' \end{cases}.$$

Since $g^1(\tau,\omega') = g^2(\tau,\omega')$ for all $\tau \geq t$ and $\omega' \in G_t(\omega)$, CP(ii) implies $g^1 \sim_{t,\omega} g^2$, while $\succsim_{t,\omega}$-nullity of ω'' implies $h^i \sim_{t,\omega} g^i$ for $i = 1, 2$. Therefore, $h^1 \sim_{t,\omega} h^2$, and

$$I_{t,\omega}(\psi^1) = I_{t,\omega}(U_t(h^1)) = V_{t,\omega}(h^1) = V_{t,\omega}(h^2) = I_{t,\omega}(\psi^2).$$

The cases $u(X) = \mathbb{R}^{++}, \mathbb{R}, \mathbb{R}^-, \mathbb{R}^{--}$ are very similar.

Conversely, if $\operatorname{dom} c_{t,\omega} \subseteq \Delta(G_t(\omega) \setminus \{\omega''\})$, then $\omega'' \in \Omega$ is $\succsim_{t,\omega}$-null. In fact, assume

$h'(\tau',\omega') = h(\tau',\omega')$ for all $\tau' \in \mathcal{T}$ and all $\omega' \neq \omega''$, then

$$\begin{aligned} V_t(\omega,h) &= \min_{p\in\Delta(\Omega)}\left(\int\sum_{\tau\geq t}\beta^{\tau-t}u(h_\tau)\,dp + c_{t,\omega}(p)\right) \\ &= \min_{p\in\Delta(G_t(\omega)\setminus\{\omega''\})}\left(\sum_{\omega'\in G_t(\omega)\setminus\{\omega''\}} p(\omega')\sum_{\tau\geq t}\beta^{\tau-t}u(h_\tau(\omega')) + c_{t,\omega}(p)\right) \\ &= \min_{p\in\Delta(G_t(\omega)\setminus\{\omega''\})}\left(\sum_{\omega'\in G_t(\omega)\setminus\{\omega''\}} p(\omega')\sum_{\tau\geq t}\beta^{\tau-t}u(h'_\tau(\omega')) + c_{t,\omega}(p)\right) \\ &= \min_{p\in\Delta(\Omega)}\left(\int\sum_{\tau\geq t}\beta^{\tau-t}u(h'_\tau)\,dp + c_{t,\omega}(p)\right) = V_t(\omega,h'), \end{aligned}$$

and hence $h \sim_{t,\omega} h'$. □

Therefore, if a state ω'' in $G_t(\omega)$ is $\succsim_{t,\omega}$-null, then $\operatorname{dom} c_{t,\omega} \subseteq \Delta(G_t(\omega)\setminus\{\omega''\})$, and

$$V_t(\omega,h) \neq \infty = \inf_{p\in\operatorname{ri}\Delta(G_t(\omega))}\left(\int\sum_{\tau\geq t}\beta^{\tau-t}u(h_\tau)\,dp + c_{t,\omega}(p)\right)$$

for some (indeed all) $h \in \mathcal{H}$. That is (i) ⇒ (ii). Conversely, if

$$V_t(\omega,h) \neq \inf_{p\in\operatorname{ri}\Delta(G_t(\omega))}\left(\int\sum_{\tau\geq t}\beta^{\tau-t}u(h_\tau)\,dp + c_{t,\omega}(p)\right)$$

for some $h \in \mathcal{H}$, then, by Lemma 2, $\operatorname{ri}\Delta(G_t(\omega)) \cap \operatorname{dom} c_{t,\omega} = \varnothing$. If, per contra, $\operatorname{dom} c_{t,\omega}$ is not contained in $\Delta(G_t(\omega)\setminus\{\omega''\})$ for some $\omega'' \in G_t(\omega)$, then for all $\omega' \in G_t(\omega)$ there exists $p^{\omega'} \in \operatorname{dom} c_{t,\omega}$ with $\omega' \in \operatorname{supp} p^{\omega'}$, then $|G_t(\omega)|^{-1}\sum_{\omega'\in G_t(\omega)} p^{\omega'} \in \operatorname{ri}\Delta(G_t(\omega)) \cap \operatorname{dom} c_{t,\omega}$, which is absurd. Then $\operatorname{dom} c_{t,\omega} \subseteq \Delta(G_t(\omega)\setminus\{\omega''\})$ for some ω'', which must be $\succsim_{t,\omega}$-null. This is (ii) ⇒ (i). The equivalence between (i) and (iii) descends immediately from Lemma 2. ■

Lemma 4 *Let $J : \mathbb{R}^\Omega \to \mathbb{R}$ be a concave, normalized niveloid, and $G \subseteq \Omega$. The following statements are equivalent:*

(a) If $\varphi^1, \varphi^2 \in \mathbb{R}^\Omega$ are such that $\varphi^1_{|G} = \varphi^2_{|G}$, then $J(\varphi^1) = J(\varphi^2)$;

(b) $J(\varphi+\psi) = J(\varphi) + \psi(G)$ if $\varphi,\psi \in \mathbb{R}^\Omega$ and ψ is constant on G;

(c) $J(\varphi 1_{G^c}) = 0$ for all $\varphi \in \mathbb{R}^\Omega$;

(d) $\operatorname{dom} J^* \subseteq \Delta(G)$.

Proof. (a) ⇒ (b) Since $(\varphi+\psi)_{|G} = (\varphi+\psi(G))_{|G}$,

$$J(\varphi+\psi) = J(\varphi+\psi(G)) = J(\varphi) + \psi(G).$$

(b) ⇒ (c) $J(\varphi 1_{G^c}) = J(0 + \varphi 1_{G^c}) = J(0) + \varphi 1_{G^c}(G) = 0$.

(c) ⇒ (d) Since J is a niveloid, $\operatorname{dom} J^* \subseteq \Delta(\Omega)$ (see [15]). Let $p^\circ \in \Delta(\Omega)$ and $p^\circ(\omega^\circ) > 0$ for some $\omega^\circ \notin G$. Since $b1_{\{\omega^\circ\}} = b1_{\{\omega^\circ\}}1_{G^c}$ for all $b \in \mathbb{R}$, then $J\left(b1_{\{\omega^\circ\}}\right) = 0$ and

$$J^*(p^\circ) \leq \inf_{b\in\mathbb{R}}\left(\left\langle b1_{\{\omega^\circ\}}, p^\circ\right\rangle - J\left(b1_{\{\omega^\circ\}}\right)\right) \leq \inf_{b\in\mathbb{R}} bp^\circ(\omega^\circ) = -\infty.$$

(d) $\Rightarrow$ (a) By the Fenchel-Moreau Theorem (see [15]),

$$\begin{aligned}J\left(\varphi^{1}\right) &= \min_{p\in\Delta(\Omega)}\left(\left\langle\varphi^{1},p\right\rangle - J^{*}(p)\right) = \min_{p\in\Delta(G)}\left(\left\langle\varphi^{1},p\right\rangle - J^{*}(p)\right)\\ &= \min_{p\in\Delta(G)}\left(\sum_{\omega\in G}p(\omega)\,\varphi^{1}(\omega) - J^{*}(p)\right) = \min_{p\in\Delta(G)}\left(\sum_{\omega\in G}p(\omega)\,\varphi^{2}(\omega) - J^{*}(p)\right)\\ &= J\left(\varphi^{2}\right)\end{aligned}$$

for all $\varphi^1,\varphi^2\in\mathbb{R}^\Omega$ such that $\varphi^1_{|G}=\varphi^2_{|G}$. ■

Lemma 5 *If $\{\succsim_{t,\omega}\}$ satisfy CP, FS, and DC, then for each t and ω, no state in $G_t(\omega)$ is $\succsim_{t,\omega}$-null provided $|G_t(\omega)|>1$.*

Proof. Assume, *per contra*, that there exist $\omega^\circ\in\Omega$ and $t^\circ\in\mathcal{T}$ such that $|G_{t^\circ}(\omega^\circ)|>1$ and $G_{t^\circ}(\omega^\circ)$ contains a $\succsim_{t^\circ,\omega^\circ}$-null state. W.l.o.g., ω° is $\succsim_{t^\circ,\omega^\circ}$-null. By FS, $t^\circ>0$ and

$$h(\tau',\omega') = h'(\tau',\omega') \text{ for all } \tau'\in\mathcal{T} \text{ and all } \omega'\neq\omega^\circ \text{ implies } h\sim_{t^\circ,\omega^\circ} h'. \tag{27}$$

Clearly, $|G_{t^\circ-1}(\omega^\circ)|\geq|G_{t^\circ}(\omega^\circ)|>1$. Next we show that ω° is $\succsim_{t^\circ-1,\omega^\circ}$-null. In a finite number of steps this leads to an absurd.

Assume that $h(\tau',\omega')=h'(\tau',\omega')$ for all $\tau'\in\mathcal{T}$ and all $\omega'\neq\omega^\circ$. By Eq. (27) and CP(i), $h\sim_{t^\circ,\omega} h'$ for all $\omega\in G_{t^\circ}(\omega^\circ)$. Moreover, if $\omega\in G_{t^\circ-1}(\omega^\circ)\setminus G_{t^\circ}(\omega^\circ)$, then $G_{t^\circ}(\omega)$ does not contain ω°, and $h(\tau',\omega')=h'(\tau',\omega')$ for all $\tau'\in\mathcal{T}$ and all $\omega'\in G_{t^\circ}(\omega)$. By CP(ii), $h\sim_{t^\circ,\omega} h'$ for all $\omega\in G_{t^\circ-1}(\omega^\circ)\setminus G_{t^\circ}(\omega^\circ)$. Therefore, $h\sim_{t^\circ,\omega} h'$ for all $\omega\in G_{t^\circ-1}(\omega^\circ)$. Since $|G_{t^\circ-1}(\omega^\circ)|>1$ and $h_{t^\circ-1}$ is $\mathcal{G}_{t^\circ-1}$ measurable, choose $\omega''\in G_{t^\circ-1}(\omega^\circ)-\{\omega^\circ\}$ to obtain

$$h(t^\circ-1,\omega^\circ) = h(t^\circ-1,\omega'') = h'(t^\circ-1,\omega'') = h'(t^\circ-1,\omega^\circ)$$

and conclude:

$$h(t^\circ-1,\omega^\circ) = h'(t^\circ-1,\omega^\circ) \text{ and } h\sim_{t^\circ,\omega} h' \text{ for all } \omega\in G_{t^\circ-1}(\omega^\circ),$$

DC implies that $h\sim_{t^\circ-1,\omega^\circ} h'$. That is ω° is $\succsim_{t^\circ-1,\omega^\circ}$-null. As wanted. ■

A.3 Proof of Proposition 1

Axiom 6 (Strong full support—SFS) *For each $(t,\omega)\in\mathcal{T}\times\Omega$, no state in $G_t(\omega)$ is $\succsim_{t,\omega}$-null.*

For technical reasons we prove the slightly more general version of Proposition 1.

Proposition 3 *The following statements are equivalent:*

(a) $\{\succsim_{t,\omega}\}$ satisfy CP, VP, RP, and no state in $G_t(\omega)$ is $\succsim_{t,\omega}$-null if $G_t(\omega)$ is not a singleton.

(b) There exist a scalar $\beta>0$, an unbounded affine function $u:X\to\mathbb{R}$, and a dynamic ambiguity index $\{c_t\}$, such that: for each $(t,\omega)\in\mathcal{T}\times\Omega$, $\succsim_{t,\omega}$ is represented by

$$V_t(\omega,h) = \inf_{p\in\mathrm{ri}\,\Delta(G_t(\omega))}\left(\int\sum_{\tau\geq t}\beta^{\tau-t}u(h_\tau)\,dp + c_{t,\omega}(p)\right)\quad\forall h\in\mathcal{H}. \tag{28}$$

(c) $\{\succsim_{t,\omega}\}$ satisfy CP, VP, RP, and SFS.

Moreover, $(\bar\beta,\bar u,\{\bar c_t\})$ represent $\succsim_{t,\omega}$ in the sense of Eq. (28) iff $\bar\beta=\beta$, $\bar u=au+b$ for some $a>0$ and $b\in\mathbb{R}$ and $\{\bar c_t\}=\{ac_t\}$.

Proof. (a) $\Leftrightarrow$ (b) immediately descends from Lemma 3.

(c) $\Rightarrow$ (a) is trivial.

(b) $\Rightarrow$ (c). Since (b) $\Rightarrow$ (a), if $G_t(\omega)$ is not a singleton, no state in $G_t(\omega)$ is $\succsim_{t,\omega}$-null. Let $G_t(\omega)$ be a singleton $\{\omega\}$. Then $\Delta(G_t(\omega)) = \{d_\omega\}$, and $V_t(\omega, h) = \sum_{\tau\geq t} \beta^{\tau-t} u(h_\tau(\omega))$ for all $h \in \mathcal{H}$. Since u is unbounded, there are $x^1, x^2 \in X$ such that $u(x^1) > u(x^2)$. Consider the acts

$$h^i(\tau, \omega') = \begin{cases} x^1 & \text{if } (\tau, \omega') \neq (T, \omega) \\ x^i & \text{if } (\tau, \omega') = (T, \omega) \end{cases}$$

$h^1(\tau, \omega') = h^2(\tau, \omega')$ for all $\tau \in \mathcal{T}$ and all $\omega' \neq \omega$. If ω where $\succsim_{t,\omega}$-null, we would have $h^1 \sim_{t,\omega} h^2$, but

$$V_t(\omega, h^1) = \sum_{\tau\geq t} \beta^{\tau-t} u(x^1) > \sum_{T>\tau\geq t} \beta^{\tau-t} u(x^1) + \beta^{T-t} u(x^2) = V_t(\omega, h^2).$$

Therefore ω is not $\succsim_{t,\omega}$-null. ■

Since clearly, for every t and ω, $\operatorname{ri}\Delta(G_t(\omega)) = \{p_{G_t(\omega)} : p \in \operatorname{ri}\Delta(\Omega)\}$, Eq. (28) is equivalent to

$$V_t(\omega, h) = \inf_{p\in \operatorname{ri}\Delta(\Omega)} \left(\int \sum_{\tau\geq t} \beta^{\tau-t} u(h_\tau)\, dp_{G_t(\omega)} + c_t\left(\omega, p_{G_t(\omega)}\right) \right) \quad \forall h \in \mathcal{H}. \tag{29}$$

Consider the $\mathcal{G}_t$ measurable functions $V_t(\cdot, h) : \Omega \to \mathbb{R}$ and $c_t(\cdot, p) : \Omega \to [0, \infty]$, Eq. (29) becomes

$$V_t(\cdot, h) = \inf_{p\in \operatorname{ri}\Delta(\Omega)} \left(\mathrm{E}^p \left(\sum_{\tau\geq t} \beta^{\tau-t} u(h_\tau) | \mathcal{G}_t \right)(\cdot) + c_t(\cdot, p | \mathcal{G}_t(\cdot)) \right) \quad \forall h \in \mathcal{H}, \tag{30}$$

or

$$V_t(h) = \inf_{p\in \operatorname{ri}\Delta(\Omega)} \left(\mathrm{E}^p \left(\sum_{\tau\geq t} \beta^{\tau-t} u(h_\tau) | \mathcal{G}_t \right) + c_t(p | \mathcal{G}_t) \right) \quad \forall h \in \mathcal{H}. \tag{31}$$

By Lemma 5, if $\{\succsim_{t,\omega}\}$ satisfies CP, VP, RP, DC and FS, then it admits this representation.

A.4 Dynamic Consistency

Lemma 6 *Let $\{\succsim_{t,\omega}\}$ be a family of preferences on $\mathcal{H}$ for which there exist a scalar $\beta > 0$, an unbounded affine function $u : X \to \mathbb{R}$, and a dynamic ambiguity index $\{c_t\}$, such that: for each $(t, \omega) \in \mathcal{T} \times \Omega$, $\succsim_{t,\omega}$ is represented by:*

$$V_t(\omega, h) = \min_{p\in\Delta(G_t(\omega))} \left(\int \sum_{\tau\geq t} \beta^{\tau-t} u(h_\tau)\, dp + c_t(\omega, p) \right) \quad \forall h \in \mathcal{H}.$$

The following statements are equivalent:

(a) $\{\succsim_{t,\omega}\}$ satisfy DC,

(b) For all $t < T$, $\omega \in \Omega$, and $q \in \Delta(G_t(\omega))$,

$$c_t(\omega, q) = \beta \sum_{\substack{G\in\mathcal{G}_{t+1} \\ q(G)>0}} q(G) c_{t+1}(G, q_G) + \min_{p\in\Delta(G_t(\omega)):p_{|\mathcal{G}_{t+1}}=q_{|\mathcal{G}_{t+1}}} c_t(\omega, p). \tag{32}$$

(c) For all $t < T$ and $\omega \in \Omega$,

$$c_t(\omega, \cdot) = \overline{\operatorname{co}}\, \varrho_t(\omega, \cdot) \tag{33}$$

where

$$\varrho_t(\omega, q) \equiv \beta \sum_{\substack{G\in\mathcal{G}_{t+1} \\ G\subseteq G_t(\omega)}} q(G) c_{t+1}(G, q_G) + \inf_{p\in \operatorname{ri}\Delta(G_t(\omega)):p_{|\mathcal{G}_{t+1}}=q_{|\mathcal{G}_{t+1}}} c_t(\omega, p),$$

for all $q \in \operatorname{ri}\Delta(G_t(\omega))$ and all $t < T$.

(d) For all $t < T$, $\omega \in \Omega$, and $h \in \mathcal{H}$

$$V_t(\omega, h) = u(h_t(\omega)) + \min_{r \in \Delta(\Omega, \mathcal{G}_{t+1})} \left(\int \beta V_{t+1}(h)\, dr + \min_{p \in \Delta(G_t(\omega)): p_{|\mathcal{G}_{t+1}} = r} c_t(\omega, p) \right).$$

Proof. W.l.o.g. $u(X) \in \{\mathbb{R}, \mathbb{R}^+, \mathbb{R}^{++}, \mathbb{R}^-, \mathbb{R}^{--}\}$. For all $t \in \mathcal{T}$, $\omega \in \Omega$, $\varphi \in \mathbb{R}^\Omega$, $y \in \mathcal{X}^T$ define

$$\begin{aligned} J_t(\omega, \varphi) &\equiv \min_{p \in \Delta(G_t(\omega))} \left(\langle \varphi, p \rangle + c_t(\omega, p) \right) = \inf_{p \in \operatorname{ri} \Delta(G_t(\omega))} \left(\langle \varphi, p \rangle + c_t(\omega, p) \right), \qquad (34) \\ U_t(y) &\equiv V_t(\omega, y) = \sum_{\tau \geq t} \beta^{\tau - t} u(y_\tau). \end{aligned}$$

Then $J_t : \mathbb{R}^\Omega \to \mathbb{R}^\Omega(\mathcal{G}_t)$, where $\mathbb{R}^\Omega(\mathcal{G}_t)$ the set of all $\mathcal{G}_t$-measurable functions. Notice that:

- Eq. (34) coincides with the property $\operatorname{dom} c_t(\omega, \cdot) \cap \operatorname{ri} \Delta(G_t(\omega)) \neq \varnothing$ of dynamic ambiguity indexes.
- $u(X)^\Omega = \{U_t \circ h : h \in \mathcal{H}\}$ (see Lemma 3).
- $V_t(\omega, h) = J_t(\omega, U_t \circ h)$ for all $(t, \omega, h) \in \mathcal{T} \times \Omega \times \mathcal{H}$.
- For all $\omega \in \Omega$, $t < T$, and $h \in \mathcal{H}$,

$$\begin{aligned} (U_t \circ h)(\omega) = U_t(h(\omega)) &= u(h_t(\omega)) + \sum_{\tau \geq t+1} \beta^{\tau - t} u(h_\tau(\omega)) \\ &= (u \circ h_t)(\omega) + \beta \sum_{\tau \geq t+1} \beta^{\tau - (t+1)} u(h_\tau(\omega)) \\ &= (u \circ h_t)(\omega) + \beta U_{t+1}(h(\omega)) = (u \circ h_t)(\omega) + \beta (U_{t+1} \circ h)(\omega), \end{aligned}$$

  that is

$$U_t \circ h = u \circ h_t + \beta (U_{t+1} \circ h). \qquad (35)$$

- For all $\omega \in \Omega$, $t < T$, and $h \in \mathcal{H}$,

$$\begin{aligned} V_t(\omega, h) = J_t(\omega, U_t \circ h) &= J_{t,\omega}(\beta(U_{t+1} \circ h) + u \circ h_t) \\ &= J_{t,\omega}\left((\beta(U_{t+1} \circ h) + u \circ h_t) 1_{G_t(\omega)} \right) \\ &= J_{t,\omega}\left((\beta(U_{t+1} \circ h) + u(h_t(\omega))) 1_{G_t(\omega)} \right) \\ &= J_{t,\omega}(\beta(U_{t+1} \circ h) + u(h_t(\omega))) = J_{t,\omega}(\beta(U_{t+1} \circ h)) + u(h_t(\omega)) \end{aligned}$$

  that is

$$V_t(h) = J_t(\beta(U_{t+1} \circ h)) + u \circ h_t. \qquad (36)$$

Step 1. Let $t < T$, $\omega \in \Omega$, and $\xi \in \mathbb{R}^\Omega(\mathcal{G}_{t+1})$, then

$$\begin{aligned} J_t(\omega, \xi) &= \min_{r \in \Delta(\Omega, \mathcal{G}_{t+1})} \left(\int \xi dr + \min_{p \in \Delta(G_t(\omega)): p_{|\mathcal{G}_{t+1}} = r} c_t(\omega, p) \right) \\ &= \inf_{r \in \Delta^{++}(G_t(\omega), \mathcal{G}_{t+1})} \left(\int \xi dr + \inf_{p \in \operatorname{ri} \Delta(G_t(\omega)): p_{|\mathcal{G}_{t+1}} = r} c_t(\omega, p) \right), \end{aligned}$$

with the convention that the minimum over the empty set is ∞.

Details. Denote by $\mathcal{G} = \{G_1, ..., G_g\}$ the set of all elements of $\mathcal{G}_{t+1}$ contained in $G_t(\omega)$, and by $\Delta\mathcal{G}$ the set $\Delta(G_t(\omega), \mathcal{G}_{t+1})$ (brutally: the probabilities on $\mathcal{G}_{t+1}$ with support in $\{G_1, ..., G_g\}$). For all $\xi = \sum_{G\in\mathcal{G}_{t+1}} \xi_G 1_G \in \mathbb{R}^\Omega(\mathcal{G}_{t+1})$,

$$\begin{aligned}
J_t(\omega,\xi) &= \min_{p\in\Delta(G_t(\omega))}\left[\sum_{\omega\in\Omega} p(\omega)\xi(\omega) + c_t(\omega,p)\right] \\
&= \min_{p\in\Delta(G_t(\omega))}\left[\sum_{i=1}^{g} \xi_{G_i} p(G_i) + c_t(\omega,p)\right] \\
&= \min_{r\in\Delta\mathcal{G}} \min_{p\in\Delta(G_t(\omega)):p_{|\mathcal{G}_{t+1}}=r}\left[\sum_{i=1}^{g} \xi_{G_i} p(G_i) + c_t(\omega,p)\right] \\
&= \min_{r\in\Delta\mathcal{G}}\left(\sum_{i=1}^{g} r(G_i)\xi_{G_i} + \min_{p\in\Delta(G_t(\omega)):p_{|\mathcal{G}_{t+1}}=r} c_t(\omega,p)\right) \\
&= \min_{r\in\Delta(\Omega,\mathcal{G}_{t+1})}\left(\int \xi dr + \min_{p\in\Delta(G_t(\omega)):p_{|\mathcal{G}_{t+1}}=r} c_t(\omega,p)\right).
\end{aligned}$$

The last equality descends from the observation that if $r \in \Delta(\Omega, \mathcal{G}_{t+1}) \setminus \Delta\mathcal{G}$ there is $G \in \mathcal{G}_{t+1}$ such that $G \nsubseteq G_t(\omega)$ with $r(G) > 0$, then there is no $p \in \Delta(G_t(\omega))$ such that $p_{|\mathcal{G}_{t+1}} = r$.

Analogously, write $\Delta^{++}\mathcal{G}$ instead of $\Delta^{++}(G_t(\omega), \mathcal{G}_{t+1})$, $\Delta^{++}\mathcal{G}$ is the subset of the restrictions of the elements of $\operatorname{ri}\Delta(G_t(\omega))$ to the algebra generated by $\mathcal{G}_{t+1}$ (brutally: the probabilities on $\mathcal{G}_{t+1}$ with support equal to $\{G_1, ..., G_g\}$). For all $\xi = \sum_{G\in\mathcal{G}_{t+1}} \xi_G 1_G \in \mathbb{R}^\Omega(\mathcal{G}_{t+1})$ we have

$$\begin{aligned}
J_t(\omega,\xi) &= \inf_{p\in\operatorname{ri}\Delta(G_t(\omega))}\left[\sum_{\omega'\in\Omega} p(\omega')\xi(\omega') + c_t(\omega,p)\right] = \inf_{p\in\operatorname{ri}\Delta(G_t(\omega))}\left[\sum_{i=1}^{g} \xi_{G_i} p(G_i) + c_t(\omega,p)\right] \\
&= \inf_{r\in\Delta^{++}\mathcal{G}} \inf_{p\in\operatorname{ri}\Delta(G_t(\omega)):p_{|\mathcal{G}_{t+1}}=r}\left[\sum_{i=1}^{g} \xi_{G_i} p(G_i) + c_t(\omega,p)\right] \\
&= \inf_{r\in\Delta^{++}\mathcal{G}}\left(\sum_{i=1}^{g} r(G_i)\xi_{G_i} + \inf_{p\in\operatorname{ri}\Delta(G_t(\omega)):p_{|\mathcal{G}_{t+1}}=r} c_t(\omega,p)\right).
\end{aligned}$$

□

Step 2. Let $t < T$ and $\omega \in \Omega$. The function $\gamma_t(\omega,\cdot) : \Delta(\Omega, \mathcal{G}_{t+1}) \to [0,\infty]$ defined by

$$\gamma_t(\omega,r) \equiv \min_{p\in\Delta(G_t(\omega)):p_{|\mathcal{G}_{t+1}}=r} c_t(\omega,p) \quad \forall r \in \Delta(\Omega,\mathcal{G}_{t+1}),$$

with the convention that the minimum over the empty set is ∞, is closed, convex, grounded, and $\operatorname{dom}\gamma_t(\omega,r) \subseteq \Delta(G_t(\omega), \mathcal{G}_{t+1})$.

Details. If $r \in \Delta(\Omega, \mathcal{G}_{t+1}) \setminus \Delta(G_t(\omega), \mathcal{G}_{t+1})$, there is $G \in \mathcal{G}_{t+1}$ such that $G \nsubseteq G_t(\omega)$ with $r(G) > 0$, then there is no $p \in \Delta(G_t(\omega))$ such that $p_{|\mathcal{G}_{t+1}} = r$ and $\gamma_t(\omega,r) = \infty$. Therefore $\operatorname{dom}\gamma_t(\omega,r) \subseteq \Delta(G_t(\omega), \mathcal{G}_{t+1}) = \Delta\mathcal{G}$. Let $\xi = \sum_{G\in\mathcal{G}_{t+1}} \xi_G 1_G \in \mathbb{R}^\Omega(\mathcal{G}_{t+1})$, by Step 1,

$$J_t(\omega,\xi) = \min_{r\in\Delta(\Omega,\mathcal{G}_{t+1})}\left(\int \xi dr + \min_{p\in\Delta(G_t(\omega)):p_{|\mathcal{G}_{t+1}}=r} c_t(\omega,p)\right) = \min_{r\in\Delta\mathcal{G}}\left(\sum_{i=1}^{g} r(G_i)\xi_{G_i} + \gamma_t(\omega,r)\right).$$

Hence, $J_t(\omega,0) = 0$ implies that $\min_{r\in\Delta\mathcal{G}} \gamma_t(\omega,r) = 0$, and $\gamma_t(\omega,\cdot)$ is grounded. Let $r,s \in \Delta\mathcal{G}$ and $\alpha \in (0,1)$, then

$$\begin{aligned}
\gamma_t(\omega,\alpha r + (1-\alpha)s) &= \min_{p\in\Delta(G_t(\omega)):p_{|\mathcal{G}_{t+1}}=\alpha r+(1-\alpha)s} c_t(\omega,p) \\
&\le \min_{p,q\in\Delta(G_t(\omega)):p_{|\mathcal{G}_{t+1}}=r \text{ and } q_{|\mathcal{G}_{t+1}}=s} c_t(\omega,\alpha p + (1-\alpha)q) \\
&\le \min_{p,q\in\Delta(G_t(\omega)):p_{|\mathcal{G}_{t+1}}=r \text{ and } q_{|\mathcal{G}_{t+1}}=s} (\alpha c_t(\omega,p) + (1-\alpha)c_t(\omega,q)) \\
&= \alpha \min_{p\in\Delta(G_t(\omega)):p_{|\mathcal{G}_{t+1}}=r} c_t(\omega,p) + (1-\alpha) \min_{q\in\Delta(G_t(\omega)):q_{|\mathcal{G}_{t+1}}=s} c_t(\omega,q).
\end{aligned}$$

Therefore $\gamma_t(\omega,\cdot)$ is convex. Let $b \in \mathbb{R}$ and $r_n \in \Delta\mathcal{G}$, be such that $r_n \to r$ and $\gamma_t(\omega, r_n) \leq b$ for all $n \geq 1$. For all n there exists $\bar{p}^n$ such that

$$\gamma_t(\omega, r_n) = \min_{p^n \in \Delta(\Omega): p^n_{|\mathcal{G}_{t+1}} = r_n} c_t(\omega, p^n) \leq c_t(\omega, \bar{p}^n) \leq b$$

and $\bar{p}^n_{|\mathcal{G}_{t+1}} = r_n$. Take a convergent subsequence $\bar{p}^{n_j} \to \bar{p}$ of $\bar{p}^n$, since $c_t(\omega,\cdot)$ is closed $c_t(\omega,\bar{p}) \leq b$, moreover, for all $G \in \mathcal{G}_{t+1}$

$$\bar{p}(G) = \lim_j \bar{p}^{n_j}(G) = \lim_j r_{n_j}(G) = r(G);$$

in turn this implies

$$\gamma_t(\omega, r) \leq c_t(\omega, \bar{p}) \leq b.$$

Therefore $\gamma_t(\omega,\cdot)$ is closed. □

Step 3. Let $t < T$ and $\omega \in \Omega$. The function $\nu_t(\omega,\cdot) : \Delta(\Omega) \to [0,\infty]$ defined by

$$\nu_t(\omega, q) \equiv \gamma_t\left(\omega, q_{|\mathcal{G}_{t+1}}\right) = \min_{p \in \Delta(G_t(\omega)): p_{|\mathcal{G}_{t+1}} = q_{|\mathcal{G}_{t+1}}} c_t(\omega, p) \quad \forall q \in \Delta(\Omega)$$

is grounded, closed and convex with $\operatorname{dom} \nu_t(\omega,\cdot) \subseteq \Delta(G_t(\omega))$.

Step 4. Let $t < T$ and $\omega \in \Omega$. The function $\eta_t(\omega,\cdot) : \Delta(G_t(\omega)) \to [0,\infty]$ defined by

$$\eta_t(\omega, q) \equiv \sum_{\substack{G \in \mathcal{G}_{t+1} \\ q(G) > 0}} q(G) c_{t+1}(G, q_G) \quad \forall q \in \Delta(G_t(\omega)),$$

is closed and convex.

Details. For later use (in the proof of Theorem 2) we just assume that c_{t+1} satisfies (i) and (ii) of the definition of dynamic ambiguity index (not that $\{c_t\}$ is an ambiguity index itself). We show that $\eta_t(\omega,\cdot)$ is the closure of its (convex) restriction $\kappa_t(\omega,\cdot)$ to $\operatorname{ri}\Delta(G_t(\omega))$. In fact, for all $q, p \in \operatorname{ri}\Delta(G_t(\omega))$, $\alpha \in (0,1)$ and $G \in \mathcal{G}_{t+1}$ such that $G \subseteq G_t(\omega)$,

$$\begin{aligned}
(\alpha q + (1-\alpha)p)_G(A) &= \frac{(\alpha q + (1-\alpha)p)(A \cap G)}{(\alpha q + (1-\alpha)p)(G)} \\
&= \frac{\alpha q(A \cap G) + (1-\alpha)p(A \cap G)}{(\alpha q + (1-\alpha)p)(G)} \\
&= \frac{\alpha q(G) q_G(A) + (1-\alpha)p(G) p_G(A)}{(\alpha q + (1-\alpha)p)(G)} \\
&= \frac{\alpha q(G)}{(\alpha q + (1-\alpha)p)(G)} q_G(A) + \frac{(1-\alpha)p(G)}{(\alpha q + (1-\alpha)p)(G)} p_G(A)
\end{aligned}$$

then

$$\begin{aligned}
\kappa_{t,\omega}(\alpha q + (1-\alpha)p) &= \sum_{\substack{G \in \mathcal{G}_{t+1} \\ (\alpha q + (1-\alpha)p)(G) > 0}} (\alpha q + (1-\alpha)p)(G)\, c_{t+1,G}\left((\alpha q + (1-\alpha)p)_G\right) \\
&= \sum_{\substack{G \in \mathcal{G}_{t+1} \\ G \subseteq G_t(\omega)}} (\alpha q + (1-\alpha)p)(G)\, c_{t+1,G}\left(\frac{\alpha q(G)}{(\alpha q + (1-\alpha)p)(G)} q_G + \frac{(1-\alpha)p(G)}{(\alpha q + (1-\alpha)p)(G)} p_G\right) \\
&\leq \sum_{\substack{G \in \mathcal{G}_{t+1} \\ G \subseteq G_t(\omega)}} (\alpha q + (1-\alpha)p)(G) \left(\frac{\alpha q(G)}{(\alpha q + (1-\alpha)p)(G)} c_{t+1,G}(q_G) + \frac{(1-\alpha)p(G)}{(\alpha q + (1-\alpha)p)(G)} c_{t+1,G}(p_G)\right) \\
&= \sum_{\substack{G \in \mathcal{G}_{t+1} \\ G \subseteq G_t(\omega)}} \alpha q(G)\, c_{t+1,G}(q_G) + (1-\alpha)p(G)\, c_{t+1,G}(p_G) = \alpha \kappa_{t,\omega}(q) + (1-\alpha)\kappa_{t,\omega}(p).
\end{aligned}$$

Then $\kappa_{t,\omega}$ is convex.

For all $\mathcal{G}_{t+1} \ni G \subseteq G_t(\omega)$, there is $p^G \in \mathrm{ri}\,\Delta(G) \cap \mathrm{dom}\, c_{t+1}(G,\cdot)$. Therefore, choosing $\{q(G) : \mathcal{G}_{t+1} \ni G \subseteq G_t(\omega)\}$ such that $\sum_{\mathcal{G}_{t+1} \ni G \subseteq G_t(\omega)} q(G) = 1$ and $q(G) > 0$ for all $\mathcal{G}_{t+1} \ni G \subseteq G_t(\omega)$, the probability

$$r \equiv \sum_{\mathcal{G}_{t+1} \ni G \subseteq G_t(\omega)} q(G)\, p^G \in \mathrm{dom}\, \kappa_{t,\omega}$$

and $\kappa_{t,\omega}$ is proper.[16]

Take $p \in \mathrm{ri}\,(\mathrm{dom}\, \kappa_{t,\omega})$ and $q \in \Delta(G_t(\omega))$. If $G \in \mathcal{G}_{t+1}$ and $q(G) > 0$ then $G \subseteq G_t(\omega)$. Moreover, the function

$$f(\alpha) \equiv \frac{\alpha q(G)}{\alpha q(G) + (1-\alpha) p(G)}$$

has first derivative w.r.t. α

$$\begin{aligned}
\dot{f}(\alpha) &= \frac{q(G)(\alpha q(G) + (1-\alpha) p(G)) - \alpha q(G)(q(G) - p(G))}{(\alpha q(G) + (1-\alpha) p(G))^2} \\
&= \frac{\alpha q(G)^2 + q(G) p(G) - \alpha q(G) p(G) - \alpha q(G)^2 + \alpha q(G) p(G)}{(\alpha q(G) + (1-\alpha) p(G))^2} \\
&= \frac{q(G) p(G)}{(\alpha q(G) + (1-\alpha) p(G))^2} > 0 \quad \forall \alpha \in (0,1)
\end{aligned}$$

moreover, $\lim_{\alpha \uparrow 1} f(\alpha) = 1$. Since $p \in \mathrm{ri}\,(\mathrm{dom}\, \kappa_{t,\omega})$, then $p_G \in \mathrm{dom}\, c_{t+1}(G,\cdot)$, and [18, Cor. 7.5.1] implies

$$\begin{aligned}
&\lim_{\alpha \uparrow 1} c_{t+1,G}\left((\alpha q + (1-\alpha) p)_G\right) = \\
&= \lim_{\alpha \uparrow 1} c_{t+1,G}\left(\frac{\alpha q(G)}{(\alpha q + (1-\alpha) p)(G)} q_G + \frac{(1-\alpha) p(G)}{(\alpha q + (1-\alpha) p)(G)} p_G\right) \\
&= \lim_{\alpha \uparrow 1} c_{t+1,G}\left(f(\alpha) q_G + (1 - f(\alpha)) p_G\right) = c_{t+1,G}(q_G).
\end{aligned}$$

If $q(G) = 0$, then for all $A \subseteq \Omega$,

$$(\alpha q + (1-\alpha) p)_G(A) = \frac{(\alpha q + (1-\alpha) p)(A \cap G)}{(\alpha q + (1-\alpha)) p(G)} = p_G(A) \quad \forall \alpha \in (0,1).$$

Then

$$\lim_{\alpha \uparrow 1} c_{t+1,G}\left((\alpha q + (1-\alpha) p)_G\right) = \lim_{\alpha \uparrow 1} c_{t+1,G}(p_G) = c_{t+1,G}(p_G),$$

and $c_{t+1,G}(p_G)$ is finite since $p_G \in \mathrm{dom}\, c_{t+1}(G,\cdot)$.

By [18, Thm. 7.5], we conclude that for all $q \in \Delta(G_t(\omega))$,

$$\begin{aligned}
(\overline{\mathrm{co}}\, \kappa_{t,\omega})(q) &= \lim_{\alpha \uparrow 1} \kappa_{t,\omega}((1-\alpha) p + \alpha q) \\
&= \lim_{\alpha \uparrow 1} \sum_{\substack{G \in \mathcal{G}_{t+1} \\ (\alpha q + (1-\alpha) p)(G) > 0}} (\alpha q + (1-\alpha) p)(G)\, c_{t+1,G}\left((\alpha q + (1-\alpha) p)_G\right) \\
&= \sum_{\substack{G \in \mathcal{G}_{t+1} \\ q(G) > 0}} q(G)\, c_{t+1,G}(q_G) = \eta_{t,\omega}(q).
\end{aligned}$$

□

Step 5. Let $t < T$ and $\omega \in \Omega$. If $\{\succsim_{t,\omega}\}$ satisfy DC, then

$$\varphi^1, \varphi^2 \in \mathbb{R}^\Omega \text{ and } J_{t+1,\omega'}(\varphi^1) = J_{t+1,\omega'}(\varphi^2)\ \forall \omega' \in G_t(\omega) \Rightarrow J_{t,\omega}(\beta \varphi^1) = J_{t,\omega}(\beta \varphi^2).$$

Details. First assume $\psi^1, \psi^2 \in u(X)^\Omega$ are such that $J_{t+1,\omega'}(\psi^1) = J_{t+1,\omega'}(\psi^2)$ for all $\omega' \in G_t(\omega)$. We want to show that $J_{t,\omega}(\beta \psi^1) = J_{t,\omega}(\beta \psi^2)$.

[16] Notice that for all $\mathcal{G}_{t+1} \ni G \subseteq G_t(\omega)$, $r(G) = q(G)$ and $r_G = p^G$.

- If $u(X) = \mathbb{R}^+$. Let $x^0 \in X$ be such that $u(x^0) = 0$. For all $\omega' \in \Omega$, there exists $x^{\psi^i(\omega')} \in X$ such that $u\left(x^{\psi^i(\omega')}\right) = \frac{\beta}{\beta^{T-t}} \psi^i(\omega')$. Consider the acts h^1, h^2 defined by

$$h^i(\tau, \omega') = \begin{cases} x^0 & \text{if } \tau < T \\ x^{\psi^i(\omega')} & \text{if } \tau = T. \end{cases}$$

For all $\omega' \in \Omega$:

$$\begin{aligned} U_t\left(h^i(\omega')\right) &= \sum_{\tau \geq t} \beta^{\tau - t} u\left(h^i_\tau(\omega')\right) = \beta^{T-t} u\left(x^{\psi^i(\omega')}\right) = \beta \psi^i(\omega'), \text{ and} \\ U_{t+1}\left(h^i(\omega')\right) &= \sum_{\tau \geq t+1} \beta^{\tau-(t+1)} u\left(h^i_\tau(\omega')\right) = \beta^{T-(t+1)} u\left(x^{\psi^i(\omega')}\right) = \psi^i(\omega') = \psi^i(\omega') + k \end{aligned}$$

with $k = 0$.

- If $u(X) = \mathbb{R}^{++}$. There exists $\varepsilon > 0$ such that $\psi^i - \varepsilon \in u(X)^\Omega$, choose $x^\varepsilon \in X$ such that $u(x^\varepsilon) = \left(\sum_{T>\tau\geq t} \beta^{\tau-t}\right)^{-1} \beta\varepsilon$. For all $\omega' \in \Omega$, there exists $x^{\psi^i(\omega')} \in X$ such that $u\left(x^{\psi^i(\omega')}\right) = \frac{\beta}{\beta^{T-t}}\left(\psi^i(\omega') - \varepsilon\right)$. Consider the acts h^1, h^2 defined by

$$h^i(\tau, \omega') = \begin{cases} x^\varepsilon & \text{if } \tau < T \\ x^{\psi^i(\omega')} & \text{if } \tau = T. \end{cases}$$

For all $\omega' \in \Omega$:

$$\begin{aligned} U_t\left(h^i(\omega')\right) &= \sum_{\tau \geq t} \beta^{\tau-t} u\left(h^i_\tau(\omega')\right) = \sum_{T>\tau\geq t} \beta^{\tau-t} u(x^\varepsilon) + \beta^{T-t} u\left(x^{\psi^i(\omega')}\right) \\ &= u(x^\varepsilon)\left(\sum_{T>\tau\geq t} \beta^{\tau-t}\right) + \beta^{T-t}\frac{\beta}{\beta^{T-t}}\left(\psi^i(\omega') - \varepsilon\right) \\ &= \frac{\beta}{\sum_{T>\tau\geq t}\beta^{\tau-t}}\varepsilon \sum_{T>\tau\geq t}\beta^{\tau-t} + \beta\psi^i(\omega') - \beta\varepsilon = \beta\psi^i(\omega'), \text{ and} \\ U_{t+1}\left(h^i(\omega')\right) &= \sum_{\tau\geq t+1}\beta^{\tau-(t+1)} u\left(h^i_\tau(\omega')\right) = \sum_{T>\tau\geq t+1}\beta^{\tau-(t+1)} u(x^\varepsilon) + \beta^{T-(t+1)} u\left(x^{\psi^i(\omega')}\right) \\ &= \frac{\beta}{\sum_{T>\tau\geq t}\beta^{\tau-t}}\varepsilon \sum_{T>\tau\geq t+1}\beta^{\tau-(t+1)} + \beta^{T-(t+1)}\frac{\beta}{\beta^{T-t}}\left(\psi^i(\omega') - \varepsilon\right) \\ &= \frac{\beta}{\sum_{T>\tau\geq t}\beta^{\tau-t}}\varepsilon \sum_{T>\tau\geq t+1}\beta^{\tau-(t+1)} + \psi^i(\omega') - \varepsilon = \psi^i(\omega') + k \end{aligned}$$

with $k = \beta\varepsilon\left(\sum_{T>\tau\geq t+1}\beta^{\tau-(t+1)}\right)\left(\sum_{T>\tau\geq t}\beta^{\tau-t}\right)^{-1} - \varepsilon$.

- The cases $u(X) = \mathbb{R}, \mathbb{R}^-, \mathbb{R}^{--}$ are very similar.

Then

$$\begin{aligned} V_{t+1,\omega'}\left(h^1\right) &= J_{t+1,\omega'}\left(U_{t+1}\left(h^1\right)\right) = J_{t+1,\omega'}\left(\psi^1 + k\right) = J_{t+1,\omega'}\left(\psi^1\right) + k, \text{ and} \\ V_{t+1,\omega'}\left(h^2\right) &= J_{t+1,\omega'}\left(U_{t+1}\left(h^2\right)\right) = J_{t+1,\omega'}\left(\psi^2 + k\right) = J_{t+1,\omega'}\left(\psi^2\right) + k \end{aligned}$$

for all $\omega' \in \Omega$. For all $\omega' \in G_t(\omega)$, $J_{t+1,\omega'}\left(\psi^1\right) = J_{t+1,\omega'}\left(\psi^2\right)$, then $h^1 \sim_{t+1,\omega'} h^2$; moreover $h^1_\tau = h^2_\tau$ for all $\tau \leq t$, then DC implies $h^1 \sim_{t,\omega} h^2$ and

$$J_{t,\omega}\left(\beta\psi^1\right) = J_{t,\omega}\left(U_t\left(h^1\right)\right) = V_{t,\omega}\left(h^1\right) = V_{t,\omega}\left(h^2\right) = J_{t,\omega}\left(\beta\psi^2\right).$$

As wanted. Now, for all $\varphi^1, \varphi^2 \in \mathbb{R}^\Omega$ such that $J_{t+1,\omega'}(\varphi^1) = J_{t+1,\omega'}(\varphi^2)$ for all $\omega' \in G_t(\omega)$, there exists $\psi^1, \psi^2 \in u(X)^\Omega$ and $b \in \mathbb{R}$ such that $\varphi^i = \psi^i + b$, then $J_{t+1,\omega'}(\psi^1) = J_{t+1,\omega'}(\psi^2)$ for all $\omega' \in G_t(\omega)$. Therefore,

$$J_{t,\omega}(\beta\varphi^1) = J_{t,\omega}(\beta\psi^1 + \beta b) = J_{t,\omega}(\beta\psi^1) + \beta b = J_{t,\omega}(\beta\psi^2) + \beta b = J_{t,\omega}(\beta\varphi^2).$$

□

Step 6. Let $t < T$. If $\{\succsim_{t,\omega}\}$ satisfy DC, then

$$J_t(\beta J_{t+1}(\varphi)) = J_t(\beta\varphi) \quad \forall\varphi \in \mathbb{R}^\Omega. \tag{37}$$

Details. Choose $\omega \in \Omega$, and remember that $J_{t+1}(\varphi) = \sum_{G\in\mathcal{G}_{t+1}} J_{t+1}(G,\varphi) 1_G \in \mathbb{R}^\Omega(\mathcal{G}_{t+1})$. For all $\omega' \in G_t(\omega)$, $\operatorname{dom} c_{t+1,\omega'} \subseteq \Delta(G_{t+1}(\omega'))$, then

$$\begin{aligned} J_{t+1}(\omega',\varphi) &= J_{t+1}(\omega', J_{t+1,\omega'}(\varphi) 1_\Omega) \\ &= J_{t+1}\left(\omega', J_{t+1}(G_{t+1}(\omega'),\varphi) 1_{G_{t+1}(\omega')}\right) = J_{t+1}(\omega', J_{t+1}(\varphi)), \end{aligned}$$

then, by Step 5, $J_t(\omega,\beta\varphi) = J_t(\omega,\beta J_{t+1}(\varphi))$. Since this is true for all ω, we obtained Eq. (37).□

Step 7. $\{\succsim_{t,\omega}\}$ satisfy DC iff

$$J_t(\beta J_{t+1}(\varphi)) = J_t(\beta\varphi) \quad \forall t < T, \varphi \in \mathbb{R}^\Omega. \tag{38}$$

Details. We only have to prove that Eq. (38) implies DC. Let $\omega \in \Omega$ and $t < T$. Assume $\varphi^1, \varphi^2 \in \mathbb{R}^\Omega$ are such that $J_{t+1,\omega'}(\varphi^1) \geq J_{t+1,\omega'}(\varphi^2)$ for all $\omega' \in G_t(\omega)$, then

$$\begin{aligned} J_{t,\omega}(\beta\varphi^1) &= J_{t,\omega}(\beta J_{t+1}(\varphi^1)) = J_{t,\omega}\left(\beta J_{t+1}(\varphi^1) 1_{G_t(\omega)}\right) \\ &\geq J_{t,\omega}\left(\beta J_{t+1}(\varphi^2) 1_{G_t(\omega)}\right) = J_{t,\omega}(\beta J_{t+1}(\varphi^2)) \\ &= J_{t,\omega}(\beta\varphi^2). \end{aligned} \tag{39}$$

Let $h^1, h^2 \in \mathcal{H}$ be such that $h^1_\tau = h^2_\tau$ for all $\tau \leq t$ and $h^1 \succsim_{t+1,\omega'} h^2$, for all $\omega' \in \Omega$, we want to show that $h^1 \succsim_{t,\omega} h^2$. Since $h^1 \succsim_{t+1,\omega'} h^2$, for all $\omega' \in \Omega$, then

$$J_{t+1,\omega'}(U_{t+1} \circ h^1) = V_{t+1,\omega'}(h^1) \geq V_{t+1,\omega'}(h^2) = J_{t+1,\omega'}(U_{t+1} \circ h^2)$$

for all $\omega' \in \Omega$, from Eq. (39) it follows that

$$J_{t,\omega}(\beta(U_{t+1} \circ h^1)) \geq J_{t,\omega}(\beta(U_{t+1} \circ h^2)).$$

But $h^1_t = h^2_t$, then, by Eq. (36),

$$\begin{aligned} V_t(\omega,h^1) &= J_{t,\omega}(\beta(U_{t+1} \circ h^1)) + u(h^1_t(\omega)) \geq J_{t,\omega}(\beta(U_{t+1} \circ h^2)) + u(h^1_t(\omega)) \\ &\geq J_{t,\omega}(\beta U_{t+1} \circ h^2) + u(h^2_t(\omega)) = V_t(\omega,h^2) \end{aligned}$$

which delivers $h^1 \succsim_{t,\omega} h^2$, as wanted. □

Step 8. $\{\succsim_{t,\omega}\}$ satisfy DC iff

$$V_t(\omega,h) = u(h_t(\omega)) + \min_{r\in\Delta(\Omega,\mathcal{G}_{t+1})}\left(\int \beta V_{t+1}(h)\, dr + \min_{p\in\Delta(G_t(\omega)):p_{|\mathcal{G}_{t+1}}=r} c_t(\omega,p)\right)$$

for all $t < T$, $\omega \in \Omega$, $h \in \mathcal{H}$. That is, **(a)** $\Leftrightarrow$ **(d)**.

Details. By Step 7, $\{\succsim_{t,\omega}\}$ satisfy DC iff

$$\begin{aligned}
&J_{t,\omega}\left(\beta J_{t+1}\left(\varphi\right)\right) = J_{t,\omega}\left(\beta\varphi\right) \quad \forall t<T, \varphi \in \mathbb{R}^{\Omega}, \omega \in \Omega \Leftrightarrow \\
&J_{t,\omega}\left(\beta J_{t+1}\left(\psi+b\right)\right) = J_{t,\omega}\left(\beta\left(\psi+b\right)\right) \quad \forall t<T, \psi \in u\left(X\right)^{\Omega}, b \in \mathbb{R}, \omega \in \Omega \Leftrightarrow \\
&J_{t,\omega}\left(\beta J_{t+1}\left(\psi\right)+\beta b\right) = J_{t,\omega}\left(\beta\psi+\beta b\right) \quad \forall t<T, \psi \in u\left(X\right)^{\Omega}, b \in \mathbb{R}, \omega \in \Omega \Leftrightarrow \\
&J_{t,\omega}\left(\beta J_{t+1}\left(\psi\right)\right)+\beta b = J_{t,\omega}\left(\beta\psi\right)+\beta b \quad \forall t<T, \psi \in u\left(X\right)^{\Omega}, b \in \mathbb{R}, \omega \in \Omega \Leftrightarrow \\
&J_{t,\omega}\left(\beta J_{t+1}\left(\psi\right)\right) = J_{t,\omega}\left(\beta\psi\right) \quad \forall t<T, \psi \in u\left(X\right)^{\Omega}, \omega \in \Omega \Leftrightarrow \\
&J_{t,\omega}\left(\beta J_{t+1}\left(U_{t+1}\circ h\right)\right) = J_{t,\omega}\left(\beta\left(U_{t+1}\circ h\right)\right) \quad \forall t<T, h \in \mathcal{H}, \omega \in \Omega \Leftrightarrow \\
&J_{t,\omega}\left(\beta V_{t+1}\left(h\right)\right) = J_{t,\omega}\left(\beta\left(U_{t+1}\circ h\right)\right) \quad \forall t<T, h \in \mathcal{H}, \omega \in \Omega \Leftrightarrow \\
&J_{t,\omega}\left(\beta V_{t+1}\left(h\right)\right)+u\left(h_t\left(\omega\right)\right) = J_{t,\omega}\left(\beta\left(U_{t+1}\circ h\right)\right)+u\left(h_t\left(\omega\right)\right) \quad \forall t<T, h \in \mathcal{H}, \omega \in \Omega \Leftrightarrow \\
&\min_{r\in\Delta(\Omega,\mathcal{G}_{t+1})}\left(\int \beta V_{t+1}\left(h\right)dr + \min_{p\in\Delta(G_t(\omega)):p_{|\mathcal{G}_{t+1}}=r} c_t(\omega,p)\right)+u\left(h_t\left(\omega\right)\right) = V_t\left(\omega,h\right) \quad \forall t<T, h\in\mathcal{H}, \omega\in\Omega.
\end{aligned}$$

Where the last equivalence descends from $\mathcal{G}_{t+1}$ measurability of $V_{t+1}\left(h\right)$ and Step 1 which deliver

$$J_t\left(\omega, \beta V_{t+1}\left(h\right)\right) = \min_{r\in\Delta(\Omega,\mathcal{G}_{t+1})}\left(\int \beta V_{t+1}\left(h\right)dr + \min_{p\in\Delta(G_t(\omega)):p_{|\mathcal{G}_{t+1}}=r} c_t(\omega,p)\right)$$

and Eq. (36) which implies $J_t\left(\omega, \beta\left(U_{t+1}\circ h\right)\right)+u\left(h_t\left(\omega\right)\right) = V_t\left(\omega,h\right)$. □

Step 9. For all $t < T$, $\omega \in \Omega$, and $\varphi \in \mathbb{R}^{\Omega}$

$$J_{t,\omega}\left(\beta J_{t+1}\left(\varphi\right)\right) = \inf_{q\in\operatorname{ri}\Delta(G_t(\omega))}\left(\langle\beta\varphi,q\rangle+\beta\sum_{\substack{G\in\mathcal{G}_{t+1}\\ G\subseteq G_t(\omega)}} q(G)c_{t+1}(G,q_G)+\inf_{p\in\operatorname{ri}\Delta(G_t(\omega)):p_{|\mathcal{G}_{t+1}}=q_{|\mathcal{G}_{t+1}}} c_t(\omega,p)\right). \tag{40}$$

Details. Denote by $\mathcal{G} = \{G_1, ..., G_g\}$ the set of all elements of $\mathcal{G}_{t+1}$ contained in $G_t(\omega)$. By Step 1,

$$
\begin{aligned}
& J_{t,\omega}\left(\beta J_{t+1}(\varphi)\right) = J_{t,\omega}\left(\sum_{G\in\mathcal{G}_{t+1}} \beta J_{t+1}(G,\varphi) 1_G\right) \\
= & \inf_{r\in\Delta^{++}\mathcal{G}}\left(\sum_{i=1}^{g} r(G_i)\,\beta J_{t+1}(G_i,\varphi) + \inf_{p\in \mathrm{ri}\,\Delta(G_t(\omega)):p_{|\mathcal{G}_{t+1}}=r} c_t(\omega,p)\right) \\
= & \inf_{r\in\Delta^{++}\mathcal{G}}\left(\sum_{i=1}^{g} r(G_i)\beta \inf_{p^i\in \mathrm{ri}\,\Delta(G_i)}\left[\sum_{\bar\omega\in\Omega} p^i(\bar\omega)\varphi(\bar\omega) + c_{t+1}(G_i,p^i)\right] + \inf_{p_{|\mathcal{G}_{t+1}}=r} c_t(\omega,p)\right) \\
= & \inf_{r\in\Delta^{++}\mathcal{G}}\left(\sum_{i=1}^{g} \inf_{p^i\in \mathrm{ri}\,\Delta(G_i)} r(G_i)\beta\left[\sum_{\bar\omega\in\Omega} p^i(\bar\omega)\varphi(\bar\omega) + c_{t+1}(G_i,p^i)\right] + \inf_{p_{|\mathcal{G}_{t+1}}=r} c_t(\omega,p)\right) \\
= & \inf_{r\in\Delta^{++}\mathcal{G}}\left(\sum_{i=1}^{g} \inf_{p^i\in \mathrm{ri}\,\Delta(G_i)}\left[\sum_{\bar\omega\in\Omega} r(G_i)\beta p^i(\bar\omega)\varphi(\bar\omega) + r(G_i)\beta c_{t+1}(G_i,p^i)\right] + \inf_{p_{|\mathcal{G}_{t+1}}=r} c_t(\omega,p)\right) \\
= & \inf_{r\in\Delta^{++}\mathcal{G}}\left(\inf_{p^1\in \mathrm{ri}\,\Delta(G_1),...,p^g\in \mathrm{ri}\,\Delta(G_g)}\sum_{i=1}^{g}\left[\sum_{\bar\omega\in\Omega} \beta r(G_i)p^i(\bar\omega)\varphi(\bar\omega) + \beta r(G_i)c_{t+1}(G_i,p^i)\right] + \inf_{p_{|\mathcal{G}_{t+1}}=r} c_t(\omega,p)\right) \\
= & \inf_{r\in\Delta^{++}\mathcal{G}}\left(\inf_{p^1\in \mathrm{ri}\,\Delta(G_1),...,p^g\in \mathrm{ri}\,\Delta(G_g)}\left(\sum_{i=1}^{g}\left[\sum_{\bar\omega\in\Omega} \beta r(G_i)p^i(\bar\omega)\varphi(\bar\omega) + \beta r(G_i)c_{t+1}(G_i,p^i)\right] + \inf_{p_{|\mathcal{G}_{t+1}}=r} c_t(\omega,p)\right)\right) \\
= & \inf_{r\in\Delta^{++}\mathcal{G}}\left(\inf_{p^1\in \mathrm{ri}\,\Delta(G_1),...,p^g\in \mathrm{ri}\,\Delta(G_g)}\left(\sum_{i=1}^{g}\sum_{\bar\omega\in\Omega} \beta r(G_i)p^i(\bar\omega)\varphi(\bar\omega) + \sum_{i=1}^{g}\beta r(G_i)c_{t+1}(G_i,p^i) + \inf_{p_{|\mathcal{G}_{t+1}}=r} c_t(\omega,p)\right)\right) \\
= & \inf_{r\in\Delta^{++}\mathcal{G}}\left(\inf_{p^1\in \mathrm{ri}\,\Delta(G_1),...,p^g\in \mathrm{ri}\,\Delta(G_g)}\left(\sum_{\bar\omega\in\Omega}\left(\sum_{i=1}^{g} r(G_i)p^i(\bar\omega)\right)\beta\varphi(\bar\omega) + \sum_{i=1}^{g}\beta r(G_i)c_{t+1}(G_i,p^i) + \inf_{p_{|\mathcal{G}_{t+1}}=r} c_t(\omega,p)\right)\right) \\
= & \inf_{r\in\Delta^{++}\mathcal{G},p^1\in \mathrm{ri}\,\Delta(G_1),...,p^g\in \mathrm{ri}\,\Delta(G_g)}\left(\sum_{\bar\omega\in\Omega}\left(\sum_{i=1}^{g} r(G_i)p^i(\bar\omega)\right)\beta\varphi(\bar\omega) + \sum_{i=1}^{g}\beta r(G_i)c_{t+1}(G_i,p^i) + \inf_{p_{|\mathcal{G}_{t+1}}=r} c_t(\omega,p)\right) \\
= & \inf_{q\in \mathrm{ri}\,\Delta(G_t(\omega))}\left(\sum_{\bar\omega\in\Omega} q(\bar\omega)\,\beta\varphi(\bar\omega) + \sum_{i=1}^{g}\beta q(G_i)c_{t+1}(G_i,q_{G_i}) + \inf_{p\in \mathrm{ri}\,\Delta(G_t(\omega)):p_{|\mathcal{G}_{t+1}}=q_{|\mathcal{G}_{t+1}}} c_t(\omega,p)\right).
\end{aligned}
$$

The last equality descends from

$$\left\{\left(r,p^1,...,p^g\right) : r\in\Delta^{++}\mathcal{G}, p^1\in \mathrm{ri}\,\Delta(G_1),...,p^g\in \mathrm{ri}\,\Delta(G_g)\right\} = \left\{\left(q_{|\mathcal{G}_{t+1}}, q_{G_1},...,q_{G_g}\right) : q\in \mathrm{ri}\,\Delta(G_t(\omega))\right\}.$$

□

Steps 7 and 9 imply that $\{\succsim_{t,\omega}\}$ satisfy DC iff for all $t<T$, $\omega\in\Omega$, and $\varphi\in\mathbb{R}^\Omega$

$$J_{t,\omega}(\beta\varphi) = \inf_{q\in \mathrm{ri}\,\Delta(G_t(\omega))}\left(\langle\beta\varphi,q\rangle + \beta\sum_{\substack{G\in\mathcal{G}_{t+1}\\ G\subseteq G_t(\omega)}} q(G)c_{t+1}(G,q_G) + \inf_{p\in \mathrm{ri}\,\Delta(G_t(\omega)):p_{|\mathcal{G}_{t+1}}=q_{|\mathcal{G}_{t+1}}} c_t(\omega,p)\right);$$

Eq. (34) and Lemma 1 guarantee that this is equivalent to $c_t(\omega,\cdot) = \overline{\mathrm{co}}\,\varrho_t(\omega,\cdot)$. That is, **(a)** ⇔ **(c)**.

Step 10. For all $t<T$, $\omega\in\Omega$, and $\varphi\in\mathbb{R}^\Omega$

$$J_{t,\omega}\left(\beta J_{t+1}(\varphi)\right) = \min_{q\in\Delta(G_t(\omega))}\left(\langle\beta\varphi,q\rangle + \beta\sum_{\substack{G\in\mathcal{G}_{t+1}\\ q(G)>0}} q(G)c_{t+1}(G,q_G) + \min_{p\in\Delta(G_t(\omega)):p_{|\mathcal{G}_{t+1}}=q_{|\mathcal{G}_{t+1}}} c_t(\omega,p)\right) \tag{41}$$

Details. Denote by $\mathcal{G} = \{G_1, ..., G_g\}$ the set of all elements of $\mathcal{G}_{t+1}$ contained in $G_t(\omega)$. By Steps 1 and 2

$$
\begin{aligned}
& J_{t,\omega}\left(\beta J_{t+1}(\varphi)\right) = J_t\left(\omega, \sum_{G \in \mathcal{G}_{t+1}} \beta J_{t+1}(G, \varphi) 1_G\right) \\
= & \min_{r \in \Delta(\Omega, \mathcal{G}_{t+1})} \left(\sum_{G \in \mathcal{G}_{t+1}} r(G) \beta J_{t+1}(G, \varphi) + \min_{p \in \Delta(G_t(\omega)): p_{|\mathcal{G}_{t+1}} = r} c_t(\omega, p) \right) \\
= & \min_{r \in \Delta \mathcal{G}} \left(\sum_{i=1}^{g} r(G_i) \beta J_{t+1}(G_i, \varphi) + \gamma_t(\omega, r) \right) \\
= & \min_{r \in \Delta \mathcal{G}} \left(\sum_{i=1}^{g} r(G_i) \beta \min_{p^i \in \Delta(G_i)} \left[\sum_{\bar{\omega} \in \Omega} p^i(\bar{\omega}) \varphi(\bar{\omega}) + c_{t+1}(G_i, p^i) \right] + \gamma_t(\omega, r) \right) \\
= & \min_{r \in \Delta \mathcal{G}} \left(\sum_{i=1}^{g} \min_{p^i \in \Delta(G_i)} r(G_i) \beta \left[\sum_{\bar{\omega} \in \Omega} p^i(\bar{\omega}) \varphi(\bar{\omega}) + c_{t+1}(G_i, p^i) \right] + \gamma_t(\omega, r) \right) \\
= & \min_{r \in \Delta \mathcal{G}} \left(\sum_{i=1}^{g} \min_{p^i \in \Delta(G_i)} \left[\sum_{\bar{\omega} \in \Omega} \beta r(G_i) p^i(\bar{\omega}) \varphi(\bar{\omega}) + \beta r(G_i) c_{t+1}(G_i, p^i) \right] + \gamma_t(\omega, r) \right) \\
= & \min_{r \in \Delta \mathcal{G}} \left(\min_{p^1 \in \Delta(G_1), ..., p^g \in \Delta(G_g)} \sum_{i=1}^{g} \left[\sum_{\bar{\omega} \in \Omega} \beta r(G_i) p^i(\bar{\omega}) \varphi(\bar{\omega}) + \beta r(G_i) c_{t+1}(G_i, p^i) \right] + \gamma_t(\omega, r) \right) \\
= & \min_{r \in \Delta \mathcal{G}} \left(\min_{p^1 \in \Delta(G_1), ..., p^g \in \Delta(G_g)} \left(\sum_{i=1}^{g} \left[\sum_{\bar{\omega} \in \Omega} \beta r(G_i) p^i(\bar{\omega}) \varphi(\bar{\omega}) + \beta r(G_i) c_{t+1}(G_i, p^i) \right] + \gamma_t(\omega, r) \right) \right) \\
= & \min_{r \in \Delta \mathcal{G}} \left(\min_{p^1 \in \Delta(G_1), ..., p^g \in \Delta(G_g)} \left(\sum_{i=1}^{g} \sum_{\bar{\omega} \in \Omega} \beta r(G_i) p^i(\bar{\omega}) \varphi(\bar{\omega}) + \sum_{i=1}^{g} \beta r(G_i) c_{t+1}(G_i, p^i) + \gamma_t(\omega, r) \right) \right) \\
= & \min_{r \in \Delta \mathcal{G}} \left(\min_{p^1 \in \Delta(G_1), ..., p^g \in \Delta(G_g)} \left(\sum_{\bar{\omega} \in \Omega} \left(\sum_{i=1}^{g} r(G_i) p^i(\bar{\omega}) \right) \beta \varphi(\bar{\omega}) + \sum_{i=1}^{g} \beta r(G_i) c_{t+1}(G_i, p^i) + \gamma_t(\omega, r) \right) \right) \\
= & \min_{r \in \Delta \mathcal{G}, p^1 \in \Delta(G_1), ..., p^g \in \Delta(G_g)} \left(\sum_{\bar{\omega} \in \Omega} \left(\sum_{i=1}^{g} r(G_i) p^i(\bar{\omega}) \right) \beta \varphi(\bar{\omega}) + \sum_{i=1}^{g} \beta r(G_i) c_{t+1}(G_i, p^i) + \min_{p \in \Delta(G_t(\omega)): p_{|\mathcal{G}_{t+1}} = r} c_t(\omega, p) \right) \\
= & \min_{q \in \Delta(G_t(\omega))} \left(\sum_{\bar{\omega} \in \Omega} q(\bar{\omega}) \beta \varphi(\bar{\omega}) + \sum_{\substack{i=1,...,g \\ q(G_i) > 0}} \beta q(G_i) c_{t+1}(G_i, q_{G_i}) + \min_{p \in \Delta(G_t(\omega)): p_{|\mathcal{G}_{t+1}} = q_{|\mathcal{G}_{t+1}}} c_t(\omega, p) \right).
\end{aligned}
$$

We report, for the sake of completeness the verification of the last equality. It is enough to show that

$$
\begin{aligned}
& \left\{ \sum_{\bar{\omega} \in \Omega} \left(\sum_{i=1}^{g} r(G_i) p^i(\bar{\omega}) \right) \beta \varphi(\bar{\omega}) + \sum_{i=1}^{g} \beta r(G_i) c_{t+1}(G_i, p^i) + \min_{p \in \Delta(G_t(\omega)): p_{|\mathcal{G}_{t+1}} = r} c_t(\omega, p) \right\}_{r \in \Delta \mathcal{G}, p^1 \in \Delta(G_1), ..., p^g \in \Delta(G_g)} \\
= & \left\{ \sum_{\bar{\omega} \in \Omega} q(\bar{\omega}) \beta \varphi(\bar{\omega}) + \sum_{\substack{i=1,...,g \\ q(G_i) > 0}} \beta q(G_i) c_{t+1}(G_i, q_{G_i}) + \min_{p \in \Delta(G_t(\omega)): p_{|\mathcal{G}_{t+1}} = q_{|\mathcal{G}_{t+1}}} c_t(\omega, p) \right\}_{q \in \Delta(G_t(\omega))}.
\end{aligned}
$$

$\subseteq$) Let $r \in \Delta\mathcal{G}, p^1 \in \Delta(G_1), ..., p^g \in \Delta(G_g)$, then

$$\sum_{\bar{\omega}\in\Omega}\left(\sum_{i=1}^{g} r(G_i)p^i(\bar{\omega})\right)\beta\varphi(\bar{\omega}) + \sum_{i=1}^{g}\beta r(G_i)c_{t+1}(G_i,p^i) + \min_{p\in\Delta(G_t(\omega)):p_{|\mathcal{G}_{t+1}}=r} c_t(\omega,p)$$

$$= \sum_{\bar{\omega}\in\Omega}\left(\sum_{\substack{i=1,...,g\\ r(G_i)>0}} r(G_i)p^i(\bar{\omega})\right)\beta\varphi(\bar{\omega}) + \sum_{\substack{i=1,...,g\\ r(G_i)>0}}\beta r(G_i)c_{t+1}(G_i,p^i) + \min_{p\in\Delta(G_t(\omega)):p_{|\mathcal{G}_{t+1}}=r} c_t(\omega,p).$$

Set $q \equiv \sum_{\substack{i=1,...,g\\ r(G_i)>0}} r(G_i)p^i$ and notice that $q_{|\mathcal{G}_{t+1}} = r$ and $q_{G_i} = p^i$ for all $i = 1,...,g$ with $q(G_i) = r(G_i) > 0$. Therefore, $\operatorname{supp} q \subseteq \bigcup_{i=1}^{g} G_i = G_t(\omega)$ and

$$\sum_{\bar{\omega}\in\Omega}\left(\sum_{\substack{i=1,...,g\\ r(G_i)>0}} r(G_i)p^i(\bar{\omega})\right)\beta\varphi(\bar{\omega}) + \sum_{\substack{i=1,...,g\\ r(G_i)>0}}\beta r(G_i)c_{t+1}(G_i,p^i) + \min_{p\in\Delta(G_t(\omega)):p_{|\mathcal{G}_{t+1}}=r} c_t(\omega,p)$$

$$= \sum_{\bar{\omega}\in\Omega} q(\bar{\omega})\beta\varphi(\bar{\omega}) + \sum_{\substack{i=1,...,g\\ q(G_i)>0}}\beta q(G_i)c_{t+1}(G_i,q_{G_i}) + \min_{p\in\Delta(G_t(\omega)):p_{|\mathcal{G}_{t+1}}=q_{|\mathcal{G}_{t+1}}} c_t(\omega,p).$$

$\supseteq$) Let $q \in \Delta(G_t(\omega))$, set $r \equiv q_{|\mathcal{G}_{t+1}} \in \Delta\mathcal{G}$, $p^i \equiv q_{G_i} \in \Delta(G_i)$ for all $i = 1,...,g$ with $q(G_i) = r(G_i) > 0$ and choose p^i arbitrarily in $\Delta(G_i)$ otherwise. Notice that, $q = \sum_{\substack{i=1,...,g\\ r(G_i)>0}} r(G_i)p^i$. Therefore

$$\sum_{\bar{\omega}\in\Omega} q(\bar{\omega})\beta\varphi(\bar{\omega}) + \sum_{\substack{i=1,...,g\\ q(G_i)>0}}\beta q(G_i)c_{t+1}(G_i,q_{G_i}) + \min_{p\in\Delta(G_t(\omega)):p_{|\mathcal{G}_{t+1}}=q_{|\mathcal{G}_{t+1}}} c_t(\omega,p)$$

$$= \sum_{\bar{\omega}\in\Omega}\left(\sum_{\substack{i=1,...,g\\ r(G_i)>0}} r(G_i)p^i(\bar{\omega})\right)\beta\varphi(\bar{\omega}) + \sum_{\substack{i=1,...,g\\ r(G_i)>0}}\beta r(G_i)c_{t+1}(G_i,p^i) + \min_{p\in\Delta(G_t(\omega)):p_{|\mathcal{G}_{t+1}}=r} c_t(\omega,p)$$

$$= \sum_{\bar{\omega}\in\Omega}\left(\sum_{i=1,...,g} r(G_i)p^i(\bar{\omega})\right)\beta\varphi(\bar{\omega}) + \sum_{i=1,...,g}\beta r(G_i)c_{t+1}(G_i,p^i) + \min_{p\in\Delta(G_t(\omega)):p_{|\mathcal{G}_{t+1}}=r} c_t(\omega,p).$$

□

Steps 7 and 10 imply that $\{\succsim_{t,\omega}\}$ satisfy DC iff for all $t < T$, $\omega \in \Omega$, and $\varphi \in \mathbb{R}^\Omega$

$$J_{t,\omega}(\beta\varphi) = \min_{q\in\Delta(G_t(\omega))}\left(\langle\beta\varphi, q\rangle + \beta\sum_{\substack{G\in\mathcal{G}_{t+1}\\ q(G)>0}} q(G)c_{t+1}(G,q_G) + \min_{p\in\Delta(G_t(\omega)):p_{|\mathcal{G}_{t+1}}=q_{|\mathcal{G}_{t+1}}} c_t(\omega,p)\right);$$

Eq. (34) and Lemma 1 guarantee that this is equivalent to $c_t(\omega,\cdot) = \overline{\text{co}}(\eta_t(\omega,\cdot) + \nu_t(\omega,\cdot))$ where $\eta_t(\omega,\cdot)$ and $\nu_t(\omega,\cdot)$ are defined in Steps 4 and 3. These steps also guarantee closure and convexity of $\eta_t(\omega,\cdot)$ and $\nu_t(\omega,\cdot)$. That is **(a)** $\Leftrightarrow$ **(b)**. ■

Remark 1 *In particular, for a dynamic ambiguity index $\{c_t\}$ conditions (32) and (33) are equivalent.*

A.5 Proof of Theorem 1

(a) $\Rightarrow$ **(b)** By Proposition 3 and Lemma 5 there exist a scalar $\beta > 0$, an unbounded affine function $u : X \to \mathbb{R}$, and a dynamic ambiguity index $\{c_t\}$, such that: for each $(t,\omega) \in \mathcal{T} \times \Omega$, $\succsim_{t,\omega}$

is represented by

$$V_t(\omega, h) = \inf_{p \in \Delta^{++}(\Omega)} \left(\int \sum_{\tau \geq t} \beta^{\tau - t} u(h_\tau) \, dp_{G_t(\omega)} + c_t\left(\omega, p_{G_t(\omega)}\right) \right) \quad \forall h \in \mathcal{H}.$$

Lemma 6 guarantees that Eq. (11) holds.

(b) ⇒ (a) Assume that there exist a scalar $\beta > 0$, an unbounded affine function $u : X \to \mathbb{R}$, and a dynamic ambiguity index $\{c_t\}$, such that: for each $(t, \omega) \in \mathcal{T} \times \Omega$, $\succsim_{t,\omega}$ is represented by

$$V_t(\omega, h) = \inf_{p \in \Delta^{++}(\Omega)} \left(\int \sum_{\tau \geq t} \beta^{\tau - t} u(h_\tau) \, dp_{G_t(\omega)} + c_t\left(\omega, p_{G_t(\omega)}\right) \right) \quad \forall h \in \mathcal{H},$$

by Proposition 1, $\{\succsim_{t,\omega}\}$ satisfy CP, VP, RP, and FS. Eq. (11) and Lemma 6 imply that $\{\succsim_{t,\omega}\}$ satisfies DC.

Uniqueness of the representation descends again from Proposition 1. ■

A.6 Proof of Proposition 2

(i) is trivial. Step 2 of the proof of Lemma 6 shows that $\gamma_t(\omega, \cdot)$ is grounded, closed and convex, with $\operatorname{dom} \gamma_t(\omega, \cdot) \subseteq \Delta(G_t(\omega), \mathcal{G}_{t+1})$, for all $t < T$ and all $\omega \in \Omega$. It only remains to show that and $\operatorname{dom} \gamma_t(\omega, \cdot) \cap \Delta^{++}(G_t(\omega), \mathcal{G}_{t+1}) \neq \varnothing$. Take $p^\circ \in \operatorname{ri} \Delta(G_t(\omega)) \cap \operatorname{dom} c_t(\omega, \cdot)$, then $r^\circ = p^\circ_{|\mathcal{G}_{t+1}} \in \Delta^{++}(G_t(\omega), \mathcal{G}_{t+1})$ and

$$\gamma_t(\omega, r^\circ) = \min_{p \in \Delta(G_t(\omega)) : p_{|\mathcal{G}_{t+1}} = r^\circ} c_t(\omega, p) \leq c_t(\omega, p^\circ) < \infty.$$

■

A.7 Proof of Theorem 2

(a) ⇒ (b) is an immediate consequence of Theorem 1 and Proposition 2.

(b) ⇒ (a) The proof that $\{c_t\}$ is a dynamic ambiguity index is by backward induction. Clearly, c_T satisfies (i) and (ii) of the definition of dynamic ambiguity index. Next we assume that c_{t+1} $(0 \leq t < T)$ satisfies (i) and (ii) of the definition of dynamic ambiguity index, and show that c_t satisfies them.

By assumption, $c_{t+1} : \Omega \times \Delta(\Omega) \to [0, \infty]$ is such that:

(i) $c_{t+1}(\cdot, p) : \Omega \to [0, \infty]$ is measurable w.r.t. $\mathcal{G}_{t+1}$ for all $p \in \Delta(\Omega)$,

(ii) $c_{t+1}(\omega, \cdot) : \Delta(\Omega) \to [0, \infty]$ is grounded, closed and convex, with $\operatorname{dom} c_t(\omega, \cdot) \subseteq \Delta(G_{t+1}(\omega))$ and $\operatorname{dom} c_{t+1}(\omega, \cdot) \cap \Delta^{++}(G_{t+1}(\omega)) \neq \varnothing$, for all $t \in \mathcal{T}$ and all $\omega \in \Omega$.

Clearly, for all $\omega \in \Omega$, the function $c_t(\omega, \cdot)$ defined in (b) is well defined (since c_{t+1} satisfies (i)) and $c_t(\omega, \cdot) = c_t(\omega', \cdot)$ if $G_t(\omega) = G_t(\omega')$ (since γ_t satisfies (i)).

Step 4 of the proof of Lemma 6 shows that for all $\omega \in \Omega$ the function $\eta_t(\omega, \cdot) : \Delta(G_t(\omega)) \to [0, \infty]$ defined by

$$\eta_t(\omega, q) \equiv \sum_{\substack{G \in \mathcal{G}_{t+1} \\ q(G) > 0}} q(G) c_{t+1}(G, q_G) \quad \forall q \in \Delta(G_t(\omega))$$

is closed and convex. Since $q \mapsto q_{|\mathcal{G}_{t+1}}$ is affine (and continuous) from $\Delta(\Omega)$ to $\Delta(\Omega, \mathcal{G}_{t,+1})$ and $\gamma_t(\omega, \cdot) : \Delta(\Omega, \mathcal{G}_{t+1}) \to [0, \infty]$ is grounded, closed and convex, with effective domain in $\Delta(G_t(\omega), \mathcal{G}_{t+1})$, then $q \mapsto \gamma_t\left(\omega, q_{|\mathcal{G}_{t+1}}\right)$ is closed and convex on $\Delta(\Omega)$ and its effective domain is contained in $\Delta(G_t(\omega))$. We conclude that, for all $\omega \in \Omega$, the function $c_t(\omega, \cdot)$ defined in (b) is closed and convex, from $\Delta(\Omega)$ to $[0, \infty]$, with $\operatorname{dom} c_t(\omega, \cdot) \subseteq \Delta(G_t(\omega))$.

Next we show that $c_t(\omega, \cdot)$ is grounded. Choose arbitrarily $\omega \in \Omega$, there exists $r \in \Delta(\Omega, \mathcal{G}_{t+1})$ such that $r(G_t(\omega)) = 1$ and $\gamma_t(\omega, r) = 0$; moreover, for all $G \in \mathcal{G}_{t+1}$ there exists $p^G \in \Delta(G)$ such that $c_{t+1}(G, p^G) = 0$, set $q \equiv \sum_{G \in \mathcal{G}_{t+1}} r(G) p^G$ to obtain

$$\begin{aligned} c_t(\omega, q) &= \sum_{\substack{G \in \mathcal{G}_{t+1} \\ q(G) > 0}} q(G) c_{t+1}(G, q_G) + \gamma_t\left(\omega, q_{|\mathcal{G}_{t+1}}\right) \\ &= \sum_{\substack{G \in \mathcal{G}_{t+1} \\ r(G) > 0}} r(G) c_{t+1}(G, p^G) + \gamma_t(\omega, r) = 0. \end{aligned}$$

It remains to show that $\operatorname{ri} \Delta(G_t(\omega)) \cap \operatorname{dom} c_t(\omega, \cdot) \neq \varnothing$ for all $\omega \in \Omega$. Choose arbitrarily $\omega \in \Omega$, there exists $r \in \Delta^{++}(G_t(\omega), \mathcal{G}_{t+1})$ such that $\gamma_t(\omega, r) < \infty$; moreover, for all $G \in \mathcal{G}_{t+1}$ there exists $p^G \in \operatorname{ri} \Delta(G)$ such that $c_{t+1}(G, p^G) < \infty$, set $q \equiv \sum_{G \in \mathcal{G}_{t+1}} r(G) p^G$ to obtain $q \in \operatorname{ri} \Delta(G_t(\omega))$ and

$$\begin{aligned} c_t(\omega, q) &= \sum_{\substack{G \in \mathcal{G}_{t+1} \\ q(G) > 0}} q(G) c_{t+1}(G, q_G) + \gamma_t\left(\omega, q_{|\mathcal{G}_{t+1}}\right) \\ &= \sum_{\substack{G \in \mathcal{G}_{t+1} \\ r(G) > 0}} r(G) c_{t+1}(G, p^G) + \gamma_t(\omega, r) < \infty. \end{aligned}$$

This concludes the proof that $\{c_t\}$ is a dynamic ambiguity index.

Finally, notice that, for all $\omega \in \Omega$, $t < T$, and $q \in \Delta(G_t(\omega))$,

$$\min_{p \in \Delta(G_t(\omega)) : p_{|\mathcal{G}_{t+1}} = q_{|\mathcal{G}_{t+1}}} \left(\sum_{\substack{G \in \mathcal{G}_{t+1} \\ q(G) > 0}} p(G) c_{t+1}(G, p_G) \right) = 0,$$

it is enough to take, for all $G \in \mathcal{G}_{t+1}$, $p^G \in \Delta(G)$ such that $c_{t+1}(G, p^G) = 0$ and set $p \equiv \sum_{G \in \mathcal{G}_{t+1}} q(G) p^G$. Therefore,

$$\begin{aligned} & \beta \sum_{\substack{G \in \mathcal{G}_{t+1} \\ q(G) > 0}} q(G) c_{t+1}(G, q_G) + \min_{p \in \Delta(G_t(\omega)) : p_{|\mathcal{G}_{t+1}} = q_{|\mathcal{G}_{t+1}}} c_t(\omega, p) \\ = {} & \beta \sum_{\substack{G \in \mathcal{G}_{t+1} \\ q(G) > 0}} q(G) c_{t+1}(G, q_G) + \min_{p \in \Delta(G_t(\omega)) : p_{|\mathcal{G}_{t+1}} = q_{|\mathcal{G}_{t+1}}} \left(\sum_{\substack{G \in \mathcal{G}_{t+1} \\ q(G) > 0}} p(G) c_{t+1}(G, p_G) + \gamma_t\left(\omega, p_{|\mathcal{G}_{t+1}}\right) \right) \\ = {} & \beta \sum_{\substack{G \in \mathcal{G}_{t+1} \\ q(G) > 0}} q(G) c_{t+1}(G, q_G) + \gamma_t\left(\omega, q_{|\mathcal{G}_{t+1}}\right) + \min_{p \in \Delta(G_t(\omega)) : p_{|\mathcal{G}_{t+1}} = q_{|\mathcal{G}_{t+1}}} \left(\sum_{\substack{G \in \mathcal{G}_{t+1} \\ q(G) > 0}} p(G) c_{t+1}(G, p_G) \right) \\ = {} & \beta \sum_{\substack{G \in \mathcal{G}_{t+1} \\ q(G) > 0}} q(G) c_{t+1}(G, q_G) + \gamma_t\left(\omega, q_{|\mathcal{G}_{t+1}}\right) = c_t(\omega, q). \end{aligned}$$

Hence $\{c_t\}$ satisfies condition (11) and it is a recursive ambiguity index. ■

A.8 Proof of Corollary 1

It is easy to see that the effect of MP(ii) on the representation provided by Proposition 1 is guaranteeing that, for every $t \in \mathcal{T}$ and $\omega \in \Omega$, $c_t(\omega, p) = \delta_{C_t(\omega)}(p)$, for a closed and convex subset $C_t(\omega) \subseteq \Delta(\Omega)$.

The relation $\operatorname{dom} c_{t,\omega} \subseteq \Delta(G_t(\omega))$ implies $C_t(\omega) \subseteq \Delta(G_t(\omega))$. Denote by $\mathcal{G} = \{G_1, ..., G_g\}$ the set of all elements of $\mathcal{G}_{t+1}$ contained in $G_t(\omega)$, and write indifferently C_i or $C_{t+1}(G_i)$. Let $\omega \in \Omega$ and $t < T$. Condition (11) is equivalent to

$$\begin{aligned} C_t(\omega) &= \left\{ q \in \Delta(G_t(\omega)) : \beta \sum_{\substack{G \in \mathcal{G}_{t+1} \\ q(G)>0}} q(G)\delta_{C_{t+1}(G)}(q_G) + \min_{p \in \Delta(G_t(\omega)): p_{|\mathcal{G}_{t+1}} = q_{|\mathcal{G}_{t+1}}} \delta_{C_t(\omega)}(p) = 0 \right\} \\ &= \left\{ q \in \Delta(G_t(\omega)) : \beta \sum_{\substack{i=1,...,g \\ q(G_i)>0}} q(G_i)\delta_{C_i}(q_{G_i}) + \min_{p \in \Delta(G_t(\omega)): p_{|\mathcal{G}_{t+1}} = q_{|\mathcal{G}_{t+1}}} \delta_{C_t(\omega)}(p) = 0 \right\}. \end{aligned}$$

But

$$\begin{aligned} \Delta(G_t(\omega)) &= \left\{ \sum_{\substack{i=1,...,g \\ r(G_i)>0}} r(G_i)p^i : r \in \Delta\mathcal{G},\ p^i \in \Delta(G_i)\ \forall i = 1, ..., g \right\} \\ &= \left\{ \sum_{i=1,...,g} r(G_i)p^i : r \in \Delta\mathcal{G},\ p^i \in \Delta(G_i)\ \forall i = 1, ..., g \right\}. \end{aligned}$$

Moreover, given $q \in \Delta(G_t(\omega))$, $q = \sum_{i=1,...,g} r(G_i)p^i$ with $r \in \Delta\mathcal{G}$ and $p^i \in \Delta(G_i)$ if and only if $r = q_{|\mathcal{G}_{t+1}}$ and $p^i = q_{G_i}$ for all $i = 1, ..., g$ with $q(G_i) = r(G_i) > 0$ (clearly p^i can be chosen arbitrarily in $\Delta(G_i)$ if $q(G_i) = r(G_i) > 0$). Therefore, condition (11) is equivalent to

$$C_t(\omega) = \left\{ \sum_{i=1,...,g} r(G_i)p^i : \begin{array}{l} r \in \Delta\mathcal{G},\ p^i \in \Delta(G_i)\ \forall i = 1, ..., g, \\ \beta \sum_{\substack{i=1,...,g \\ r(G_i)>0}} r(G_i)\delta_{C_i}(p^i) + \min_{p \in \Delta(G_t(\omega)): p_{|\mathcal{G}_{t+1}} = r} \delta_{C_t(\omega)}(p) = 0 \end{array} \right\}$$

$$C_t(\omega) = \left\{ \sum_{i=1,...,g} r(G_i)p^i : \begin{array}{l} r \in \Delta\mathcal{G},\ p^i \in \Delta(G_i)\ \forall i = 1, ..., g, \\ \sum_{\substack{i=1,...,g \\ r(G_i)>0}} r(G_i)\delta_{C_i}(p^i) = 0, \\ \min_{p \in \Delta(G_t(\omega)): p_{|\mathcal{G}_{t+1}} = r} \delta_{C_t(\omega)}(p) = 0 \end{array} \right\}$$

$$C_t(\omega) = \left\{ \sum_{i=1,...,g} r(G_i)p^i : \begin{array}{l} r \in \Delta\mathcal{G},\ p^i \in \Delta(G_i)\ \forall i = 1, ..., g, \\ \delta_{C_i}(p^i) = 0 \text{ for all } i \text{ s.t. } r(G_i) > 0, \\ \delta_{C_t(\omega)}(p) = 0 \text{ for some } p \in \Delta(G_t(\omega)) : p_{|\mathcal{G}_{t+1}} = r \end{array} \right\}$$

$$C_t(\omega) = \left\{ \sum_{i=1,...,g} r(G_i)p^i : \begin{array}{l} r \in \Delta\mathcal{G},\ p^i \in \Delta(G_i)\ \forall i = 1, ..., g, \\ p^i \in C_i \text{ for all } i \text{ s.t. } r(G_i) > 0, \\ \text{there exists } p \in C_t(\omega) \text{ s.t. } p_{|\mathcal{G}_{t+1}} = r \end{array} \right\}$$

$$C_t(\omega) = \left\{ \sum_{i=1,...,g} r(G_i)p^i : \begin{array}{l} r \in \Delta\mathcal{G},\ p^i \in \Delta(G_i)\ \forall i = 1, ..., g, \\ p^i \in C_i \text{ for all } i = 1, ..., g, \\ r \in C_t(\omega)_{|\mathcal{G}_{t+1}} \end{array} \right\}$$

$$C_t(\omega) = \left\{ \sum_{i=1,...,g} r(G_i)p^i : \begin{array}{l} p^i \in C_{t+1}(G_i) \text{ for all } i = 1, ..., g, \\ r \in C_t(\omega)_{|\mathcal{G}_{t+1}} \end{array} \right\}$$

$$C_t(\omega) = \left\{ \sum_{G \in \mathcal{G}_{t+1}} p^G r(G) : p^G \in C_{t+1}(G)\ \forall G \in \mathcal{G}_{t+1} \text{ and } r \in C_t(\omega)_{|\mathcal{G}_{t+1}} \right\}.$$

■

A.9 Proof of Proposition 3

W.l.o.g., set $\theta = 1$. Moreover, we denote by q° the reference probability of the statement. The properties of the relative entropy (see, e.g., [13]) guarantee that $\{c_t\}$ is a dynamic ambiguity index.

By Theorem 1, we only have to show that $\{c_t\}$ satisfies (11) or the equivalent (33), see Remark 1.

Next we show that, for all $t < T$, $\omega \in \Omega$, and $q \in \operatorname{ri}\Delta(G_t(\omega))$,

$$c_{t,\omega}(q) = \beta \sum_{\substack{G\in\mathcal{G}_{t+1}\\ G\subseteq G_t(\omega)}} q(G)c_{t+1,G}(q_G) + \inf_{p\in\operatorname{ri}\Delta(G_t(\omega)):p_{|\mathcal{G}_{t+1}}=q_{|\mathcal{G}_{t+1}}} c_{t,\omega}(p). \tag{42}$$

For all $p \in \operatorname{ri}\Delta(G_t(\omega))$,

$$c_{t,\omega}(p) = \frac{1}{\beta^t}\sum_{\omega'\in G_t(\omega)} p_{G_t(\omega)}(\omega')\log\frac{p_{G_t(\omega)}(\omega')}{q^\circ_{G_t(\omega)}(\omega')} = \frac{1}{\beta^t}\sum_{\omega'\in G_t(\omega)} p_{\omega'}\log\frac{p_{\omega'}}{q^\circ_{G_t(\omega),\omega'}}$$

where $p_{\omega'} \equiv p(\omega')$ and $q^\circ_{G_t(\omega),\omega'} \equiv q^\circ_{G_t(\omega)}(\omega')$. For all $G \in \mathcal{G}_{t+1}$ such that $G \subseteq G_t(\omega)$ and all $p \in \operatorname{ri}\Delta(G)$

$$c_{t+1,G}(p) = \frac{1}{\beta^{t+1}}\sum_{\omega'\in G} p_{G,\omega'}\log\frac{p_{G,\omega'}}{q^\circ_{G,\omega'}} = \frac{1}{\beta^{t+1}}\sum_{\omega'\in G} p_{\omega'}\log\frac{p_{\omega'}}{\left(q^\circ_{G_t(\omega)}\right)_{G,\omega'}}.$$

To simplify the notation, set $S = G_t(\omega)$, $\bar{q} = q^\circ_S$, $\mathcal{G} = \{G \in \mathcal{G}_{t+1} : G \subseteq G_t(\omega)\}$ (notice that $\mathcal{G}$ is a partition of S), and choose arbitrarily $q \in \operatorname{ri}\Delta(G_t(\omega))$:

$$\begin{aligned}
\beta^{t+1}c_{t+1,G}(q_G) &= \sum_{s\in G}\frac{q_s}{q(G)}\log\frac{q_s}{q(G)}\frac{\bar{q}(G)}{\bar{q}_s} = \frac{1}{q(G)}\sum_{s\in G} q_s\log\frac{q_s}{\bar{q}_s} + \frac{1}{q(G)}\sum_{s\in G} q_s\log\frac{\bar{q}(G)}{q(G)}\\
&= \frac{1}{q(G)}\sum_{s\in G} q_s\log\frac{q_s}{\bar{q}_s} - \log\frac{q(G)}{\bar{q}(G)}.
\end{aligned}$$

Then, for all $q \in \operatorname{ri}\Delta(G_t(\omega))$,

$$\begin{aligned}
\beta \sum_{\substack{G\in\mathcal{G}_{t+1}\\ G\subseteq G_t(\omega)}} q(G)c_{t+1,G}(q_G) &= \beta\sum_{G\in\mathcal{G}} q(G)\frac{1}{\beta^{t+1}}\left(\frac{1}{q(G)}\sum_{s\in G} q_s\log\frac{q_s}{\bar{q}_s} - \log\frac{q(G)}{\bar{q}(G)}\right)\\
&= \frac{1}{\beta^t}\left(\sum_{G\in\mathcal{G}}\sum_{s\in G} q_s\log\frac{q_s}{\bar{q}_s} - \sum_{G\in\mathcal{G}} q(G)\log\frac{q(G)}{\bar{q}(G)}\right)\\
&= \frac{1}{\beta^t}\sum_{s\in S} q_s\log\frac{q_s}{\bar{q}_s} - \frac{1}{\beta^t}\sum_{G\in\mathcal{G}} q(G)\log\frac{q(G)}{\bar{q}(G)},
\end{aligned}$$

that is

$$\beta \sum_{\substack{G\in\mathcal{G}_{t+1}\\ G\subseteq G_t(\omega)}} q(G)c_{t+1,G}(q_G) = c_{t,\omega}(q) - \frac{1}{\beta^t}\sum_{G\in\mathcal{G}} q(G)\log\frac{q(G)}{\bar{q}(G)}. \tag{43}$$

Moreover, for all $q \in \operatorname{ri}\Delta(G_t(\omega))$

$$\inf_{p\in\operatorname{ri}\Delta(G_t(\omega)):p_{|\mathcal{G}_{t+1}}=q_{|\mathcal{G}_{t+1}}} c_{t,\omega}(p) = \inf_{p\in\operatorname{ri}\Delta(S):p_{|\mathcal{G}}=q_{|\mathcal{G}}} c_{t,\omega}(p)$$

is the value of the problem

$$\begin{cases}
\inf \frac{1}{\beta^t}\sum_{s\in S} p_s\log\frac{p_s}{\bar{q}_s}\\
\text{sub}\\
p_s > 0 \quad \forall s\in S\\
\sum_{s\in S} p_s = 1\\
\sum_{s\in G} p_s = q(G) \quad \forall G\in\mathcal{G}.
\end{cases} \tag{44}$$

We solve the easier problem

$$\begin{cases}
\inf \sum_{s\in S} p_s\log\frac{p_s}{\bar{q}_s}\\
\text{sub}\\
\sum_{s\in G} p_s = q(G) \quad \forall G\in\mathcal{G}
\end{cases} \tag{45}$$

and observe that the solution p° is unique, it is a strictly positive vector (this is also required for the existence of $\log \frac{p_s^\circ}{\bar{q}_s}$),

$$\sum_{s\in S} p_s^\circ = \sum_{G\in\mathcal{G}} \sum_{s\in G} p_s^\circ = \sum_{G\in\mathcal{G}} q(G) = 1,$$

and obviously the constant β^t has no effect. Thus p° is the solution of problem (44). The Lagrangean of problem (45) is

$$\mathcal{L}(p,\lambda) = \sum_{s\in S} p_s \log \frac{p_s}{\bar{q}_s} - \sum_{G\in\mathcal{G}} \lambda_G \left(\sum_{s\in G} p_s - q(G) \right),$$

denoting by $G(s)$ the element of G containing s, the first order conditions are

$$\begin{cases} \log \frac{p_s}{\bar{q}_s} + 1 - \lambda_{G(s)} = 0 & \forall s \in S \\ \sum_{s\in G} p_s = q(G) & \forall G \in \mathcal{G} \end{cases} \tag{46}$$

simple manipulation yields

$$\begin{cases} p_s = \bar{q}_s \exp\left(\lambda_{G(s)} - 1\right) & \forall s \in S \\ \sum_{s\in G} p_s = q(G) & \forall G \in \mathcal{G} \end{cases} \tag{47}$$

then the observation that $G(s) = G(w)$ for all $s \in G(w)$ implies

$$\begin{cases} p_s = \bar{q}_s \exp\left(\lambda_{G(s)} - 1\right) & \forall s \in S \\ \sum_{s\in G(w)} \bar{q}_s \exp\left(\lambda_{G(w)} - 1\right) = q(G(w)) & \forall w \in S \end{cases} \tag{48}$$

and

$$\begin{cases} \exp\left(\lambda_{G(w)} - 1\right) = \frac{q(G(w))}{\bar{q}(G(w))} & \forall w \in S \\ p_s = \bar{q}_s \frac{q(G(s))}{\bar{q}(G(s))} & \forall s \in S. \end{cases} \tag{49}$$

The solution is

$$\begin{cases} \lambda_G^\circ = 1 + \log \frac{q(G)}{\bar{q}(G)} & \forall G \in \mathcal{G} \\ p_s^\circ = \bar{q}_s \frac{q(G(s))}{\bar{q}(G(s))} & \forall s \in S \end{cases} \tag{50}$$

which plugged into the value function $\frac{1}{\beta^t} \sum_{s\in S} p_s \log \frac{p_s}{\bar{q}_s}$ delivers

$$\begin{aligned} \inf_{p\in \mathrm{ri}\,\Delta(G_t(\omega)):p_{|\mathcal{G}_{t+1}}=q_{|\mathcal{G}_{t+1}}} c_{t,\omega}(p) &= \frac{1}{\beta^t} \sum_{s\in S} p_s^\circ \log \frac{p_s^\circ}{\bar{q}_s} \\ &= \frac{1}{\beta^t} \sum_{s\in S} \bar{q}_s \frac{q(G(s))}{\bar{q}(G(s))} \log \frac{q(G(s))}{\bar{q}(G(s))} \\ &= \frac{1}{\beta^t} \sum_{G\in\mathcal{G}} \sum_{s\in G} \bar{q}_s \frac{q(G)}{\bar{q}(G)} \log \frac{q(G)}{\bar{q}(G)}, \end{aligned}$$

finally

$$\inf_{p\in \mathrm{ri}\,\Delta(G_t(\omega)):p_{|\mathcal{G}_{t+1}}=q_{|\mathcal{G}_{t+1}}} c_{t,\omega}(p) = \frac{1}{\beta^t} \sum_{G\in\mathcal{G}} q(G) \log \frac{q(G)}{\bar{q}(G)} \tag{51}$$

which together with Eq. (43) delivers Eq. (42).

Setting, for all $t < T$, $\omega \in \Omega$, and $q \in \Delta(\Omega)$,

$$\varrho_{t,\omega}(q) = \begin{cases} \beta \sum_{\substack{G\in\mathcal{G}_{t+1} \\ G\subseteq G_t(\omega)}} q(G) c_{t+1,G}(q_G) + \inf_{p\in \mathrm{ri}\,\Delta(G_t(\omega)):p_{|\mathcal{G}_{t+1}}=q_{|\mathcal{G}_{t+1}}} c_{t,\omega}(p) & \text{if } q \in \mathrm{ri}\,\Delta(G_t(\omega)) \\ \infty \quad \text{otherwise} \end{cases}$$

the function $\varrho_{t,\omega}$ coincides with the closed and convex function $c_{t,\omega}$ on $\mathrm{ri}\,(\Delta(G_t(\omega)))$. Take $q \in \mathrm{ri}\left(\mathrm{dom}\,\varrho_{t,G_t(\omega)}\right) = \mathrm{ri}\,(\Delta(G_t(\omega)))$, by [18, Thm. 7.5], for all $p \in \Delta(G_t(\omega))$,

$$\overline{\mathrm{co}}\,\varrho_{t,\omega}(p) = \lim_{\lambda\uparrow 1} \varrho_{t,\omega}((1-\lambda)q + \lambda p) = \lim_{\lambda\uparrow 1} c_{t,\omega}((1-\lambda)q + \lambda p) = c_{t,\omega}(p). \tag{52}$$

Since Eq. (52) is *a fortiori* true if $p \notin \Delta(G_t(\omega))$, condition (33) holds, as wanted.

To complete the proof we need to prove (17). Let $c_{t,\omega}(p) \equiv \beta^{-t} R\left(p_{G_t(\omega)} \| q^{\circ}_{G_t(\omega)}\right)$ for all $(t,\omega,p) \in \mathcal{T} \times \Omega \times \Delta(\Omega)$. Fix $\omega \in \Omega$ and $t < T$. Step 4 of the proof of Lemma 6 shows that there is a suitable $p \in \mathrm{ri}\,(\Delta(G_t(\omega)))$ such that for all $q \in \Delta(G_t(\omega))$

$$\sum_{\substack{G \in \mathcal{G}_{t+1} \\ q(G)>0}} q(G)\, c_{t+1,G}(q_G) = \lim_{\alpha\uparrow 1} \sum_{\mathcal{G}_{t+1} \ni G \subseteq G_t(\omega)} (\alpha q + (1-\alpha)p)(G)\, c_{t+1,G}\left((\alpha q + (1-\alpha)p)_G\right).$$

By definition of $c_{t,\omega}$, $p \in \mathrm{dom}\, c_{t,\omega}$, we also have

$$\infty > c_{t,\omega}(q) = \lim_{\alpha\uparrow 1} c_{t,\omega}(\alpha q + (1-\alpha)p).$$

Since $\{c_t\}$ is a recursive ambiguity index we have

$$\gamma_{t,\omega}\left(q_{|\mathcal{G}_{t+1}}\right) = c_{t,\omega}(q) - \sum_{\substack{G \in \mathcal{G}_{t+1} \\ q(G)>0}} q(G)\, c_{t+1,G}(q_G),$$

since both summands are finite, setting $q\alpha p = \alpha q + (1-\alpha)p$,

$$\gamma_{t,\omega}\left(q_{|\mathcal{G}_{t+1}}\right) = \lim_{\alpha\uparrow 1} \left(c_{t,\omega}(q\alpha p) - \sum_{\mathcal{G}_{t+1} \ni G \subseteq G_t(\omega)} (q\alpha p)(G)\, c_{t+1,G}\left((q\alpha p)_G\right) \right),$$

but $q\alpha p \in \mathrm{ri}\,(\Delta(G_t(\omega)))$ and Eq. (43) delivers

$$\begin{aligned}
\gamma_{t,\omega}\left(q_{|\mathcal{G}_{t+1}}\right) &= \lim_{\alpha\uparrow 1} \frac{1}{\beta^t} \sum_{\mathcal{G}_{t+1} \ni G \subseteq G_t(\omega)} (\alpha q + (1-\alpha)p)(G) \log \frac{(\alpha q + (1-\alpha)p)(G)}{q^{\circ}_{G_t(\omega)}(G)} \\
&= \frac{1}{\beta^t} \sum_{\substack{\mathcal{G}_{t+1} \ni G \subseteq G_t(\omega) \\ q(G)>0}} q(G) \log \frac{q(G)}{q^{\circ}_{G_t(\omega)}(G)} \\
&= \frac{1}{\beta^t} \sum_{G \in \mathcal{G}_{t+1}} q(G) \log \frac{q(G)}{q^{\circ}_{G_t(\omega)}(G)}
\end{aligned}$$

for all $q \in \Delta(G_t(\omega))$ and by Proposition 2 $\gamma_{t,\omega}\left(q_{|\mathcal{G}_{t+1}}\right) = \infty$ if $q \in \Delta(\Omega) \setminus \Delta(G_t(\omega))$. Therefore,

$$\gamma_{t,\omega}(r) = R_{\mathcal{G}_{t+1}}\left(r \left\| \left(q^{\circ}_{G_t(\omega)}\right)_{|\mathcal{G}_{t+1}}\right.\right) \quad \forall r \in \Delta(\Omega, \mathcal{G}_{t+1}).$$

By (12) and (13), this implies (17). ■

B More on Rectangularity

In this appendix the definition of $\{\mathcal{G}_t\}$-*rectangular* set of beliefs is discussed and compared with the definition of recursive ambiguity index.

If $\mathcal{C}$ is a subset of $\Delta(\Omega)$ and G a subset of Ω such that $p(G) > 0$ for all $p \in \mathcal{C}$,

$$\mathcal{C}_G \equiv \{p_G : p \in \mathcal{C}\}$$

is the set of Bayesian updates of $\mathcal{C}$. Epstein and Schneider [5] consider the following:

Definition 1 *The set $\mathcal{C}$ is $\{\mathcal{G}_t\}$-*rectangular *if,*

(R.1) $\mathcal{C}$ is a non-empty, compact and convex subset of $\Delta^{++}(\Omega)$;

(R.2) for all $(t,\omega) \in \mathcal{T} \times \Omega$ with $t < T$,

$$\mathcal{C}_{G_t(\omega)} = \left\{ \sum_{G \in \mathcal{G}_{t+1}} p^G r(G) : p^G \in \mathcal{C}_G \ \forall G \in \mathcal{G}_{t+1}, r \in \mathcal{C}_{G_t(\omega)|\mathcal{G}_{t+1}} \right\}.$$

For the sake of comparison, consider the following:

Definition 2 *The family $\{C_t(\omega) : (t,\omega) \in \mathcal{T} \times \Omega\}$ of subsets of $\Delta(\Omega)$ is $\{\mathcal{G}_t\}$-*rectangular *if $\{\delta_{C_t}\}$ is a recursive ambiguity index such that $\operatorname{dom} \delta_{C_t(\omega)} \subseteq \Delta^{++}(G_t(\omega))$ for all $(t,\omega) \in \mathcal{T} \times \Omega$.*

The stronger constraint on the effective domains matches the requirement that $\mathcal{C} \subseteq \Delta^{++}(\Omega)$.

Fact 4 *The family $\{C_t(\omega) : (t,\omega) \in \mathcal{T} \times \Omega\}$ of subsets of $\Delta\Omega$ is $\{\mathcal{G}_t\}$-rectangular if and only if:*

(R'.1) for all $t \in \mathcal{T}$ and $\omega, \omega' \in \Omega$, $C_t(\omega)$ is a non-empty, compact and convex subset of $\Delta^{++}(G_t(\omega))$, and $C_t(\omega) = C_t(\omega')$ if $G_t(\omega) = G_t(\omega')$;

(R'.2) for all $(t,\omega) \in \mathcal{T} \times \Omega$ with $t < T$,

$$C_t(\omega) = \left\{ \sum_{G \in \mathcal{G}_{t+1}} p^G r(G) : p^G \in C_{t+1}(G) \ \ \forall G \in \mathcal{G}_{t+1}, r \in C_t(\omega)_{|\mathcal{G}_{t+1}} \right\}. \tag{53}$$

(R'.1) translates the fact that $\{\delta_{C_t}\}$ is a dynamic ambiguity index, (R'.2) is the specialization of the "no-gain" condition (11) to indicator functions (see the proof of Corollary 1).

Proposition 4 *Let $\{C_t(\omega)\}$ be a family of subsets of $\Delta(\Omega)$ satisfying (R'.1). For all $(t,\omega) \in \mathcal{T} \times \Omega$ with $t < T$, set*

$$D_t(\omega) \equiv \left\{ \sum_{G \in \mathcal{G}_{t+1}} p^G r(G) : p^G \in C_{t+1}(G) \ \ \forall G \in \mathcal{G}_{t+1}, r \in C_t(\omega)_{|\mathcal{G}_{t+1}} \right\}.$$

$D_t(\omega)$ is closed, convex, and $(D_t(\omega))_G = C_{t+1}(G)$ for all $G \in \mathcal{G}_{t+1}$ such that $G \subseteq G_t(\omega)$.

Proof. Let $\{G_1, ..., G_g\} \equiv \{G \in \mathcal{G}_{t+1} : G \subseteq G_t(\omega)\}$, and write $C_0 \equiv C_t(\omega)$, and $C_i \equiv C_{t+1}(G_i)$ for $i = 1, ..., g$, then

$$D_t(\omega) = \left\{ \sum_{i=1}^{g} p^i r(G_i) : p^i \in C_i \ \forall i = 1, ..., g, r \in C_{0|\mathcal{G}_{t+1}} \right\}.$$

Closure immediately follows from compactness of $C_0, C_1, ..., C_g$.

Let $\alpha \in (0,1)$ and $\sum_{i=1}^{g} p^i r(G_i), \sum_{i=1}^{g} \bar{p}^i \bar{r}(G_i) \in D_t(\omega)$. Set

$$\begin{aligned} \hat{p}^j &\equiv \frac{\alpha r(G_j)}{\alpha r(G_j) + (1-\alpha)\bar{r}(G_j)} p^j + \frac{(1-\alpha)\bar{r}(G_j)}{\alpha r(G_j) + (1-\alpha)\bar{r}(G_j)} \bar{p}^j \in C_j \quad \forall j = 1, ..., g \text{ and} \\ \hat{r} &\equiv \alpha r + (1-\alpha)\bar{r} \in C_{0|\mathcal{G}_{t+1}}. \end{aligned}$$

It is simple to check that for all $\omega \in G_t(\omega)$ (say $\omega \in G_j$)

$$\begin{aligned} \left(\alpha \sum_{i=1}^{g} p^i r(G_i) + (1-\alpha) \sum_{i=1}^{g} \bar{p}^i \bar{r}(G_i) \right)(\omega) &= \sum_{i=1}^{g} \left(\alpha r(G_i) p^i(\omega) + (1-\alpha)\bar{r}(G_i)\bar{p}^i(\omega) \right) \\ &= \alpha r(G_j) p^j(\omega) + (1-\alpha)\bar{r}(G_j)\bar{p}^j(\omega) \\ &= \hat{r}(G_j)\hat{p}^j(\omega) = \left(\sum_{i=1}^{g} \hat{p}^i \hat{r}(G_i) \right)(\omega) \end{aligned}$$

and

$$\left(\alpha\sum_{i=1}^{g}p^{i}r\left(G_{i}\right)+\left(1-\alpha\right)\sum_{i=1}^{g}\bar{p}^{i}\bar{r}\left(G_{i}\right)\right)\left(\omega\right)=\left(\sum_{i=1}^{g}\hat{p}^{i}\hat{r}\left(G_{i}\right)\right)\left(\omega\right)=0$$

if $\omega\notin G_{t}\left(\omega\right)$. In sum,

$$\alpha\sum_{i=1}^{g}p^{i}r\left(G_{i}\right)+\left(1-\alpha\right)\sum_{i=1}^{g}\bar{p}^{i}\bar{r}\left(G_{i}\right)=\sum_{i=1}^{g}\hat{p}^{i}\hat{r}\left(G_{i}\right)\in D_{t}\left(\omega\right),$$

and $D_{t}\left(\omega\right)$ is convex.

For all $j=i,...,g$ and $\sum_{i=1}^{g}p^{i}r\left(G_{i}\right)\in D_{t}\left(\omega\right)$, we have $\left(\sum_{i=1}^{g}p_{i}r\left(G_{i}\right)\right)_{G_{j}}=p_{j}\in C_{t+1}\left(G_{j}\right)$. Conversely, if $p_{j}\in C_{t+1}\left(G_{j}\right)$, arbitrarily choose $p_{i}\in C_{i}$ for $i\neq j$ and $r\in C_{0|\mathcal{G}_{t+1}}$, to obtain

$$p_{j}=\left(\sum_{i=1}^{g}p_{i}r\left(G_{i}\right)\right)_{G_{j}}\in\left(D_{t}\left(\omega\right)\right)_{G_{j}}.$$

We conclude $\left(D_{t}\left(\omega\right)\right)_{G_{j}}=C_{t+1}\left(G_{j}\right)$ and the proof. ■

Proposition 5 *If the family $\{C_{t}\left(\omega\right)\}$ is $\{\mathcal{G}_{t}\}$-rectangular, then*

$$\mathcal{C}\equiv C_{0}\left(\Omega\right)\ \text{ is }\ \{\mathcal{G}_{t}\}\text{-rectangular and }C_{t}\left(\omega\right)=\mathcal{C}_{G_{t}\left(\omega\right)}\ \text{ for all }\ \left(t,\omega\right)\in\mathcal{T}\times\Omega.$$

Conversely, if $\mathcal{C}$ is $\{\mathcal{G}_{t}\}$-rectangular, then

$$C_{t}\left(\omega\right)\equiv\operatorname{co}\mathcal{C}_{G_{t}\left(\omega\right)}\ \text{ for all }\ \left(t,\omega\right)\in\mathcal{T}\times\Omega\ \text{ defines a }\ \{\mathcal{G}_{t}\}\text{-rectangular family.}$$

Proof. By (R'.1), $\mathcal{C}$=$C_{0}\left(\omega\right)$ is a non-empty, compact and convex subset of $\Delta^{++}\left(G_{0}\left(\omega\right)\right)=\Delta^{++}\left(\Omega\right)$; that is (R.1) holds. For all $\omega\in\Omega$ and all $t<T$, by Proposition 4 and rectangularity of $\{C_{t}\left(\omega\right)\}$,

$$C_{t+1}\left(\omega\right)=C_{t+1}\left(G_{t+1}\left(\omega\right)\right)=\left(D_{t}\left(\omega\right)\right)_{G_{t+1}\left(\omega\right)}=\left(C_{t}\left(\omega\right)\right)_{G_{t+1}\left(\omega\right)},$$

that is $C_{t+1}\left(\omega\right)=\left(C_{t}\left(\omega\right)\right)_{G_{t+1}\left(\omega\right)}$. By induction, this implies $C_{t}\left(\omega\right)=\mathcal{C}_{G_{t}\left(\omega\right)}$ for all $\left(t,\omega\right)\in\mathcal{T}\times\Omega$. In fact, for all $\omega\in\Omega$, $C_{1}\left(\omega\right)=\left(C_{0}\left(\omega\right)\right)_{G_{1}\left(\omega\right)}=\mathcal{C}_{G_{1}\left(\omega\right)}$; and $C_{\tau}\left(\omega\right)=\mathcal{C}_{G_{\tau}\left(\omega\right)}$ for $\tau<T$ implies

$$C_{\tau+1}\left(\omega\right)=\left(C_{\tau}\left(\omega\right)\right)_{G_{\tau+1}\left(\omega\right)}=\left(\mathcal{C}_{G_{\tau}\left(\omega\right)}\right)_{G_{\tau+1}\left(\omega\right)}=\mathcal{C}_{G_{\tau+1}\left(\omega\right)}$$

since $G_{t+1}\left(\omega\right)\subseteq G_{t}\left(\omega\right)$. $C_{t}\left(\omega\right)=\mathcal{C}_{G_{t}\left(\omega\right)}$ for all $\left(t,\omega\right)\in\mathcal{T}\times\Omega$ and (R'.2) imply (R.2).

For the converse, (R.1) implies that for every non-empty subset A of Ω the set $\mathcal{C}_{A}$ is a non-empty compact subset of $\Delta^{++}\left(A\right)$. Therefore, for all $\left(t,\omega\right)\in\mathcal{T}\times\Omega$, $\mathcal{C}_{G_{t}\left(\omega\right)}$ is a non-empty compact subset of $\Delta^{++}\left(G_{t}\left(\omega\right)\right)$, and $\left\{\operatorname{co}\mathcal{C}_{G_{t}\left(\omega\right)}\right\}$ satisfy (R'.1). Taking an unbounded and affine $u:X\to\mathbb{R}$ and $\beta>0$, the results of Epstein and Schneider [5] guarantee that the preferences represented by

$$V_{t}\left(\omega,h\right)=\inf_{p\in\Delta^{++}\left(\Omega\right)}\left(\int\sum_{\tau\geq t}\beta^{\tau-t}u\left(h_{\tau}\right)dp_{G_{t}\left(\omega\right)}+\delta_{\mathcal{C}_{G_{t}\left(\omega\right)}}\left(p_{G_{t}\left(\omega\right)}\right)\right)\quad\forall h\in\mathcal{H}$$

are recursive variational preferences. But $C_{t}\left(\omega\right)=\operatorname{co}\mathcal{C}_{G_{t}\left(\omega\right)}$ implies

$$V_{t}\left(\omega,h\right)=\inf_{p\in\Delta^{++}\left(\Omega\right)}\left(\int\sum_{\tau\geq t}\beta^{\tau-t}u\left(h_{\tau}\right)dp_{G_{t}\left(\omega\right)}+\delta_{C_{t}\left(\omega\right)}\left(p_{G_{t}\left(\omega\right)}\right)\right)\quad\forall h\in\mathcal{H}.$$

Finally, Corollary 1 guarantees that (R'.2) holds. ■

References

[1] K. Detlefsen, G. Scandolo, Conditional and dynamic convex risk measures, mimeo, 2005.

[2] Dolecki, S. and G.H. Greco (1995). Niveloids, *Topological Methods in Nonlinear Analysis,* 5, 1–22.

[3] P. Dupuis and R. S. Ellis, *A weak convergence approach to the theory of large deviations*, Wiley, New York, 1997.

[4] D. Ellsberg, Risk, ambiguity, and the Savage axioms, *Quarterly Journal of Economics*, 75, 643–669, 1961.

[5] L.G. Epstein and M. Schneider, Recursive multiple-priors, *Journal of Economic Theory*, 113, 1-31, 2003.

[6] Ghirardato, Maccheroni, and Marinacci, Ambiguity from the differential viewpoint, ICER WP 17, 2002.

[7] I. Gilboa and D. Schmeidler, Maxmin expected utility with a non-unique prior, *Journal of Mathematical Economics*, 18, 141-153, 1989.

[8] E. Hanany and P. Klibanoff, Updating preferences with multiple priors, mimeo, 2005.

[9] L. Hansen and T. Sargent, Robust control and model uncertainty, *American Economic Review*, 91, 60-66, 2001.

[10] S. Hart, S. Modica, and D. Schmeidler, A neo^2 Bayesian foundation of the maxmin value for two-person zero-sun games, *International Journal of Game Theory*, 23, 347-358, 1994.

[11] T. Hayashi, Intertemporal substitution, risk aversion, and ambiguity aversion, *Economic Theory*, 25, 2005.

[12] J.P. Hiriart-Urruty and C. Lemaréchal, *Convex analysis and minimization algorithms, I and II*, Springer, New York, 1996.

[13] F. Liese and I. Vajda, *Convex Statistical Distances*, Teubner, Leipzig, 1987.

[14] F. Maccheroni, M. Marinacci, and A. Rustichini, Ambiguity aversion, robustness, and the variational representation of preferences, mimeo, 2004.

[15] F. Maccheroni, M. Marinacci and A. Rustichini, Niveloids and their extensions, mimeo, 2005.

[16] F. Maccheroni, M. Marinacci, A. Rustichini, and M. Taboga, Portfolio selection with monotone mean-variance preferences, mimeo, 2005.

[17] E. Ozdenoren and J. Peck, Ambiguity aversion, games against Nature, and dynamic consistency, mimeo, 2005.

[18] R. T. Rockafellar, *Convex analysis*, Princeton Univ. Press, Princeton, 1970.

[19] M. Siniscalchi, Dynamic choice under ambiguity, mimeo, 2004.

[20] C. Skiadas, Robust control and recursive utility, *Finance and Stochastics*, 7, 475-489, 2003.

[21] T. Wang Conditional preferences and updating, *Journal of Economic Theory*, 108, 286-321, 2003.