Structural holes in social networks

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Abstract

We consider a setting where every pair of players that interact (e.g. exchange goods or information) create a surplus. An interaction can take place only if the players involved have a connection. If the connection is direct the two players split the surplus equally while if it is indirect then intermediate players also get a share of the surplus. Thus individuals form links with others to create surplus, to gain intermediation rents and to circumvent others who are trying to become intermediary.

Our principal result is that strategic link formation in such a setting leads to the star network. In a star a single agent acts as an intermediary for all transactions and there is significant payoff inequality across ex-ante identical players.

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1 Introduction

It is now widely agreed that knowledge of the structure of interaction among individuals is important for a proper understanding of a number of important questions in economics, such as the spread of new ideas and technologies, the patterns of employment and wage inequality, competitive strategies in dynamic markets, and career profiles of managers.[1]

Connections facilitate timely access to important information – on trade opportunities, job vacancies, project deadlines, and novel ideas for research. In some important instances – e.g., trade opportunities, the combining of novel research ideas – the payoffs an individual entity gets in a network will clearly depend on his relative importance in bridging gaps in the network between others.[2] The potential gains from bridging different parts of a network were important in the early work of Granovetter (1974) and are central to the notion of *structural holes* developed by Burt (1994). In recent years, a number of empirical studies have shown that individuals or organizations who bridge 'structural holes' in networks gain significant payoff advantages.[3] For instance, the work on promotions and performance evaluation argues that the differences in structural location of individuals – in particular whether they bridge structural holes in the social network – explains a significant part of the variation in promotion timing of otherwise similar people. Given these significant payoffs effects, it seems natural that an individual will make investments in connections so as to become structurally important, while other individuals will likewise form connections to circumvent such attempts. Are special structural positions and the corresponding large payoff differences sustainable when individual entities form connections strategically?

[1]We mention a small sample of this work. See Bala and Goyal (1998), Coleman, Katz, and Menzel (1966) and Ellison and Fudenberg (1992) for the role of communication networks in technological diffusion; Rees (1966), Granovetter (1974), Montgomery (1992) and Calvo and Jackson (2004) for effects of social networks on employment and wage inequality; Ahuja (2000), Goyal and Moraga (2001) and Gulati, Nohria and Zaheer (2000) on the role of strategic partnering in generating competitive advantage in markets; Burt (1994) for the effects of organizational networks on individual promotions and salaries, and Rauch (2001) on the role of social and economic networks in internation trade.

[2]Burt (2004) explores the influence of individual position in social networks in shaping the generation of creative ideas.

[3]See Burt (1994) and Mehra, Kilduff and Brass (2001) for influence of structural positions on promotions and performance evaluation, Podolny and Baron (1997) for work on network positions and mobility, and Ahuja (2000) for the influence of a firm's position in inter-organizational networks on its innovativeness and overall performance.

We develop a simple model of network formation to address this question. We consider a setting where interaction between every pair of individuals generates a surplus. If two individuals are directly linked then they split this surplus equally, while if they are indirectly connected – there are other players in the 'path' between them – then the division of surplus depends on the competition between these intermediaries. In this setting, there are three types of incentives for individuals to form links with others. The first incentive is the desire to create surpluses: individuals would like to join the network so as to create exchange possibilities which in turn create surpluses. The second incentive is related to the rewards from intermediation: players would like to place themselves between others in order to extract rents from intermediation. The third incentive arises out of the desire to avoid sharing surpluses with intermediaries; in other words, individuals will try to circumvent intermediate players to retain more of the surplus for themselves.

Our principal finding is that strategic link formation leads to the star network. This is a network in which one player (referred to as the central player) forms links with all the other players (who are referred to as the peripheral players) and there are no other links in the network. There are two aspects of the result that we would like to stress: *one*, the star is the unique non-empty network that arises in equilibrium and *two*, the star entails significant payoff inequality among ex-ante identical players. Thus an extreme version of strategic positioning – with the central player earning significantly larger payoffs – is the only possible outcome in a setting with ex-ante identical individuals. We now briefly sketch the main arguments underlying this result.

The first observation is that, due to the role played by access benefits – the surplus generated from direct and indirect interaction– an equilibrium network must be either empty or connected. Thus, among non-empty networks, we only need to check which connected networks are robust with respect to individual incentives. The second observation is that if the network is minimally connected (i.e. has no cycles) then long paths cannot be stable. This is because players located at the 'end' of the network benefit from connecting to a "central player" in order to save on intermediation costs (cutting path lengths) and, on the other hand, a central player is also ready to incur the cost of an additional link because this enhances her intermediation payoffs. Thus there is a natural tendency for minimally connected networks to become agglomerated around central players. But what about networks with cycles? In principle, a cycle containing all players appears to be very attractive. Every player is in a symmetric position and each of them can access every other. Such a cycle network, however, is vulnerable to the incentives of individuals towards becoming uniquely intermediate and reaping the entailed intermediation benefits. To see this, consider two players that are far apart in the cycle and establish a direct link. By simultaneously breaking one link each, they can produce a "line" and become central in it. In a line, they must pay intermediation costs to a number of others, but their prominent centrality more than offsets these costs through even larger intermediation benefits. A similar argument can be used to break any network with cycles, thus leaving the star network as the only (non-empty) equilibrium network.

We examine the robustness of these arguments in a number of directions. In the basic model intermediation rents arise only if a player is essential. We consider an alternative formulation where intermediation rents increase smoothly in the extent of the criticality of a player and find that stars arise in that setting as well. In the basic model stars arise in societies with a large number of players. This leads us to examine the nature of networks when individuals are constrained in the number of links they can form. We find that these constraints give rise to networks with a group of inter-connected central players who are local stars each with their own distinct group of peripheral players.

We now clarify the contribution of our paper by discussing its relationship with research in economics, sociology and the literature on complex systems. In recent years, much attention has been devoted by economists and game theorists to the theory of network formation.[4] In existing work it is assumed that the goods transacted via the links are non-rival — see e.g. the connections model studied by Jackson and Wolinsky (1996) and the version with one-sided link formation proposed by Bala and Goyal (2000). Here we consider a setting with *rival* goods and explicitly incorporate the idea of *intermediation rents* and payments. The allocation of intermediation rents in turn requires a model of competition between intermediaries and a related contribution of our paper is a method to assign these benefits as a function of the network structure. We introduce the idea of *essential* players – players without whom an interaction cannot take place – to determine which intermediaries will earn rents. The incorporation of intermediary rents and payments has powerful implications for equilibrium networks. For example, in earlier work on the two-sided connections model, stars can be sustained only over a small range of parameter values, and the central player who is bridging the structural holes actually earns

[4]See Aumann and Myerson, (1989), Boorman (1975), Bala and Goyal (2000), Dutta, Tijs and van den Nouweland (1998), Jackson and Wolinsky (1996) and Kranton and Minehart (2001).

a lower payoff than the peripheral players (see Proposition 2 in Jackson and Wolinsky, 1996). By contrast, in the present model, the star is sustainable for practically the entire range of parameters and the central player earns a much higher payoff as compared to the peripheral players.[5]

The ideas of access advantages and strategic positioning are important building blocks for the notion of structural holes, introduced by Burt (1994), and have been an important part of the tradition in sociology since the work of Granovetter (1974). As we mentioned earlier, empirical research by sociologists shows that differences in structural location of otherwise similar individuals – in particular whether they bridge structural holes in the social network – explains a significant part of this variance. Our paper shows that the presence of structural holes and corresponding payoffs differences is consistent with a model of rational players who strategically seek to create positional advantages for themselves and also have an incentive in preventing others from becoming central. To the best of our knowledge this is the first formal model to develop an explanation for structural holes based on intermediation rents and the large inequality in payoffs that go with them.

In recent years statistical physicists studying complex networks have documented a number of empirical properties – small average distances, high inequality in number of links across nodes of the network, among others – of large social, biological and technological networks.[6] One of the leading models in this literature is that proposed by Barabasi and Albert (1999). They formulate a simple dynamic model of network formation in which, at every point in time, a new node arrives and forms new links with existing nodes. The probability that these links connect to any given existing node is taken to be increasing in –typically, proportional to – the number of links the node has; this is referred to as *preferential attachment*. The property of preferential attachment is critical for the derivation of unequal link distributions in the long run – in particular, the so/called scale-free

[5]Recent papers by Feri (2004) and Hojman and Szeidl (2004) also explore the emergence of star networks. Their approach, however, is different from the one pursued here in two important respects. First, as in the connections model, the transacted goods are non-rival; second, links are one-sided (i.e. they are fully payed by the player who initiates them). It is precisely this second feature that allows for the establishment of periphery-sponsored stars in large populations: all links are initiated by peripheral players and thus the center incurs no cost at all for any of her links. Another model where stars may arise at equilibrium is studied by Galeotti and Melendez (2004). Their approach, however, has agents involved in an infinitely repeated Prisoner's Dilemma that determines how linking costs are shared.

[6]For a survey of this work see Albert and Barabasi (2002), Newman (2003), and Vega-Redondo (2005).

(or power-law) degree distributions. This literature, however, *does not provide us with any reasons for why a highly linked older player and a newly arriving player should want to connect with each other*. We show that large intermediation rents for the older player and the desire of the new player to minimize the number of intermediaries (and thereby reduce payments) jointly provide a simple explanation for preferential attachment.

The paper is organized in five sections. Section 2 lays out the basic model. Section 3 presents the main results on equilibrium and efficient networks. Section 4 discusses the role of different assumptions in our analysis, while section 5 concludes. All the proofs are given in an Appendix at the end of the paper.

2 The Model

We consider a population composed of finite set of ex-ante identical agents, $N = \{1, 2, ..., n\}$ where $n \geq 3$. These agents play a network-formation game where every one of them makes a simultaneous announcement of *intended* links. An intended link $s_{i,j} \in \{0, 1\}$, where $s_{ij} = 1$ means that player i intends to form a link with player j, while $s_{ij} = 0$ means that player i does not intend to form such a link. Thus a strategy of player i is given by $s_i = [s_{ij}]_{j \in N \setminus \{i\}}$, with S_i denoting the strategy set of player i.

A link between two players i and j is formed if and only if $s_{ij} = s_{ji} = 1$. We denote the formed (undirected) link by $g_{i,j} \equiv g_{j,i} = 1$ and the absence of a link by $g_{i,j} \equiv g_{j,i} = 0$. Any given strategy profile $s = (s_1, s_2, ..., s_n)$ therefore induces a network $g(s)$. The *network* $g(s) = \{(g_{i,j})\}_{i,j \in N}$ is a formal description of the pair-wise links that exist between the players. There exists a *path* between i and j in a network g if either $g_{i,j} = 1$ or if there is a distinct set of players $\{i_1, \ldots, i_n\}$ such that $g_{i,i_1} = g_{i_1,i_2} = g_{i_2,i_3} = \cdots = g_{i_n,j} = 1$. All players with whom i has a path defines the component of i in g, which is denoted by $C_i(g)$.

We shall suppose that any pair of players j and k who are connected by a path generate a unit of surplus. The allocation of this surplus depends on whether there are intermediaries between j and k and on the competition between the intermediaries. In the basic model we will take the view that any two paths between any two players j and k fully compete away all the surplus. This is in the spirit of Bertrand competition between paths. In general, the unit of surplus may be divided among all agents in N according to some imputation $z^{jk} = (z_i^{jk})_{i \in N}$ of non-negative shares. In the Appendix we show that the above payoff division is the unique allocation in the kernel of the corresponding cooperative game (see, Davis and Maschler (1965)).

We are then led to the notion of essential players: a player i is said to be *essential* for j and k if i lies on *every* path that joins j and k in the network. For example, the number of essential players between j and k is zero if the players have a direct link, or if players are located around a circle – in the latter case, for every pair of players j and k and every other player i, there is always a path joining j and k that does *not* include i. On the other hand, note that in a star every pair of peripheral players has a single and common essential player, namely the center of the star.

We suppose that agents have to pay a fixed (marginal) cost c for each the links they establish. Denote by $E(j,k;g)$ the set of players who are essential to connect j and k in network g and let $e(j,k;g) = |E(j,k;g)|$. Then, for every strategy profile $s = (s_1, s_2, ..., s_n)$, the (net) payoffs to player i are given by:

$$\Pi_i(s_i, s_{-i}) = \sum_{j \in C_i(g)} \frac{1}{e(i,j;g)+2} + \sum_{j,k \in N} \frac{I_{\{i \in E(j,k)\}}}{e(j,k;g)+2} - \eta_i(g)c, \tag{1}$$

where $I_{\{i \in E(j,k)\}} \in \{0,1\}$ stands for the indicator function specifying whether i is essential for j and k, and $\eta_i(k,g) \equiv |\{j \in N : j \neq i,\ g_{ij} = 1\}|$ denotes the number of players with whom player i has a link.

We note that even if a player lies on several paths between two players she may still get no intermediation payoffs if she is not essential. In Section 4 we examine in detail the case where a player i's intermediation rents from connecting j and k are an increasing function of the number of shortest paths (between j and k that) she lies on. We also note that the payoff function above assumes that the costs of link formation are constant. In section 4 we study the implications of increasing costs by considering the case of capacity constraints on the number of links an individual can form.

The main objective of the paper is studying the architecture of networks that are strategically stable and assess their efficiency. Our notion of strategic stability is a refinement of Nash equilibrium that allows for coordinated two-person deviations.

Definition 1 *A strategy profile* $\mathbf{s}^*$ *is a* Bilateral Equilibrium (BE) *if the following conditions hold:*

- *for any* $i \in N$, *and every* $s_i \in S_i$, $\Pi_i(\mathbf{s}^*) \geq \Pi_i(s_i, \mathbf{s}^*_{-i})$;

- *for any pair of players $i, j \in N$, and every strategy pair (s_i, s_j),*

$$\Pi_i(s_i, s_j, \mathbf{s}^*_{-i-j}) > \Pi_i(s^*_i, s^*_j, \mathbf{s}^*_{-i-j}) \Rightarrow \Pi_j(s_i, s_j, \mathbf{s}^*_{-i-j}) < \Pi_j(s^*_j, s^*_j, \mathbf{s}^*_{-i-j}).$$

We apply the term 'bilateral equilibrium' to a strategy profile where no player or pair of players can deviate (unilaterally or bilaterally, respectively) and benefit from the deviation (at least one of them strictly, for bilateral deviations). This notion *refines* the original formulation of pair-wise stability due to Jackson and Wolinsky (1996) by allowing pairs of players to form and delete links simultaneously.

In fact, our analysis will focus on a refinement of BE that we call *strict.* It rules out the existence of deviations (unilateral or bilateral) that have some consequence (i.e. affect the network) but nevertheless are payoff indifferent for the agents involved. In part, the motivation of this equilibrium concept is dynamic: a gradual adjustment process that allows pairs of players to revise the network in sequence will (only) be absorbed by equilibria of this sort – see below for an elaboration.

Definition 2 *A strategy profile* $\mathbf{s}^*$ *is a* Strict Bilateral Equilibrium (SBE) *if the following conditions hold:*

- *for any $i \in N$, and every $s_i \in S_i$ such that $g(s_i, \mathbf{s}^*_{-i}) \neq g(\mathbf{s}^*)$, $\Pi_i(\mathbf{s}^*) > \Pi_i(s_i, \mathbf{s}^*_{-i})$;*
- *for any pair of players, $i, j \in N$ and every strategy pair (s_i, s_j) with $g(s_i, s_j, \mathbf{s}^*_{-i-j}) \neq g(\mathbf{s}^*)$,*

$$\Pi_i(s_i, s_j, \mathbf{s}^*_{-i-j}) \geq \Pi_i(s^*_i, s^*_j, \mathbf{s}^*_{-i-j}) \Rightarrow \Pi_j(s_i, s_j, \mathbf{s}^*_{-i-j}) < \Pi_j(s^*_j, s^*_j, \mathbf{s}^*_{-i-j}).$$

We note that a strict bilateral equilibrium is a bilateral equilibrium.

Finally, we introduce the notion of efficiency. In line with the assumption that the bargaining setup involves transferable utility, different networks g are assessed in terms of the total surplus generated, $W(g) \equiv \sum_{i \in N} \Pi_i(g)$. Let $\mathcal{G}$ denote the set of all possible networks (i.e. all undirected graphs with n vertices).

Definition 3 *A network $\tilde{g}$ is efficient if $W(\tilde{g}) \geq W(g)$ for all $g \in \mathcal{G}$.*

Before undertaking the analysis of the model, it might useful to review some standard graph-theoretic notions that will be used repeatedly. A network is said to be *connected* if there exists a path between any pair $i, j \in N$. Given any $g' \subset g$, let $N(g') \equiv \{i \in N : g'_{ij} = g'_{ji} = 1 \text{ for some } j\}$ be the subset of nodes which display some link in g'. Then, a network, $g' \subset g$, is a *component* of g if for all $i, j \in N(g')$, $i \neq j$, there exists a path in g' connecting i and j, and for all $i \in N(g')$ and $k \in N$, $g_{i,k} = 1$ implies $k \in N(g')$. A component $g' \subset g$ is *complete* if $g_{i,j} = 1$ for all $i, j \in N(g')$.

Two networks g and g' are said to have the same architecture if one network can be obtained from the other by a permutation of the players' labels. A network is said to be symmetric if all players have the same number of links, say η. The *complete* network, g^c, is a symmetric network in which $\eta = n - 1$, $\forall i \in N$, while the *empty* network, g^e, is a symmetric network in which $\eta = 0$, $\forall i \in N$. Another example of a symmetric network is a *cycle* where $\eta = 2$ and the whole set of nodes can be ordered in a list $i_1, i_2, ...i_n$ with $g_{i_1,i_2} = g_{i_2,i_3} = ... = g_{i_n,i_1} = 1$.

Finally, we shall say that a network is asymmetric if there is at least one pair of players who have a different number of links. One important example is the *star* where there is a single node, i_c, with $\eta_{i_c} = n - 1$ while $\eta_i = 1$ for all other $i \neq i_c$. An interesting mixture is what we shall call the *hybrid cycle-star* network where there is a subset of k nodes $C = \{i_1, i_2, ...i_k\}$ arranged in a cycle and some particular player $i_x \in C$ such that $g_{j,i_x} = 1$ for all $j \in N \backslash C$. Figure 1 illustrates these different types of networks.

3 Analysis

Our main result is that equilibrium networks must be stars. This result shows that structural holes arise endogenously in the starkest possible manner – a single individual (the center of the star) is *essential* to the value generated in the *whole* network. This in turn implies that the center also appropriates a correspondingly large part of the total surplus generated by the network.

We start by noting a property of equilibrium networks.

Proposition 1 *A bilateral equilibrium network is either empty or connected.*

Proof: See Appendix.

Suppose that, contrary to what is asserted, an equilibrium network is split into two or more components. Consider two individuals, i and j, in different components. First, we observe that their marginal payoffs of establishing a link between them (thereby merging the two components) are exactly the same for both players. This happens because the additional "access payoffs" of i from connecting to the individuals in the other component are identical to the corresponding intermediation payoffs earned by j. Therefore, since gross payoffs are the sum of access and intermediation payoffs, both players enjoy the same marginal gross benefit from linking to each other. Now, we argue that it cannot be optimal for these players to remain in separate components. The reason is as follows. Suppose that i's component contains more than one player. We show that the component must have some agent (let us suppose that it is i herself) that enjoys no intermediation payoffs (either because she is extremal or an "inessential" part of a cycle). Then, the payoffs of j from linking with i are, in addition to the payoff from the direct link, a "scaled down replica" of the access payoffs of i. It then follows that the gross payoffs to j must exceed the linking cost if, as assumed, it is optimal for player i to incur the cost of *two* links to have her (pure access) payoffs.

What type of equilibrium networks are then possible? We start by noting that two types of architectures are always possible, unless the cost is very low or the population is quite small. On the one hand, the empty network is SBE if $c > 1/2$, even if the population is arbitrarily large. This reflects a simple instance of "coordination failure": if no links are formed, then the creation of any link has to be judged on a stand-alone basis, which is unprofitable if the linking cost exceeds half of the unit surplus earned by an isolated pair of players.

On the other hand, it is also clear that the star is a SBE if the population is large enough and the linking cost is no so low as to justify a direct connection with every other player. Specifically, suppose that $1/6 < c < 1/2 + (n-2)/6$. Then, in a star, the payoffs to the center are positive and equal to

$$\frac{n-1}{2} + \frac{1}{3}\frac{(n-1)(n-2)}{2} - (n-1)c$$

whereas if she were to keep only k of the $n-1$ links she would earn the lower payoffs:

$$\frac{k}{2} + \frac{1}{3}\frac{k(k-1)}{2} - kc.$$

As for the incentives of the peripheral players, in the star they earn $1/2+(n-2)/3-c > 0$ while if they created an additional link the entailed payoffs would be $1/2 + 1/2 + (n-3)/3 - 2c$. The latter is smaller than the former if $c > 1/6$.

The star is a (strict) bilateral equilibrium network (SBE) for a wide range of parameters and this is due to the fact that centrality generates large payoffs from essentialness. However, a star also exhibits an extreme form of such essentialness, with only one player being essential for all pairs of players. This raises the question: are there other – perhaps more egalitarian – network architectures that can arise in equilibrium? The following Theorem, the main result of the paper, provides a complete (negative) answer to this question for large societies.

Theorem 1 *Suppose that n is large. If $1/6 < c < 1/2$ then the unique SBE network is the star, while if $1/2 < c < 1/2 + (n-2)/6$ then the star and the empty network are the only SBE networks. If $c > 1/2 + (n-2)/6$ then the empty network is the unique bilateral equilibrium network.*

Proof: See Appendix.

The proof presented in the Appendix proceeds by showing that all networks other than the empty network and the star are not sustainable in a strict equilibrium. The arguments rely on the three kinds of incentives that arise in our model: accessing others, gaining intermediation rents, and avoiding intermediation payments. The intuition underlying the main steps can be outlined as follows.

1. *If a SBE network is minimally connected (i.e. a tree), it must be a star*

   This follows from two observations. First, the number of essential players between any two players is bounded above, independently of the population size (if it were too large, the incentives of these players to establish a link and thus gain shorter access to each of them as well as to many others would certainly exceed the cost of the link). Secondly, "extremal" and "central" players always have much to gain by connecting directly since by so doing the gains are proportional to population size but (opportunity) costs are bounded. A simple illustration of the latter point is depicted in Figure 2.

2. *There can be at most one cycle in a SBE network*

   Here, the main point is that, if several cycles exist in a network, it is always possible for two players to establish a new link and at least remain equally well off. Two possible cases need to be distinguished in this respect, illustrated in turn by Figures 3A and 3B. In the first case, the two cycles have no players in common and players such as 3 and 4 in Figure 3A.1 have a strict incentive to connect and destroy their links to essential players 1 and 2. This change leads to the network in Figure 3A.2, where players 3 and 4 access the same number of other players, avoid the need of sharing some payoffs with players 1 and 2, and incur the same linking costs. In the second case, there are common players in both cycles (i.e. player 1 in Figure 3.B.1), so that a deviation by two players (1 and 2) can transform those two cycles into just one (Figure 3.B.2) where the payoffs to both of them are exactly the same as before.[7] This means that the original network fails to be a strict bilateral equilibrium.

3. *A single cycle cannot be a SBE network*

   A cycle provides incentives for players lying on opposite sides of it to connect and, by also deleting two of their links, become markedly central and thus earn large intermediation rents (cf. Figure 4). It is true that by so doing the two players in question have to make intermediation payments to others where none of these existed under the cycle. We find however that the intermediation rents dominate the intermediation payments and as a result the cycle is not sustainable.

4. *If a SBE network includes a cycle, all other players not belonging to it are connected to the same player in the cycle, i.e. it is a hybrid cycle-star network*

   The starting point of the argument is the observation that, given the linking cost c, the cycle cannot include more than a certain number of players – otherwise, creating a link that bypasses the "gate player" that is essential to accessing the cycle would become profitable. Then, since almost all players (if the population is large) must lie outside the cycle, we can rely on an adaptation of the ideas used in Step 1 above to argue that, in the minimally connected part of the network that lies outside the cycle, players must be arranged in a star configuration.

[7]This change in link pattern, however, strictly increases the payoffs of player 3 since player 3 has the same access benefits, no essential players, but forms one link less in the new network.

5. *The only hybrid cycle-star network that defines a SBE is the "degenerate" star network*

   As explained above, in a SBE network that involves a unique cycle most of the players in the (large) population must lie outside the cycle and connect to it through a single player. Therefore, much as it was argued for Step 3, that configuration opens the possibility that the cycle be broken by the concerted deviation of two players in it, who can then obtain high intermediation payoffs. The nature of such a deviation is illustrated in Figure 5. It rules out that a hybrid cycle-star SBE network may include any genuine cycle, thus leaving us with the strict star network as the only possibility for a non-empty SBE. Since it is clear that a star network is a SBE (see the discussion preceding the statement of the theorem), the conclusion follows

The above result provides us with a complete characterization of equilibrium networks for large societies. What can we say about equilibrium networks in small societies? We first note that a star is an equilibrium for small societies so long as $1/6 < c < 1/2 + (n-2)/6$, and that any minimal network contains structural holes and corresponding payoff inequality. Second, we note that steps 2 and 3 do not use the size of society, and so a single cycle or a network with several cycles cannot arise even in small societies. Finally, step 4 shows that, in a network with one cycle and some players outside it, there is an upper bound to the number of players in the cycle. In other words, most of the players are peripheral to a 'local' star. These considerations lead us to conclude that equilibrium networks in small societies also have a tendency to generate structural holes and there will be players who will earn large intermediation payoffs by spanning them.

We end this section by discussing briefly the architecture of socially optimal networks. First, we observe that from a social point of view there is no gain in adding links in a connected network, while there is a cost to adding links since $c > 0$. So efficient networks must be minimal or empty. On the other hand, in our setting where positive surpluses are earned from every direct and indirect connection, it is easy to see that networks with distinct (non-trivial) components cannot be efficient. Thus, an efficient network must either be empty or minimally connected. Finally, we note that the aggregate net payoff in the latter case is $\frac{1}{2}n(n-1) - 2c(n-1)$, which is positive if, and only if, $c < n/4$. Based on these observations, the following result readily follows.

Proposition 2 *If $c < n/4$ then an efficient network is minimally connected, while if $c > n/4$ then an efficient network is empty.*

Clearly, the star is a minimally connected network and is therefore efficient for large n. Therefore, combining Proposition 2 and Theorem 1, we conclude that efficiency is guaranteed at a nonempty SBE network, provided the population is large.

4 Discussion

We have shown that in a setting with access benefits and intermediation rents strategic network formation generates sharp predictions: for large societies, all non-empty equilibrium networks are stars. In this section we will examine how sensitive this result is to some of the specific assumptions we made in the analysis. First, In Subsection 4.1, we explore a variant of the model where payoffs depend on the length of connecting paths rather than on the existence of competing alternatives. Second, in Subsection 4.2, we discuss the implications of capacity constraints in the number of links that each player can individually support. Finally, in Subsection 4.3, we check whether our analysis is robust to the requirement of credibility "coalition-proofness" in bilateral deviations.

4.1 Essentiality and intermediation

In the model we have assumed that a player gets intermediation payoffs if and only if she is essential. This formulation has intuitive appeal in terms of a form of "Bertrand-like" competition between paths connecting individuals. In fact, it is such form of competition that underlies the intuition that any allocation in the Core gives zero payoffs to non-essential players, and it is also partly reflected in the selection of the Core afforded by the Kernel. In some settings, however, it may be argued that a player's intermediation payoffs should depend on the nature of the connecting path on which she lies – i.e. not only on whether there are some alternatives. For example, if we abstract from inter-path competition, a natural assignment rule would be one prescribing that all players in the path *actually* used to connect any two players are to receive an equal share of the surplus. Indeed, this is the outcome to be expected if the decision of what path to use must be determined *ex ante* in an irreversible manner. In that case, once such a decision has been taken, the existence of alternative paths should play no role in the assignment of the surplus and all players involved should receive an equal share.

Here, we discuss briefly the implications of this alternative approach. Note that an immediate implication of it is that only shortest paths will be used. That is, any player i aiming to connect to j will always choose to start the endeavor by contacting a direct neighbor that lies on one of the shortest path to j. Then, subsequently, since all other players in that path share the same objective, a shortest path will indeed be used. This induces payoff functions $\hat{\Pi}_i$ that, for any strategy profile $s = (s_1, s_2, ..., s_n)$, are defined as follows:

$$\hat{\Pi}_i(s_1, s_2, ..., s_n) = \sum_{j \in C_i(g)} \frac{1}{d(i, j; g) + 1} + \sum_{j,k \in C_i(g)} \frac{b_i(j, k; g)}{d(j, k; g) + 1}, \tag{2}$$

where $d(i, j; g)$ denotes the (geodesic) distance between i and j in network g, whereas $b_i(j, k; g)$ stands for the fraction of shortest paths in g joining j and k that pass through i.

We now argue that many of the insights obtained in the original model carry over to this setting. For simplicity, let us restrict attention to the collection of networks where the tension between symmetry and polarization arises in a simple but stark manner: the hybrid cycle-star networks. These networks include the pure cycle – reflecting no intermediation rents – on one end, and pure stars – reflecting extreme intermediation power – on the other end. Within the original model, the tension was resolved in favor of extreme polarization (i.e. stars). We claim that the same occurs as well for the present version where payoffs are given by (2).

To see this, consider any hybrid cycle-star network where there are m peripheral players connected to a single player i in a cycle, which consists of the remaining $n - m$ players. We now show that this network cannot be sustained at a SBE if $m < n$, i.e. whenever the network "falls short" of being a pure star. (Note that this statement covers the cycle as well, which arises when $m = 0$.) Let j be the player opposite to i in the cycle. Suppose that i and j are given the opportunity of forming a link. We claim that they will always form it, since they can profit from so doing if they also delete one link each and transform the network into a hybrid line-star network with i and j central. (Here, the deviation is akin to that illustrated in Figure 4 concerning the instability of the cycle in the original model.) This follows from the following observations:

1. Player i has the access benefits unchanged, i.e. it accesses the same number of players with correspondingly the same number of intermediate players as before.

2. Player j has weakly more access benefits than before: she accesses the same number of players as before and (if $m > 0$) the peripheral ones with fewer intermediaries (now, only player i).

3. Player i has more intermediation payoffs than before: she keeps matters unchanged concerning the connections between agents in the line and the peripheral players, but she earns more intermediation payoffs than before by connecting agents in the line (she is now relied upon for more bilateral connections).

4. Player j has the same intermediation payoffs as before concerning connections between individuals in the line but has additional such payoffs (whereas before she had none) concerning connection of some agents in the line and the peripheral players.

Since the number of links by players i and j do not change by the deviation, 1-4 implies that the bilateral deviation considered is jointly profitable. This shows that none of the networks considered other than the pure star is sustainable at a SBE. This suggests that in the alternative model proposed the same tendency towards polarization arises, and again materializes in an extreme fashion, within a parametrized family of networks i.e. the collection of hybrid cycle-star networks. Finally, we note that an analogous insight is gained if we focus our attention on minimally connected networks (i.e. trees). Circumscribed to these networks, the two payoff functions (Π_i and $\hat{\Pi}_i$, as given respectively by (1) and (2)) coincide, which implies that Point 1 in the discussion following Theorem 1 (i.e. Lemma 3) applies unchanged, i.e. among the minimally connected networks, all SBE are stars.

4.2 Capacity constraints

In the basic model we assumed that the marginal costs of linking between players are constant and in particular do not depend on the number of links a player forms. In some settings it seems more natural to suppose that the costs per link are increasing in the number of links. Or, analogously, we should expect that an individual player will be subject to some capacity constraints. In this section we briefly discuss the implications of such constraints on equilibrium networks.

Let us suppose that any individual can form a maximum of $K << n - 1$ links. This capacity constraint means that the agglomeration forces at work in Theorem 1 are now constrained. How does this affect the arguments underlying this result? Reconsidering for this case the logic organized in Points 1-5 above (cf. the discussion following the statement of Theorem 1, as well as the lemmas in the Appendix), we arrive at the following set of observations. First, note that the forces towards agglomeration spelled out in Point 1 (Lemma 3) still operate unchanged up to the point where capacity constraints are reached. Next, we observe that the arguments used in Points 2, 3 and 5 (lemmas 4, 5, and 7) do not involve players forming any additional links and therefore carry over to a setting with capacity constraints. This means that networks with two or more cycles, networks with a single cycle, or a hybrid cycle-star network cannot arise in equilibrium. Therefore, the only possibility that remains is a network with a cycle and some players outside this cycle. Of course, many different network architectures are still consistent with this description. But all of them must display significant agglomeration polarized at some players, as allowed by their capacity constraints. More specifically, one expects that equilibrium networks include a cycle composed of highly connected capacity-constrained individuals, who act as local hubs and are also the gates through which peripheral players (outside the cycle) access most of the population. In the end, this suggests that, even when players are capacity constrained, high centrality and significant payoff inequality still arise as important features of equilibrium networks.

4.3 Two-person coalition proof networks

In our analysis so far we have assumed that a deviation by two players is credible so long as it yields higher payoffs to both players. We have not looked at profitable deviations from the agreed upon deviation, thus ignoring considerations that are in the spirit of coalition proofness. In this section we will examine the implications of this further requirement, which of course can only enlarge the set of equilibria – in general, only a subset of bilateral deviations qualify as valid. We start with a definition of *bilateral proofness* in our context, focusing directly on the "strict" version of this concept.

Definition 4 *A strategy profile* $\mathbf{s}^*$ *is a* Strict Bilateral-Proof Equilibrium (SBPE) *if the following conditions hold:*

1. *for any* $i \in N$*, and every* $s_i \in S_i$ *such that* $g(s_i, \mathbf{s}^*_{-i}) \neq g(\mathbf{s}^*)$, $\Pi_i(\mathbf{s}^*) > \Pi_i(s_i, \mathbf{s}^*_{-i})$;
2. *For any pair of players* $i, j \in N$*, and every strategy pair* (s_i, s_j) *with* $g(s_i, s_j, \mathbf{s}^*_{-i-j}) \neq g(\mathbf{s}^*)$ *and* $\Pi_i(s_i, s_j, \mathbf{s}^*_{-i-j}) \geq \Pi_i(s^*_i, s^*_j, \mathbf{s}^*_{-i-j})$*, one of the following two conditions hold:*

(a) $\Pi_j(s_i, s_j, \mathbf{s}^*_{-i-j}) < \Pi_j(s^*_j, s^*_j, \mathbf{s}^*_{-i-j})$

(b) $\exists k, \ell \in \{i, j\}$, $k \neq \ell$, *and some* $\tilde{s}_k \in S_k$, *such that* $\Pi_k(\tilde{s}_k, s_\ell, \mathbf{s}^*_{-k-\ell}) > \Pi_k(s_k, s_\ell, \mathbf{s}^*_{-k-\ell})$.

Most of the insights obtained in Section 3 through the SBE concept are maintained if we consider the less demanding SBPE concept. Indeed, reviewing the Propositions and Lemmata that underlie the proof of Theorem 1, it is straightforward to check that all of them continue to apply for the SBPE concept, except for Lemmas 5 and 7 (corresponding to Points 3 and 5 in the discussion following Theorem 1). In particular, it is possible to show that a cycle and hybrid cycle-star networks can be sustained in strict bilateral proof equilibrium. The following result shows it for the cycle.

Proposition 3 *Given any $c > 0$, there exists an $\hat{n}(c)$ such that for all $n \geq \hat{n}(c)$, a cycle containing all players is a strict bilateral proof equilibrium.*[8]

The proof of this result is presented in the Appendix. It shows that the deviation by two distant players in a cycle which takes the cycle to a line is not credible since each of the players has an incentive to renege on the deviation and retain both their erstwhile links, under the assumption that the other player will delete one her links. This incentive is clarified in Figure 6. Recall that, in Theorem 1 (cf. Point 4 following its statement), a cycle was broken by a deviation in which, say, players 1 and 2 in Figures 6A and 6B move and create a line. The proof of Proposition 3 shows that this deviation is vulnerable to a further deviation in which player 1 retains both the links that she had. This deviation yields network 6C. By moving to this network, player 1 is able to circumvent a large number of essential players in the line. This deviation is worthwhile if there are enough players on the line going from 1 to 3. An analogous reasoning can be used to discard bilateral deviations from a hybrid cycle-star network: a joint deviation by two players in the cycle is not in the interest of any one of them, under the assumption that the other abides by it. In fact, it is not difficult to construct examples (see Remark 1 in the Appendix) where a hybrid cycle-star network is SBPE for a large enough population.

The above discussion indicates that, under the requirement of bilateral proofness, a richer set of network architectures can be supported at equilibrium – specifically, the cycle and a hybrid cycle-star network are also possible, in addition to full stars. Thus, in this sense, the force towards polarization that proved so acute under unqualified bilateral stability

[8] The stability of the cycle (which is not minimally connected) is an instance of the inefficiency discussed in Jackson (2003) where players over-connect to avoid positional disadvantages.

becomes mitigated when we insist that bilateral deviations be internally consistent. We now argue, however, that such alternative architectures (a cycle and a hybrid cycle-star) are, in effect, rather fragile. Specifically, they should not be expected to last under the pressure of occasional perturbations that affect some of the links. A natural way of formulating precisely this idea is to check the performance of some suitably modelled dynamics of adjustment under the pressure of occasional random decay of links. In what follows, we outline such a dynamic model and explain the gist of the argument. While we dispense with the formal details, they are available from the authors upon request.

Consider a dynamic process of bilateral adjustment in which, at every point in (discrete) time $t = 1, 2, ...,$ a pair of players is selected at random and are given the option to create a link between them (if this link is not already in place) and, simultaneously, destroy any subset of the existing links whose maintenance they control. In line with the considerations that underlie the SBPE concept, let us suppose that, in evaluating any joint move by the two players involved in the adjustment, only those that are internally consistent are admitted as valid. That is, joint moves are only judged valid if they are immune to a unilateral deviation (that would always involve keeping a link that should otherwise be maintained under the contemplated move). In addition to this adjustment dynamics, suppose that there is also a "slow" dynamics of link decay. Specifically, postulate that, at (the end of) every t, each of the existing links vanishes with some probability ε, conceived as small.

The dynamics just described may be parameterized by ε. If $\varepsilon = 0$, we simply have the pure adjustment dynamics, whose stationary points are SBPE networks – they include cycle, hybrid cycle-star, or star networks, together with the empty network if and only if $c > 1/2$. To check the robustness of these different architectures, suppose $\varepsilon > 0$ and consider what would happen if *one* of the existing links, randomly selected, simply vanishes. The implicit assumption here is that such a perturbation is very infrequent so that no more than one instance of it may occur with significant probability before the adjustment dynamics restores the stationarity of the situation. For concreteness, we focus our discussion on the cycle, since the arguments pertaining to a hybrid cycle-star network are very similar.

After a link in the cycle has vanished, a line network obtains. We argue that, thereafter, there is positive probability that the adjustment dynamics leads to a star, which is not only absorbing for this dynamics but robustly so. To show positive probability of convergence, it is enough to construct a finite admissible path of adjustment that leads to a star. The details are somewhat involved, and spelled in some detail in the Appendix (cf. Remark 2). The main idea, however, is simple, being again a reflection of the forces towards agglomeration that arise naturally in our model, even under the requirement of bilateral proofness. The convergent paths considered are different, depending on the cost of linking c. If $c < 1/2$, the process is illustrated in Figure 7. It consist of a series of adjustments by which extremal players gradually close their distance to the center of the line, eventually conforming a single star encompassing all players. In the alternative case where $c > 1/2$, those aforementioned paths no longer embody bilateral-proof adjustments and must be replaced by adjustments in which the lines stretching to either side of the central player become progressively shorter by second-neighbors of that central player establishing a direct connection. This process is illustrated in Figure 8.

The former considerations show the fragility of the cycle to the removal of a *single* link. Now we argue that the star is a much more robust architecture, in that no single perturbation can trigger an adjustment process away from it. Commence with the star and suppose that a link between the center, say player 1, and some other player i is deleted. If player i then receives an opportunity to link to player 1, both players want to form the link and the star is restored. So suppose that there is a linking opportunity between i and another peripheral player j. Then, both will want to form the link as well and this is a bilateral-proof adjustment. Of course, if player i next receives a linking opportunity to 1, this link will be formed and the previously established link between i and j deleted (since $c > 1/6$, which means that the original star is restored. Finally, consider the situation where i and some other peripheral player k receive a linking opportunity. It can be checked that these players will *not* form a link if $c > 5/12$.[9] Thus, in this case, no link between i and any other peripheral player (different from j) will be created. Eventually, i and the central player 1 will receive a linking opportunity, in which case the star will restored.

[9]If $c \leq 5/12$, the link between i and k would be formed and a hybrid cycle-line network would be reached that is nevertheless not stationary for the adjustment dynamics. In fact, based on considerations used in Step 4 of our main Theorem, it can be shown that the cost is so low that the cycle may expand steadily until it encompasses the whole population. In this case, therefore, one expects that, in the presence of perturbations, the network architecture should switch indefinitely between the cycle and the star.

The above discussion shows that a star is relatively more robust than a cycle. It is however silent on whether, and how, a non-empty network might be formed through gradual adjustment when the linking cost $c > 1/2$. In general, this would require that the perturbations/mutations also affect link creation, thus generating a sufficient "critical mass." A natural modelling option, for example, would be to posit that not only existing links are destroyed with some small (and say, independent) probability $\varepsilon > 0$, but that every non-existent link may arise under the same conditions. The adjustment process would then become ergodic, thus guaranteeing the existence of a unique invariant probability measure summarizing its long-run behavior. Following standard evolutionary literature, the aim would be to identify the so-called stochastically stable states that arise with significant long-run probability as $\varepsilon \downarrow 0$. We conjecture that in such a dynamic model stars would be uniquely stable for a range of cost values $1/6 < c < \bar{c}$.

5 Conclusion

This paper has studied a simple model of network formation where agents may exploit positional advantages if these provide them with the ability to block profitable bilateral interaction between two players who are not direct neighbors. We show that the strategic struggle for these advantages leads to a polarized star architecture where a single player becomes essential to connect every other pair of players. This represents a clear cut formalization of the notion found in the sociological literature that structural holes opens the potential for large benefits to those individuals who succeed in bridging them.

Appendix

The surplus bargaining game

Consider any pair of players, i and j, which may generate a unit of surplus (i.e. have at least a network path joining them). Given the prevailing network g, the considerations explained in the text induce a coalitional-form with transferrable utility given by the characteristic function $v : 2^N \to \mathbb{R}^n$ defined, for each $S \subset N$ as follows:

$$v(S) = \begin{cases} 1 & \text{if } \exists \{i_1, \ldots, i_n\} \subset S \text{ s.t. } g_{i,i_1} \cdot g_{i_1,i_2} \cdot g_{i_2,i_3} \cdots g_{i_n,j} = 1 \\ 0 & \text{otherwise} \end{cases} . \tag{3}$$

Given any imputation $z \in \mathbb{R}^n$ define the *excess payoff* that can be earned by any *coalition* S by:

$$U(S,z) = v(S) - \sum_{i \in S} z_i$$

and the *excess payoff* that can be earned by some player k *against* other player l by:

$$u_{k\ell}(z) = \max\{U(S,z) : S \subset N,\ k \in S,\ \ell \notin S\}.$$

Then the Kernel of the game induced by the network g (which, in this case, also happens to be in the Core[10]) is defined as the set of imputations $\hat{z}$ that satisfy, $\forall k, \ell \in N$, one of the following two conditions:

$$u_{k\ell}(\hat{z}) \geq u_{\ell k}(\hat{z}); \tag{4}$$

$$u_{k\ell}(\hat{z}) < u_{\ell k}(\hat{z}) \Rightarrow \hat{z}_k = v(\{k\}) = 0. \tag{5}$$

The intuitive basis for (4)-(5) is the idea that each player evaluates any imputation $\hat{z}$ contemplated throughout the bargaining process on the basis of the induced excess payoffs $u_{k\ell}(\hat{z})$. Thus, if there is a bilateral "imbalance" in these magnitudes for some pair of players, this is sure to trigger a "reasonable objection" from the unfavored party (i.e. the agent with the higher excess) unless the other is already at her minimum (individually rational) payoff.

We now argue that, given the characteristic function (3) associated to a particular pair of players i and j, the induced Kernel consists of the *unique* imputation vector $\hat{z}$ satisfying:

$$\hat{z}_k = \begin{cases} \frac{1}{e(i,j)+2} & \text{if } k \in E(i,j) \cup \{i,j\} \\ 0 & \text{otherwise.} \end{cases}$$

thus inducing the payoff function specified in (1). This conclusion follows from the following two claims.

Claim 2 *Consider any pair of players $k, \ell \in E(i,j) \cup \{i,j\}$. Then, any Kernel imputation $\hat{z}$ satisfies $\hat{z}_k = \hat{z}_\ell$.*

Claim 3 *Consider any player $k \notin E(i,j) \cup \{i,j\}$. Then, any Kernel imputation $\hat{z}$ satisfies $\hat{z}_k = 0$.*

[10] It is easy to check that the Core of the game associated to the surplus generated by i and j consists of *all* those imputations where inessential players receive a null share.

To show Claim 2, suppose that $k, \ell \in E(i,j) \cup \{i,j\}$ but $\hat{z}_k > \hat{z}_\ell$. Given that for any $S \subset N\backslash\{k\}$ and for any $S' \subset N\backslash\{\ell\}$, we have $v(S) = v(S') = 0$. Therefore,

$$\begin{aligned} u_{k\ell}(\hat{z}) &= \max\{U(S,\hat{z}) : S \subset N,\ k \in S,\ \ell \notin S\} = U(\{k\},\hat{z}) = -\hat{z}_k \\ &< -\hat{z}_\ell = U(\{\ell\},\hat{z}) \leq \max\{U(S,\hat{z}) : S \subset N,\ k\ell \in S,\ k \notin S\} = U_{\ell k}(\hat{z}). \end{aligned}$$

By virtue of (5), this requires that $\hat{z}_k = 0$, which is a contradiction with the fact that $\hat{z}_\ell \geq 0$.

Next, to establish Claim 3, suppose that $k \notin E(i,j) \cup \{i,j\}$ but $\hat{z}_k > 0$. Consider then some other $\ell \in E(i,j) \cup \{i,j\}$. Again, since $v(S) = 0$ for any $S \subset N\backslash\{\ell\}$, we have that

$$u_{k\ell}(\hat{z}) = -\hat{z}_k < 0.$$

Considering now the reciprocal excess payoff $u_{\ell k}(\hat{z})$, note that $S_0 \equiv E(i,j) \cup \{i,j\} \subset N\backslash\{k\}$ so that $v(S_0) = 1$ and, therefore,

$$\begin{aligned} u_{\ell k}(\hat{z}) &= \max\{u(S,z) : S \subset N,\ k \in S,\ \ell \notin S\} \\ &\geq u(S_0,z) \geq 1 - \sum_{u \neq k} \hat{z}_u = \hat{z}_k > 0. \end{aligned}$$

Therefore, (5) requires that $\hat{z}_k = 0$, a contradiction. ■

Proof of Proposition 1: The proof of the result requires two preliminary lemmas. The first one concerns critical links, i.e. links that define the only path between the two end players (and whose deletion, therefore, would increase the number of components). It establishes that the marginal payoff of any such critical link is equal for the two players involved. The second lemma shows that in any component of a network, there are at least two non-essential players, i.e. players who are not essential for any interaction (and therefore enjoy only access payoffs).

Lemma 1 *Consider any network g. If $g_{i,j} = 1$ and the link is critical then $M_i(g_{i,j}; g) = M_j(g_{i,j}; g)$.*

Proof: We show that the number By hypothesis $g_{i,j}$ is critical, and so it follows that i and j lie in different components in the network $g - g_{i,j}$. Let $C_i(g)$, $C_j(g)$ be the components that contain i and j, respectively, where we shall usually dispense with an explicit account of the dependence and simply write C_i and C_j. The marginal payoff of the link ij for player i, is given by:

$$
\begin{aligned}
&M_i(g_{i,j};g) \\
&= \frac{1}{2} + \sum_{k\in C_j\setminus\{j\}} \frac{1}{e(i,k)+2} + \sum_{l\in C_i\setminus\{i\}} \sum_{k\in C_j\setminus\{j\}} \frac{1}{e(l,k)+2} + \sum_{l\in C_i\setminus\{i\}} \frac{1}{e(l,j)+2} - c
\end{aligned}
$$

where the first two terms refer to access benefits while the latter two terms refer to essentiality benefits. Similarly, we can write the marginal payoffs of player j form link $g_{i,j}$ as:

$$
\begin{aligned}
&M_j(g_{i,j};g) \\
&= \frac{1}{2} + \sum_{l\in C_i\setminus\{i\}} \frac{1}{e(j,l)+2} + \sum_{l\in C_i\setminus\{i\}} \sum_{k\in C_j\setminus\{j\}} \frac{1}{e(l,k)+2} + \sum_{k\in C_j\setminus\{j\}} \frac{1}{e(i,k)+2} - c
\end{aligned}
$$

It follows then that $M_i(g_{i,j};g) = M_j(g_{i,j};g)$ and the proof of the Lemma is complete.

Lemma 2 *In a network g, a component with m players has at least 2 non-essential players.*

Proof: Consider any arbitrary component of the network with m nodes. First, we note that from any arbitrary connected network one can reach a line network by a series of steps that involve only two operations: (i) removal of links; (ii) reconnection of existing links to extremal players (i.e. players with only one link). It is easy to see that any such operation can (weakly) increase the number of essential players. Since in the line network the two extremal players are non-essential, it follows that the maximum number of essential players in the original component can be no higher than $m-2$. Thus any component in a network with m players must have at least 2 non-essential players.

Equipped with Lemmas 1 and 2, we proceed with the proof of the proposition. Let g be a non-empty BE network, and suppose it is not connected. Let $\hat{C}$ be the largest component in g, which must therefore contain at least 2 players. We claim that there is a player $j \notin \hat{C}$ that can establish a mutually profitable link with some player in $\hat{C}$. For simplicity, we shall consider the extreme (and less favorable) case where j has no connections, i.e. defines a singleton component.

By Lemma 2, we know that $\hat{C}$ has some non-essential player $i \in \hat{C}$. Two possibilities need to be considered separately. One is that i is extremal, i.e. she has only one link that connects her to some other player ℓ in the component. Then, it is clear that, since i and ℓ both find it profitable to keep their link, player ℓ would find it optimal to create a link with j if given the opportunity, and so would player j. This contradicts that the network g may be a BE network.

The second possibility is that i is non-extremal and therefore $\eta_i \geq 2$. Let $\mathcal{N}_i^m(g) = |\{j \in C_i : e(i,j) = m)\}|$ be the players whom i accesses via m essential players and let $\eta_i^m(g) = |\mathcal{N}_i^m(g)|$. The payoffs of this player i in network g are then given by:

$$\frac{\eta_i^0(g)}{2} + \frac{\eta_i^1(g)}{3} + \frac{\eta_i^2(g)}{4} + ... + \frac{\eta_i^r(g)}{r+2} - \eta_i(g)c \tag{6}$$

for some $r \leq n-2$.

Since g is an equilibrium, it follows then

$$\frac{1}{\eta_i(g)}\left[\frac{\eta_i^0(g)}{2} + \frac{\eta_i^1(g)}{3} + \frac{\eta_i^2(g)}{4} + ... + \frac{\eta_i^r(g)}{r+2}\right] \geq c. \tag{7}$$

Now let us examine marginal returns for player $j \notin \hat{C}$ from a link with player i. Suppose, for simplicity, that player j is a singleton component. Then the *marginal returns* to j are given as follows:

$$\frac{\eta_j^0(g+g_{i,j})}{2} + \frac{\eta_j^1(g+g_{i,j})}{3} + \frac{\eta_i^2(g+g_{i,j})}{4} + ... + \frac{\eta_i^r(g+g_{i,j})}{r+2} - c, \tag{8}$$

where, by analogy with previous notation, $g + g_{i,j}$ simply denotes the network obtained by replacing $g_{i,j}$ in network g by a new $g_{i,j} = 1$.

Note now that for every $k \in \hat{C}\backslash\{i\}$, $e(j,k;g+g_{i,j}) = e(i,k;g)+1$. Thus $\eta_j^m(g+g_{i,j}) = \eta_i^{m-1}(g)$ for every $m \geq 1$ and $\eta_j^0(g+g_{i,j}) = 1$. Using these facts we can write the marginal returns of player j from the link with i as follows:

$$\frac{1}{2} + \frac{\eta_i^0(g)}{3} + \frac{\eta_i^1(g)}{4} + ... + \frac{\eta_i^r(g)}{r+3} - c. \tag{9}$$

Now we argue that

$$
\begin{aligned}
&\frac{1}{2}+\frac{\eta_i^0(g)}{3}+\frac{\eta_i^1(g)}{4}+....+\frac{\eta_i^r(g)}{r+3} \\
&\quad > \quad \frac{1}{2}\left[\frac{\eta_i^0(g)}{2}+\frac{\eta_i^1(g)}{3}+\frac{\eta_i^2(g)}{4}+....+\frac{\eta_i^r(g)}{r+2}\right] \\
&\quad \geq \quad \frac{1}{\eta_i(g)}\left[\frac{\eta_i^0(g)}{2}+\frac{\eta_i^1(g)}{3}+\frac{\eta_i^2(g)}{4}+....+\frac{\eta_i^r(g)}{r+2}\right] \\
&\geq c
\end{aligned}
$$

The first inequality is immediate, while we use $\eta_i(g) \geq 2$ in deriving the second inequality and equation (7), in deriving the final inequality. We now apply Lemma 1 to conclude that player i also has a strict incentive to form a link with j, given that all existing links are retained. But note that given that link $g_{i,j}$ is formed, player i has no incentive to delete any of his erstwhile links since (roughly speaking) the marginal returns from each of these links has actually increased. Thus players i and j have a strict incentive to form an additional link. These arguments extend directly to cover the case where j belongs to a non-singleton component. Thus g is not a BE network, a contradiction that completes the proof. ■

Proof of Theorem 1: As indicated in the text, the proof can be decomposed into five steps. In what follows, each of these steps is formally embodied by a corresponding lemma. All of them assume that $c > 1/6$ and $n \geq \hat{n}(c)$ for some suitable $\hat{n}(c)$.

Lemma 3 *The star is the only minimal network which can be sustained in a SBE.*

Proof: Consider a g that is minimally connected but not a star. Then, there are two players, say i and j, such that (a) $E(i,j;g) \geq 2$; (b) they are "end players," i.e. $\eta_i(g) = \eta_j(g) = 1$. Then, it readily follows that for at least one of them, say i, the following holds: There is a player x with $e(i,x;g) = 1$ (thus, x is two steps away from i in g) and the set of players k for whom x is essential in connecting i and k has a cardinality that is at least $(n-4)/2$, i.e. $|\{k \in N : x \in E(i,k;g)| \geq (n-4)/2$.

We argue that such a network g cannot be induced by an equilibrium. First note that, given the linking cost c, the number of essential players that can be supported in equilibrium between any two players has some (finite) upper bound $\hat{e}(c)$, independent of n. Consider then the possibility that i and x were to form a link. The gross gains, $\Delta\pi_i$ and

$\Delta\pi_x$, induced by that change for i and x (if all other links were to remain in place) is bounded below as follows:

$$\min\{\Delta\pi_i, \Delta\pi_x\} \geq \frac{(n-4)/2}{\hat{e}(c)-1} - \frac{(n-4)/2}{\hat{e}(c)} = \frac{n-4}{2\hat{e}(c)(\hat{e}(c)-1)}.$$

This expression is larger than c if n is large enough, which implies that both i and x benefit from a deviation that creates a link between them and keeps all other links.

Lemma 4 *There can be at most one cycle in a SBE network.*

Proof: Suppose g is an equilibrium network and there are two or more cycles in it. Let $\chi_1 = (i_1, i_2, ...i_n)$ be ordered set of players in one cycle, and let $\chi_2 = (j_1, j_2, ...j_m)$ be those in the other cycle. Since g is connected it follows that there are two possibilities: (1) cycles have common players and (2) cycles have no common players. We take these up in turn.
(1). Cycles have players in common: If there is a single common player i_1 in the two cycles then it is easy to see that the partners of i_1 (say) $i_2 \in \chi_1$ and $j_2 \in \chi_2$ have a strict incentive to delete their links with i_1 and instead form a link with each other. (Throughout, we shall abuse notation and write $i \in \chi$ if i is one of the nodes in the ordered collection of nodes specifying χ.) Consider next the case with two or more players in common. Let $(i_1, i_2, ...i_k)$ be the players in common. Suppose that $k \geq 3$; the case of $k = 2$ is simple and omitted. Then there exist players i_1, i_x, and j_y with the following properties: $i_x \in \chi_1$ but $i_x \notin \chi_2$, while $j_y \in \chi_2$ and $j_y \notin \chi_1$ and $g_{i_1,i_x} = g_{i_1,j_y} = g_{i_1,i_2} = ... = g_{i_{k-1},i_k} = 1$. Note also that like player i_1, i_k must again have links with a player who belongs to one of χ_1 and χ_2 only. It then follows that i_{k-1} and j_y have at least a weak incentive to delete their current link with i_k and i_1 respectively and instead form a link with each other. It then follows that g cannot be sustained by a SBE.
(2). Cycles have no common players. Since g is a SBE network it is connected and so there exists a path between the two cycles. Let $(i_1, i_2, ..., i_k)$ be members of such a path with $i_1 \in \chi_1$ while $i_k \in \chi_2$. Suppose $g_{i_1,i_x} = 1$ and $g_{i_k,j_y} = 1$, where $i_x \in \chi_1$ and $j_y \in \chi_2$. Now it is easy to use a variant of the earlier argument for case 1 above to show that players i_x and j_y have a strict incentive to delete their link with i_1 and i_k and instead form a link with each other. The proof is complete. ■

Lemma 5 *A cycle containing all players cannot be sustained in a BE.*[11]

Proof: Consider two player i and j who are furthest apart in terms of geodesic distance in the cycle. Now consider the deviation in which each of the players deletes one link and they form a link with each other in such a way that they create a line. Assume, for simplicity, that n is even, so that there are $(n-2)/2$ players to one side of player i and $(n-2)/2$ players to the other side of player j in the line created. We now show that players i and j will strictly increase their payoff with this coordinated deviation.
We proceed in two steps: the first step is to show that individual payoffs are strictly increasing as we move toward the center of the line. The payoffs of an individual player consist of two components, the returns from accessing others and the returns from being essential on paths between pairs of other players. Number the players on a line as 1, 2, ...n. The access returns to player l are given by

$$\frac{1}{l}+\frac{1}{l-1}+\ ...\ +\frac{1}{2}+\frac{1}{2}+\ ...+\ \frac{1}{n-l+1} \tag{10}$$

while the access returns to player $l+1$ are given by

$$\frac{1}{l+1}+\frac{1}{l}+\ ...\ \frac{1}{2}+\frac{1}{2}+\ ...+\ \frac{1}{n-l} \tag{11}$$

It now follows that access returns for player $l+1$ are larger than access returns for player l if $l<n/2$.
We now turn to the returns from being essential. The essentialness payoff to player l can be written as follows:

$$\sum_{i=1}^{l-1}\sum_{j=l+2}^{n}\frac{1}{e(i,j)+2}+\sum_{i=1}^{l-1}\frac{1}{e(i,l+1)+2} \tag{12}$$

Similarly, the essentialness payoffs to player $l+1$ can be written as follows:

$$\sum_{i=1}^{l-1}\sum_{j=l+2}^{n}\frac{1}{e(i,j)+2}+\sum_{j=l+2}^{n}\frac{1}{e(l,j)+2} \tag{13}$$

The first part of the essentialness payoffs to the two players are equal, while the second part of the payoffs are greater for player $l+1$ if $l<n/2$.

[11]Here, we need that $\hat{n}(c)\geq 4$, since for $n=3$ a complete network can be sustained in equilibrium for $c<1/6$.

Let g^C and g^L denote the networks prevailing before the contemplated deviation and after it (i.e. the cycle and the line with i and j at the center, respectively). To show that i and j indeed obtain higher payoffs under g^L, note that the aggregate gross payoffs obtained in both cases are the same. The above argument implies that i and j enjoy a higher share of total gross value in the line as compared to the other players. This implies that players i and j earn a higher gross payoff in the line. Since their linking cost is the same in both cases (i.e. $2c$), it follows that they obtain higher net payoffs as well, which completes the proof. ■

Lemma 6 *A SBE network with a cycle has the hybrid star-cycle architecture, for large enough n.*

Proof: Suppose that g is a non-empty SBE network with a cycle. Let X be the set of players and x the number of players who are outside the cycle, while Y is the set of players in cycle and define $y = n - x$ to be number of players in cycle. Note that $y \geq 4$ since $y = 3$ cannot be sustained in equilibrium given that $c > 1/6$. Next we argue that, for any such c, there is a $y(c)$ such that $y \leq y(c)$ in equilibrium. Given step 2 above, clearly $x \geq 1$. Suppose $g_{i,j} = 1$ for some $i \in X$ and $j \in Y$. Since $j \in Y$, there is some $k \in Y$ such that $g_{j,k} = 1$. Clearly, player k prefers strictly to switch link from j to i. This reduces the essentialness payoffs he has to pay out to j and keeps his costs constant. On the other hand, i benefits from such an adjustment if $(y-1)/6 > c$. So given a c, in order for the cycle to be stable, we must have $y \leq y(c) \equiv 6c + 1$.
Next we show that all nodes that do not belong to the cycle must be connected to a *single* node in the cycle. The initial observation is that, if n is large enough, there cannot be two distinct trees with different roots lying in the cycle. To see this, consider one of those subtrees that has a number of nodes at least equal to $[n - y(c)]/y(c) = (n/y(c)) - 1$. At least one such tree must exist since there are at most $y(c)$ nodes in the cycle. Let i_1 be the root of this tree and i_2 be the root of any other cycle. Now consider any node j in the second tree different from its root, i_2. A straightforward adaptation of the arguments used in the proof of Lemma 3 lead to the conclusion that, if n is large, j and i_1 can both profit from forming a link, whether or not j maintains its link with i_2. The reason is that, since the number of essential players $e(\ell, \ell')$ for any pair of players, ℓ and ℓ', is bounded by some $\hat{e}(c)$, independently of n, the gross gains from the link between j and i_1 grow

linearly with n for both players, independently of whether player j maintains the link to i_2.

Once established that there can be at most one tree connected to the cycle through its root, again relying on the arguments introduced in Lemma 3 we arrive at the conclusion that, for any two nodes in the tree, u and v, the number of essential players $e(u,v) \leq 1$. This still leaves open the possibility that the tree consists of a star with a center at some node i_c that is not part of the cycle but has a link to a node in the cycle. But, in that case, the node i_c and any of the neighbors of k in the cycle, say k', both profit from establishing a direct link, for large enough n. ■

Lemma 7 *The star is the unique hybrid-cycle equilibrium.*

Proof: As argued in the previous step, $y \leq y(c)$ for a function $y(c)$ that is independent of n. Thus, fixing some c, consider the class of hybrid networks g in which $y \leq y(c)$. Let i be the player in the center of the star and suppose that $j,k \in Y$ with $g_{i,j} = g_{i,k} = 1$. We now show that j and k have a strict incentive to form a link if number of peripheral players $x \geq n - y(c)$ is sufficiently large. The payoffs of players j and k in the hybrid network g are given by

$$\pi_j(g^H) = \pi_k(g^H) = \frac{x}{3} + \frac{y-1}{2} - 2c \tag{14}$$

Now consider a deviation by players j and k in which player k deletes his link with player i and player j deletes his link with player m in the cycle and instead players j and k form a link with each other. The resulting network is a minimal network g' in which there are x peripheral players and a line starting with player i which consists of y players. The payoffs of player j in g' are given by

$$\pi_j(g') = \frac{x}{3} + \frac{1}{2} + \sum_{k=2}^{y-1} \frac{1}{k} + x \sum_{k=4}^{y+1} \frac{1}{k} + \sum_{k=3}^{y} \frac{1}{k} - 2c \tag{15}$$

These payoffs are bounded below by

$$\frac{x}{3} + x\frac{y-2}{y+1} - 2c = M \tag{16}$$

Next note that $M > x/3 + (y-1)/2 - 2c$ if

$$x > \frac{y-1}{2}\frac{y+1}{y-2} \tag{17}$$

Since $y \geq 4$, the right hand side is increasing in y and bounded above by $[y(c) - 1][y(c)+1]/2[y(c)-2]$. The final step is to note that $x = n - y \geq n - y(c)$ and so (17) applies for sufficiently large n. Thus player j has a strict incentive to switch links to player k for large n. We now turn to the incentives of player k.

The payoffs of player k in g' are given by

$$\pi_k(g') = \frac{x}{4} + \frac{1}{3} + \frac{1}{2} + \sum_{k=2}^{y-2} \frac{1}{k} + x \sum_{k=5}^{y+1} \frac{1}{k} + \sum_{k=4}^{y} \frac{1}{k} - 2c \tag{18}$$

These payoffs are bounded below by

$$\frac{x}{4} + x\frac{y-3}{y+1} - 2c = M' \tag{19}$$

Note that $M' > x/3 + (y-1)/2 - 2c$ if

$$x :>: \frac{6(y-1)(y+1)}{11y - 37} \tag{20}$$

Since $y \geq 4$, the right hand side is positive and increasing in y and so is bounded above by $6[y(c)-1][y(c)+1]/[11y(c)-37]$. Note that $x = n - y \geq n - y(c)$ is larger than this term for sufficiently large n. Thus player k has a strict incentive to switch links to player j, for sufficiently large n.

Combining Lemmas 3-7 the proof of Theorem 1 is complete. ■

Proof of Proposition 3: In a cycle, all players are symmetrically located, so we need only consider the incentives for a typical player, say some player i. First note that the payoffs of player i in a cycle are $(n-1)/2 - 2c$. The payoffs from deleting both links are 0 and clearly for a given c, there is always a large enough n such that deleting both links is not optimal. Consider next the deviation of deleting one link. If player i deletes one link and keeps the other link as in the cycle then he becomes an end-player of the line network. In this network his payoffs is given by

$$\sum_{k=2}^{n} \frac{1}{k} - c \tag{21}$$

The payoff from both links in cycle are higher if

$$\frac{n-1}{2} - \sum_{k=2}^{n} \frac{1}{k} > c \tag{22}$$

Clearly this inequality holds for large enough n. We next consider the deviation in which player i deletes both links but forms a new link (say) with the center of the line network that arises. Again, a variation of the above argument shows that for large n this reduces payoffs.

We finally consider deviations by player i which involve coordinated deviation with one other player. Suppose that in the cycle he has a link with $i-1$ and $i+1$. In the deviation, he maintains the link with $i-1$ but deletes the link with $i+1$ and instead forms a link with some player k. If player k retains all his links as in the cycle, it is easy to see that the payoffs of the player go down strictly since the costs remain the same (he maintains two links) while the gross payoffs decline since player k is essential for accessing at least one player, namely $i+1$. So this deviation is not profitable. If player k deletes both his links then the payoffs from the deviation are still lower and so it clearly not profitable. We turn to the final case, where players i and k coordinate and link up with each other but delete a link each so that the new network is a line. From Lemma 5 it follows that players such a deviation is profitable. However, now we need to check whether this deviation is credible: do the players have an incentive to actually delete one of their links? We show that for large n, at least on the players has a strict incentive to retain their links in the cycle. Number the players in the line as $1, 2...., n$, from left to right. So there are $i-1$ players to the left of player i while there are $n-k$ players to the right of player k in the line. Suppose without loss of generality that $i-1 \geq n-k$. Player k gets a payoff $\sum_{l=2}^{i+1} 1/l$ in the line network from these players $1, 2..., i$. The payoffs can be increased to $i/2$ if player k forms a link with player 1. Clearly, it is profitable for player k to deviate from the deviation and form a link with player 1, if n is large. It is similarly possible to show that player 1 would have an incentive to form a link with player k, if n is large. Thus the deviation in which i and k deviate to create a line is not two-person coalition proof. Finally, we note that starting from a cycle it is not profitable for player i to form any additional links. The proof for cycle being a SBPE network is complete. ■

Remark 1 ***Hybrid cycle-star networks are SBPE***

Consider a hybrid cycle-star network with x nodes out of the cycle and y nodes in the cycle. We know from the Lemma 6 that, in order for such a configuration to be an equilibrium, it must be that $y \leq 6c+1$. Under the latter inequality, it is clear that no (bilateral) deviation involving a player outside the cycle can be profitable for this player.

Thus, we are left with deviations involving pairs of non-neighboring players in the cycle, as those considered in Lemma 7. Let i and j be any such players. The only way in which they can gain is by attempting to become "central." This must involve establishing a link between them and, simultaneously, destroying one of their respective links to the cycle. Now we argue that any such deviation is not immune to a unilateral omission by at least one of the players in her link deletion. By keeping her link, player i can access one half of the players in the cycle without any intermediation costs. Thus, for large y the gain $\Delta\Pi_i$ from retaining the link (assuming j abides by the deviation) can be approximated as follows:

$$\Delta\Pi_i \simeq \frac{1}{2}\frac{y}{2} - \ln\frac{y}{2} - c.$$

Now suppose $y = c/\alpha$ for some $\alpha \in (1/6, 1/4)$. Then $y < 6c + 1$ but $\Delta\Pi_i > 0$ for large enough c and y. Similarly, it is possible to show that a member of the cycle has no incentives to delete links with cycle members and switch to a link with the center of star. These computations put together show that there is a SBPE that supports a hybrid cycle-star network. ■

Remark 2 ***The cycle is not dynamically robust***

Starting from a line network we claim that there is positive probability that the adjustment dynamics converges to a star network. We need to consider separately two cases: $c < 1/2$ and $c \geq 1/2$.

Case 1: $c < 1/2$**:** After the deletion of the link, the resulting network is a line, so let players be indexed consecutively from one end to the other, $i = 1, 2, ..., n$, with n odd for simplicity. Consider the following inductive argument. Assume that, at some stage of the process, the network has, on the one hand, the players $1, 2, ..., m-1$ ($m \leq (n+1)/2$) displaying a *single* link that connects them to player m and, on the other hand, the players $r+1, ..., n$ ($r \geq (n+1)/2$) with a *single* link connecting them to player r. Furthermore, suppose that players $j = m, m+1, ..., r$ are all connected through a single path. We claim that, with positive probability, a suitable chain of bilateral adjustments may bring the network to a situation such as the starting one with the indices of players m and r replaced, respectively, by $m' = m+1$ and $r' = r-1$. This inductive procedure stops when $m = r$, in which case a star network already prevails.

To see this, suppose first that all players $i = 1, 2, ...m-1$ receive in turn a bilateral revision opportunity with player $m+1$. We argue that, if the population is large, a new link between i and $m+1$ will be formed, irrespectively of whether or not these players want to delete as well some of their other links. The reason is that, keeping their current links, the fresh link to be created generates an additional benefit to the two players involved (i and $m+1$) that can be bounded below by a number arbitrarily close to $1/2$ when n is large. Specifically, the gross marginal benefit $\Delta\Pi$ for either player of connecting directly (rather than through player m) is bounded as follows:

$$\begin{aligned} \Delta\Pi &\geq (\frac{1}{2}-\frac{1}{3})+(\frac{1}{3}-\frac{1}{4})+(\frac{1}{4}-\frac{1}{5})+...+(\frac{1}{n-m+1}-\frac{1}{n-m+2}) \\ &= \frac{1}{2}-\frac{1}{n-m+2} \\ &\geq \frac{1}{2}-\frac{2}{n+3} \end{aligned}$$

which is indeed larger than c ($< 1/2$) for large enough n. This implies that by creating (and paying for) a link, players i and $m+1$ can benefit, even if they hold fixed all other links they currently support. If any one or them were to simultaneously destroy some of their other links, the marginal profitability of the link in question can only increase. Thus, if i deletes her (only) link to m, then $m+1$ becomes the unique (and therefore "most valuable") partner of i, whereas if $m+1$ denotes any subset of her other links, the marginal gains of the link can only increase as well. In sum, the adjustment is bilateral-proof.

After this round of bilateral revisions is completed, there is a link between every $i = 1, 2, ..., m-1$ and $m+1$. But this implies that none of the links between i and m are profitable for the latter player if $c > 1/6$. Thus, they will be removed if a further round of revision is given to the player m (say, as part of a bilateral adjustment between this player and $m+1$). At this point, we arrive at a network where all players $i = 1, 2, ..., m$ have their links as desired. But, clearly, the same considerations may be applied to players $i = r, r+1, ..., n$, which establishes the inductive argument.

Case 2: $c \geq 1/2$**:** Again, after the deletion of the link, the resulting network is a line, so we continue to index players consecutively from one end to the other, $i = 1, 2, ..., n$, with n odd for simplicity. Suppose that player $\frac{n-3}{2}$ is given the option of linking to the central player in the line, $\frac{n+1}{2}$ and deleting her link to player $\frac{n-1}{2}$. Both players strictly benefit from this adjustment if the population is large enough. Concerning player $\frac{n-3}{2}$, this happens because the payoffs she obtains from players $\frac{n-1}{2}$ and $\frac{n+1}{2}$ are simply permuted, whereas in the modified network the number of essential players that $\frac{n-3}{2}$ requires to access each $i = \frac{n+3}{2}, \frac{n+5}{2}, ..., n$ is one less. On the other hand, to see that player $\frac{n+1}{2}$ also benefits strictly from the adjustment note that her gain in *gross* payoffs $\Delta\Pi$ is given by:

$$\begin{aligned}
\Delta\Pi &= \left[(\frac{1}{2}-\frac{1}{3})+(\frac{1}{3}-\frac{1}{4})+...+(\frac{1}{\frac{n-1}{2}-1}-\frac{1}{\frac{n-1}{2}})\right] \\
&\quad +\left[(\frac{1}{2+1}-\frac{1}{3+1})+(\frac{1}{3+1}-\frac{1}{4+1})+...+(\frac{1}{\frac{n-1}{2}-1+1}-\frac{1}{\frac{n-1}{2}+1})\right]+... \\
&\quad +\left[(\frac{1}{2+\frac{n-1}{2}}-\frac{1}{3+\frac{n-1}{2}})+(\frac{1}{3+\frac{n-1}{2}}-\frac{1}{4})+...+(\frac{1}{\frac{n-1}{2}-1+\frac{n-1}{2}}-\frac{1}{\frac{n-1}{2}+\frac{n-1}{2}})\right] \\
&= \left[\frac{1}{2}-\frac{1}{\frac{n-1}{2}}\right]+\left[\frac{1}{2+1}-\frac{1}{\frac{n-1}{2}+1}\right]+\left[\frac{1}{2+\frac{n-1}{2}}-\frac{1}{\frac{n-1}{2}+\frac{n-1}{2}}\right] \\
&= \sum_{r=0}^{\frac{n}{2}}\left[\frac{1}{2+r}-\frac{1}{\frac{n-1}{2}+r}\right].
\end{aligned}$$

As n grows, the asymptotic behavior of the above expression can be approximated by

$$\int_0^{\frac{n}{2}}\left[\frac{1}{2+r}-\frac{1}{\frac{n-1}{2}+r}\right]\,dr = \ln(2+\frac{n-1}{2})$$

which increases unboundedly as $n \uparrow \infty$. The net gains of the contemplated adjustment for player $\frac{n+1}{2}$ consist of the gross gain $\Delta\Pi$ net of the cost c for the additional link she supports with $\frac{n-3}{2}$. It follows, therefore, that, given any c, the net gains $\Delta\Pi - c > 0$ as long as n is large enough.

However, in connection to the previous bilateral adjustment by players $\frac{n-3}{2}$ and $\frac{n+1}{2}$, the question remains as to whether it is *bilateral proof.* Clearly, no deviation from such an adjustment can be optimal for player $\frac{n+1}{2}$. On the other hand, for player $\frac{n-3}{2}$, it is still conceivable that she might want to deviate and maintain the link to $\frac{n-1}{2}$, since this saves the need to pay intermediation costs to player $\frac{n+1}{2}$ when player $\frac{n-1}{2}$ accesses $\frac{n-3}{2}$ and all players $i < \frac{n-3}{2}$. The gross gain derived from such a deviation would be:

$$\begin{aligned}
\Delta\Pi &= (\frac{1}{2}-\frac{1}{3})+(\frac{1}{3}-\frac{1}{4})+(\frac{1}{4}-\frac{1}{5})+...+(\frac{1}{\frac{n-1}{2}}-\frac{1}{\frac{1}{n+1}}) \\
&= \frac{1}{2}-\frac{2}{n+1},
\end{aligned}$$

which is always lower than c ($\geq 1/2$). Thus, such a deviation is not profitable and, consequently, the contemplated adjustment is bilateral-proof. Now we may proceed iteratively with players that are two steps apart from the central player in the current network (e.g. either player $\frac{n-5}{2}$ or $\frac{n+3}{2}$ at the next iteration). It is immediate to see that none of the former considerations concerning the profitability and bilateral-proofness of the adjustment until a star centered at player $\frac{n+1}{2}$ is finally reached. At this point, no further adjustment possibilities will change the network. ■

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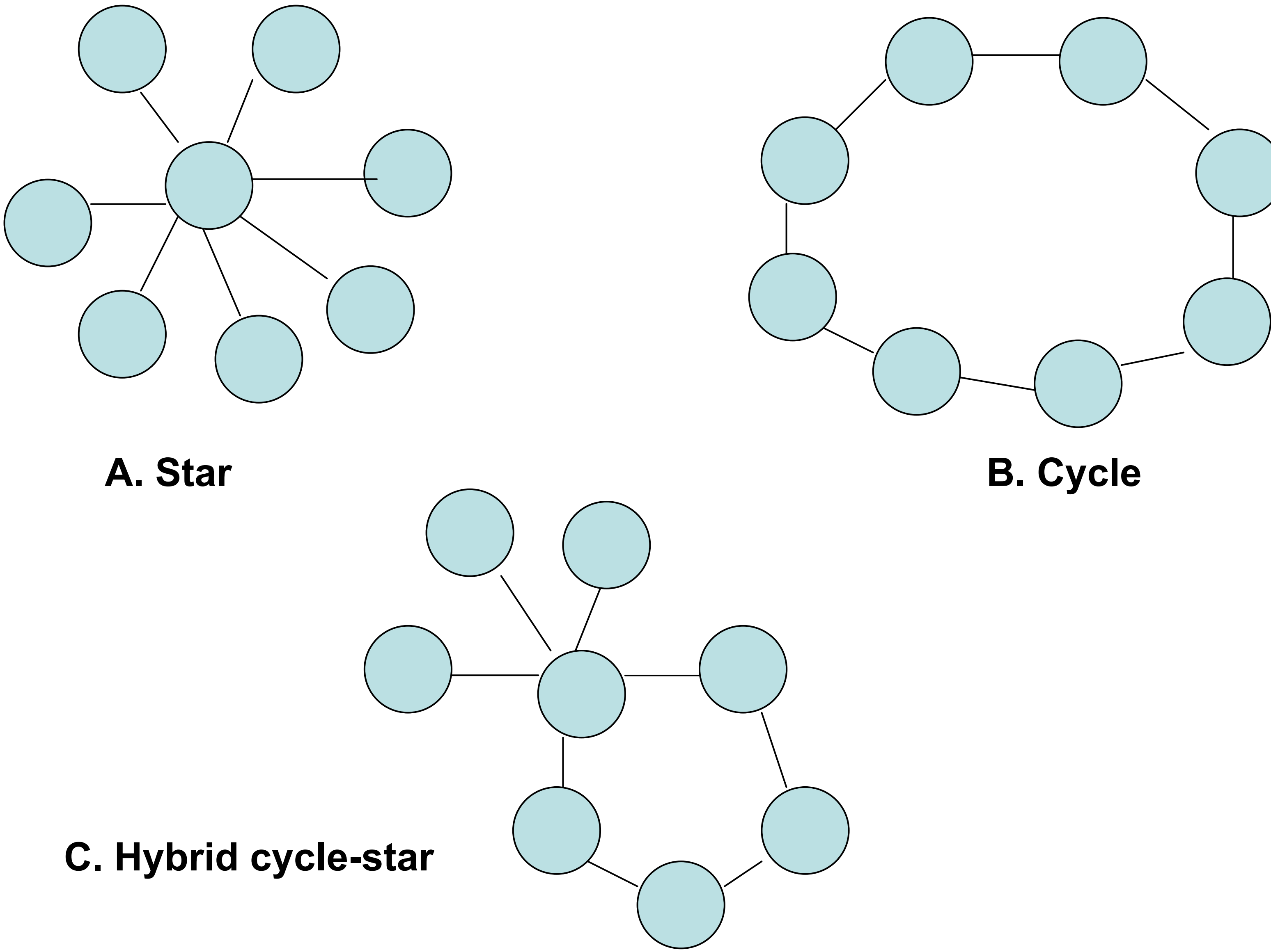
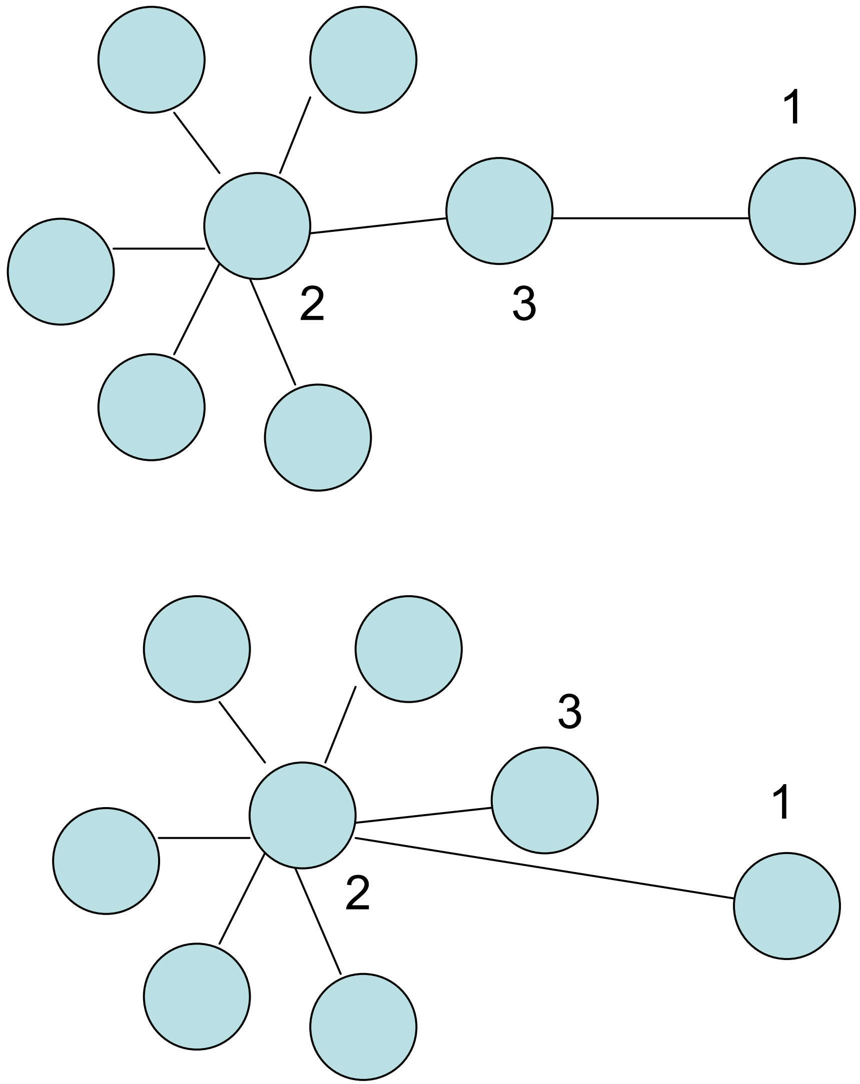


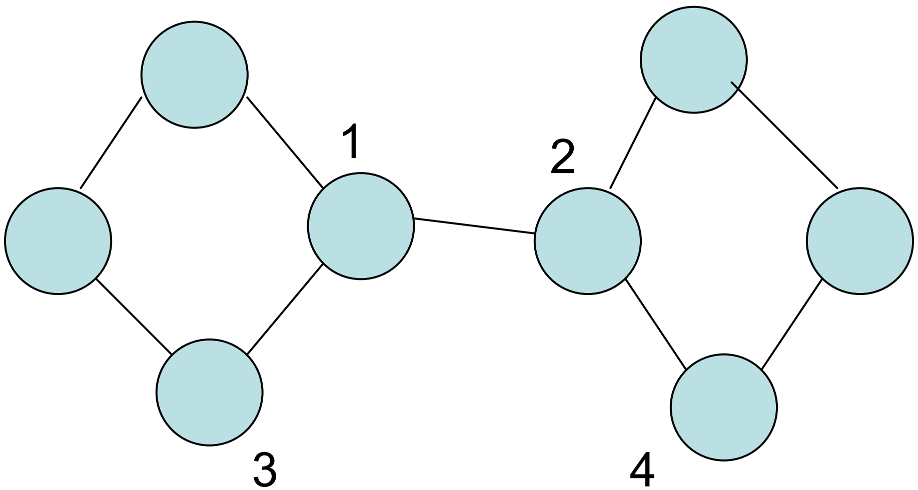
Figure 1: Main Architectures



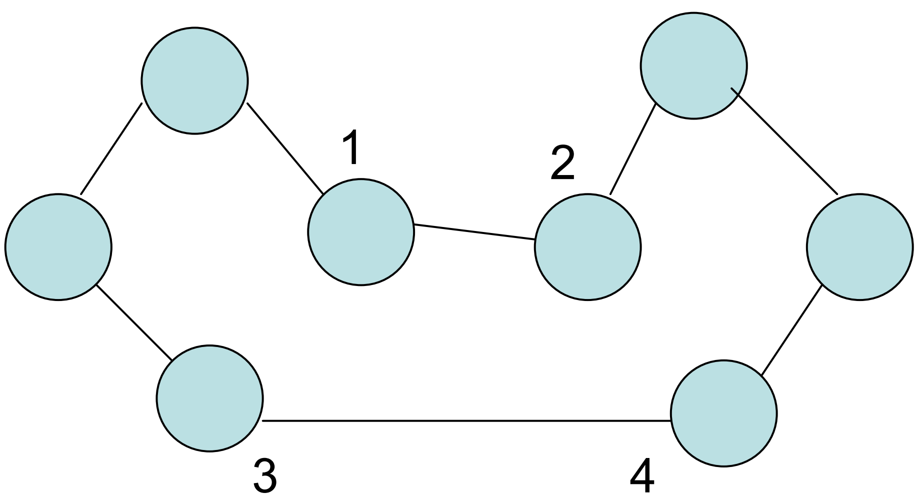
A. Minimal network with 2 essential players

B. Players 1 & 2 deviate yielding a star

Figure 2: Incentives in minimal networks

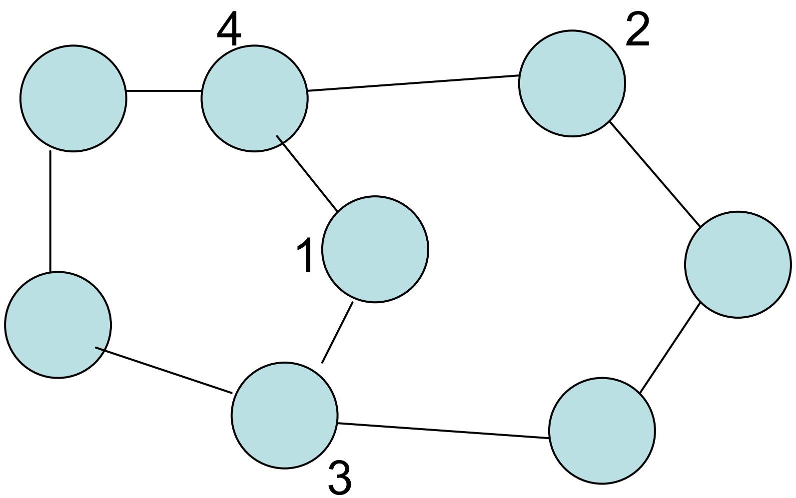


I. Cycles with no common players

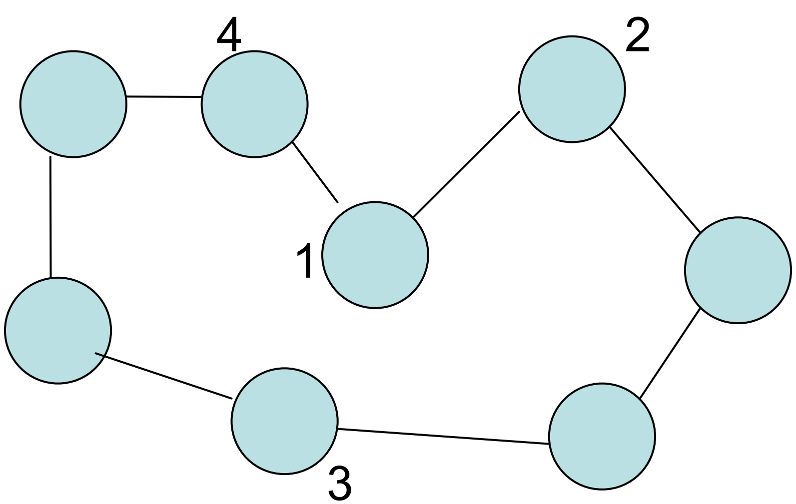


II. Players 3 and 4 deviate yielding one cycle

Figure 3A: Instability of networks with two cycles



I. Network with two cycles



II. Players 1 and 2 deviate yielding one cycle

Figure 3B. Instability of networks with two cycles

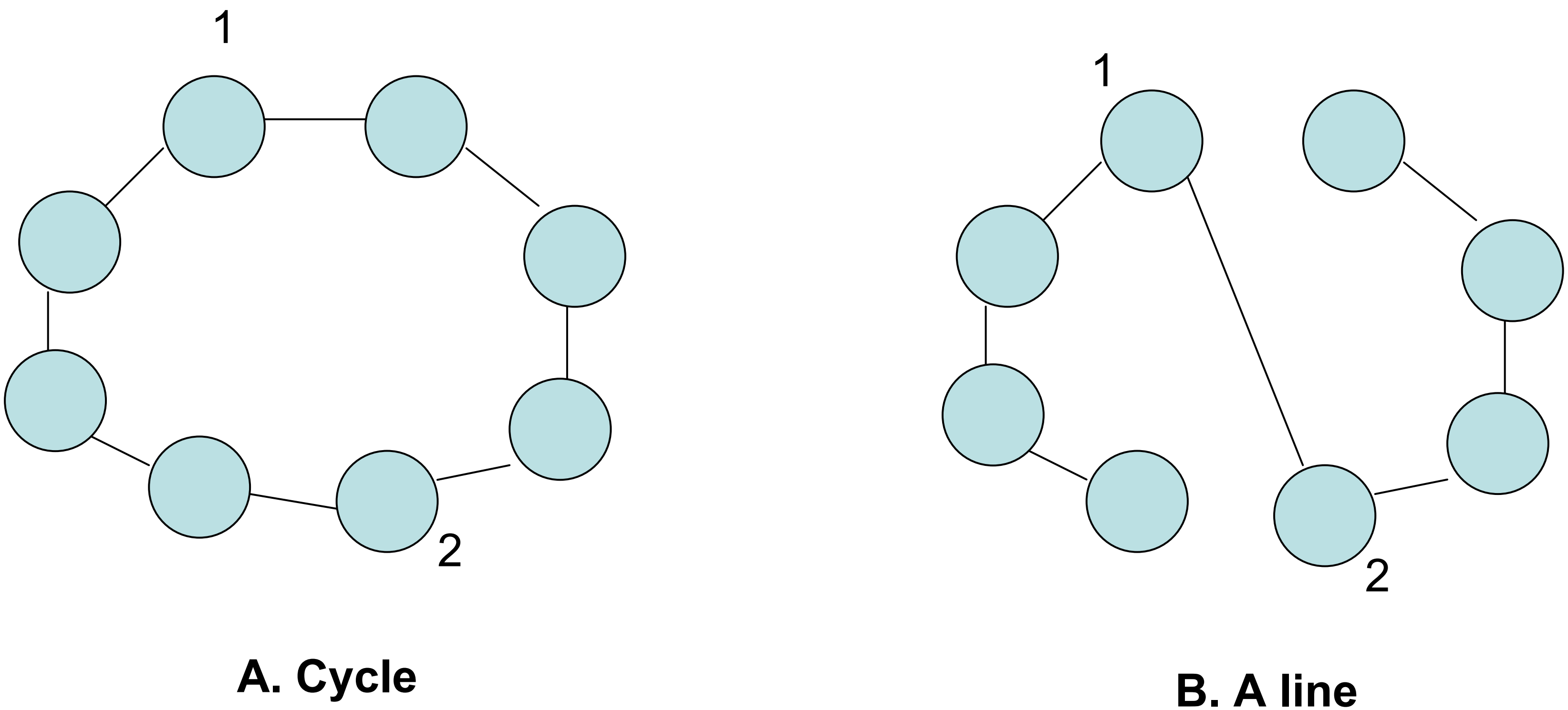
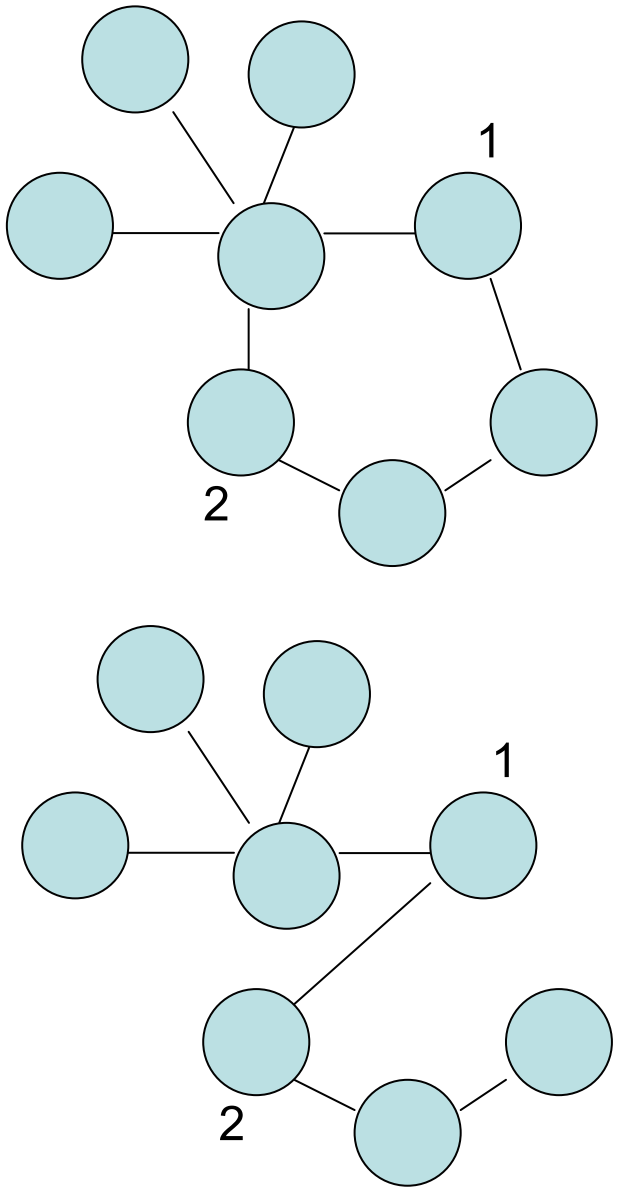


Figure 4: Bilateral deviations from cycle



A. Hybrid cycle-star

B. Players 1 and 2 have an incentive to deviate

Figure 5: Instability of hybrid cycle-star

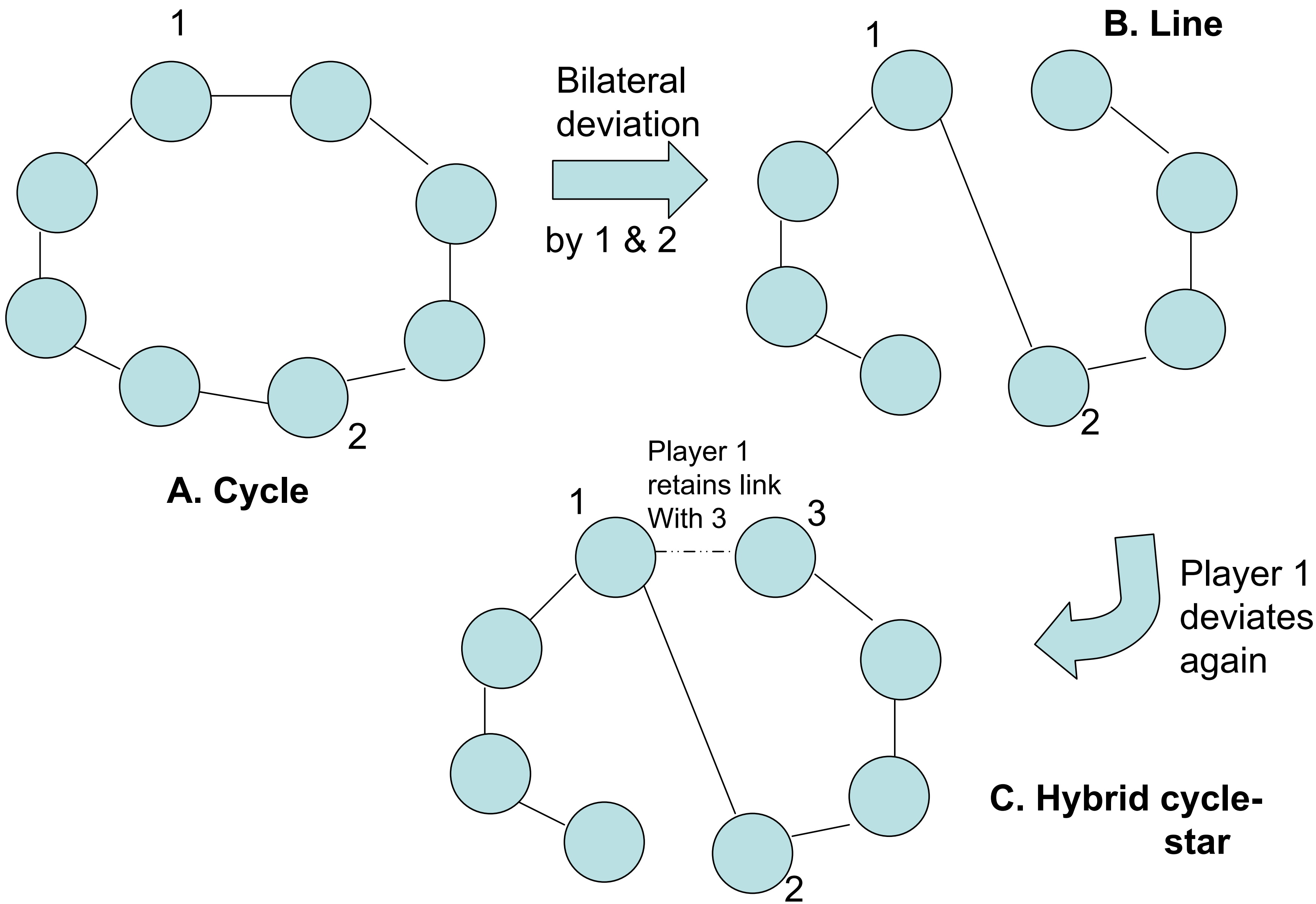


Figure 6: Bilateral proof deviations

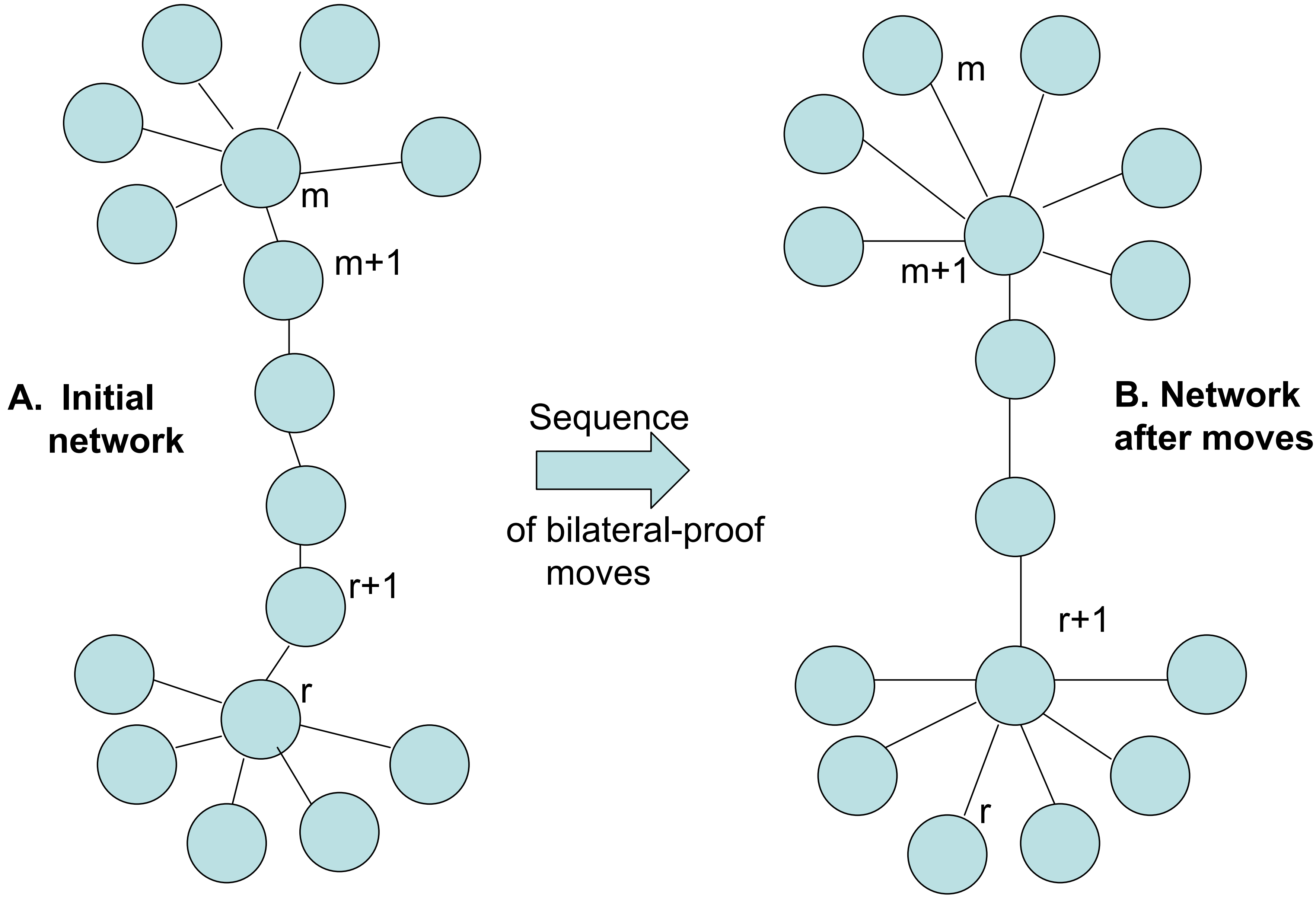


Figure 7: Process of agglomeration, c < 1/2

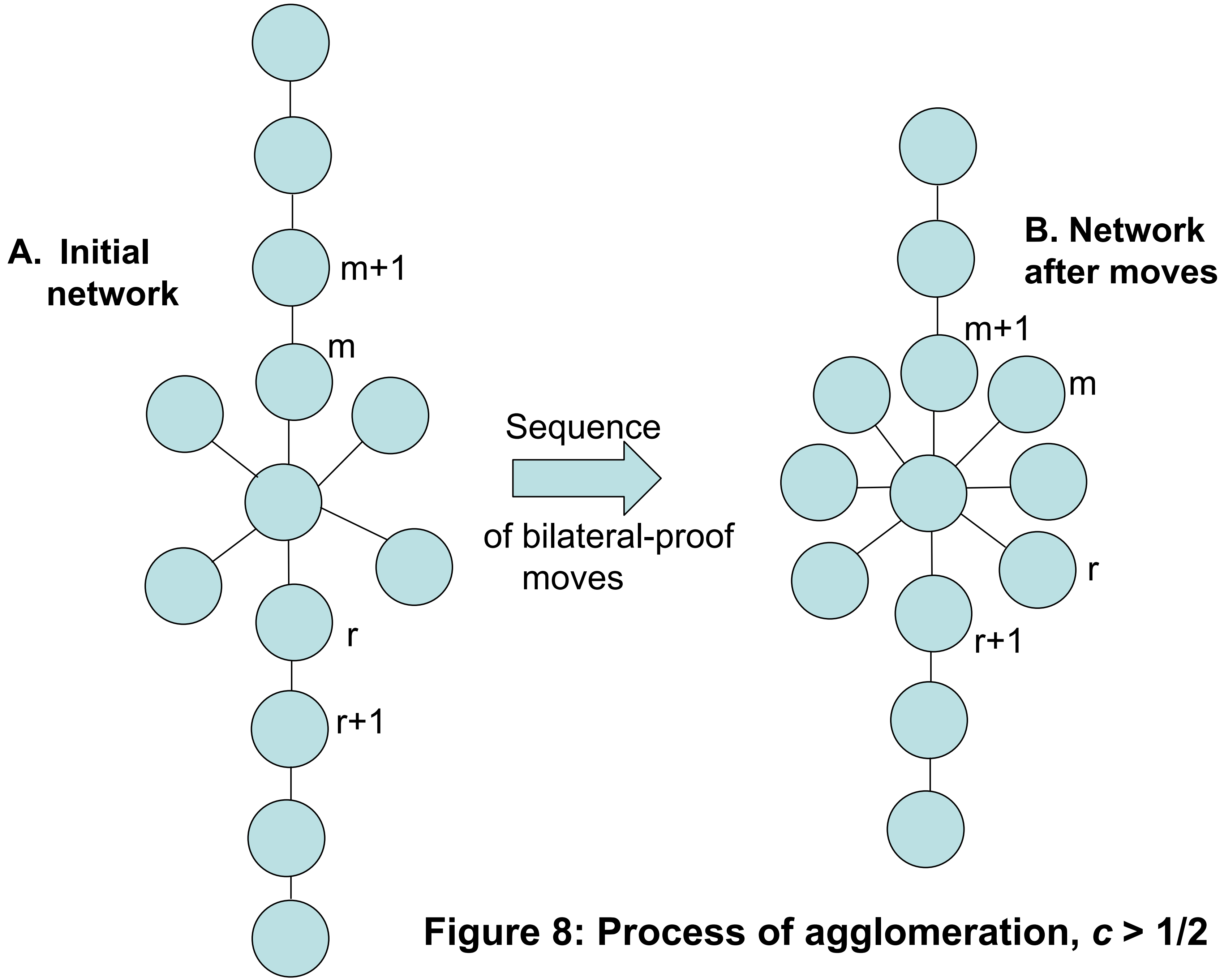


Figure 8: Process of agglomeration, *c* > 1/2