(CHOQUET-) INTEGRATING OVER PRIORS: $\alpha(f)$-MEU

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Abstract. Every invariant biseparable preference relation can be represented by an integral on the set of continous affine mappings over a set of priors, where the integral is taken in the sense of Choquet. In looser terms: the representation takes the form of an integration over priors. As a by-product, we provide a novel interpretation of the $\alpha(f)$-MEU functional and of its relation with the Choquet integral.

1. Introduction

Invariant biseparable preferences (see [5], for an explanation of this terminology) constitute a wide class of preferences which includes as special cases several popular models of decision making: Subjective Expected Utility (SEU), Choquet Expected Utility (CEU), Maxmin Expected Utility (MEU), Maxmax Expected Utility as well as any fixed convex combination of Maxmin and Maxmax (α-MEU; $\alpha \in [0,1]$). Recently, Ghirardato, Maccheroni and Marinacci [3] (henceforth, GMM) have shown that every invariant biseparable preference is represented by a functional I which is convex combination of the Maxmin and the Maxmax functional, but the coefficient in the convex combination is allowed to vary with the act being evaluated. Besides this restriction, the functional I appears to be fairly arbitrary. In fact, an additional axiom (to the ones identifying invariant biseparable preferences) is called upon to pin down the exact form of the functional. Suggestively, though a bit imprecisely, this conforms to the interpretation provided by GMM: the set of priors appearing in the representation describes the ambiguity faced by the decision maker, while the additional axiom describes his attitudes toward ambiguity, thus leading to various criteria for decision making.

In this paper, we study the representation of invariant biseparable preferences provided by GMM. In Section 3, we observe that the additional axiom is really an axiom on the way the decision maker deals with the set of priors, thus confirming GMM interpretation. For instance, SEU corresponds to integrating over priors with respect to some probability measure, while other models correspond to other types of operations. Equipped with this observation, we look depeer into the correspondence between models of decision making and operations on the set priors. Our main result is perhaps surprising: only one type of operation suffices to characterize invariant biseparable preferences. We find that every invariant biseparable preference corresponds to the operation of integrating over priors with respect to some capacity, with integration being performed in the sense of Choquet. The converse is true as well: the operation of (Choquet-) integrating over priors produces an invariant biseparable preference.

Clearly, SEU is a special case of this procedure, which obtains when the capacity is a probability measure. Equivalently, all non-SEU models appear to be the generalization, moving from the case of a probability to that of a capacity, of the idea of integrating over priors. Different models of decision making thus correspond to different properties of the capacity with respect to which integration is performed. One of these models, the CEU model, looks, however, quite different from the others mentioned above. Because of this, in the final section we go back to the idea of integrating over priors. We show that the case of integrating with respect to a probability can be described in two equivalent ways. Each of these admit a fairly natural extension to the case of a capacity, but these two extensions are not equivalent: one, more permissive, leads to the functional form of GMM, the other to CEU.

2. Setting and notation

We consider the usual framework of decision making under uncertainty. This consists of four primitives: (1) A measurable space (S, Σ) – Σ a σ-algebra of subsets of S – which is called the state space; (2) A prize space X, assumed to be a mixture space ([2], [4]); (3) A set $\mathcal{F}$ of alternatives available to the decision maker, which are viewed as mappings $S \longrightarrow X$, and are called acts; (4) A preference relation $\succsim$ on $\mathcal{F}$.

Existence of a utility function $u : X \longrightarrow \mathbb{R}$ (guaranteed by the assumptions below) allows us to identify the set of acts with the set $B(S, \Sigma)$ of bounded real-valued Σ-measurable functions on S by means of the mapping $g \longmapsto u \circ g$. By a mild abuse of notation, in what follows we simply write g in the place of $u \circ g$.

Sometimes, we just write $B(S)$ in the place of $B(S, \Sigma)$. The notation $ba_1(\Sigma)$ stands for the set of finitely additive probability measures on Σ, and the weak*-topology on $ba_1(\Sigma)$ is the one produced by the duality $(ba(\Sigma), B(S))$. Let $M \subset ba_1(\Sigma)$. In the remainder of the paper, we will encounter three other spaces: (i) the space of all bounded, measurable functions on M, which is denoted by $B(M)$; it is understood that the σ-algebra on M is the one generated by the weak*-topology; (ii) the space of all weak*-continuous affine mappings on M, which is denoted by $A(M)$; and, finally (iii) the space of all weak*-continuous mappings on M, which is denoted by $\mathcal{C}(M)$.

3. Invariant biseparble preferences

Let $\mathcal{F}_c \subset \mathcal{F}$ denote the set of constant acts, that is acts h such that $h(s) = x \in X$ for any $s \in S$. The class of invariant biseparable preferences is identified by the following axioms:

A1 $\succsim$ is complete and transitive.

A2 (C-independence) For all $f, g \in \mathcal{F}$ and $h \in \mathcal{F}_c$ and for all $\gamma \in (0,1)$

$$f \succ g \qquad \Longleftrightarrow \qquad \gamma f + (1-\gamma)h \succ \gamma g + (1-\gamma)h$$

A3 (Archimedean property) For all $f, g, h \in \mathcal{F}$, if $f \succ g$ and $g \succ h$ then $\exists \gamma, \beta \in (0,1)$ such that $\gamma f + (1-\gamma)h \succ g$ and $g \succ \beta f + (1-\beta)h$.

A4 (Monotonicity) For all $f, g \in \mathcal{F}$, $f(s) \succsim g(s) \Longrightarrow f \succsim g$.

A5 (Non-degeneracy) $\exists x, y \in X$ such that $x \succ y$.

By adding one of the following alternative axioms to the five above, one obtains several well-known models:

A6 (a) (SEU, Anscombe and Aumann [2]) For all $f, g \in \mathcal{F}$ such that $f \sim g$, $\frac{1}{2}f + \frac{1}{2}g \sim g$;

A6 (b) (CEU, Schmeidler [9]) For all $f, g \in \mathcal{F}$ such that $f \sim g$, $\frac{1}{2}f + \frac{1}{2}g \sim g$ if f and g are comonotonic;

A6 (c) (MEU, Gilboa and Schmeidler [6]) For all $f, g \in \mathcal{F}$ such that $f \sim g$, $\frac{1}{2}f + \frac{1}{2}g \succsim g$;

A6 (d) (α-MEU, GMM [3]) For all $f, g \in \mathcal{F}$, $C^*(f) = C^*(g)$ implies $f \sim g$.[1]

3.1. The structure of invariant biseparable preferences.

GMM [3] have shown that every invariant biseparable preference is represented by a sup-norm continuous functional $I : B(S) \longrightarrow \mathbb{R}$ having the form

$$I(f) = (1 - \hat{\alpha}(f)) \min_{m \in M} \int f dm + \hat{\alpha}(f) \max_{m \in M} \int f dm \tag{3.1}$$

where $M \subseteq ba_1(\Sigma)$ is weak*-closed and convex.

That is, decision makers who obey axioms A1 to A5 are described by a criterion which is a convex combination of maxmin expected utility and maxmax expected utility. The coefficient in the convex combination may, however, depend on the act which is evaluated. As such, the functional I appears to satisfy very few restrictions: it is a sup-norm continuous functional minorized by the concave MEU functional and majorized by the convex Maxmax Expected Utility functional.

As it turns out, however, the mapping $\hat{\alpha} : B(S) \longrightarrow [0,1]$ in the above representation has an interesting property ([3], Theorem 11):

(1) If $f, g \in B(S)$ are such that $\left(\int f dm\right)_{m \in M} = \left(\int g dm\right)_{m \in M}$, then $\hat{\alpha}(f) = \hat{\alpha}(g)$.

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[1] For a definition of $C^*(\cdot)$, see GMM.

That is, the coefficient $\hat{\alpha}(\cdot)$ depends only on the profile of expected utilities associated to an act rather than on the act itself. In fact, more is true ([3], Theorem 11):

(2) If $f, g \in B(S)$ are such that $\left(\int f dm\right)_{m \in M} = \gamma \left(\int g dm\right)_{m \in M} + \delta \mathbf{1}$, where $\gamma > 0$, $\delta \in \mathbb{R}$ and $\mathbf{1}$ is the function $M \longrightarrow \mathbb{R}$ which is identically equal to 1, then $\hat{\alpha}(f) = \hat{\alpha}(g)$.

Let $A(M)$ be the set of weak*-continuous affine mappings on M and let $\kappa : B(S) \longrightarrow A(M)$ be defined by $f \longmapsto \psi_f$, where $\psi_f : M \longrightarrow \mathbb{R}$ is the function that at point $m \in M$ takes the value $\psi_f(m) = \int f dm$ (it is readily checked that $\psi_f \in A(M)$). Then, property (1) allows us to rewrite the functional I as

$$I(f) = V(\kappa(f)) = (1 - \alpha(\kappa(f))) \min_{m \in M} \kappa(f) + \alpha(\kappa(f)) \max_{m \in M} \kappa(f) \tag{3.2}$$

where $\alpha : A(M) \longrightarrow [0, 1]$ is defined by $\alpha(\kappa(f)) = \hat{\alpha}(f)$, which can be done in view of property (1). The mapping κ, which is the canonical mapping $B(S) \longrightarrow A(M)$, is sup-norm to sup-norm continuous and the functional $V : A(M) \longrightarrow \mathbb{R}$ – which we will refer to as the *GMM functional* – is sup-norm continuous (see Amarante [1] for details):

$$\begin{array}{ccc} B(S) & \xrightarrow{\kappa} & A(M) \\ & I \searrow & \downarrow V \\ & & \mathbb{R} \end{array}$$

This decomposition clarifies the nature of GMM representation: it shows that additional axioms of type A6 $(\cdot)$ affect the form of the functional I only in that they affect the form of the functional V. In other words, in the context of invariant biseparable preferences, we should be viewing axioms of type A6 $(\cdot)$ as restrictions on the form the functional V or, in more intuitive terms, as restrictions on the way the decision maker deals with the set of priors M. We can rephrase this in a more suggestive way: the representation of invariant biseparable preferences always takes the form of an "integration of priors". For instance, if we demand that V be linear, then $I = V \circ \kappa$ is precisely the usual integration over prior procedure, $I(f) = \displaystyle\int \kappa(f) dP$, which leads to SEU by virtue of a well-known argument. In all other cases, we may then give the same interpretation as long as we think of V as corresponding, loosely speaking, to some other notion of integration.

3.2. **Main result.** The interpretation of GMM's representation as an integration over priors theorem is much more compelling than what we said above. In the remainder of this paper, we are going to show that if $I : B(S) \longrightarrow \mathbb{R}$ represents an invariant biseparable preference, then it is always the case that

$$I(f) = \int \kappa(f) dC \tag{3.3}$$

for some capacity C on M. That is, I has exactly the form of an integration over priors but integration is performed in the sense of Choquet.

We will use this result to provide a new rationale for the $\alpha(f)$-MEU functional. In Section 7, we will show that there are two equivalent procedures describing integration over priors when the capacity C in (3.3) is a probability measure. Both of these admit a formal extension to the case of a capacity, but their equivalence breaks down, generally speaking, when we do so. Thus we can think of them as of two alternative, but equally legitimate, extensions of the usual notion of integrating over priors. We will show that one of these extensions corresponds precisely to the $\alpha(f)$-MEU functional, while the other corresponds to the notion of Choquet functional on $B(S)$ (the one that obtains in correspondence of axiom A6 (b).

4. Preliminary observations

The above considerations about the structure of invariant biseparable preferences tell us that all the action is in the GMM functional $V : A(M) \longrightarrow \mathbb{R}$, which thus becomes the object of our study. For reasons that will become clear momentarily, it will be convenient to consider functionals defined on a larger domain. Let $\mathcal{M}$ denote the Borel σ-algebra generated by the weak*-topology on M, and let $B(M)$ denote the Banach space (sup-norm) of all bounded, $\mathcal{M}$-measurable functions. Notice that $A(M)$ is a sup-norm closed linear subspace of $B(M)$. We are going to focus on functionals $W : B(M) \longrightarrow \mathbb{R}$ having the form

$$W(\psi) = (1 - \alpha(\psi)) \inf_{m \in M} \psi + \alpha(\psi) \sup_{m \in M} \psi \tag{4.1}$$

where $\psi \in B(M)$ and $\alpha : B(M) \longrightarrow [0, 1]$.

4.1. Choquet Integrals.

We have already seen that any SEU model obtains by means of an integration over priors. A similar fact is immediately seen to hold for many other models. In fact, if the mapping $\hat{\alpha}(\cdot)$ in (3.1) is constant, then the functional $V : A(M) \longrightarrow \mathbb{R}$ takes the form

$$V(\psi) = (1 - \alpha) \min_{m \in M} \psi + \alpha \max_{m \in M} \psi$$

for $\psi \in A(M)$. Clearly, one such a V admits an obvious (purely formal) extension to the whole $B(M)$, and it is immediate to verify that such an extension is monotone and comonotonic additive. Hence, as a (trivial) consequence of Schmeidler's theorem [8], we have

Proposition 1. *Every α-MEU is an integration over prior. That is,*

$$I(f) = V(\kappa(f)) = \int \kappa(f)dC$$

for some capacity $C : \mathcal{M} \longrightarrow [0,1]$.

Notice that, in particular, every Maxmin Expected Utility model obtains as a Choquet integration over priors. Proposition 1 motivates the following question:

(Q): Which $\alpha(f)$-MEU are Choquet integrations over priors?

4.2. **Capacities.** When V is extended to the whole $B(M)$ in a way that makes it monotone and comonotonic additive – like in the cases seen above – the capacity C is defined by the restriction of V to the indicator functions. For functionals having the form (4.1), this restriction coincides with the restriction of the function $\alpha(\cdot)$ to the indicator functions. In other words, for $E \in \mathcal{M}$ and for functionals having the form (4.1), we have

$$C(E) = V(\chi_E) = \alpha(\chi_E)$$

Looking at this from the reverse angle, a necessary condition for a V of the form (4.1) to be representable by a Choquet integral is that

$$V(\psi) = \int \psi dC \qquad \textit{with } C(E) = \alpha(\chi_E) \text{ for every } E \in \mathcal{M}$$

Remark 1. *Models of the type α-MEU are associated with the capacity C on $\mathcal{M}$ defined by*

$$\begin{aligned} C(E) &= \alpha \qquad \textit{if } E \notin \{\varnothing, M\} \\ C(\varnothing) &= 0 \qquad C(M) = 1 \end{aligned}$$

It is easily seen that for $\alpha \in (0,1)$ these capacities are neither concave nor convex, while $\alpha = 0$ and $\alpha = 1$ produce capacities that are convex and concave, respectively.

4.3. **Monotone, comonotonic additive $W : B(M) \longrightarrow \mathbb{R}$.** We now give necessary and sufficient conditions for a functional W of the form (4.1) to be comonotonic additive. Notice that since the functionals inf and sup are both monotone and comonotonic additive on $B(M)$ (this is so because $\mathcal{M}$ contains all the singletons), conditions guaranteeing that W is either monotone or comonotonic additive are conditions on the function $\alpha(\cdot)$ only. We have already observed that if $\alpha(\cdot)$ is constant, then W is both monotone and comonotonic additive.

In order to economize on notation, for $\psi \in B(M)$, let us set $si(\psi) = \sup\psi - \inf\psi$.

Proposition 2. *W is comonotonic additive if and only if*

(a) for any pair of non-constant comonotonic functions ψ, φ on $B(M)$, $\alpha(\cdot)$ is such that

$$\alpha(\psi + \varphi) = \alpha(\psi)\frac{si(\psi)}{si(\psi) + si(\varphi)} + \alpha(\varphi)\frac{si(\varphi)}{si(\psi) + si(\varphi)}$$

(b) for any non-constant $\psi \in B(M)$ and for any $\varphi \in B(M)$ such that $\varphi(m) \equiv x$ for each $m \in M$

$$\alpha(\psi + x) = \alpha(\psi)$$

Proof. The condition $W(\psi + \varphi) = W(\psi) + W(\varphi)$ is immediately seen to be equivalent to condition (a). The condition $W(\psi + x) = W(\psi) + x$ immediately seen to be equivalent to condition (b). □

Every comonotonic additive functional which is also monotone is positively homogeneous. It turns out that for functionals of the form (4.1) the monotonicity condition is not needed. That is, comonotonic functionals of the form (4.1) are automatically positively homogeneous (see Appendix).

5. The GMM functional $V : A(M) \longrightarrow \mathbb{R}$

The GMM functional $V : A(M) \longrightarrow \mathbb{R}$ defined in (3.2) enjoys the following properties:

(1) V is monotone ([3], Proposition 5);

(2) For any $\psi \in A(M)$, $y > 0$ and $x \in \mathbb{R}$, $\alpha(y\psi + x) = \alpha(\psi)$ ([3], Theorem 11).

It is immediate to check that property (2) of Section 3 implies that if $\varphi = y\psi + x$, then $V(\psi + \varphi) = V(\psi) + V(\varphi)$. That is, V is additive on isotonic (see below) mappings.

5.1. Comonotonic vs isotonic mappings on $B(M)$.

Two mappings, $\psi, \varphi \in B(M)$ are comonotonic if and only if $\forall m, m' \in M$

$$[\psi(m) - \psi(m')][\varphi(m) - \varphi(m')] \geq 0$$

Two mappings $\psi, \varphi \in B(M)$ are isotonic if and only if

$$\psi(m) \geq \psi(m') \qquad \Longleftrightarrow \qquad \varphi(m) \geq \varphi(m')$$

that is, if they order elements of M in the same way.

Clearly, two isotonic mappings are comonotonic but two comonotonic mappings, ψ and φ, may fail to be isotonic because we may have $\psi(m) = \psi(m')$ while $\varphi(m) < \varphi(m')$ or vice versa. However, the following simple lemma shows that converse is approximately true, in the sense that if ψ and φ are comonotonic and non-constant, then there exist mappings γ and γ' which are arbitrarily close to ψ and φ, respectively, and that are isotonic to each other. In particular, if ψ and φ are in $A(M)$, then the converse is exactly true.

Lemma 1. *If ψ and φ are non-constant and comonotonic, then $\forall x, x' \in (0, 1)$, the mappings $\gamma = x\psi + (1 - x)\varphi$ and $\gamma' = x'\psi + (1 - x')\varphi$ are isotonic. In particular, if ψ and φ are in $A(M)$, then ψ and φ are comonotonic if and only if they are isotonic.*

Proof. Let ψ and φ be non-constant and comonotonic, and let $\gamma(m)-\gamma(m')\geq 0$. Observe that we cannot have $\psi(m)-\psi(m')<0$ and $\varphi(m)-\varphi(m')>0$ or vice versa because ψ and φ are comonotonic. Hence, $\gamma(m)-\gamma(m')\geq 0$ implies both $\psi(m)-\psi(m')\geq 0$ and $\varphi(m)-\varphi(m')\geq 0$, which imply $\gamma'(m)-\gamma'(m')\geq 0$. In the other direction, $\gamma(m)-\gamma(m')<0$ implies that at least one between $\psi(m)-\psi(m')$ and $\varphi(m)-\varphi(m')$ is strictly negative and the other is non-positive because of the comonotonicity if ψ and φ. Hence, $\gamma'(m)-\gamma'(m')<0$, which completes the proof of the first part.

Now, suppose that ψ and φ are in $A(M)$. Then, both γ and γ' are in $A(M)$ as well. Since they are affine and isotonic, $\exists a>0$ and $b\in\mathbb{R}$ such that

$$\gamma' = a\gamma + b \tag{5.1}$$

In particular,[2]

$$a = \frac{\gamma'(\bar{m})-\gamma'(\underline{m})}{\gamma(\bar{m})-\gamma(\underline{m})}$$

where $\bar{m}$ and $\underline{m}$ are point of maximum and minimum, respectively ($\bar{m}$ and $\underline{m}$ exist because both γ and γ' are continuous on the compact set M; they can be taken to be the same for both functions because γ and γ' are isotonic). By setting $\hat{\psi}=\psi(\bar{m})-\psi(\underline{m})$ and $\hat{\varphi}=\varphi(\bar{m})-\varphi(\underline{m})$, the expression for a can be rewritten as $a=\frac{x'\hat{\psi}+(1-x')\hat{\varphi}}{x\hat{\psi}+(1-x)\hat{\varphi}}$, from which we see that a satisfies

$$(\hat{\psi}-\hat{\varphi})(ax-x') = (1-a)\hat{\varphi}$$

From (5.1), we have

$$\psi(x'-ax) = (x'-ax)\varphi - (1-a)\varphi + b$$

and combining the last two equations

$$\psi = \frac{\hat{\psi}}{\hat{\varphi}}\varphi + b$$

as claimed. □

The equivalence of comonotonicity and isotonicity for mappings in $A(M)$ now implies

Corollary 1. *The GMM functional $V: A(M)\longrightarrow\mathbb{R}$ defined in (3.2) is monotone and comonotonic additive.*

The latter result leads us to reformulate question (Q) asked at the end of subsection 2.1 as follows:

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[2]The expression of a is well-defined: neither γ nor γ' can be constant because of the comonotonicity and non-constancy of ψ and φ.

(Q'): Can V be represented by a Choquet integral with respect to some capacity?

The main difficulty in answering this question is that the subspace $A(M)$ does not contain the indicator functions on M. As a consequence, we cannot define the capacity by using V only. In fact, the trouble is greater than that: $A(M)$ is neither a lattice – which prevents us to use Zhou's theorem [10] – nor is a filtering family with respect to the space of continuous functions on M (in fact, $A(M)$ is norm-closed subspace in $\mathcal{C}(M)$), which prevents us to mimic Zhou's argument. These considerations lead us to yet another reformulation of question (Q) as follows:

(Q"): Does there exist a monotone, comonotonic additive extension $\bar{V}$ of V to the whole $B(M)$?

For if this is the case, then $\bar{V}$ would be representable by a Choqet integral, and we would get the conclusion that any $\alpha(f)$-MEU model can be represented by means of a Choquet integration over a set of priors.

6. Choquet-Integrating over priors: $\alpha(f)$-MEU

The above question is answered in the affirmative in the course of the proof of the theorem below. Thus, we have

Theorem 1. *A functional $I : B(S) \longrightarrow \mathbb{R}$ represents an invariant biseparable preference if and only if for any $f \in B(S)$*

$$I(f) = \int \kappa(f) dC$$

for some capacity $C : \mathcal{M} \longrightarrow [0,1]$, where κ is the canonical mapping $B(S) \longrightarrow A(M)$.

We are going to show that the GMM functional $V : A(M) \longrightarrow \mathbb{R}$ in (3.2) has a monotone, comonotonic additive extension to the whole $B(M)$. Then, the result will follow by applying Schmeidler's theorem [8]. Let K_0 denote the intersection of the positive cone with the unit ball in $B(M)$. In order to obtain the desired extension, it will be convenient to begin by restricting the mapping $\alpha : A(M) \longrightarrow [0,1]$ appearing in (3.2) to the domain $A(M) \cap K_0$, and then extend both α and the corresponding functional to the whole $B(M)$.

Proof. Begin with the functional V of (3.2), and consider the mapping $\alpha : A(M) \cap K_0 \longrightarrow [0,1]$. Now, let $\varphi \in K_0 \backslash A(M) \cap K_0$ and observe that φ cannot be constant because constant mappings are in $A(M)$. Hence, it follows that $si(\varphi) > 0$. Moreover, for $\psi \in A(M) \cap K_0$ and $x \in [0,1)$ we have $si(x\psi + (1-x)\varphi) > 0$ (because $si(x\psi + (1-x)\varphi) = 0$ implies $x\psi + (1-x)\varphi = y$ for some constant y, and this in turn implies $\varphi = -\frac{x}{1-x}\psi + \frac{1}{1-x}y$. Since the RHS is an element of $A(M)$, this contradicts $\varphi \notin A(M)$).

Now define

$$\tilde{\alpha}(\varphi) = \sup\left\{\frac{\inf\psi - \inf\varphi}{si(\varphi)} + \alpha(\psi)\frac{si(\psi)}{si(\varphi)}\right\} \tag{6.1}$$

where the sup is taken over all $\psi \in A(M) \cap K_0$ such that $\psi \leq \varphi$ [the set of $\psi \in A(M) \cap K_0$ such that $\psi \leq \varphi$ is always nonempty because the function identically equal to 0 is in $A(M) \cap K_0$]. Notice that $\tilde{\alpha}(\varphi) \in [0,1]$.

Now, $\forall x \in [0,1]$ and $\psi \in A(M) \cap K_0$ define

$$\begin{aligned} \tilde{\alpha}(x\psi + (1-x)\varphi) &= \alpha(\psi)\frac{si(x\psi)}{si(x\psi + (1-x)\varphi)} + \tilde{\alpha}(\varphi)\frac{si((1-x)\varphi)}{si(x\psi + (1-x)\varphi)} \\ & \quad \textit{and} \\ \tilde{\alpha}(\psi) &= \alpha(\psi) \qquad\qquad \textit{if } \psi \textit{ is constant} \end{aligned}$$

and notice that for $x = 1$ and ψ non-constant this coincides with the given $\alpha : A(M) \cap K_0 \longrightarrow [0,1]$. Notice also that for $\psi \equiv y \in \mathbb{R}$, we have $\tilde{\alpha}(xy + (1-x)\varphi) = \tilde{\alpha}(\varphi)$, where $0 \leq y < 1$.

Now, for all convex combinations of the form $x\psi + (1-x)\varphi$, define the functional

$$\begin{aligned} & \tilde{V}(x\psi + (1-x)\varphi) \qquad (6.2) \\ = \; & (1 - \tilde{\alpha}(x\psi + (1-x)\varphi)) \inf_{m \in M}(x\psi + (1-x)\varphi) + \tilde{\alpha}(x\psi + (1-x)\varphi) \sup_{m \in M}(x\psi + (1-x)\varphi) \end{aligned}$$

and observe that

(a) $\tilde{V}$ coincides with the original functional V on $A(M) \cap K_0$;

(b) $\tilde{V}$ is monotone: In light of the observation (a), it suffices to show that $\psi \leq \varphi$ implies $\tilde{V}(\psi) \leq \tilde{V}(\varphi)$ and $\psi \geq \varphi$ implies $\tilde{V}(\psi) \geq \tilde{V}(\varphi)$ (the case of convex combinations of the form $x\psi + (1-x)\varphi$ obviously reduces to one of these two cases). For the case $\psi \leq \varphi$, notice that the desired property is immediately delivered by the very definition (6.1) of $\tilde{\alpha}(\varphi)$. For the case $\psi \geq \varphi$, from (6.1) we have that $\forall \varepsilon > 0$, there exists a $\psi' \leq \varphi$, $\psi' \in A(M) \cap K_0$ such that

$$\tilde{V}(\psi') + \varepsilon \cdot si(\varphi) > \tilde{V}(\varphi)$$

Hence, $\forall \varepsilon > 0$

$$V(\psi) - V(\psi') - \varepsilon \cdot si(\varphi) = \tilde{V}(\psi) - \tilde{V}(\psi') - \varepsilon \cdot si(\varphi) < \tilde{V}(\psi) - \tilde{V}(\varphi)$$

and for $\varepsilon \to 0$, we have

$$0 \leq V(\psi) - V(\psi') = \tilde{V}(\psi) - \tilde{V}(\psi') \leq \tilde{V}(\psi) - \tilde{V}(\varphi)$$

where the first inequality follows from the fact that the functional V is monotone on its domain.

(c) For $0 \le y < 1$, $x \in [0,1]$ and $\psi \in A(M) \cap K_0$, $\tilde{V}(xy + (1-x)\psi) = xy + (1-x)\tilde{V}(\psi)$. In particular, $\tilde{V}((1-x)\psi) = (1-x)\tilde{V}(\psi)$.

(d) If $\psi \in A(M) \cap K_0$ and φ are comonotonic, then $\forall x \in [0,1]$, $\tilde{V}(x\psi + (1-x)\varphi) = x\tilde{V}(\psi) + (1-x)\tilde{V}(\varphi)$, for conditions (a) and (b) in Proposition (2) are satisfied for all convex combinations of the form $x\psi + (1-x)\varphi$ whenever ψ and φ are comonotonic.

By transfinite induction, the mapping $\tilde{\alpha}(\cdot)$ has an extension to the whole K_0 such that the corresponding functional $\tilde{V}$, as defined by (6.2), has all the properties (a) to (d) listed above. We still denote such extensions by $\tilde{\alpha}(\cdot)$ and $\tilde{V}$, respectively.

Now, **for any** $\psi \in B(M)$, ψ non-constant, define

$$
\begin{aligned}
\bar{\alpha}(\psi) &= \tilde{V}\left(\frac{\psi - \inf \psi}{si(\psi)}\right) = \tilde{V}(\psi') \\
&\quad \textit{and} \\
\bar{V}(\psi) &= (1 - \bar{\alpha}(\psi)) \inf_{m \in M} \psi + \bar{\alpha}(\psi) \sup_{m \in M} \psi
\end{aligned}
\tag{6.3}
$$

Since the function $\psi' = \left(\frac{\psi - \inf \psi}{si(\psi)}\right) \in K_0$, this is well-defined. Now, observe that

(a') $\bar{V} \mid_{A(M)} = V$: By definition for any non-constant $\psi \in A(M)$

$$
\begin{aligned}
\bar{\alpha}(\psi) &= \tilde{V}\left(\frac{\psi - \inf \psi}{si(\psi)}\right) = \tilde{V}(\psi') \\
&= V(\psi') \qquad \text{(by (a) above)} \\
&= \alpha(\psi') \qquad \text{(by definition of } V) \\
&= \alpha(\psi) \qquad \text{(by property (2) of } \alpha \text{, Section 3)}
\end{aligned}
$$

Hence, for any non-constant $\psi \in A(M)$, we have $\bar{V}(\psi) = V(\psi)$. Now, we can extend $\bar{V}$ to the subspace of constant functions by setting $\bar{V}(x) = V(x)$ for any $x \in \mathbb{R}$.

(b') $\bar{V}$ is positively homogeneous and constant-additive: For $x > 0$, $y \in \mathbb{R}$ and $\psi \in B(M)$,

$$
\begin{aligned}
\bar{V}(x\psi + y) &= \inf(x\psi + y) + \bar{\alpha}(x\psi + y) \cdot si(x\psi + y) \\
&= \inf(x\psi + y) + \tilde{V}\left(\frac{x\psi + y - \inf(x\psi + y)}{si(x\psi + y)}\right) \cdot si(x\psi + y) \\
&= \inf(x\psi + y) + \tilde{V}\left(\frac{\psi - \inf \psi}{si(\psi)}\right) \cdot si(x\psi + y) \\
&= y + x\bar{V}(\psi)
\end{aligned}
$$

(c') $\bar{V}$ is comonotonic additive: Let ψ and φ be comonotonic. In view of (b'), we can assume that they are both non-constant. We have

$$\begin{aligned}
\bar{V}(\psi+\varphi) &= \inf(\psi+\varphi)+\bar{\alpha}(\psi+\varphi)\cdot si(\psi+\varphi)\\
&= \inf(\psi+\varphi)+\tilde{V}\left(\frac{\psi+\varphi-\inf(\psi+\varphi)}{si(\psi+\varphi)}\right)\cdot si(\psi+\varphi)\\
&= \inf(\psi+\varphi)+\tilde{V}\left(\frac{\psi-\inf\psi}{si(\psi)}\frac{si(\psi)}{si(\psi+\varphi)}+\frac{\varphi-\inf\varphi}{si(\varphi)}\frac{si(\varphi)}{si(\psi+\varphi)}\right)\cdot si(\psi+\varphi)\\
&\quad \text{(by comonotonicity of } \psi \text{ and } \varphi)\\
&= \inf(\psi+\varphi)+\left\{\frac{si(\psi)}{si(\psi+\varphi)}\tilde{V}\left(\frac{\psi-\inf\psi}{si(\psi)}\right)+\frac{si(\varphi)}{si(\psi+\varphi)}\tilde{V}\left(\frac{\varphi-\inf\varphi}{si(\varphi)}\right)\right\}\cdot si(\psi+\varphi)\\
&\quad \text{(by property (d) above)}\\
&= \bar{V}(\psi)+\bar{V}(\varphi)
\end{aligned}$$

(d) $\bar{V}$ is monotone: Let $\psi \geq \varphi$. By using the definition of $\bar{V}$, it is easily checked that the statement is trivially true if either ψ or φ is constant. Hence, we can assume that they are both non-constant. There exists an $n \in \mathbb{N}$, such that both

$$0 \leq \frac{1}{n}\frac{\inf\psi-\inf\varphi}{si(\psi)} < 1 \qquad and \qquad 0 < \frac{1}{n}\frac{si(\varphi)}{si(\psi)} < 1$$

For such an n, $\frac{1}{n}\frac{\psi-\inf\psi}{si(\psi)} \in K_0$. Moreover, $\psi \geq \varphi$ is equivalent to

$$\frac{1}{2}\frac{1}{n}\frac{\psi-\inf\psi}{si(\psi)}+\frac{1}{2}\frac{1}{n}\frac{\inf\psi-\inf\varphi}{si(\psi)} \geq \frac{1}{2}\frac{1}{n}\frac{si(\varphi)}{si(\psi)}\frac{\varphi-\inf\varphi}{si(\varphi)}$$

Since both sides are elements in K_0, monotonicity of $\tilde{V}$ on K_0 implies

$$\tilde{V}\left(\frac{1}{2}\frac{1}{n}\frac{\psi-\inf\psi}{si(\psi)}+\frac{1}{2}\frac{1}{n}\frac{\inf\psi-\inf\varphi}{si(\psi)}\right) \geq \tilde{V}\left(\frac{1}{2}\frac{1}{n}\frac{si(\varphi)}{si(\psi)}\frac{\varphi-\inf\varphi}{si(\varphi)}\right)$$

By using property (c) above, this implies

$$\tilde{V}\left(\frac{\psi-\inf\psi}{si(\psi)}\right)+\frac{\inf\psi-\inf\varphi}{si(\psi)} \geq \frac{si(\varphi)}{si(\psi)}\tilde{V}\left(\frac{\varphi-\inf\varphi}{si(\varphi)}\right)$$

which, in turn, is equivalent to

$$\bar{V}(\psi) \geq \bar{V}(\varphi)$$

Summarizing, the functional $\bar{V} : B(M) \longrightarrow \mathbb{R}$ defined by

$$\bar{V}(\psi) = (1-\bar{\alpha}(\psi))\inf_{m\in M}\psi+\bar{\alpha}(\psi)\sup_{m\in M}\psi$$

is an extension of the GMM functional V in (3.2), and $\bar{V}$ is monotone and comonotonic additive. From Schmeidler's theorem [8], it follows that $\bar{V}(\psi) = \int \psi dC$, where the capacity C is defined by the restriction of $\bar{V}$ to the indicator functions. Hence, any invariant biseparable preference functional $I : B(S) \longrightarrow \mathbb{R}$ can be written as

$$I(f) = \int \kappa(f) dC$$

where κ is the canonical mapping $B(S) \longrightarrow A(M)$.

Conversely, any functional $B(M) \longrightarrow \mathbb{R}$ of the type $\bar{V}(\psi) = \int \psi dC$, defines a functional $B(S) \longrightarrow \mathbb{R}$ by means of $I(f) = \int \kappa(f) dC$, and the preference relation induced by the functional I thus defined satisfies axioms 1 to 5 of Section 3. □

6.1. **Monotone continuous preferences.** In [3], GMM introduced an additional axiom, called Monotone Continuity ([3], Sec B.3), which guarantees that all the priors in the representation (3.1) are countably additive. Moreover, the new axiom guarantees ([3], Sec B.3) that the set M is weak-compact and there exists a measure $\hat{m} \in ca(\Sigma)$ (ca = countably additive measures) such that all the priors in M are absolutely continuous with respect to $\hat{m}$. From these results, it follows (by using Radon-Nikodym) that the set of priors M is isometrically isomorphic to a subset of $\mathcal{L}^1(\hat{m})$, and hence metrizable. In such a case, the space $\mathcal{C}(M)$ of continuous functions on M is separable, and we can prove the same result as in Theorem 1 by using simple induction instead of transfinite induction. The proof goes as follows. First, one extends the GMM functional $V : A(M) \longrightarrow \mathbb{R}$ in (3.2) to a functional $\mathcal{C}(M) \longrightarrow \mathbb{R}$, which is monotone and comonotonic additive. The extension obtains exactly as in the proof of Theorem 1, but simple induction is now sufficient due to the separability of $\mathcal{C}(M)$. $\mathcal{C}(M)$ does not contain indicator functions, but now the capacity and the associated Choquet integral can be obtained by using Zhou's theorem [10].

7. The interpretation of the coefficient $\alpha(f)$: barycenters and adjoint

By Theorem 1, all models of decision making compatible with axioms 1 to 5 of Section 3 are straightforward generalizations of SEU: just replace a probability by a capacity when you integrate over priors. Yet, as we already noticed in Section 3, there is a wide variety of models compatible with axioms 1 to 5. Thus, it makes sense to take a closer look at the process of passing from a probability to a capacity. To this end, the notions of *adjoint of a linear operator* and of *barycenter of a probability measure* will come handy.

Let X and Y be Banach spaces and let $T : X \longrightarrow Y$ be a linear operator. Recall that the adjoint T^* of T is the linear operator $T^* : Y^* \longrightarrow X^*$ (where the $*$ denotes dual spaces) defined as follows: if $L \in Y^*$, then $T^*(L) = L \circ T$. We are going to be interested in the adjoint of the canonical mapping $\kappa : B(S) \longrightarrow A(M)$.

Since $\kappa(B(S)) = A(M)$ is a Banach space, the adjoint κ^* of κ is then the linear operator $\kappa^* : A^*(M) \longrightarrow B(S)^*$ defined by $\kappa^*(F) = F \circ \kappa$ for $F \in A^*(M)$. Notice that, in particular, for F represented by a probability measure P on M, we have $\kappa^*(P)(f) = \int \kappa(f)dP$, for any $f \in B(S)$.

The barycenter of a probability measure is defined as follows. Recall that the compact, convex set of priors M is a subset of $B(S)^*$, the dual of $B(S)$. By definition(see [7]), the barycenter of a probability P on M is a point $\mu \in M$ such that for any linear functional L in $B(S)^{**}$, we have

$$L(\mu) = \int LdP$$

Notice that, since in our setting every measure P on M has a unique barycenter ([7], Ch. 2), we have that for any $f \in B(S)$

$$\int \kappa(f)dP = \kappa(f)(\mu) = \int fd\mu$$

7.1. **The $\alpha(f)$-MEU representation of SEU.** In this subsection, we are going to see that there are two equivalent ways of describing SEU, that is the process of integrating over priors with respect to a probability. The first obtains by considering the adjoint κ^* of κ, while the second obtains by requiring that the coefficient $\alpha(f)$ in the GMM functional (3.2) be given a certain meaning. Later in the section, we will extend these two procedures by passing from the case of a probability to that of a general capacity. We will see that the equivalence between the two procedures breaks down when we do so: one extension, which turns out to be more permissive, leads to a general $\alpha(f)$-MEU model, while the other leads to the CEU model.

When $\alpha(f)$-MEU reduces to SEU, we have that for any $f \in B(S)$

$$I(f) = \int \kappa(f)dP$$

where P is a probability on M. By using the definitions of barycenter and adjoint given above, we also see that for any $f \in B(S)$

$$I(f) = \int \kappa(f)dP = \kappa(f)(\mu) = \int fd\mu = \kappa^*(P)(f)$$

The second equality says that by "integrating over priors" we get the barycenter μ of P, which thus defines the SEU functional. The last equality says that this is exactly what the adjoint does: it associates P with its barycenter μ, which again defines the SEU functional.

There is another way of describing the mechanics of SEU which calls into play another barycenter. This is intimately related to the coefficient $\alpha(f)$.

In fact, from the identities

$$\alpha(f) = \frac{\displaystyle\int \kappa(f)dP - \inf \kappa(f)}{\sup \kappa(f) - \inf \kappa(f)}$$

and

$$\int \kappa(f)dP - \inf \kappa(f) = \int P\{\kappa(f) > t\}\, dt$$

we see that $\alpha(f)$ is the *mean value* of the survival function associated to $\kappa(f)$. By letting F_P denote the distribution function of the random variable $\kappa(f)$, we also see that

$$\alpha(f) = \frac{\displaystyle\int t dF_P}{\sup \kappa(f) - \inf \kappa(f)} = \frac{mean\ \ of\ \ \kappa(f)}{\sup \kappa(f) - \inf \kappa(f)} = \frac{E(\kappa(f))}{\sup \kappa(f) - \inf \kappa(f)}$$

that is $\alpha(f)$ is the barycenter of the measure generated by F_P. Hence, the $\alpha(f)$-MEU functional can be written as

$$I = \inf \kappa(f) + bc(F_P) \cdot [\sup \kappa(f) - \inf \kappa(f)] = \inf \kappa(f) + E(\kappa(f))$$

where $bc(F_P)$ stands for the barycenter of the measure generated by F_P.

With a wild (yet suggestive) abuse of notation, we can summarize these two equivalent views by means of the diagram below: for each $f \in B(S)$

$$\begin{array}{ccc} F_P & \longleftarrow & P \\ \downarrow & & \downarrow \kappa^* \\ bc(F_P) & \xrightarrow{I} & \displaystyle\int f d\mu \end{array}$$

The RHS of the diagram describes the job of the adjoint: it associated P with its barycenter, and this defines a linear functional on $B(S)$. The other part of the diagram describes the second process: P determines the distribution of the r.v. $\kappa(f)$, which leads to the barycenter of the associated measure; then, the $\alpha(f)$-MEU functional is applied. For each $f \in B(S)$ the diagram commutes, thus expressing that these ways are equivalent.

7.2. **General $\alpha(f)$-MEU.** In the general case, by virtue of Theorem 1, we have $I(f) = \displaystyle\int \kappa(f)dC$, for some capacity C on M. This suggests the following question: starting from a functional $\displaystyle\int \kappa(f)dC$ on $A(M)$, how do we extend (from the case of a probability to that of a capacity) the two procedures described in the diagram above?

The second procedure (the one described by the LHS of the diagram) admits a straightforward generalization. In fact, we still have the identities

$$\alpha(f) = \frac{\int \kappa(f)dC - \inf \kappa(f)}{\sup \kappa(f) - \inf \kappa(f)}$$

and

$$\int \kappa(f)dC - \inf \kappa(f) = \int C\{\kappa(f) > t\}\, dt$$

Then, we can define – by (formal) analogy with the case of a probability – the same concepts of mean and barycenter, and proceed like above. By doing so, we obtain the $\alpha(f)$-MEU functional. In other words, *the $\alpha(f)$-MEU theorem appears to be exactly the generalization (to the case of a capacity) of the procedure described by LHS of the above diagram.*

7.3. **CEU preferences.** The procedure described by the adjoint seems, however, to naturally lend itself to a different type of generalization. To see this, let us first reconsider the case of a probability P on M. In such a case, we can think of the job of the adjoint as of consisting of two parts. First, the probability P is associated to its barycenter μ; then μ is extend to $B(S)$ by means of the formula

$$\int f d\mu = \int_0^{+\infty} \mu(f > t)dt + \int_{-\infty}^{0} [\mu(f > t) - 1]dt \tag{7.1}$$

The first part corresponds to computing, for each $A \in \Sigma$, $\int \chi_A dP$ which is equal (by definition of barycenter) to $\chi_A(\mu) = \int \chi_A d\mu = \mu(A)$. Such computation can be readily extended to the case of a capacity C on M by defining a capacity ν on Σ by means of the formula $\nu(A) = \int \chi_A dC$. Once this is done, the extension of ν to $B(S)$ can be obtained by means of the same formula (7.1). Clearly, this process leads to a Choquet integral on $B(S)$, that is to a CEU preference. As such, generally speaking, it does not coincide with the $\alpha(f)$-MEU functional (which corresponds to the "other" generalization).

The corollary below states an easy necessary and sufficient condition for an $\alpha(f)$-MEU preference to be of the CEU type. Let C be a capacity on M. Also let $\nu : \Sigma \longrightarrow [0,1]$ be defined by $\nu(A) = \int \chi_A dC$, for any $A \in \Sigma$, and set $E_\nu(f) = \int \nu(f > t)dt$ and $E_C(\kappa(f)) = \int C(\kappa(f) > t)dt$ (i.e., mean=integral of survival function). We have

Corollary 2 (CEU preferences). *An $\alpha(f)$-MEU preference relation*

$$f \succsim g \qquad iff \qquad \int \kappa(f) dC \geq \int \kappa(g) dC$$

for C a capacity on M, is a CEU preference relation if and only if

$$E_\nu(f) - E_C(\kappa(f)) = \inf \kappa(f) - \inf f$$

for every $f \in B(S)$.

Appendix

The following lemma and subsequent proposition describe a few more properties of functionals of the form (4.1).

Lemma 2. *If a comonotonic additive functional $W : B(M) \longrightarrow \mathbb{R}$ has the form (4.1), then for any non-constant $\psi \in B(M)$, we have $\alpha(y\psi) = \alpha(\psi)$ for any $y > 0$. In particular, for any any $y > 0$ and $x \in \mathbb{R}$, $\alpha(y\psi + x) = \alpha(\psi)$.*

Proof. For W is comonotonic additive, take $\psi = \varphi$ in condition (a) Proposition 2 to get $2\alpha(2\psi) = 2\alpha(\psi) \Longrightarrow \alpha(2\psi) = \alpha(\psi)$. In the same condition, take $\varphi = n\psi$, $n \in \mathbb{N}$, to get

$$(n+1)\alpha((n+1)\psi) = n\alpha(\psi) + \alpha(\psi)$$

Now, this and the previous one imply $\alpha(n\psi) = \alpha(\psi)$, $\forall n \in \mathbb{N}$. Hence,

$$\alpha(\psi) = \alpha(\frac{n}{n}\psi) = \alpha(\frac{1}{n}\psi) \qquad \forall n \in \mathbb{N}$$

and for $q \in \mathbb{Q}_{++}$

$$\alpha(q\psi) = \alpha(\frac{m}{n}\psi) = \alpha(\frac{1}{n}\psi) = \alpha(\psi)$$

Now, for $\lambda \in B(M)$ and non-constant, let $\psi = t_0\lambda$, $\varphi = h\lambda$ and let $t_0 > 0$. Consider the function $F_{t_0,\lambda} : \mathbb{R} \to \mathbb{R}$ defined by

$$F_{t_0,\lambda}(h) = t_0[\alpha((t_0 + h)\lambda) - \alpha(t_0\lambda)] + h[\alpha((t_0 + h)\lambda) - \alpha(h\lambda)]$$

Notice that $F_{t_0,\lambda}$ is defined on the whole line because $\alpha(\cdot)$ is defined on the whole $B(M)$, and that $F_{t_0,\lambda}(0) = 0$ (because $\alpha(\cdot)$ is bounded).

Now, notice that if $h > 0$, then $\psi = t_0\lambda$ and $\varphi = h\lambda$ are comonotonic, and condition (a) Proposition 2 implies that $F_{t_0,\lambda}(h) = 0$. This implies that $\lim_{h\to 0^+} F_{t_0,\lambda}(h) = 0$. Next, we are going to show that $\lim_{h\to 0^-} F_{t_0,\lambda}(h) = 0$. To this end, observe that the second addendum in the definition of $F_{t_0,\lambda}$ always goes to zero as h goes to zero because $\alpha(\cdot)$ is bounded. Hence,

$$\begin{aligned}
& \lim_{h\to 0^-} t_0[\alpha((t_0+h)\lambda)-\alpha(t_0\lambda)]+h[\alpha((t_0+h)\lambda)-\alpha(h\lambda)] \\
= & \lim_{h\to 0^-} t_0[\alpha((t_0+h)\lambda)-\alpha(t_0\lambda)]+h[\alpha(2(t_0+h)\lambda)-\alpha((t_0+2h)\lambda)] \\
& \text{(by the previous observation)} \\
= & \lim_{h\to 0^-} t_0[\alpha(2(t_0+h)\lambda)-\alpha(t_0\lambda)]+h[\alpha(2(t_0+h)\lambda)-\alpha((t_0+2h)\lambda)] \\
& \text{(by homogeneity over } \mathbb{N}\text{)}
\end{aligned}$$

Now, notice that for h sufficiently close to zero, $t_0+2h>0$. That is, for h sufficiently close to zero, the functions $\psi=t_0\lambda$ and $\varphi=(t_0+2h)\lambda$ are comonotonic, and $F_{t_0,\lambda}(h)=0$. This establishes $\lim_{h\to 0^-} F_{t_0,\lambda}(h)=0$. Since t_0 is an arbitrary positive number, this implies that $\forall t_0>0$

$$0=\lim_{h\to 0} F_{t_0,\lambda}(h)=\lim_{h\to 0}[\alpha((t_0+h)\lambda)-\alpha(t_0\lambda)]$$

That is, the function $\tilde{\alpha}_\lambda:\mathbb{R}_{++}\to\mathbb{R}$ defined by $\tilde{\alpha}_\lambda(t)=\alpha(t\lambda)$ is continuous at each $t>0$. From the preceding, we know that such a function is constant on $\mathbb{Q}_{++}$ and equal to $\alpha(\lambda)$. It follows, that the function is constant and equal to $\alpha(\lambda)$ on the whole $\mathbb{R}_{++}$. This completes the proof of the first statement. The second follows immediately by using condition (b) in Proposition 2. □

Proposition 3. *If W is comonotonic additive, then W is (i) positively homogeneous; and (ii) constant additive. If, in addition, W is monotone, then W is sup-norm continuous.*

Proof. For the first part, just observe that, in light of the previous lemma, for $y>0$, $x\in\mathbb{R}$ and $\forall\psi\in B(M)$

$$\begin{aligned}
W(y\psi+x) &= (1-\alpha(y\psi+x))\inf_{m\in M}(y\psi+x)+\alpha(y\psi+x)\sup_{m\in M}(y\psi+x) \\
&= (1-\alpha(\psi))\inf_{m\in M}(y\psi+x)+\alpha(\psi)\sup_{m\in M}(y\psi+x) \\
&= yW(\psi)+x
\end{aligned}$$

For the second, it is well-known (and easy to check) that monotone, constant additive functionals are sup-norm continuous. □

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