Supermodularity is not Falsifiable*

Christopher P. Chambers and Federico Echenique †

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Abstract

Supermodularity has long been regarded as a natural notion of complementarities in individual decision making; it was introduced as such in the nineteenth century by Edgeworth and Pareto. But supermodularity is not an ordinal property. We study the ordinal content of assuming a supermodular utility, i.e. what it means for the individual's underlying preferences. We show that supermodularity is a very weak property, in the sense that many preferences have a supermodular representation. As a consequence, supermodularity is not testable using data on individual consumption. We also study supermodularity in the Choquet expected utility model, and show that uncertainty aversion is generally not testable when only bets are observed.

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†Division of the Humanities and Social Sciences, California Institute of Technology, Pasadena CA 91125, USA. Email: `chambers@hss.caltech.edu` (Chambers) and `fede@hss.caltech.edu` (Echenique)

1 Introduction

Economists have long regarded supermodularity as the formal expression of complementarity in preference; according to Samuelson (1947), the use of supermodularity as a notion of complementarities dates back to Fischer, Pareto and Edgeworth. Our goal in this paper is to show that this approach is misguided by establishing that the assumption of supermodularity lacks empirical content in many ordinal economic models.

Supermodularity is a cardinal property of a function defined on a lattice. It roughly states that a function has "increasing differences." For this reason, it is usually interpreted as modeling complementarities. For example, consider a consumer with a utility function over two goods. A natural notion of complementarity is that the two goods are complementary if the marginal utility of consuming one of the goods is increasing in the consumption of the other; for smooth functions, if the cross-partial derivatives are non-negative. This notion is equivalent to supermodularity of the utility function.

Because it is a cardinal property, a number of authors, including Allen (1934), Hicks and Allen (1934), Samuelson (1947), and Stigler (1950), have already rejected supermodularity as a notion of complementarities (see Samuelson (1974) for a summary of the historical debate). They thought that, because it is a cardinal and not an ordinal notion, it would have no testable implications. While these authors had intuitively the right idea, their argument was incomplete. Their argument ran as follows: A supermodular function has positive cross-derivatives. Fix any given point. We may take an ordinal transformation of the function, preserving the preferences it represents, such that some cross-derivatives at this point become negative. While this is correct, it only demonstrates the lack of testable implications of supermodularity as *a local property.* Supermodularity at any given point (in the form of nonnegative cross-derivatives) is not refutable. However, they believed that, as a consequence, supermodularity has no global implications.

The argument in Samuelson, Hicks, Allen and Stigler is weak, as local violations of supermodularity have no obvious consequences for consumer behavior. More importantly, they were incorrect in believing that supermodularity has no ordinal implications. In an influential paper, Milgrom and Shannon (1994) introduced quasisupermodularity, an ordinal implication of supermodularity. Thus, when one assumes a function is supermodular, one is also making the assumption that the preference represented by this function is quasisupermodular.

Milgrom and Shannon characterized quasisupermodular functions as a class of functions with a monotone comparative statics property. Quasisupermodularity is a strict generalization of the concept of supermodularity, in the sense that there exist quasisupermodular preferences which cannot be represented by supermodular utility functions. Nevertheless, supermodularity has dominated quasisupermodularity as an assumption in economic applications. This is true even in the case where the function assumed to be supermodular has only ordinal significance. Our work should help to understand why this is the case.

Our main results will characterize the ordinal content of supermodularity and show that, in many situations, it is equivalent to quasisupermodularity. We show that supermodularity has few empirical implications for general individual decision-making environments, and no implications for data on consumption decisions, thus rigorously confirming the intuition of Samuelson, Hicks, Allen and Stigler's.

Consider a consumer with preferences over a finite set of consumption bundles. We characterize the preferences that can be represented by a supermodular utility function. Our primary result shows that weakly increasing and quasisupermodular preferences which are representable can be represented by a supermodular utility function. We also show a dual statement, that weakly decreasing, quasisupermodular and representable preferences have a supermodular representation. Thus, for many economic applications (where there is free disposal, or scarcity is modeled by monotonicity of preference relations), assuming an agent has a supermodular utility is no stronger than assuming his preference is quasisupermodular.

If quasisupermodularity is an ordinal implication of supermodularity, how can one make the claim that supermodularity is not falsifiable? The answer is simple: any *strictly monotonic* function is automatically quasisupermodular. So our results imply that any consumer with strictly increasing preferences can be modeled as having a supermodular utility function. This suggests that supermodularity may not be an appropriate mathematical interpretation of the notion of complementarity, as it is implied by selfish behavior alone.

We also fully characterize the class of all preferences (monotonic or otherwise) which can be represented by a supermodular utility; in this sense we are the first to establish the ordinal content of assuming supermodularity. We present an analogous characterization of quasisupermodularity, and compare it with the characterization of supermodularity.

Our results have direct and negative implications for the refutability of supermodularity in economic environments. We present these implications for two models: Afriat's model of consumption data and Schmeidler's model of uncertainty aversion.

First, we consider Afriat's (1967) model of data generated by an individual choosing consumption bundles at different prices. Arguably, this is the right model for the question of whether supermodularity has empirical implications in consumer theory. We show that the choices are either irrational, and cannot be rationalized by any utility function, or they can be rationalized by a supermodular utility function. Our result is reminiscent of Afriat's (1967) (see also Varian (1982)) finding that the concavity of utility functions has no testable implications; we find that supermodularity has no testable implications. In addition, we investigate the joint assumption of supermodularity and concavity. We show that the joint assumption has testable implications by presenting an example that can be rationalized by a supermodular and by a concave utility, but not by a utility with both properties.

Its lack of testable implications is further evidence that supermodularity is problematic as a notion of complementarity in consumption. Our finding is, in a sense, more problematic than Afriat's was for concavity, because one possible interpretation of concavity is as an implication of risk-aversion, and as such it is testable using lotteries. The interpretation of supermodularity as complementarity suggests statements about a consumer choosing—to use a textbook example—coffee and sugar. For such environments, we show that supermodularity is vacuous.

Our second application is to decision-making under uncertainty. Supermodularity is used to model uncertainty aversion in the Choquet expected utility model (Schmeidler, 1989). We study the refutability of uncertainty aversion when observing preferences over binary acts (bets) alone. We show that in many situations, uncertainty aversion has no testable implications. In particular, uncertainty aversion is not refutable using data on choices over binary acts when any event is considered more likely than any subevent. It is only refutable using data on more complicated acts, but such acts also entail attitudes towards risk.

Do our results imply that supermodularity is completely useless and has no place in economic theory? The answer is clearly no. Supermodularity has proved a useful assumption in very different areas in economics. It has been useful because it has strong theoretical implications.[1] But the environments in which supermodularity is useful are usually *cardinal* environments. To illustrate this, let us consider two examples.

First, in Becker's (1973)'s marriage market model, there are two types of agents, say, men and women. Men and women are ordered according to some characteristic, say height. When a man matches to a woman, they generate some numerical surplus. Becker assumes that this surplus function is supermodular in height: thus, the extra surplus generated when a given man matches with a taller woman is higher for a taller man. Becker's primary interest is in maximizing the aggregate surplus of a given matching. Becker shows that the matchings which maximize aggregate surplus are *assortative*: the tallest man is matched with the tallest woman, and so forth, in decreasing order. One might be tempted to think that our results therefore imply that any increasing surplus function leads to assortative matchings. However, this conclusion is incorrect. The matching-maximizing aggregate surplus is not preserved under arbitrary ordinal transformation. In Becker's model, the surplus is cardinally measurable and freely transferable.

Another model in which supermodularity proves extremely useful is in the theory of transferable-utility games. Shapley (1971) showed that any transferable-utility game which is supermodular has a nonempty core. Are we to therefore conclude that any game which is strictly monotonic (and can be ordinally transformed into a supermodular game) also has a nonempty core? Obviously not. Again, the solution concept of the core is a *cardinal* con-

[1]See the books by Topkis (1998) and Vives (1999) and the recent survey by Vives (2005).

cept, and the core is obviously not preserved under ordinal transformations. Hence, there are important economic applications in which supermodularity is meaningful and has implications.

We should also emphasize that the joint assumption of supermodularity and other properties can also have implications. Chipman (1977) showed that a differentiable, strongly concave and supermodular utility implies a normal demand. Quah (2004) shows that a concave and supermodular utility implies a class of monotone comparative statics; among other results, Quah generalizes Chipman's theorem. It seems important to study the testable implications of the joint assumptions of supermodularity and concavity. Neither our nor Afriat's results give an answer to this problem. The problem is that Afriat's separation argument is in the so-called "Afriat numbers," while ours is in the utility indexes themselves. However, two recent papers, one by Richter and Wong (2004), the other by Kannai (2005), state necessary and sufficient conditions for an order on a finite subset of some Euclidean space to be representable by a concave function. It is possible that combining their condition with ours in some way will lead to a necessary and sufficient condition for supermodular and concave representation.

We comment on the existing results that are related to ours.

Li Calzi (1990) presents classes of functions f such that a continuous increasing transformation of f is supermodular. He is also the first to note a connection between monotonicity and supermodularity, although his framework is different. While we establish our results for binary relations on a finite lattice, he shows that any twice differentiable strictly increasing function on a compact product lattice in $\mathbb{R}^n$ is ordinally equivalent to a supermodular function. Li Calzi does not study the testability of supermodularity (but see the comment in Section 6 on how his results can be used).

In a classical paper on the theory of preference over nonempty subsets (strictly speaking, not a lattice), Kreps (1979) introduces a condition on monotonic preferences which is both necessary and sufficient for the existence of a "totally submodular" functional representation. The condition is referred to later in the literature (specifically by Nehring (1999)) as ordinal submodularity. Submodularity is a dual notion of supermodularity, essentially equivalent to "decreasing differences." Total submodularity is a cardinal property which is intuitively interpreted as "decreasing differences of all orders." Now, Kreps' results can all be restated in a framework of supermodularity (simply by considering the dual of his binary relation). What we might call "ordinal supermodularity" bears a strong resemblance to quasisupermodularity; and is in fact implied by it under the assumption that the binary relation is weakly decreasing. Now, restating Kreps' theorem in this context, his theorem shows that any weakly decreasing and quasisupermodular binary relation on the set of nonempty subsets has a supermodular representation. We establish this result as well, although we have shown it to hold on general finite lattices. Kreps's result has no implications, for example, for lattices in $\mathbb{R}^n$. Moreover, it is not clear that Kreps' techniques can be adapted to yield a result on weakly *increasing* binary relations on the set of nonempty subsets, which is arguably the more interesting case.

We adapt Milgrom and Shannon's (1994) Monotonicity Theorem to provide a characterization of supermodularity in terms of a class of monotone comparative statics. There are previous characterizations of supermodularity in terms of monotone comparative statics for all linear perturbations of an objective function; see Milgrom and Shannon (1994), Topkis (1998) and Athey, Milgrom, and Roberts (1998). These are not characterizations of the ordinal content of supermodularity because the family of comparative statics is obtained by linear perturbations.

The seminal papers on supermodularity and lattice programming in economics are Topkis (1978), Topkis (1979), Vives (1990), Milgrom and Roberts (1990) and Milgrom and Shannon (1994). Most of our proofs apply an integer version of the Theorem of the Alternative (Aumann, 1964; Fishburn, 1970). Fostel, Scarf, and Todd (2004) and Chung-Piaw and Vohra (2003) present proofs of Afriat's theorem using similar ideas.

In Section 2 we present basic definitions. In Sections 3, 4 and 5 we present results on the representation of binary relations by supermodular and quasisupermodular functions. We present our results for Afriat's model in Section 6. We present some implications for a model of decision-making under uncertainty in Section 7.

2 Definitions.

Let X be a set and R be a binary relation on X. We write xRy for $(x, y) \in R$ and $xR^{\tau}y$ if there is a sequence

$$x = x_1, x_2, \ldots, x_K = y$$

with $K > 1$ and $x_k R x_{k+1}$, for $k = 1, \ldots K-1$. Define the binary relation P_R by $xP_R y$ if xRy and not yRx.

An **R-cycle** is a set $\{x_1, ..., x_K\}$, where $K > 1$, for which for all $i = 1, ..., K-1$, $x_i R x_{i+1}$ and $x_K R x_1$.

A **representation of** R is a function $u : X \to \mathbb{R}$ for which $i)$ for all $x, y \in X$, if xRy, then $u(x) \geq u(y)$, and $ii)$ for all $x, y \in X$, if $xP_R y$, then $u(x) > u(y)$.

A **partial order** on X is a reflexive, transitive and antisymmetric binary relation on X. A **partially-ordered set** is a pair $(X, \preceq)$ where X is a set and $\preceq$ is a partial order on X. A **lattice** is a partially-ordered set $(X, \preceq)$ such that for all $x, y \in X$, there exists a unique greatest lower bound $x \wedge y$ and a unique least upper bound $x \vee y$ according to $\preceq$. We write $x \parallel y$ if neither $x \preceq y$ or $y \preceq x$. Let $(X, \preceq)$ be a lattice and S and S' be subsets of X. We say that S is smaller than S' in the **strong set order** if, for any $x \in S$ and $y \in S'$, $x \wedge y \in S$ and $x \vee y \in S'$. Let ∂S be the **order boundary** of S, defined as $\partial S = \{x \in S : y \not\succeq x \text{ for all } y \in S \setminus \{x\}\}$.

Say that a function $u : X \to \mathbb{R}$ is **weakly increasing** if for all $x, y \in X$, $x \preceq y$ implies $u(x) \leq u(y)$. It is **strongly increasing** if for all $x, y \in X$, $x \preceq y$ and $x \neq y$ imply $u(x) < u(y)$. Say it is **weakly decreasing** if for all $x, y \in X$, $x \preceq y$ implies $u(y) \leq u(x)$. It is **weakly monotonic** if it is either weakly increasing or weakly decreasing.

A function $u : X \to \mathbb{R}$ is **quasisupermodular** if, for all $x, y \in X$, $u(x) \geq u(x \wedge y)$ implies $u(x \vee y) \geq u(y)$ and $u(x) > u(x \wedge y)$ implies $u(x \vee y) > u(y)$. A function $u : X \to \mathbb{R}$ is **supermodular** if, for all $x, y \in X$, $u(x \vee y) + u(x \wedge y) \geq u(x) + u(y)$.

A property of functions $u : X \to \mathbb{R}$ is **ordinal** if it is preserved by any strongly increasing transformation of u. A property is **cardinal** if it is preserved by any positive affine transformation of u. Quasisupermodularity, and all properties above with the words "strongly" or "weakly," are ordinal. Supermodularity, on the other hand, is a cardinal property which is not ordinal.

Finally, when $x \in X$, we denote by 1_x the vector in $\mathbb{R}^X$ that takes the value zero everywhere except at x.

3 Supermodular Representation.

We present results on when R has a supermodular representation. We first present the equivalence of supermodularity and quasisupermodularity for weakly monotonic binary relations. This result says that quasisupermodularity is exactly the ordinal meaning of supermodularity in the weakly increasing case. We then provide a necessary and sufficient condition for supermodular representation for arbitrary binary relations. The condition is less transparent than in the weakly increasing case.

Our results have implications for the refutability of supermodularity on finite sets of data. Some immediate consequences of the results in this section are that supermodularity of utility cannot be tested with data on expenditure (Proposition 11, Section 6), and that supermodularity of belief-capacities is difficult to refute with data on choices over bets (Proposition 13, Section 7). Supermodularity of belief-capacities is the model of uncertainty aversion in Schmeidler (1989).

Let $(X, \preceq)$ be a finite lattice. Our primary result is the following,

> THEOREM 1: *A binary relation on X has a weakly increasing and quasisupermodular representation if and only if it has a weakly increasing and supermodular representation.*

The proof of Theorem 1 is in Section 9. We present an intuitive argument for a related (weaker) result in Section 4.

> REMARK: *In Theorem 1, the term "weakly increasing" can be replaced by "weakly decreasing" by a straightforward modification of its proof.*

Theorem 1 has an important immediate corollary. The corollary follows because any strictly increasing function is quasisupermodular.

Corollary 2: *If a binary relation has a strongly increasing representation, then it has a supermodular representation.*

Proof: Let R be a binary relation with a strongly increasing representation u. We claim that u is quasisupermodular. Let $x, y \in X$. First, $x \succeq x \wedge y$ and $x \vee y \succeq y$, so $u(x) \geq u(x \wedge y)$ and $u(x \vee y) \geq u(y)$ hold. Second, if $u(x) > u(x \wedge y)$, then $x \succeq x \wedge y$ and $x \neq x \wedge y$. Hence, $x \vee y \succeq y$ and $x \vee y \neq y$, so $u(x \vee y) > u(y)$. By Theorem 1, R has a supermodular representation.$\square$

If we interpret R as a preference relation, Corollary 2 says that any consumer who has strictly monotonic preferences can be assumed to have a supermodular utility without loss of generality. This observation implies that interpreting supermodularity as modeling complementarities between goods may be problematic.

Milgrom and Shannon (1994) characterized quasisupermodular functions as the class of objective functions of maximization problems that present monotone comparative statics. Theorem 1 implies that we can recast their result as a characterization of supermodular functions. We present some definitions and then reinterpret Milgrom and Shannon's monotonicity theorem as our Corollary 3.

For a function $u : X \to \mathbb{R}$, let $M^u(S)$ be the set of maximizers of u on $S \subseteq X$. So $M^u(S) = \{x \in S : u(y) \leq u(x) \text{ for all } y \in S\}$. Say that $M^u(S)$ exhibits **non-satiation** if, $\partial S \cap M^u(S) \neq \emptyset$. We say that $M^u(S)$ is weakly increasing in S if $M^u(S)$ is smaller than $M^u(S')$ in the strong set order when S is smaller than S' in the strong set order.

Corollary 3: *There exists a strictly increasing $f : \mathbb{R} \to \mathbb{R}$ such that the function $f \circ u : X \to \mathbb{R}$ is weakly increasing and supermodular if and only if $M^u(S)$ is weakly increasing in S and exhibits non-satiation.*

The proof of Corollary 3 is immediate from Milgrom and Shannon's Monotonicity Theorem (Theorem 4 in Milgrom and Shannon (1994)) and our Theorem 1.

To see that weak monotonicity alone is not sufficient for supermodularity, consider the following example.

EXAMPLE 4: Let $X = \{1,2\}^2$ with the usual ordering. Let R be representable by the function $u : X \to \mathbb{R}$ for which $u((1,1)) = 0$, and $u((1,2)) = u((2,1)) = u((2,2)) = 1$. Clearly, R cannot be represented by a supermodular function (any such function v would require $v((1,2)) = v((2,1)) = v((2,2)) > v((1,1))$, so that $v((1,1)) + v((2,2)) < v((2,1)) + v((1,2))$. Nevertheless, R is weakly monotonic. R cannot be represented by a quasisupermodular function, as $u((1,2)) > u((1,1))$, yet $u((2,1)) \geq u((2,2))$.

To see that the monotonicity in Corollary 2 is not necessary for a supermodular representation (even with an antisymmetric R, see Theorem 8), consider the following example.

EXAMPLE 5: Let $X = \{1,2\}^2$ with the usual ordering. Let R be representable by the function $u : X \to \mathbb{R}$ for which $u((1,1)) = 0$, $u((1,2)) = -1$, $u((2,1)) = 2$, and $u((2,2)) = 1.5$. Clearly, u is (strictly) supermodular. However, note that R is not monotonic.

In light of Examples 4 and 5, one may ask for a necessary and sufficient condition for supermodular representation. We obtain one from an application of the Theorem of the Alternative.

THEOREM 6: *Let $(X, \preceq)$ be a finite lattice. There exists a supermodular representation of R if and only if, for all $N, K \in \mathbb{N}$, for all $\{x_i\}_{i=1}^N$, $\{y_i\}_{i=1}^N$,*

$\{z_l\}_{l=1}^K$, $\{w_l\}_{l=1}^K \subseteq X$ *for which for all* $l = 1, ..., K-1$, $z_l R w_l$ *and for which*

$$\sum_{i=1}^{N} (1_{x_i \vee y_i} + 1_{x_i \wedge y_i}) + \sum_{l=1}^{K} 1_{z_l} = \sum_{i=1}^{N} (1_{x_i} + 1_{y_i}) + \sum_{l=1}^{K} 1_{w_l},$$

$z_K P_R w_K$ *does not hold.*

PROOF: The existence of a supermodular representation is equivalent to the existence of a vector $u \in \mathbb{R}^X$ for which $i)$ for all $x, y \in X$ for which $x \parallel y$, $(1_{x\vee y} + 1_{x\wedge y} - 1_x - 1_y) \cdot u \geq 0$, and $ii)$ for all $x, y \in X$ for which $x P_R y$, $(1_x - 1_y) \cdot u > 0$. By the integer version of the Theorem of the Alternative (Aumann, 1964; Fishburn, 1970), such a vector fails to exist if and only if for all $x, y \in X$ for which $x \parallel y$, there exists some $n_{\{x,y\}} \in \mathbb{Z}_+$ and for all $x, y \in X$ for which xRy, there exists some $n_{(x,y)} \in \mathbb{Z}_+$, and there exists at least one $n_{(x,y)} > 0$ for which $xP_R y$, such that

$$\sum_{\{\{x,y\}:x\|y\}} n_{\{x,y\}} (1_{x\vee y} + 1_{x\wedge y} - 1_x - 1_y) + \sum_{\{(x,y):xRy\}} n_{(x,y)} (1_x - 1_y) = 0.$$

Separating terms, we obtain

$$\begin{aligned} & \sum_{\{\{x,y\}:x\|y\}} n_{\{x,y\}} (1_{x\vee y} + 1_{x\wedge y}) + \sum_{\{(x,y):xRy\}} n_{(x,y)} 1_x \\ = & \sum_{\{\{x,y\}:x\|y\}} n_{\{x,y\}} (1_x + 1_y) + \sum_{\{(x,y):xRy\}} n_{(x,y)} 1_y. \end{aligned}$$

It is easy to see that this is equivalent to the existence of $N, K \in \mathbb{N}$, $\{x_i\}_{i=1}^N, \{y_i\}_{i=1}^N$, $\{z_l\}_{l=1}^K$, $\{w_l\}_{l=1}^K \subseteq X$ such that for all $l = 1, ..., K-1$, $z_l R w_l$ and for which

$$\sum_{i=1}^{N} (1_{x_i \vee y_i} + 1_{x_i \wedge y_i}) + \sum_{l=1}^{K} 1_{z_l} = \sum_{i=1}^{N} (1_{x_i} + 1_{y_i}) + \sum_{l=1}^{K} 1_{w_l},$$

and $z_K P_R w_K$. $\square$

4 Monotonic Representation

One of the main implications of Theorem 1 is that a strongly increasing representation implies a supermodular representation (Corollary 2). Here, we present a simple direct proof of this fact for the case of antisymmetric binary relations. We believe that this proof is instructive, especially in light of how involved the proof of Theorem 1 is. The direct proof shows that the link between supermodularity and monotonicity is quite intuitive. As a by-product, we characterize the relations that have a monotonic representation; to the best of our knowledge, this is the first such characterization.

Let $(X, \preceq)$ be a finite lattice and R be a binary relation on X. We first reproduce, without proof, the standard result on when R has a representation (see e.g. Richter (1966)).

> Theorem 7: *There is a representation of R if and only if $xR^{\tau}y$ implies that $yP_R x$ is false.*

We obtain monotonic and supermodular representations by strengthening the condition in Theorem 7. The strengthening consists in "augmenting" R with binary relations associated to the order structure $\preceq$ on X. Define the following binary relations on X.

- xBy if either $x \succeq y$ or xRy.
- $x \trianglerighteq y$ if there exists $z \parallel y$ for which $x = y \vee z$.
- xTy if either $x \trianglerighteq y$ or xRy.

We obtain analogous results if we define $x \trianglerighteq y$ to hold when there exists $z \parallel x$ for which $y = x \wedge z$.

Theorem 8: *There exists a monotonic representation of R if and only if $xB^\tau y$ implies that $yP_R x$ is false. If R is antisymmetric, then R has a supermodular representation if $xT^\tau y$ implies that $yP_R x$ is false.*

It is instructive to compare the conditions in Theorems 7 and 8. Theorem 7 says that a "better than" relation R has a representation if it has no cycles. We show that it has a monotonic representation if the relation B, which means "either better than or larger than," has no cycles. The condition for a supermodular representation (in the antisymmetric case) is similar, but with a *weaker* "larger than" relation.

Theorem 8 implies that an antisymmetric and monotonic relation has a supermodular representation: since the $\trianglerighteq$ order is weaker than $\succeq$, $xT^\tau y$ implies $xB^\tau y$.

Proof: The existence of a monotonic representation is equivalent to the existence of a vector $u \in \mathbb{R}^X$ satisfying the following three properties: *i*) for all $x, y \in X$ for which $x \succeq y$ and $x \neq y$, $(1_x - 1_y) \cdot u \geq 0$, *ii*) for all $x, y \in X$ for which xRy and $x \neq y$, $(1_x - 1_y) \cdot u \geq 0$, and *iii*) for all $x, y \in X$ for which $xP_R y$, $(1_x - 1_y) \cdot u > 0$. Clearly, these inequalities are satisfied if and only if for all x, y for which xBy and $x \neq y$, $(1_x - 1_y) \cdot u \geq 0$, with the inequality strict in the case of $xP_R y$. By the Theorem of the Alternative, such a u does not exist if and only if for all $x, y \in X$ for which xBy and $x \neq y$, there exists $n_{x,y} \in \mathbb{Z}_+$ so that

$$\sum_{\{(x,y):xBy\}} n_{x,y} \left(1_x - 1_y\right) = 0,$$

where there exists a pair $xP_R y$ for which $n_{x,y} > 0$. By a straightforward modification of a well-known result in graph theory (e.g. Diestel's (2000) Theorem 1.9.7 or Berge's (2001) Theorem 15.5, see also Lemma 14 below), there exist a collection of B-cycles, say

$$\{x_1^1, ..., x_{K_1}^1\}, ..., \{x_1^N, ..., x_{K_N}^N\}$$

for which $n_{x,y}$ is the number of times xBy appears in one of the cycles. In other words, the equality can only hold if and only if there exists a set $\{x_1, ..., x_K\} \subseteq X$ for which for all $i = 1, ..., K-1$, $x_i B x_{i+1}$ and $x_K P_R x_1$.

We now prove the second statement in the theorem. The existence of a supermodular representation is equivalent to the existence of a vector $u \in \mathbb{R}^X$ for which $i)$ for all $x, y \in X$ for which $x \parallel y$, $(1_{x\vee y} + 1_{x\wedge y} - 1_x - 1_y) \cdot u \geq 0$, and $ii)$ for all $x, y \in X$ for which $xP_R y$, $(1_x - 1_y) \cdot u > 0$; note that, by antisymmetry, xRy and $x \neq y$ imply $xP_R y$.

Suppose that a supermodular u does not exist. Then, by the Theorem of the Alternative, there exist a pair of collections, $n_{\{x,y\}}$ and $n_{(x,y)}$, each in $\mathbb{Z}_+$, such that

$$\sum_{\{\{x,y\}:x\|y\}} n_{\{x,y\}} (1_{x\vee y} + 1_{x\wedge y} - 1_x - 1_y) + \sum_{\{(x,y):xP_R y\}} n_{(x,y)} (1_x - 1_y) = 0, \quad (1)$$

□

where some $n_{(x,y)} > 0$.

We now construct a sequence $\{y_k\}$ such that, for all k, there is z with either $n_{\{z,y_k\}} > 0$ or $n_{(z,y_k)} > 0$. First, there exists some $n_{(x,y)} > 0$; let $y_1 = y$. Now proceed by induction. Suppose that y_k is defined such that there is z with either $n_{\{z,y_k\}} > 0$ or $n_{(z,y_k)} > 0$. In the first case, $(z \vee y_k) T y_k$, so define $y_{k+1} = z \vee y_k$. In the second case, $zP_R y_k$, so define $y_{k+1} = z$. In either case, the fact that (1) is satisfied implies that there exists a w for which either $n_{\{w,y_{k+1}\}} > 0$ or $n_{(w,y_{k+1})} > 0$. This completes the construction of the sequence. Note that $y_{k+1} \neq y_k$ and $y_{k+1} T y_k$.

As X is finite, this demonstrates the existence of a T-cycle. Since $\preceq$ is asymmetric and transitive, we conclude that there must exist a set

$\{x_1, ..., x_K\} \subseteq X$ for which for all $i = 1, ..., K-1$, $x_i T x_{i+1}$, and $x_K P_R x_1$.

5 Quasisupermodular Representation.

Milgrom and Shannon (1994) introduce the notion of quasisupermodularity as an ordinal generalization of supermodularity. They show that quasisupermodularity is necessary and sufficient for a class of monotone comparative statics. Here we state a simple characterization of when R can be represented by a quasisupermodular function. The result is simply a restatement of the definition of quasisupermodularity. We include it as a comparison with Theorem 6.

Note that the condition in Theorem 6 implies that the relation R has a representation. The analogous condition for quasisupermodularity will have no such implication; so we assume outright that R is complete and transitive.

THEOREM 9: *Let $(X, \preceq)$ be a finite lattice and R be complete and transitive. Then there exists a quasisupermodular $u : X \to \mathbb{R}$ which represents R if and only if for all x, y, $\{z_l\}_{l=1}^2$, $\{w_l\}_{l=1}^2 \subseteq X$ for which $z_1 R w_1$,and for which*

$$(1_{x \vee y} + 1_{x \wedge y}) + \sum_{l=1}^{2} 1_{z_l} = (1_x + 1_y) + \sum_{l=1}^{2} 1_{w_l},$$

$z_2 P_R w_2$ does not hold.

PROOF: A complete and transitive R is representable by a quasisupermodular function u if and only if $xR\,(x \wedge y)$ implies $(x \vee y)\,Ry$ and $xP_R\,(x \wedge y)$ implies $(x \vee y)\,P_R y$. This is equivalent to the condition displayed in the statement of the Proposition. $\square$

Milgrom and Shannon (1994) present an example of a quasisupermodular function for which no monotonic transformation yields a supermodular function. We reproduce their example as Example 10 and show how it fails the condition in Theorem 6. The function in Milgrom and Shannon's example is not weakly monotonic.

EXAMPLE 10: Let $X = \{1,2\} \times \{1,2,3,4\}$ and $f : X \to \mathbb{R}$ be given by the following values:

	1	2	3	4
1	1	2	2	1
2	3	4	5	3

It is easy to verify that f is quasisupermodular. Let R be the order on X induced by f.

Let $x_1 = (2,1)$, $y_1 = (1,2)$, and $x_2 = (2,3)$, $y_2 = (1,4)$. Then the vector

$$\sum_{i=1}^{2} (1_{x_i \vee y_i} + 1_{x_i \wedge y_i}) - \sum_{i=1}^{2} (1_{x_i} + 1_{y_i})$$

is

$$\begin{array}{ccccc} & 1 & 2 & 3 & 4 \\ 1 & 1 & -1 & 1 & -1 \\ 2 & -1 & 1 & -1 & 1 \end{array} \tag{2}$$

Let $w_1 = x_2 \vee y_2 = (2,4)$ and $z_1 = x_1$; so $z_1 R w_1$. Let $w_2 = x_1 \wedge y_1 = (1,1)$ and $z_2 = y_2$; so $z_2 R w_2$. Let $w_3 = x_2 \wedge y_2 = (1,3)$ and $z_3 = y_1$; so $z_3 R w_3$. Let $w_4 = x_1 \vee y_1 = (2,2)$ and $z_4 = x_2$; so $z_4 P_R w_4$. It is clear that (2) equals

$$\sum_{k=1}^{4} (1_{z_k} - 1_{w_k}).$$

Theorem 6 thus implies that there is no supermodular function that represents R.

6 Application 1: Afriat's Model

Afriat (1967) studies how data on consumption choices at different prices can refute assumptions on consumer preferences (see also Varian (1982)). Afriat shows that the data can arise from a rational consumer if and only if one can model the consumer with a monotone and concave utility function. Concavity of utility is thus not refutable with data on consumption expenditures. In a similar vein, we prove that supermodularity is not refutable with data on consumption.[2]

The are long-standing doubts about the empirical content of supermodularity in consumption theory. Allen (1934), Hicks and Allen (1934), Samuelson (1947), and Stigler (1950) believed that supermodularity has no empirical implications for consumer behavior. Their doubts come from the realization that supermodularity is not a cardinal property and that, at any point, there is a representation of utility for which marginal utility is locally increasing and one for which it is locally decreasing.

These doubts prove to be misleading. First, Chipman (1977), addressing explicitly the critique of Allen, Hicks, Samuelson and Stigler, shows that a consumer with a supermodular and strongly concave utility has a normal demand. Chipman suggests that, as a result, supermodularity has testable implications. Second, Milgrom and Shannon (1994) show that quasisupermodularity is an ordinal notion of complementarity and that it is implied by supermodularity. So supermodularity has an ordinal implication which has a natural interpretation as a complementarity property.

We address the testability of supermodularity using an explicit model of

[2]This section is better thanks to Ed Schlee's and Chris Shannon's suggestions. Ed pointed us to the implications of the joint assumption of concavity and supermodularity and Chris suggested the proof using Chiappori and Rochet's and Li Calzi's results.

the data one might use to test it: Let $X \subseteq \mathbb{R}^n_+$ be a finite lattice. Let the pairs (x^k, S^k), $k = 1, \ldots, K$, be such that $S^k \subseteq X$ and $x^k \in S^k$ for all k. Interpret each (x^k, S^k) as one observation of the consumption bundle x^k chosen by an individual out of the budget set S^k. Arguably, this is the right model for the question of whether supermodularity has empirical implications in consumer theory. The model is Matzkin's (1991) generalization of Afriat's model.

We make three assumptions on the data:

1. For all k, $x^k \in \partial S^k$; a "non-satiation" assumption.

2. If $x \in S^k$ and $y \leq x$ then $y \in S^k$; if x is "affordable" in budget S^k, and y is weakly less consumption of all goods, then y is also affordable.

3. If $x^{k'} \in S^k$ and $x^{k'} \neq x^k$, then $x^k \notin S^{k'}$; a version of the weak axiom of revealed preference.

Consider the binary relation R on X defined by xRy if there is k such that $x = x^k$ and $y \in S^k$. Note that R is the standard revealed-preference relation. Note that xP_Ry if xRy and $x \neq y$.

Say that $u : X \to \mathbb{R}$ **rationalizes** the data $(x^k, S^k)_{k=1}^K$ if it represents R. Hence, a consumer with utility u would rationally choose x^k out of S^k, for each k. The function u is called a rationalization of the data. Under Property 3, this notion of rationalization coincides with Afriat's (1967).

PROPOSITION 11: *The data $(x^k, S^k)_{k=1}^K$ has a rationalization if and only if it has a supermodular rationalization.*

PROOF: Let B be the binary relation defined as xBy if xRy or $x \geq y$. It is easily verified that xP_By if xP_Ry or $x > y$. We show below that R has a representation if and only if B has a representation. The proposition then follows from Corollary 2, because any representation of B must be strongly monotone.

By Theorem 7, if B has a representation, then so does R. Suppose, by means of contradiction, that B does not have a representation. We will show that R does not have a representation.

If B does not have a representation, then there exist $x, y \in X$ for which $xB^{\tau}y$ and yP_Bx. Suppose that $x \geq y$. Then yP_Bx implies that there exists k for which $y = x^k$. But $y \in \partial S^k$, so $x \geq y$ implies $x = y$, contradicting the fact that yP_Bx. Therefore, $x \not\geq y$. Hence, $xB^{\tau}y$ implies that there exists $\{x_1, \ldots x_L\} \subseteq X$ for which $xBx_1B \ldots Bx_LBy$, where at least one B corresponds to R and not $\geq$.

We claim that there exists $x' \neq y$ for which $x \geq x'$ and $x'R^{\tau}y$. To see this, note that for any collection of data $\{z_1, ..., z_m\} \subseteq X$ for which $z_1Rz_2 \geq z_3 \geq ... \geq z_m$, it follows by the fact that $z_2 \geq z_m$ and Property 2 of the data that z_1Rz_m. From this fact, we establish that there is $x' \neq y$ with $x \geq x'$ and $x'R^{\tau}y$.

As yP_Bx, either yP_Rx or $y > x$. If yP_Rx, then yP_Rx' by $x \geq x'$ and Property 2 of the data. In this case, we may conclude that R has no representation (Theorem 7). If instead $y > x$, then we have $x'R^{\tau}y$ and $y > x'$. Then, $x'Ry$ contradicts Property 1 of the data, so there exists $x'' \in X$, $x'' \neq x'$ for which $x'R^{\tau}x''Ry$. Now, $y \geq x'$ and Property2 of the data imply that $x''Rx'$. In fact, by Property 3 and $x'' \neq x'$, $x''P_Rx'$. So R does not have a representation. □

Our model is related to Afriat's in the following way. Afriat's data consists of pairs (x^k, p^k), $k = 1, \ldots, K$, such that $x^k \in X \subseteq \mathbb{R}^n_+$ and $p^k \in \mathbb{R}^n_{++}$, for all k. Each pair is an observed consumption choice x^k at prices p^k. Afriat's data are obtained from ours by setting

$$S^k = \left\{ x : p^k \cdot x \leq p^k \cdot x^k \right\}.$$

Note that the weak axiom, Property 3 of the data, is implicit in Afriat's results. If we do not assume Property 3 we would need to distinguish between representability of R and rationalization; the result in Proposition 11 continues to hold.

Our model allows us to more generally accommodate non-linear budgets sets; Matzkin (1991) introduced this model as a way of discussing the issues raised by Afriat's results in situations where consumers have monopsony power, or where the consumer is a social planner facing an economy's production possibility set.

With Afriat's data, there is a very direct proof of our result: Afriat's theorem implies that, if the data are rationalizable, R has a strongly monotonic representation (the utility he constructs is the lower envelope of a finite number of strongly monotonic linear functions). Then, by Corollary 2, R has a supermodular representation.

Two remarks are in order. First, under an additional restriction on Afriat's data, rationalizability implies that there is a smooth, strongly monotonic rationalization (Chiappori and Rochet, 1987). Corollary 20 in Li Calzi (1990) then implies the existence of a supermodular rationalization. So one can use existing results to prove a version of Proposition 11 for the Afriat data under Chiappori and Rochet's assumptions.

The second remark refers to concave rationalizations. Afriat shows that having a rationalization is equivalent to having a rationalization by a concave function and Proposition 11 says that it is equivalent to a rationalization by a supermodular function. One might conjecture that any rationalizable data con be rationalized by a function that is concave and supermodular. This turns out to be false, essentially because supermodularity and concavity imply that demand is normal. So, while concavity and supermodularity have no testable implications as individual assumptions, they are refutable as joint assumptions.

That supermodularity and concavity imply normal demand is shown in Quah (2004); the earlier result of Chipman (1977) requires additional smoothness assumptions on utility. Quah's result does not apply to functions that are supermodular on a finite domain. We present a very simple adaptation of Quah's argument in Example 12.

EXAMPLE 12: Consider the data $(x^k, p^k)_{k=1}^2$, where $p^1 = (2, 1.1)$, $x^1 = (1, 2)$, $p^2 = (2.1, 1)$, and $x^2 = (2, 1)$. This collection of data is rationalizable (as it satisfies Afriat's condition) and thus has both a concave and a supermodular rationalization. We show that it has no concave and supermodular rationalization.

Let $C \subseteq \mathbb{R}^2$ be convex and $X \subseteq C$ be a sublattice such that

$$\{(1, 2), (2, 1), (3/2, 1), (3/2, 2), (2, 2)\} \subseteq X.$$

Suppose that $u : C \to \mathbb{R}$ is concave and that $u|_X$ is supermodular. We shall prove that u cannot rationalize the data.

We first note that $p^1 \cdot (3/2, 1) < p^1 \cdot (1, 2)$ so that we need $u(3/2, 1) < u(1, 2)$ for u to rationalize the data. We then prove that $u(2, 1) < u(3/2, 2)$, which is inconsistent with u rationalizing the data, as $p^2 \cdot (3/2, 2) < p^2 \cdot (2, 1)$.

Start from $u(3/2,2) - u(2,1) = u(3/2,2) - u(2,2) + u(2,2) - u(2,1)$. Then,

$$\begin{aligned} u(3/2,2) - u(2,2) &= u\left((2,2) - (1/2)(1,0)\right) - u(2,2) \\ &\geq u\left((2,2) - (1/2)(1,0) - (1/2)(1,0)\right) \\ &\quad -u\left((2,2) - (1/2)(1,0)\right) \\ &= u(1,2) - u(3/2,2); \end{aligned}$$

the inequality above follows from concavity (Quah, 2004). Supermodularity on X implies that $u(2,2) - u(2,1) \geq u(3/2,2) - u(3/2,1)$. Hence,

$$\begin{aligned} u(3/2,2) - u(2,1) &\geq u(1,2) - u(3/2,2) + u(3/2,2) - u(3/2,1) \\ &= u(1,2) - u(3/2,1). \end{aligned}$$

This implies that u cannot rationalize the data because $u(3/2,1) < u(1,2)$ implies $u(2,1) < u(3/2,2)$.

Note that Quah's result on normal demand is not directly applicable because the domain on which u is supermodular is finite, and because prices change, as well as expenditure, between observations $k = 1$ and $k = 2$. A straightforward modification of Quah's arguments still gives the result.

7 Application 2: Uncertainty aversion and the Choquet expected utility model.

We now turn to a model of decision under uncertainty where supermodularity models uncertainty aversion.

An individual faces risk when probabilities are exogenously specified. If she is not given these probabilities, she faces uncertainty. When the events are not given probabilities, there is no reason to suspect that the individual will assign probabilities to them. We study a model introduced by Schmeidler (1989); in this model, the individual in fact need not assign probabilities to events, but assigns some measure of likelihood to them. This measure is called a capacity. Supermodularity of the capacity in this model is interpreted as uncertainty aversion.

Our results imply that Schmeidler's notion of uncertainty aversion places few restrictions on the individual's preferences over bets. As a consequence, if one can elicit her likelihood ranking of events, it is very difficult to refute that she is uncertainty averse (for example if larger events are always perceived as strictly more likely, uncertainty aversion cannot be refuted). We briefly describe Schmeidler's model and explain the implications of our results.

Let Ω be a finite set of possible states of the world and let Y be a set of possible outcomes. The set of (Anscombe and Aumann, 1963) **acts** is the set of functions $f : \Omega \to \Delta(Y)$. Denote the set of acts by $\mathcal{F}$. A **capacity** is a function $\nu : 2^{\Omega} \to \mathbb{R}$ for which $\nu(\varnothing) = 0$, $\nu(\Omega) = 1$, and $A \subseteq B$ implies $\nu(A) \leq \nu(B)$. A capacity is **supermodular** if it is supermodular when 2^{Ω} is endowed with the set-inclusion order.

A binary relation R over $\mathcal{F}$ conforms to the **Choquet expected utility model** if there exists some $u : \Delta(Y) \to \mathbb{R}$ conforming to the von Neumann-Morgenstern axioms and a capacity ν on Ω for which the function $U : \mathcal{F} \to \mathbb{R}$ represents R, where

$$U(f) \equiv \int_{\Omega} u(f(\omega))\, d\nu(\omega);\ ^{3} \tag{2}$$

Schmeidler (1989) axiomatizes those R conforming to the Choquet expected utility model. A binary relation R which conforms to the Choquet expected

[3] The Choquet integral with respect to ν is defined as:

$$\begin{aligned} & \int_{\Omega} g(\omega)\, d\nu(\omega) \\ = \ & \int_{0}^{+\infty} \nu(\{\omega : g(\omega) > t\})\, dt + \int_{-\infty}^{0} [\nu(\{\omega : g(\omega) > t\}) - 1]\, dt \end{aligned}$$

utility model exhibits **Schmeidler uncertainty aversion** if and only if ν is supermodular.

For a given binary relation R over $\mathcal{F}$ conforming to the Choquet expected utility model, define the likelihood relation R^* over 2^{Ω} by ER^*F if there exist $x, y \in X$ for which $xP_R y$[4] and

$$\left[\begin{array}{c} x \text{ if } \omega \in E \\ y \text{ if } \omega \notin E \end{array}\right] R \left[\begin{array}{c} x \text{ if } \omega \in F \\ y \text{ if } \omega \notin F \end{array}\right].$$

The likelihood relation reflects a "willingness to bet." If ER^*F, then the individual prefers to place stakes on E as opposed to F. For the Choquet expected utility model, this relation is complete. We will write the asymmetric part of R^* by P^* and the symmetric part by I^*. The following is an immediate corollary to Theorem 1, using the fact that all likelihood relations are weakly increasing.

> PROPOSITION 13: *Suppose that R conforms to the Choquet expected utility model. Then the likelihood relation R^* is incompatible with Schmeidler uncertainty aversion if and only if there exist events $A, B, C \subseteq \Omega$ for which $A \subseteq B$ and $B \cap C = \varnothing$ for which $(A \cup C)\, P^* A$ and $(B \cup C)\, I^* B$.*

First, note that Schmeidler uncertainty aversion can never be refuted when the likelihood relation R^* is strictly monotonic. This observation follows from Proposition 13 for the same reason that Corollary 2 follows from Theorem 1. Hence, if an individual always strictly prefers to bet on larger events, one cannot refute that she is uncertainty averse.

Second, even in the case when R^* is not strictly monotonic, uncertainty aversion is difficult to refute. Using the notation in Proposition 13, it requires

[4]Here we are abusing notation by identifying a constant act with the value that constant act takes.

that adjoining C to A results in an event which is strictly more likely than A, yet adjoining C to a larger event B results in an event which is equally as likely as B. There do, however, exist such decision makers, and one example is a decision maker who views all nonempty events as equally likely and strictly more likely than the empty event.

Proposition 13 implies that, in many situations, there are no testable implications of Schmeidler uncertainty aversion from a purely ordinal standpoint, meaning it is not testable using only preferences over bets. If one could completely uncover R, then uncertainty aversion is testable because Schmeidler (1989) shows that there is a unique capacity that works in the representation. But to elicit R, one needs to observe preferences over acts that are not bets, and therefore entail attitudes towards risk. Only bets entail attitudes purely towards uncertainty. Schmeidler's definition of uncertainty aversion therefore requires observing preferences over risky acts. To some extent, this is well-known; indeed, Schmeidler's formal definition of uncertainty aversion makes explicit use of mixtures of acts.[5] But we believe our observation, that *most* likelihood relations are compatible with uncertainty aversion, is new. The first to argue that uncertainty aversion should be observable solely through preferences over bets was Epstein (1999); his work discusses this issue at length.

There are other theories of uncertainty aversion, due to Epstein (1999) and Ghirardato and Marinacci (2002). The theories of Epstein and Ghirardato and Marinacci are more general and are based on comparative notions of uncertainty aversion. The differences in the two theories are as to what they take to be the benchmark of "uncertainty neutral." Both theories have implications for the Choquet expected utility model.

[5]Schmeidler's definition in terms of R states that for any two acts $f, g \in \mathcal{F}$, and $\alpha \in [0,1]$, if fRg, then $\alpha f + (1-\alpha) gRg$.

8 Conclusion

We provide a characterization of the preferences which have a supermodular representation. For weakly monotonic preferences, supermodularity is equivalent to quasisupermodularity, Milgrom and Shannon's (1994) ordinal notion of complementarities.

Our results confirm the intuition of Allen (1934), Hicks and Allen (1934), Samuelson (1947), and Stigler (1950) that supermodularity is void of empirical meaning in ordinal economic environments. We have established this in two important economic models—a generalized version of Afriat's model and the Choquet expected utility model. We have shown that any strictly increasing function on a finite lattice can be ordinally transformed into a supermodular function. One issue that we have not dealt with is the theory of supermodular functions on infinite lattices. Obviously, not all strictly increasing binary relations on infinite lattices are representable by supermodular functions–just consider the standard lexicographic order on $\mathbb{R}^2$, which is not even representable. However, our results are meant to apply to testable environments, or environments in which a finite set of data can be observed. Thus, from an empirical perspective, we do not believe the restriction to finite lattices is problematic.

9 Proof of Theorem 1

In this section, we provide a proof of Theorem 1. We emphasize that the proof can be adapted to show the equivalence of quasisupermodularity and supermodularity for weakly decreasing binary relations.

In one direction, the theorem is trivial: if R has a weakly increasing supermodular representation, then this representation is also quasisupermodular.

In the rest of this section we show that, if R is weakly increasing and quasisupermodular, then it has a supermodular representation; the property of being weakly increasing is ordinal, so the resulting supermodular representation will be weakly increasing.

Let R be monotonic, quasisupermodular, and representable. We need some preliminary definitions.

A **multigraph** is a pair $G = (X, E)$, where X is a finite set and E is a matrix with $|X|$ columns; each column is identified with an element of X, and each row can be written as $1_x - 1_y$ for two distinct $x, y \in X$. Abusing notation, we identify rows $1_x - 1_y$ with the pair (x, y) and refer to (x, y) as an **edge**. Note that there may be more than one edge (x, y), as there may be more than one copy of the row $1_x - 1_y$ in E (herein lies the notational abuse).

A **cycle** for a multigraph (X, E) is a sequence $\{z_i\}_{i=1}^n$ with $(z_{i+1}, z_i) \in E$ (modulo n). We say that z_{i+1} **follows** z_i in the cycle.

Say that (X, E) can be **partitioned into cycles** $\left\{z_i^k\right\}_{i=1}^{n_k}$, $k = 1, \ldots, K$ if,

- for each k, $\left\{z_i^k\right\}_{i=1}^{n_k}$ is a cycle,
- $X = \cup\left\{z_i^k : i = 1, \ldots, n_k, k = 1, \ldots, K\right\}$,
- for each two distinct $x, y \in X$, the number of rows $1_x - 1_y$ equals the number of times x follows y in one of the cycles.

We state the following lemma without proof. The lemma follows from a straightforward modification of, for example, Diestel's (2000) Theorem 1.9.7 or Berge's (2001) Theorem 15.5.

Lemma 14: *If the sum of the rows of E equals the null vector in $\mathbb{R}^X$, then (X, E) has a partition into cycles.*

We define a **canonical multigraph** G as a multigraph, the edges of which can be partitioned into four edge sets: $E_R^G, E_P^G, E_\vee^G$, and $E_\wedge^G$, which satisfy the following seven properties:

i) If $(x, y) \in E_R^G$, then $(x, y) \in R \setminus P$

ii) If $(x, y) \in E_P^G$, then $(x, y) \in P$

iii) If $(x, y) \in E_\wedge^G$, then there exists $z \in X$ for which $y \parallel z$ and $x = z \wedge y$

iv) If $(x, y) \in E_\vee^G$, then there exists $z \in X$ for which $y \parallel z$ and $x = z \vee y$,

v) G can be partitioned into cycles

vi) $E_P^G \neq \varnothing$

vii) If $(y \wedge x, x) \in E_\wedge^G$ and $xP\,(y \wedge x)$, then $(x \vee y, y) \in E_\vee^G$.

The proof proceeds by contradiction. We first outline the steps involved. First, we show that if there does not exist a supermodular representation, then there exists a canonical multigraph. We then show that, for any canonical multigraph, there must exist $(y \wedge x, x) \in E_\wedge^G$ for which $xP\,(y \wedge x)$. We also show that if G is a canonical multigraph, there exists another canonical multigraph G' for which there are strictly less elements $(y \wedge x, x) \in E_\wedge^G$ for which $xP\,(y \wedge x)$. These latter two statements taken together are directly contradictory.

The new canonical multigraph will be constructed by adding a collection of edges to E_R^G which themselves can be partitioned into cycles. These additional edges are used to construct a large cycle which includes elements $(y \wedge x, x) \in E_\wedge^G$ for which $xP\,(y \wedge x)$. The large cycle will never contain an element $(x \vee y, y) \in E_{\vee}^{G}$ for which $(x \vee y)\, Py$ without containing a corresponding element $(x \wedge y, x) \in E_{\wedge}^{G}$. By removing this large cycle, we will have a new multigraph satisfying all of the appropriate conditions.

Step 1: Existence of a canonical multigraph

The following system of inequalities defines the problem of existence of a supermodular representation: For each $(x, y) \in R \setminus P$, $(1_x - 1_y) \cdot u \geq 0$. For each $(x, y) \in P$, $(1_x - 1_y) \cdot u > 0$. For each x and y with $x \parallel y$, $(1_{x \vee y} + 1_{x \wedge y} - 1_x - 1_y) \cdot u \geq 0$.

Suppose, by means of contradiction, that there is no supermodular representation of R. By the integer version of the Theorem of the Alternative, there exists a collection of non-negative integers $\left\{\eta_{(x,y)}\right\}$ for $(x, y) \in R \setminus P$, $\left\{\eta_{(x,y)}\right\}$ for $(x, y) \in P$ and $\left\{\eta_{\{x,y\}}\right\}$ for x and y with $x \parallel y$ such that

$$\begin{aligned} &\sum_{(x,y) \in R \setminus P} (1_x - 1_y)\, \eta_{(x,y)} \\ &+ \sum_{(x,y) \in P} (1_x - 1_y)\, \eta_{(x,y)} \\ &+ \sum_{\{x,y\} : x \parallel y} (1_{x \vee y} + 1_{x \wedge y} - 1_x - 1_y)\, \eta_{\{x,y\}} = 0, \end{aligned}$$

and such that $\eta_{(x,y)} > 0$ for at least one $(x, y) \in P$.

Construct a matrix B with one column for each element of X, and rows constructed as follows. For each $(x, y) \in R \setminus P$, there are $\eta_{(x,y)}$ rows equal to $1_x - 1_y$. For each $(x, y) \in P$, there are $\eta_{(x,y)}$ rows equal to $1_x - 1_y$. For each x and y with $x \parallel y$, there are *either* $\eta_{\{x,y\}}$ rows equal to $1_{x \wedge y} - 1_x$ and $\eta_{\{x,y\}}$ rows equal to $1_{x \vee y} - 1_y$ *or* $\eta_{\{x,y\}}$ rows equal to $1_{x \vee y} - 1_x$ and $\eta_{\{x,y\}}$ rows equal to $1_{x \wedge y} - 1_y$, but not both (we may choose arbitrarily).

By construction, to each row in B of type x and $x \wedge y$ there corresponds exactly one row with non-zero entries for $x \vee y$ and y. We denote this relation as $(x \vee y, y) = f(x \wedge y, x)$.

The matrix B defines a multigraph $G = (X, B)$. We show that G is a canonical multigraph. By construction, the rows of B sum to the zero vector, so Lemma 14 implies that the vertexes of G can be partitioned into a collection of cycles $\{z_i^k\}_{i=1}^{n_k}$, $k = 1, \ldots, K$. This verifies condition (v) in the definition of canonical multigraph. Denote the edges of G by E_R^G if they correspond to $(x, y) \in R \setminus P$ in the original system of inequalities. Similarly, denote by E_P^G the edges that correspond to P. Denote by $E_\vee^G$ and $E_\wedge^G$ the edges that correspond to a $(x \vee y, y)$ and $(x \wedge y, x)$ row in matrix B, respectively. Clearly, conditions $(i) - (iv)$ in the definition of canonical multigraph are now satisfied. Clearly, f is a one-to-one function from $E_\wedge^G$ onto $E_\vee^G$. This verifies that condition (vii) in the definition of canonical multigraph is satisfied.

There exists $(x, y) \in E_P^G$, because there exists x, y for which $\eta_{(x,y)} > 0$. This verifies condition (vi) in the definition.

Step 2: For any canonical multigraph, there exists $(y, x) \in E_\wedge^G$ for which xPy.

Let G be a canonical multigraph. Then there exists $(x, y) \in E_P^G$. Moreover, G can be partitioned into cycles, say $\{z_i^k\}_{i=1}^{n_k}$, $k = 1, ..., K$. Without loss of generality, $(x, y) = (z_2^1, z_1^1)$. The cycle $\{z_i^1\}_{i=1}^{n_1}$ must involve at least one pair (y', x') where y' follows x' in the cycle, and $E_\wedge^G$ with $x'Py'$. If it did not involve such a pair, then by the monotonicity of R, we would have $z_1^1 R^\tau z_2^1$, as all other edges in the cycle would correspond to elements of either E_P^G,E_R^G, $E_\vee^G$, or pairs $(\tilde{y}, \tilde{x}) \in E_\wedge^G$ for which $\tilde{y} R \tilde{x}$. But then $z_2^1 P z_1^1$ would contradict the representability of R, by Theorem 7. Thus, we have established that for a canonical multigraph G, there always exists at least one edge $(y, x) \in E_\wedge^G$ for which xPy.

Step 3: Construction of the new canonical multigraph (by constructing and deleting a large cycle).

Let $G = (X, B)$ be a canonical multigraph. The proof now proceeds by establishing that, any time there is an edge $(y, x) \in E_{\wedge}^{G}$, for which xPy, one can always construct a new matrix from B by deleting rows corresponding to elements of $E_{\wedge}^{G}$, $E_{\vee}^{G}$, or E_{R}^{G} containing some $(y, x) \in E_{\wedge}^{G}$ for which xPy, and which sum to zero. We will never remove a row corresponding to an element of E_{P}^{G}, but we may add new rows to B corresponding to x, y for which xRy.

For each $(y \wedge x, x) \in E_{\wedge}^{G}$ with $xP\,(y \wedge x)$, let $f\,(y \wedge x, x)$ denote $(x \vee y, y)$. Note that $f\,(y \wedge x, x) \in E_{\vee}^{G}$ by item (vii) of the definition of canonical multigraphs.

By Step 2, there exists $(y^*, x^*) \in E_{\wedge}^{G}$ for which x^*Py^*. We now construct a sequence of edges of G, $\{(x_l, y_l)\}_{l=1}^{\infty}$ such that $(x_1, y_1) = (x^*, y^*)$ and:

(a) If l is odd, $(y_l, x_l) \in E_{\wedge}^{G}$ and x_lPy_l.

(b) If l is even, $(x_l, y_l) \in E_{\vee}^{G}$ and x_lPy_l.

(c) For all l, $x_{l+1}Rx_l$.

We construct the sequence by induction: Let $(x_1, y_1) = (x^*, y^*)$, note that it satisfies (a). Given (x_l, y_l) that satisfies (a) and (b):

Suppose l is odd. Since $(y_l, x_l) \in E_{\wedge}^{G}$, $f(y_l, x_l)$ is well defined. Let $(x_{l+1}, y_{l+1}) = f(y_l, x_l)$. Then, quasisupermodularity of R and x_lPy_l imply that $x_{l+1}Py_{l+1}$. Here, by monotonicity of R, $x_{l+1}Rx_l$ because $x_{l+1} = x_l \vee y_l$.

Suppose now that l is even. Then there exists some cycle $\left\{z_i^k\right\}_{i=1}^{n_k}$ in which (x_l, y_l) lies. Without loss of generality, suppose that $x_l = z_2^k$ and $y_l = z_1^k$. Since x_lPy_l, by the same argument we used in Step 2, the representability of R implies that there is an edge $(y, x) \in E_{\wedge}^{G}$ with xPy and $y = z_{i_m+1}^k$, $x = z_{i_m}^k$

for some $i_m \geq 1$. Consider the smallest i_m corresponding to such an edge. Define $(y_{l+1}, x_{l+1}) = \left(z_{i_m+1}^k, z_{i_m}^k\right)$. For all $i \leq i_m$, $\left(z_{i+1}^k, z_i^k\right) \in E_R^G \cup E_P^G \cup E_\vee^G$, so it follows by monotonicity of R that

$$x_{l+1} = z_{i_m}^k R z_{i_{m-1}}^k \ldots z_{i_2}^k R z_2^k = x_l. \tag{3}$$

To show (c), note that we have $x_{l+1} R x_l$ when l is even by statement (3).

The sequence is infinite, and runs through the edges of $E_\vee^G$ and $E_\wedge^G$ (which are finite), so there must exist a cycle; without loss of generality, let us suppose that this cycle is $\{(x_l, y_l)\}_{l=1}^L$. (Note that there is no loss of generality in assuming that the cycle begins with $(y_1, x_1) \in E_\wedge^G$ for which $x_1 P y_1$ as the method in which the sequence was defined ensures that L is even, with $L/2$ elements corresponding to edges in $E_\wedge^G$ and $L/2$ elements corresponding to edges in $E_\vee^G$.)

Because $\{x_l\}$ is an R-cycle and R is representable, for all $l = 1, ..., L$, $(x_{l+1}, x_l) \in R \setminus P$ (here, $L + 1 = 1$). Moreover, this also implies that for all (x_{l+1}, x_l), the corresponding sequence lying in between the two

$$x_{l+1} = z_{i_m}^{k_l} R z_{i_{m-1}}^{k_l} \ldots z_3^{k_l} R z_2^{k_l} = x_l$$

also satisfies $\left(z_i^{k_l}, z_{i-1}^{k_l}\right) \in R \setminus P$.

When l is odd, $y_l = x_l \wedge y_{l+1}$ so $y_{l+1} R y_l$ by the monotonicity of R. But we must have $(y_{l+1}, y_l) \in R \setminus P$; if this were false, then $(y_{l+1}, y_l) \in P$ and quasisupermodularity of R would imply $x_{l+1} P x_l$, contradicting that $(x_{l+1}, x_l) \in R \setminus P$.

Construct a new matrix B' by adding a row corresponding to $1_{y_{l+1}} - 1_{y_l}$ and another row corresponding to $1_{y_l} - 1_{y_{l+1}}$ for all $l = 1, ...L$. The new rows of B' also sum to zero, and correspond to $R \setminus P$ relations. This new matrix B' corresponds to a new multigraph G' which is the same as G but with additional edges, (y_{l+1}, y_l), and (y_l, y_{l+1}), each lying in $E_R^{G'}$. The rows of B' also clearly sum to zero, as the rows we have added sum to zero.

We now exhibit a specific cycle in G'. The purpose of adding rows to G is to ensure that $(y_{l+1}, y_l) \in E_R^{G'}$ for all odd l. This allows us to construct the cycle represented in the following diagram, and described formally below.

$$\begin{tikzcd}
x_1 \arrow[d] & x_2 \arrow[r] & z_{i_2}^k \arrow[r] & \cdots & z_{i_{m-1}}^k \arrow[r] & x_3 \arrow[d] & x_4 \arrow[r] & \cdots & x_1 \arrow[d] \\
y_1 \arrow[r] & y_2 \arrow[u] & & & & y_3 \arrow[r] & y_4 \arrow[u] & \cdots & y_1
\end{tikzcd}$$

Where: $x_2 = x_1 \vee y_2$, $y_1 = x_1 \wedge y_2$, $x_4 = x_3 \vee y_4$, $y_3 = x_3 \wedge y_4$, $\ldots$

We have a collection of rows in B' that add to the null vector: the row corresponding to $(y_1, x_1) \in E_\wedge^{G'}$, $(y_2, y_1) \in E_R^{G'}$, $(x_2, y_2) \in E_\vee^{G'}$, $(z_2^k, x_2) \in E_R^{G'}, \ldots, (x_3, z_{i_{m-1}}^k) \in E_R^{G'}$, $(y_3, x_3) \in E_\wedge^{G'}$, and so on. The rows correspond to a cycle, as in the diagram above, so they add to the null vector and may be deleted from the matrix B', resulting in a new matrix B^*. The new matrix corresponds to a new multigraph G^* which is clearly canonical. In other words, we have not deleted any edges from $E_P^{G'}$, and the only edges $(x \vee y, x) \in E_\vee^{G'}$ deleted from $E_\vee^{G'}$ without deleting the corresponding $(x \wedge y, y) \in E_\wedge^{G'}$ satisfied $xR\,(x \vee y)$. By quasisupermodularity, it is therefore impossible that we have deleted $f(x \wedge y, x)$ corresponding to some $xP\,(x \wedge y)$ without deleting $(x \wedge y, y) \in E_\wedge^{G'}$. However, we have deleted at least one element $(x \wedge y, y) \in E_\wedge^{G'}$ corresponding to $yP\,(y \wedge x)$, as the cycle contained at least one such element.

$\square$

References

Afriat, S. N. (1967): "The Construction of Utility Functions from Expenditure Data," *International Economic Review*, 8(1), 67–77.

Allen, R. G. D. (1934): "A Comparison Between Different Definitions of Complementary and Competitive Goods," *Econometrica*, 2(2), 168–175.

Anscombe, F., and R. Aumann (1963): "A Definition of Subjective Probability," *Annals of Mathematical Statistics*, 34, 199–205.

Athey, S., P. Milgrom, and J. Roberts (1998): "Robust Comparative Statics," Mimeo, Stanford University.

Aumann, R. J. (1964): "Subjective Programming," in *Human Judgments and Optimality*, ed. by M. Shelley, and G. L. Bryan. Wiley. New York.

Becker, G. S. (1973): "A Theory of Marriage: Part I," *Journal of Political Economy*, 81(4), 813–846.

Berge, C. (2001): *Theory of Graphs*. Dover.

Chiappori, P.-A., and J.-C. Rochet (1987): "Revealed Preferences and Differentiable Demand," *Econometrica*, 55(3), 687–691.

Chipman, J. S. (1977): "An empirical implication of Auspitz-Lieben-Edgeworth-Pareto complementarity," *Journal of Economic Theory*, 14(1), 228–231.

Chung-Piaw, T., and R. Vohra (2003): "Afriat's Theorem and Negative Cycles," Mimeo, Northwestern University.

Diestel, R. (2000): *Graph Theory*. Springer.

Epstein, L. G. (1999): "A Definition of Uncertainty Aversion," *Review of Economic Studies*, 66(3), 579–608.

Fishburn, P. C. (1970): *Utility Theory for Decision Making.* Wiley, New York.

Fostel, A., H. Scarf, and M. Todd (2004): "Two New Proofs of Afriat's Theorem," *Economic Theory*, 24(1), 211–219.

Ghirardato, P., and M. Marinacci (2002): "Ambiguity Made Precise: A Comparative Foundation," *Journal of Economic Theory*, 102(2), 251–289.

Hicks, J. R., and R. G. D. Allen (1934): "A Reconsideration of the Theory of Value. Part I," *Economica*, 1(1), 52–76.

Kannai, Y. (2005): "Remarks concerning concave utility functions on finite sets," *Economic Theory,*, 26(2), 333–344.

Kreps, D. M. (1979): "A Representation Theorem for "Preference for Flexibility"," *Econometrica*, 47(3), 565–578.

Li Calzi, M. (1990): "Generalized Symmetric Supermodular Functions," Mimeo, Stanford University.

Matzkin, R. L. (1991): "Axioms of Revealed Preference for Nonlinear Choice Sets," *Econometrica*, 59(6), 1779–1786.

Milgrom, P., and J. Roberts (1990): "Rationalizability, Learning and Equilibrium in Games with Strategic Complementarities," *Econometrica*, 58(6), 1255–1277.

Milgrom, P., and C. Shannon (1994): "Monotone Comparative Statics," *Econometrica*, 62(1), 157–180.

Nehring, K. (1999): "Preference for Flexibility in a Savage Framework," *Econometrica*, 67(1), 101–119.

Quah, J. K.-H. (2004): "The Comparative Statics of Constrained Optimization Problems," Mimeo, Oxford University.

Richter, M. K. (1966): "Revealed Preference Theory," *Econometrica*, 34(3), 635–645.

Richter, M. K., and K.-C. Wong (2004): "Concave utility on finite sets," *Journal of Economic Theory*, 115(2), 341–357.

Samuelson, P. A. (1947): *Foundations of Economic Analysis*. Harvard University Press.

——— (1974): "Complementarity: An Essay on The 40th Anniversary of the Hicks-Allen Revolution in Demand Theory," *Journal of Economic Literature*, 12(4), 1255–1289.

Schmeidler, D. (1989): "Subjective probability and expected utility without additivity," *Econometrica*, 57(3), 571–587.

Shapley, L. S. (1971): "Cores of Convex Games," *International Journal of Game Theory*, 1(1), 11–26.

Stigler, G. J. (1950): "The Development of Utility Theory, II," *Journal of Political Economy*, 58(5), 373–396.

Topkis, D. M. (1978): "Minimizing a Submodular Function on a Lattice," *Operations Research*, 26(2), 305–321.

Topkis, D. M. (1979): "Equilibrium Points in Nonzero-Sum n-Person Submodular Games," *SIAM Journal of Control and Optimization*, 17(6), 773–787.

——— (1998): *Supermodularity and Complementarity.* Princeton University Press, Princeton, New Jersey.

Varian, H. R. (1982): "The Nonparametric Approach to Demand Analysis," *Econometrica*, 50(4), 945–974.

Vives, X. (1990): "Nash Equilibrium with Strategic Complementarities," *Journal of Mathematical Economics*, 19(3), 305–321.

——— (1999): *Oligopoly Pricing.* MIT Press, Cambridge, Massachusetts.

——— (2005): "Complementarities and Games: New Developments," *Journal of Economic Literature*, 43(2), 437–479.