Revenue Maximization in the Dynamic Knapsack Problem

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Abstract

We characterize the revenue maximizing policy in the dynamic and stochastic knapsack problem where a given capacity needs to be allocated by a given deadline to sequentially arriving agents. Each agent is described by a two-dimensional type that reflects his capacity requirement and his willingness to pay per unit of capacity. Types are private information. We first characterize implementable policies. Then we solve the revenue maximization problem for the special case where there is private information about per-unit values, but weights are observable. After that we derive two sets of additional conditions on the joint distribution of values and weights under which the revenue maximizing policy for the case with observable weights is implementable, and thus optimal also for the case with two-dimensional private information. Finally, we analyze a simple policy for which per-unit prices vary with requested weight but do not vary with time. Its implementation requirements are similar to those of the optimal policy and it turns out to be asymptotically revenue maximizing when available capacity/ time to the deadline both go to infinity.

1 Introduction

The knapsack problem is a classic combinatorial optimization problem with numerous practical applications: several objects with given, known capacity requests (or weights) and given, known, values must be packed in a "knapsack" of given capacity in order to maximize the value of the included objects. In the dynamic and stochastic version (see Ross and Tsang [22]) objects sequentially arrive over time and their weight/value combination is stochastic but becomes known to the designer at arrival times.

In the present paper we add incomplete information to the dynamic and stochastic setting: there is a finite number of periods, and at each period a request for capacity

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arrives from an agent that is privately informed about both his valuation per unit of capacity and the needed capacity[1]. Each agent derives positive utility if he gets the needed capacity (or more), and zero utility otherwise. The designer accepts or rejects the requests in order to maximize the revenue obtained from the allocation.

The dynamic and stochastic knapsack problem with complete information about values and requests has been analyzed by Papastavrou, Rajagopalan and Kleywegt [18] and by Kleywegt and Papastavrou [13]. These authors have characterized optimal policies in terms of thresholds. Kincaid and Darling [11] , and Gallego and van Ryzin [7] look at a model that can be re-interpreted as having (one dimensional) incomplete information about values, but in their frameworks all requests have the same known weight[2]. In particular, Gallego and van Ryzin show that revenue is concave in capacity in the case of equal weights. Kleywegt and Papastavrou have examples showing that total value may not be concave in capacity if the weight requests are heterogeneous. Gershkov and Moldovanu [8] generalize the Gallego-van Ryzin model to incorporate objects with the same weight but with several qualities that are equally ranked by all agents, independently of their types (which are also one-dimensional).

The theory of multidimensional mechanism design is relatively complex: the main problem is that incentive compatibility - which in the one-dimensional case often reduces to a monotonicity constraint - imposes, besides a monotonicity requirement, an integrability constraint that is not easily included in maximization problems (see examples in Rochet [20], Armstrong [2], Jehiel, Moldovanu and Stacchetti [10], and the survey of Rochet and Stole [21]). Our implementation problem is special though because useful deviations in the weight dimension can only be one-sided (upwards). This feature allows us a less cumbersome characterization of implementable policies that can be embedded in the dynamic analysis under certain conditions on the joint distribution of values and weights of the arriving agents. Other multidimensional mechanism design problems with restricted deviations in one or more dimensions have been studied by Blackorby and Szalay [4], Iyengar and Kumar [9], Kittsteiner and Moldovanu [12], and Pai and Vohra [17].

Our main results characterize the revenue maximizing policy for the knapsack problem in several cases. The logic of the construction is as follows: We first characterize implementable policies, as explained above. Then we solve the revenue maximization problem for the special case where there is private information about per-unit values, but weights are observable: under a standard monotonicity assumption on virtual values, we show that this policy is Markovian, deterministic, and has a threshold property. It is important to emphasize that the resulting optimal policy need not be implementable for the case where both values and weights are unobservable, unless additional conditions are imposed. We then derive two sets of additional conditions

[1] The results are easily extended to the setting where arrivals are stochastic and/or time is continuous.

[2] We refer the reader to the book by Talluri and Van Ryzin [23] for references to the large literature on revenue (or yield) management that adopts variations on these models.

on the joint distribution of values and weights under which the revenue maximizing policy for the case with observable weights is implementable, and thus optimal also for the case with two-dimensional private information. These conditions - which are satisfied in a variety of intuitive settings - involve a form of positive correlation between weights and values expressed by a hazard rate ordering of conditional values, and a weakening of the first set of sufficient conditions in combination with the non-primitive assumption of concave revenues respectively.

Finally, we analyze a simple policy for which per-unit prices vary with requested weight but not with time. Its implementation requirements are similar to those of the optimal policy and it turns out to be asymptotically revenue maximizing when available capacity/ time to the deadline both go to infinity. This is particularly valuable since policies which lead to prices that sometimes decrease in time create incentive issues if agents are strategic with respect to their arrival times. We also point out that a policy that varies with time but not with requested weight (whose asymptotic optimality in the complete information case has been established by Lin, Lu and Yao [14]) is usually not optimal under incomplete information.

The paper is organized as follows: In Section 2 we present the dynamic model and the informational assumptions about values and weights. In Section 3 we characterize incentive compatible allocation policies. In Section 4 we focus on dynamic revenue maximization. We first characterize the revenue maximizing policy for the case where values are private information but weight requests are observable. We then offer two results that exhibit conditions under which the above policy is incentive compatible, and thus optimal also for the case where both values and weights are private information. In Section 5 we introduce a simpler time-independent policy as described above, analyze the limit case where the capacity and time to deadline become very large and demonstrate asymptotic optimality.

2 The Model

The designer has a "knapsack" of given capacity $C \in \mathbb{R}$ that he wants to allocate in a revenue-maximizing way to several agents in at most $T < \infty$ periods. In each period, an impatient agent arrives with a demand for capacity characterized by a weight or quantity request w and by a per-unit value v[3]. While the vector (w, v) is private information to the arriving agent, the designer is assumed to know the distribution of the random vector (w, v) which is given by the joint cumulative distribution function $F(w, v)$, with continuously differentiable density $f(w, v)$, defined on $[0, \infty)^2$. Demands are independent across different periods.

In each period, the designer decides how much capacity to allocate to the arriving agent (possibly none) and on a monetary payment. Type (w, v)'s utility is given by $wv - p$ if at price p he is allocated a capacity $w' \geq w$ and by $-p$ if he is assigned

[3]It is an easy extension to assume that the arrival probability per period is given by $p < 1$.

an insufficient capacity $w' < w$. Each agent observes the remaining capacity of the designer.[4] Finally, we assume strict monotonicity of the conditional virtual values, more precisely: for all w, $\hat{v}(v,w) := v - \frac{1-F(v|w)}{f(v|w)}$ is increasing in v with strictly positive derivative.

3 Incentive Compatible Policies

In this section, we characterize incentive compatible allocation policies. Without loss of generality, we restrict attention to direct mechanisms where every agent, upon arrival, reports a type (w, v) and where the mechanism then specifies an allocation and a payment. The schemes we develop also have an obvious and immediate interpretation as indirect mechanisms, where the designer sets a time- and capacity-dependent menu of per-unit prices, one for each weight demand.

An allocation policy is called *deterministic* and *Markovian* if, at any period $t = 1, ..., T$ and for any possible type of agent arriving at t, it uses a non-random allocation rule that only depends on the arrival time t, on the declared type of the arriving agent, and on the still available capacity at period t, denoted by c. The restriction to these policies is innocuous as shown in Section 4.

We can assume without loss of generality that a deterministic Markovian allocation policy for time t with remaining capacity c has the form $\alpha_t^c : [0, +\infty)^2 \to \{1, 0\}$ where 1 (0) means that the reported capacity demand w is satisfied (not satisfied). Indeed, it never makes sense to allocate an insufficient quantity $0 < w' < w$ because individually rational agents are not willing to pay for this. On the other hand, allocating more capacity than the reported demand is useless as well: Such allocations do not further increase agents' utility while they may decrease continuation values for the designer. Let $q_t^c : [0, +\infty)^2 \to \mathbb{R}$ be the associated payment rule.

Proposition 1 *A deterministic, Markovian allocation policy $\{\alpha_t^c\}_{t,c}$ is implementable if and only if for every t and every c it holds that:*

1. $\forall\, (w, v), v' \geq v,\ \alpha_t^c(w, v) = 1 \Rightarrow \alpha_t^c(w, v') = 1.$
2. *The function $wp_t^c(w)$ is non-decreasing in w, where* $p_t^c(w) = \inf\{v \ / \ \alpha_t^c(w, v) = 1\}$[5]

Proof. See Appendix. ■

The threshold property embodied in condition 1 of the above Proposition is standard, and is a natural feature of welfare maximizing rules under complete information. When there is incomplete information in the value dimension, this condition imposes

[4] Alternatively, we can assume that each agent observes the entire history of the previous allocations.

[5] We set $p_t^c(w) = \infty$ if the set $\{v/\alpha_t^c(w, v) = 1\}$ is empty.

limitations on the payments that can be extracted in equilibrium. Condition 2 is new: it reflects the limitations imposed in our model by the incomplete information in the weight dimension.

4 Dynamic Revenue Maximization

In this section, we first demonstrate how the dynamic revenue maximization problem may be solved if w is observable. Hence we first assume that there is incomplete information only about v. We then identify a set of conditions ensuring that the corresponding optimal policy is implementable even if w is not observable. The logic of the derivation for solving the revenue maximization problem is somewhat involved, and we now detail it below:

1. Without loss of generality, we can restrict attention to Markovian policies. The optimality of Markovian, possibly randomized, policies is standard for all models where, as is the case here, the per-period rewards and transition probabilities are history-independent - see for example Theorem 11.1.1 in Puterman [19] which shows that, for any history-dependent policy, there is a Markovian, possibly randomized, policy with the same payoff.

2. If there is incomplete information about v, but complete information about the weight requirement w, then Markovian, deterministic and implementable policies are characterized for each t and c by the threshold property of Condition 1 in Proposition 1.

3. Naturally, in the given revenue maximization problem with complete information about w we need to restrict attention to interim individually-rational policies where no agent ever pays more than the utility obtained from her actual capacity allocation. It is easy to see that, for any Markov, deterministic and implementable allocation policy α_t^c, the maximal, individually-rational payment function which supports it is given by

$$q_t^c(w,v) = \begin{cases} wp_t^c(w) & \text{if } \alpha_t^c(w,v) = 1 \\ 0 & \text{if } \alpha_t^c(w,v) = 0 \end{cases}$$

   where $p_t^c(w) = \inf\{v \;/\; \alpha_t^c(w,v) = 1\}$ as defined in the above section. Otherwise, the designer pays some positive subsidy to the agent, and this cannot be revenue-maximizing.

4. At each period t, and for each remaining capacity c, the designer's problem under complete information about w is equivalent to a simpler, one-dimensional static problem where a known capacity needs to be allocated to the arriving agent, and where the seller has a salvage value for each remaining capacity: the

salvage values in the static problem correspond to the continuation values in the dynamic version. Analogously to the analysis of Myerson [16], each static revenue-maximization problem has a monotone (in the sense of Condition 1 in Proposition 1), non-randomized solution as long as, for any weight w, the agent's conditional virtual valuation $v - \frac{1-F(v|w)}{f(v|w)}$ is increasing in v. Indeed, the expected revenue $R(c, T+1-t)$ if per-unit prices are set at $p_t^c(w)$ in period $t \leq T$ with remaining capacity c and if the optimal Markovian policy is followed from time $t+1$ onwards can be written as:

$$R(c, T+1-t) = \int_0^c w\, p_t^c(w)\,(1 - F(p_t^c(w)|w))\, \bar{f}_w(w)\, dw$$
$$+ \int_0^c [(1 - F(p_t^c(w)|w))R^*(c-w, T-t) + F(p_t^c(w)|w)R^*(c, T-t)]\, \bar{f}_w(w)\, dw,$$

where $\bar{f}_w$ denotes the marginal density in w, and where R^* denotes optimal revenues with $R^*(c, 0) = 0$ for all c. The first-order conditions for the revenue-maximizing unit prices $p_t^c(w)$ are given by:

$$w\left(p_t^c(w) - \frac{1 - F(p_t^c(w)|w)}{f(p_t^c(w)|w)}\right) = R^*(c, T-t) - R^*(c-w, T-t).$$

5. By backward induction, and by the above reasoning, the seller has a Markov, non-randomized optimal policy in the dynamic problem with complete information about w. Note also that, by a simple duplication argument, $R^*(c, T+1-t)$ must be monotone non-decreasing in c.

Points 1, 4 and 5 above imply that the restriction to the deterministic and Markovian allocation problems is without loss of generality. If the above solution satisfies the incentive compatibility constraint in the weight dimension, i.e. if $wp_t^c(w)$ happens to be monotone as required by Condition 2 of Proposition 1, then the associated allocation where $\alpha_t^c(w, v) = 1$ if and only if $v \geq p_t^c(w)$ is also implementable in the original problem with incomplete information about both v and w. It then constitutes the revenue maximizing scheme that we are after. The next example illustrates that Condition 2 of Proposition 1 can be binding.

Example 1 *Assume that $T = 1$. The distribution of the agents' types is given by the following stochastic process. First, the weight request w is realized according to an exponential distribution with parameter λ. Next, the per-unit value of the agent is sampled from the following distribution*

$$F(v|w) = \begin{cases} 1 - e^{-\overline{\lambda} v} & \text{if} \quad w > w^* \\ 1 - e^{-\underline{\lambda} v} & \text{if} \quad w \leq w^* \end{cases}$$

where $\overline{\lambda} > \underline{\lambda}$ and $w^* \in (0, c)$.

In this case, for an observable weight request, the seller charges the take-it-or-leave-it offer of $\frac{1}{\underline{\lambda}}$ ($\frac{1}{\overline{\lambda}}$) *per unit if the weight request is smaller (larger) than or equal to* w^*. *This implies that*

$$wp_t^c(w) = \begin{cases} \frac{w}{\overline{\lambda}} & \text{if} \quad w > w^* \\ \frac{w}{\underline{\lambda}} & \text{if} \quad w \leq w^* \end{cases} .$$

and therefore, $wp_t^c(w)$ *is not monotone.*

We next proceed to identify conditions on the distribution of types ensuring the monotonicity of $wp_t^c(w)$.

4.1 The Hazard Rate Stochastic Ordering

A key condition guaranteeing implementability is a stochastic ordering of the conditional distributions of per-unit values: the conditional distribution given a higher weight should be (weakly) statistically higher in the hazard rate order than the conditional distribution given a lower weight.

Theorem 1 *For each* c, t, *and* w *let* $p_t^c(w)$ *denote the solution to the revenue maximizing problem under complete information about* w, *determined recursively by the Bellman equation*

$$w\left(p_t^c(w) - \frac{1 - F(p_t^c(w)|w)}{f(p_t^c(w)|w)}\right) = R^*(c, T - t) - R^*(c - w, T - t). \tag{1}$$

Assume that the following conditions hold:

1. *For any* w, *the conditional hazard rate* $\frac{f(v|w)}{1-F(v|w)}$ *is non-decreasing in* v[6].
2. *For any* $w' \geq w$, *and for any* v, $\frac{f(v|w)}{1-F(v|w)} \geq \frac{f(v|w')}{1-F(v|w')}$.

Then, $wp_t^c(w)$ *is non-decreasing in* w, *and, consequently, the underlying allocation where* $\alpha_t^c(w, v) = 1$ *if and only if* $v \geq p_t^c(w)$ *is implementable. In particular, equations* (1) *characterize the revenue maximizing scheme under incomplete information about both values and weights.*

Proof. See Appendix. ■

An important special case for which the conditions of the above Theorem hold is the one where the distribution of per-unit values is independent of the distribution of weights, and has an increasing hazard rate.

[6]Note that this condition already implies the needed monotonicity in v of the conditional virtual value for all w.

4.2 Concavity of Expected Revenue in Capacity

A major result for the case where capacity comes in discrete units, and where all weights are equal is that expected revenue is concave in capacity (see Gallego and van Ryzin [7] for a continuous time framework with Poisson arrivals and Bitran and Mondschein [3] for a discrete time setting). This is a very intuitive property since it says that additional capacity is more valuable to the designer when capacity itself is scarce. Due to the more complicated combinatorial nature of the knapsack problem with heterogenous weights, concavity need not generally hold (see Papastavrou, Rajagopalan and Kleywegt [18] for examples where concavity of expected welfare in the framework with complete information fails).

Our main result in this subsection identifies a condition on the distribution of types that, together with concavity of the expected revenue in the remaining capacity, ensures that, for each t and c, $wp_t^c(w)$ is increasing, hence rendering the underlying distribution implementable. Afterwards we provide conditions on the models' primitives that are sufficient for the concavity of the expected revenue.

Theorem 2 *Assume that*

1. *The expected revenue $R^*(c, T+1-t)$ is a concave function of c for all times t.*
2. *For any $w \leq w'$, $v - \frac{1-F(v|w)}{f(v|w)} \geq \frac{vw}{w'} - \frac{1-F(\frac{vw}{w'}|w')}{f(\frac{vw}{w'}|w')}$.*

For each c, t, and w let $p_t^c(w)$ denote the solution to the revenue maximizing problem under complete information about w, determined recursively by equations (1)*. Then $wp_t^c(w)$ is non-decreasing in w, and hence the underlying allocation where $\alpha_t^c(w, v) = 1$ if and only if $v \geq p_t^c(w)$ is implementable. In particular, equation* (1) *characterizes the revenue maximizing scheme under incomplete information about both values and weights.*

Proof. See Appendix. ■

Remark 1 *The sufficient conditions for implementability used in Theorem 1 are, taken together, stronger than Condition 2 in Theorem 2. To see this, assume that, for any w, the conditional hazard rate $\frac{f(v|w)}{1-F(v|w)}$ is increasing in v, and that for any $w' \geq w$ and for all v, $\frac{f(v|w)}{1-F(v|w)} \geq \frac{f(v|w')}{1-F(v|w')}$. This yields:*

$$v - \frac{1-F(v|w)}{f(v|w)} \geq \frac{vw}{w'} - \frac{1-F(\frac{vw}{w'}|w)}{f(\frac{vw}{w'}|w)} \geq \frac{vw}{w'} - \frac{1-F(\frac{vw}{w'}|w')}{f(\frac{vw}{w'}|w')}$$

where the first inequality follows by the monotonicity of the hazard rate, and the second by the stochastic order assumption.
Theorem 2 and in particular its Condition 2 will also be useful when discussing the implementability of the simple policy in Section 5.2. Note also that Condition 2 of Theorem 2 can be formulated as requiring that the functions $\alpha v - \frac{1-F(\alpha v|\frac{w}{\alpha})}{f(\alpha v|\frac{w}{\alpha})}$ are non-decreasing in α.

Our next result identifies conditions on the joint distribution $F(w, v)$ that imply concavity of expected revenue with respect to c for all periods, as required by the above Theorem. It is convenient to introduce the joint distribution of weight and total valuation $u = vw$, which we denote by $G(w, u)$ with density $g(w, u)$. By means of a transformation of variables, the densities f and g are related by $w\, g(w, wv) = f(w, v)$. In particular, marginal densities in w coincide, i.e.

$$\bar{f}_w(w) = \int_0^\infty f(w, v)\, dv = \int_0^\infty g(w, u)\, du = \bar{g}_w(w).$$

An increasing virtual value implies that the virtual total value is increasing in u with strictly positive derivative for any given w:

$$\hat{u}(u, w) := u - \frac{1 - G(u|w)}{g(u|w)} = wv - \frac{1 - F(v|w)}{f(v|w)/w} = w\hat{v}(v, w)$$

We write $\hat{u}^{-1}(\hat{u}, w)$ for the inverse of $\hat{u}(u, w)$ with respect to u and define a distribution $\hat{G}(\hat{u}, w)$ by both $\hat{G}(\hat{u}|w) := G(\hat{u}^{-1}(\hat{u}, w)|w)$ for all w and $\bar{\hat{g}}_w(w) := \bar{g}_w(w)$. On the level of $\hat{v}$, this corresponds to $\hat{F}(\hat{v}|w) = F(\hat{v}^{-1}(\hat{v}, w)|w)$ and $\bar{\hat{f}}_w(w) = \bar{f}_w(w)$.

Theorem 3 *Assume that the conditional distribution $\hat{G}(w|\hat{u})$ is concave in w for all $\hat{u}$, that both $\hat{g}(w|\hat{u})$ and $\frac{d}{dw}\hat{g}(w|\hat{u})$ are bounded, and that the total virtual value $\hat{u}$ has a finite mean. Then, in the revenue maximization problem where the designer has complete information about w, the expected revenue $R^*(c, T + 1 - t)$ is concave as a function of c for all times t.*

Proof. See Appendix. ■

Example 2 *A simple example where the conditions of Theorem 2 are satisfied is obtained by assuming that $G(u, w)$ is such that u and w are independent, $u - \frac{1-G_u(u)}{g_u(u)}$ is differentiable with strictly positive derivative, and G_w is concave*[7]*. Condition 1 in that Theorem is satisfied by Theorem 3, while Condition 2 is satisfied since by independence $w'\hat{v}(\frac{vw}{w'}, w') = \hat{u}(vw, w') = \hat{u}(vw, w) = w\hat{v}(v, w)$ and $w \leq w'$ by assumption.*

5 Simple, Asymptotically Optimal Policies

The optimal policy characterized above seems too sensitive to be used in practice since it requires price adjustments in every period, and for any quantity request w. Our main result in the present section suggests that while exploiting dependency between w and v - if there is any - may be important for revenue maximization, carefully chosen dynamics are superfluous if both capacity and time go to infinity. As above, we start by focusing on the case of observable weights. We then show that the sufficient conditions

[7] We also assume that the other very mild technical conditions of Theorem 3 are satisfied.

identified in Theorem 1 are also applicable here, ensuring the implementability of the obtained policy.

Instead of solving the stochastic problem, we first solve a simpler, suitably chosen deterministic maximization problem. The revenue obtained in the solution to that problem provides an upper bound for the optimal expected revenue of the stochastic problem, and suggests the use of per-unit prices that depend on w, but that are constant in time. We next show that the derived policy is asymptotically optimal also in the original stochastic problem where both capacity and time go to infinity: the ratio of expected revenue from following the considered policy over expected revenue from the optimal Markovian policy converges to one. Moreover, there are various ways to quantify this ratio for moderately large capacities and time horizons. The basic logic hence follows a suggestion made by Gallego and van Ryzin [7]. However, our knapsack problem with a general distribution $F(w, v)$ is substantially more complex than the model tackled in their paper.

Let us first recall some assumptions, and introduce further notation. The marginal density $\bar{f}_w(w)$ and the conditional densities $f(v|w)$ pin down the distribution of (independent) arriving types $(w_t, v_t)_{t=1}^T$. Given w, the demanded per-unit price p and the probability λ^w of a request being accepted are related by $\lambda^w(p) = 1 - F(p|w)$. Let $p^w(\lambda)$ be the inverse of λ, and note that this is well defined on $(0, 1]$. Because of monotonicity of conditional virtual values, the instantaneous (expected) per-unit revenue functions $r^w(\lambda) := \lambda\, p^w(\lambda)$ are strictly concave, and each one attains a unique interior maximum. Indeed, $p^w(\lambda) = F(\cdot|w)^{-1}(1 - \lambda)$ and hence

$$\begin{aligned}
\frac{d}{d\lambda} r^w(\lambda) &= p^w(\lambda) - \lambda \frac{1}{f(p^w(\lambda)|w)} = p^w(\lambda) - \frac{1 - F(p^w(\lambda)|w)}{f(p^w(\lambda)|w)} = \hat{v}(p^w(\lambda), w); \\
\frac{d^2}{d\lambda^2} r^w(\lambda) &= -\left(\frac{\partial}{\partial v}\hat{v}\right)(p^w(\lambda), w)\frac{1}{f(p^w(\lambda)|w)} < 0.
\end{aligned}$$

Consequently, r^w is strictly concave, strictly increasing up to the $\lambda^{w,*}$ that satisfies $\hat{v}(p^w(\lambda^{w,*}), w) = 0$ and strictly decreasing from there on.

5.1 The Deterministic Problem

We now formulate an auxiliary deterministic problem. Let Cap : $(0, \infty) \to (0, \infty)$, $w \mapsto$ Cap(w) be a measurable function. Consider the problem:

$$\max_{\mathrm{Cap}(\cdot)} \int_0^\infty \max_{(\lambda_t^w)_{t=1,\ldots,T}} \left(\sum_{t=1}^T r^w(\lambda_t^w)\right) w \bar{f}_w(w)\, dw, \tag{2}$$

subject to

$$\sum_{t=1}^T \lambda_t^w\, w \bar{f}_w(w) \le \mathrm{Cap}(w) \text{ a.s. } \textbf{and} \int_0^\infty \mathrm{Cap}(w)\, dw \le C. \tag{3}$$

In words, we analyze a problem where:

1. The capacity C needs to be divided into capacities $\text{Cap}(w)$, one for each w.

2. In each w - subproblem, a deterministic quantity request of $w\bar{f}_w(w)$ arrives in each period, and λ_t^w determines a share (not a probability!) of this request that is accepted and sold at per-unit price $p^w(\lambda_t^w)$.

3. In each sub-problem, the allocated capacity over time cannot exceed $\text{Cap}(w)$, and total allocated capacity in all sub-problems $\int_0^\infty \text{Cap}(w)\,dw$, cannot exceed C.

4. The designer's goal is to maximize total revenue. We call the revenue at the solution $R^d(C,T)$.

As r^w is strictly concave and increasing up to $\lambda^{w,*}$, it is straightforward to verify that, given a choice $\text{Cap}(w)$, the solution to the w - subproblem,

$$\max_{(\lambda_t^w)_{t=1,\dots,T}} \left(\sum_{t=1}^{T} r^w(\lambda_t^w)\right) w\bar{f}_w(w) \text{ such that} \sum_{t=1}^{T} \lambda_t^w\, w\bar{f}_w(w) \le \text{Cap}(w)$$

is given by:

$$\lambda_t^w \equiv \lambda^{w,d} := \begin{cases} \lambda^{w,*} & \text{if } \lambda^{w,*} \le \frac{\text{Cap}(w)}{Tw\bar{f}_w(w)} \\ \frac{\text{Cap}(w)}{Tw\bar{f}_w(w)} & \text{else} \end{cases} \tag{4}$$

Accordingly, the revenue in the w-subproblem is $r^w(\lambda^{w,d})Tw\bar{f}_w(w)$.

Proposition 2 *The solution to the deterministic problem given by* (2) *and* (3) *is characterized by :*

1. $\hat{v}(p^w(\lambda^{w,d}), w) = \beta(C,T) = const$

2. $\lambda_t^w = \lambda^{w,d} = \frac{\text{Cap}(w)}{Tw\bar{f}_w(w)}$,

3. $\int_0^\infty \text{Cap}(w)\,dw = \min(C, T\int_0^\infty \lambda^{w,*} w\bar{f}_w(w)\,dw)$

Proof. See Appendix. ■

To get an intuition for the above result, observe that the marginal increase of the optimal revenue for the w-subproblem from marginally increasing $\text{Cap}(w)$ is:

$$\left(\frac{d}{d\lambda} r^w\right)\left(\frac{\text{Cap}(w)}{Tw\bar{f}_w(w)}\right) = \hat{v}(p^w(\lambda^{w,d}), w) \text{ if } \lambda^{w,*} > \frac{\text{Cap}(w)}{Tw\bar{f}_w(w)},$$

and 0 else.

Proposition 2 says that, optimally, the capacity should be split in such a way that the marginal revenue from increasing $\text{Cap}(w)$ is the same for all w. Actually solving the problem amounts to the simple static exercise of determining the constant $\beta(C,T)$ in accordance with the integral feasibility constraint.

The above construction is justified by the following result, showing that the optimal revenue in the deterministic problem bounds from above the optimal revenue in our original stochastic problem.

Since we assume here that weights are observable, a Markovian policy α for the original stochastic problem is characterized by the acceptance probabilities $\lambda_t^{w_t}[c_t]$ contingent on current time t, remaining capacity c_t and weight request w_t. Expected revenue from policy α at the beginning of period t (i.e. when there are $(T-t+1)$ periods left) with remaining capacity c_t is given by:

$$\begin{aligned} R_\alpha(c_t, T-t+1) &= E_\alpha\left[\sum_{s=t}^{T} w_s\, p^{w_s}(\lambda_s^{w_s}[c_s])\, I_{\{v_s \geq p^{w_s}(\lambda_s^{w_s}[c_s])\}}\right] \\ s.t.\ \sum_{s=t}^{T} w_s\, I_{\{v_s \geq p^{w_s}(\lambda_s^{w_s}[c_s])\}} &\leq c_t. \end{aligned}$$

Here, the constraint must hold almost surely when following α. As before, we write $R^*(c_t, T-t+1)$ for the optimal revenue, i.e. the supremum of expected revenues taken over all feasible Markovian policies α.

Theorem 4 *For any capacity C and deadline T, it holds that $R^*(C,T) \leq R^d(C,T)$.*

Proof. See Appendix. ■

5.2 A Simple Policy for The Stochastic Problem

Theorem 4 above suggests that a w-contingent yet time-independent pricing policy may be able to yield close to optimal revenues in the stochastic problem. To construct such a Markovian time-independent policy for the stochastic problem, α_{TI}, we proceed as follows:

1. Given C and T, solve the deterministic problem to obtain $\beta(C,T)$, $\lambda^{w,d}$ and thus $p^{w,d} := p^w(\lambda^{w,d}) = \hat{v}^{-1}(\beta(C,T), w)$.

2. In the stochastic problem charge these weight-contingent prices $p^{w,d}$ for the entire time horizon, provided that the quantity request does not exceed the remaining capacity. Else, charge a price equal to $+\infty$ (i.e., reject the request).

An important observation is that, under the conditions of Theorem 1, the time independent policy α_{TI} defined above is implementable also for the case that interests us, where weights are not observable. This follows immediately by recalling that the weight-contingent prices $p^{w,d}$ satisfy the equation $\hat{v}(p^w(\lambda^{w,d}), w) = \beta(C,T)$. Indeed, under the conditions of Theorem 1, the solution to this equation is monotonic in w, and hence $wp^{w,d}$ is also monotonic in w, as required for implementability. Moreover, implementability is even satisfied under the strictly weaker Condition 2 of Theorem 2, since setting all virtual valuation thresholds equal to a constant is like setting them optimally for linear and hence concave salvage values.

We now determine how well the time-independent policy constructed above performs compared to the optimal Markovian policy. We do this by comparing its expected revenue, $R^{TI}(C,T)$, with the optimal revenue in the deterministic problem, $R^d(C,T)$, which, as we know by Theorem 4, provides an upper bound for the optimal revenue in the stochastic problem, $R^*(C,T)$.

Theorem 5 *1. For any joint distribution of values and weights,*

$$\lim_{C,T\to\infty,\frac{C}{T}=const} \frac{R^{TI}(C,T)}{R^d(C,T)} = 1$$

2. Assume that w and v are independent. Then,

$$\frac{R^{TI}(C,T)}{R^d(C,T)} \geq \left(1 - \frac{\sqrt{E[w^2]/E[w]}}{2\sqrt{\min(C,\lambda^* E[w]T)}}\right).$$

In particular, $\lim_{\min(C,T)\to\infty} \frac{R^{TI}(C,T)}{R^d(C,T)} = 1$

Proof. See Appendix. ■

We have chosen to focus on these two general limit results. Various others could be proven by similar techniques at the expense of slightly more technical effort and possibly some further assumptions on F. As we indicated in the introduction, an interesting remark is that, since the policy α_{TI} is stationary, it does not generate incentives to postpone arrivals even in a more complex model where buyers are patient and can choose their arrival time.

Remark 2 *In a complete information knapsack model, Lin, Lu and Yao [14] study policies which start by accepting only high value requests, and then switch-over to accepting also lower values as time goes by. They establish asymptotic optimality of such policies (with carefully chosen switch-over times) as available capacity and time go to infinity. In other words, their prices are time-dependent but do not condition on the weight request. It is easy to show that, in our incomplete information model such policies are, in general, suboptimal. Consider first a one-period example where the seller has capacity 2, and where the arriving agent has either a weight request of 1 or 2 (equally likely). If the weight request is 1(2), the agent's per-unit value distributes uniformly between 0 and 1 (between 1 and 2). The optimal mechanism in this case is as follows: if the buyer requests one unit, the seller sells it for a price of 0.5, and if the buyer requests two units, the seller sells each unit at a price of 1. Note that this policy is implementable since the requested per-unit price is monotonically increasing in the weight request. The expected revenue is 9/8. If, however, the seller is forced to sell* *all units at the same per-unit price without conditioning on the weight request, he will charge the price of 1 for each unit, yielding an expected revenue of 1, and thus loose 1/8 versus the optimal policy. Replicate now this problem so that there are T periods and capacity C=2T. Then, the expected revenue from the optimal mechanism is 9/8T, while the expected revenue from the constrained mechanism is only T. Obviously, the constrained mechanism is not asymptotically optimal.*

6 Appendix

Proof of Proposition 1. **I)** $\Longrightarrow$ So assume that conditions 1 and 2 are satisfied and define for any t, c:

$$q_t^c(w,v) \;=\; \begin{cases} wp_t^c(w) & \text{if } \alpha_t^c(w,v) = 1 \\ 0 & \text{if } \alpha_t^c(w,v) = 0 \end{cases}$$

Consider then an arrival of type (w, v) in period t with remaining capacity c. There are two cases:

a) $\alpha_t^c(w,v) = 1$. In particular, $v \geq p_t^c(w)$. Then, truth-telling yields utility $w(v - p_t^c(w)) \geq 0$. Assume that the agent reports instead $(\widehat{w}, \widehat{v})$. If $\alpha_t^c(\widehat{w}, \widehat{v}) = 0$, then the agent's utility is zero and the deviation is not profitable. Assume then that $\alpha_t^c(\widehat{w}, \widehat{v}) = 1$. By the form of the utility function, a report of $\widehat{w} < w$ is never profitable. But, for $\widehat{w} \geq w$, the agent's utility is $wv - \widehat{w}p_t^c(\widehat{w}) \leq w(v - p_t^c(w))$, where we used condition 2. Therefore, such a deviation is also not profitable.

b) $\alpha_t^c(w,v) = 0$. In particular, $v \leq p_t^c(w)$. Truth-telling yields here utility of zero. Assume that the agent reports instead $(\widehat{w}, \widehat{v})$. If $\alpha_t^c(\widehat{w}, \widehat{v}) = 0$, then the agent's utility remains zero, and the deviation is not profitable. Assume then that $\alpha_t^c(\widehat{w}, \widehat{v}) = 1$. By the form of the utility function, a report of $\widehat{w} < w$ is never profitable. Thus, consider the case where $\widehat{w} \geq w$. In this case, the agent's utility is $wv - \widehat{w}p_t^c(\widehat{w}) \leq w(v - p_t^c(w)) \leq 0$, where we used condition 2. Therefore, such a deviation is also not profitable.

II) $\Longleftarrow$ Consider now an implementable, deterministic and Markovian allocation policy $\{\alpha_t^c\}_{t,c}$. Assume first, by contradiction, that condition 1 in the statement of the Proposition is not satisfied. Then, there exist (w, v) and (w, v') such that $v' > v$, $\alpha_t^c(w,v) = 1$ and $\alpha_t^c(w,v') = 0$. We obtain the chain of inequalities $wv' - q_t^c(w,v) > wv - q_t^c(w,v) \geq -q_t^c(w,v')$ where the second inequality follows by incentive compatibility for type (w, v). This shows that a deviation to a report (w, v) is profitable for type (w, v'), a contradiction to implementability. Therefore, condition 1 must hold.

In particular, note that for any two types who have the same weight request, (w, v) and (w, v'), if both are accepted, i.e. $\alpha_t^c(w,v) = \alpha_t^c(w,v') = 1$, the payment must be the same (otherwise the type which needs to make the higher payment would deviate and report the other type). Denote this payment by $r_t^c(w)$. Note also that any two types (w, v) and (w', v') such that $\alpha_t^c(w,v) = \alpha_t^c(w',v') = 0$ must also make the same payment (otherwise the type that needs to make the higher payment would deviate and report the other type) and denote this payment by s.

Assume now, by contradiction, that condition 2 does not hold. Then there exist w and w' such that $w' > w$ but $w'p_t^c(w') < wp_t^c(w)$. In particular, $w'p_t^c(w') < \infty$, and therefore $p_t^c(w') < \infty$.

Assume first that $p_t^c(w) < \infty$. We have $w'p_t^c(w') - r_t^c(w') = wp_t^c(w) - r_t^c(w) = -s$ because, by incentive compatibility, both types $(w, p_t^c(w))$ and $(w', p_t^c(w'))$ must be indifferent between getting their request and not getting it. Since by assumption $w'p_t^c(w') < wp_t^c(w)$, we obtain that $r_t^c(w') < r_t^c(w)$. Consider now a type (w, v) for which $v > p_t^c(w)$. By reporting truthfully, this type gets utility $wv - r_t^c(w)$, while by deviating to (w', v) he gets utility $wv - r_t^c(w') > wv - r_t^c(w)$, a contradiction to incentive compatibility.

Assume now that $p_t^c(w)$ is infinite, and therefore $wp_t^c(w)$ is infinite. Consider a type (w', v) where $v > p_t^c(w')$. The utility of this type is $w'v - r_t^c(w') > w'p_t^c(w') - r_t^c(w') = -s$. In particular, $r_t^c(w')$ must be finite. By reporting truthfully, a type (w, v) gets utility $-s$, while by deviating to a report of (w', v) he gets $wv - r_t^c(w')$. For v large enough, we obtain $wv - r_t^c(w') > -s$, a contradiction to implementability.

Thus, condition 2 must hold and, in particular, the payment $r_t^c(w)$ is monotonic in w. ■

Proof of Theorem 1. Let $w < w'$. We need to show that $wp_t^c(w) - w'p_t^c(w') \leq 0$. If $p_t^c(w) \leq p_t^c(w')$ the result is clear. Assume then that $p_t^c(w) > p_t^c(w')$. We obtain the following chain of inequalities:

$$\begin{aligned}
& w\left(\frac{1 - F(p_t^c(w)|w)}{f(p_t^c(w)|w)}\right) - w'\left(\frac{1 - F(p_t^c(w')|w')}{f(p_t^c(w')|w')}\right) \\
\leq \; & w'\left(\frac{1 - F(p_t^c(w)|w)}{f(p_t^c(w)|w)} - \frac{1 - F(p_t^c(w')|w')}{f(p_t^c(w')|w')}\right) \\
\leq \; & w'\left(\frac{1 - F(p_t^c(w')|w)}{f(p_t^c(w')|w)} - \frac{1 - F(p_t^c(w')|w')}{f(p_t^c(w')|w')}\right) \; \leq \; 0,
\end{aligned}$$

where the second inequality follows by the monotonicity of the hazard rate, and the third by the hazard rate ordering condition.

Since $R^*(c - w, T - t)$ is monotonically decreasing in w, we obtain that

$$\begin{aligned}
w\left(p_t^c(w) - \frac{1 - F(p_t^c(w)|w)}{f(p_t^c(w)|w)}\right) \; & \leq \; w'\left(p_t^c(w') - \frac{1 - F(p_t^c(w')|w')}{f(p_t^c(w')|w')}\right) \Leftrightarrow \\
wp_t^c(w) - w'p_t^c(w') \; & \leq \; w\left(\frac{1 - F(p_t^c(w)|w)}{f(p_t^c(w)|w)}\right) - w'\left(\frac{1 - F(p_t^c(w')|w')}{f(p_t^c(w')|w')}\right) \leq 0
\end{aligned}$$

where the last inequality follows by the derivation above. Hence $wp_t^c(w) - w'p_t^c(w') \leq 0$ as desired. ■

Proof of Theorem 2. For any concave function ϕ, and for any $x < y < z$ in its domain, the well known "Three Chord Lemma" asserts that

$$\frac{\phi(y)-\phi(x)}{y-x} \geq \frac{\phi(z)-\phi(x)}{z-x} \geq \frac{\phi(z)-\phi(y)}{z-y}$$

Consider then $w < w'$ and let $x = c - w' < y = c - w < z = c$. For the case of a concave revenue, the Lemma yields then:

$$\begin{aligned}\frac{R^*(c-w,T-t)-R^*(c-w',T-t)}{w'-w} &\geq \frac{R^*(c,T-t)-R^*(c-w',T-t)}{w'} \\ &\geq \frac{R^*(c,T-t)-R^*(c-w,T-t)}{w}.\end{aligned}$$

We obtain in particular

$$\begin{aligned}p_t^c(w') - \frac{1-F(p_t^c(w')|w')}{f(p_t^c(w')|w')} &= \frac{R^*(c,T-t)-R^*(c-w',T-t)}{w'} \\ \geq \frac{R^*(c,T-t)-R^*(c-w,T-t)}{w} &= p_t^c(w) - \frac{1-F(p_t^c(w)|w)}{f(p_t^c(w)|w)},\end{aligned}$$

which yields

$$p_t^c(w') - \frac{1-F(p_t^c(w')|w')}{f(p_t^c(w')|w')} \geq p_t^c(w) - \frac{1-F(p_t^c(w)|w)}{f(p_t^c(w)|w)} \geq \frac{w}{w'}p_t^c(w) - \frac{1-F(\frac{w}{w'}p_t^c(w)|w')}{f(\frac{w}{w'}p_t^c(w)|w')}$$

where the last inequality follows by the condition in the statement of the Theorem. Since virtual values are increasing, this yields $p_t^c(w') \geq \frac{w}{w'}p_t^c(w) \Leftrightarrow w'p_t^c(w') \geq wp_t^c(w)$ as desired. ■

For the proof of Theorem 3, we first need a Lemma on maximization of expected welfare under complete information. The result appears (without proof) in Papastavrou, Rajagopalan and Kleywegt [18].

Lemma 1 . *Assume that the total value u has finite mean, and that both $g(w|u)$ and $\frac{d}{dw}g(w|u)$ are bounded and continuous. Consider the allocation policy that maximizes expected welfare under complete information (i..e, upon arrival the agent's type is revealed to the designer). If $G(w|u)$ is concave in w for all u, then the optimal expected welfare, denoted U_t^c is twice continuously differentiable and concave in the remaining capacity c for all periods $t \leq T$.*

Proof. Note that, for notational convenience throughout this proof, we index optimal expected welfare by the current time t and not by periods remaining to deadline. By standard arguments, the optimal policy for this unconstrained dynamic optimization problem is deterministic and Markovian, and U_t^c is non-decreasing in remaining capacity c by a simple strategy duplication argument. Moreover, the optimal policy can be characterized by weight thresholds $w_t^c(u) \leq c$: If c remains at time t and a request whose acceptance would generate value u arrives, then it is accepted if and only if $w \leq w_t^c(u)$. If $U_{t+1}^c \geq u$, then the weight threshold must satisfy the indifference condition

$$u = U_{t+1}^c - U_{t+1}^{c-w_t^c(u)}. \tag{5}$$

Otherwise, we have $w_t^c(u) = c$.

We now prove the Lemma by backward induction. At time $t = T$, i.e. in the deadline period, it holds that

$$U_T^c = \int_0^\infty G(c|u) u\, \bar{g}_u(u)\, du.$$

This is concave in c because all $G(c|u)$ are concave by assumption, because $u\,\bar{g}_u(u)$ is positive, and because the distribution of u has a finite mean. Since both $g(w|u)$ and $\frac{d}{dw} g(w|u)$ are bounded and continuous, U_T^c is also twice continuously differentiable.

Suppose now that the Lemma has been proven down to time $t+1$. The optimal expected welfare at t provided that capacity c remains may be written as:

$$U_t^c = \int_0^\infty \left[uG(w_t^c(u)|u) + \int_0^{w_t^c(u)} U_{t+1}^{c-w} g(w|u)\, dw + (1 - G(w_t^c(u)|u)) U_{t+1}^c \right] \bar{g}_u(u)\, du. \tag{6}$$

We proceed to show concavity with respect to c of the term in brackets, for all u. This in turn implies concavity of U_t^c and hence, with a short additional argument for differentiability, is sufficient to conclude the induction step. We distinguish the cases $u > U_{t+1}^c$ for which the indifference condition (5) does not hold, and $u \leq U_{t+1}^c$ for which it does. For both cases, we demonstrate that the second derivative (one-sided if necessary) of the bracket term with respect to c is non-positive, and thus establish global concavity.

Case 1: $u > U_{t+1}^c$. The bracket term becomes $uG(c|u) + \int_0^c U_{t+1}^{c-w} g(w|u)\, dw + (1 - G(c|u)) U_{t+1}^c$. By continuity of U_{t+1}^c, this representation also holds in a small interval around c. We find

$$\begin{aligned}
&\frac{d}{dc} \left[uG(c|u) + \int_0^c U_{t+1}^{c-w} g(w|u)\, dw + (1 - G(c|u)) U_{t+1}^c \right] \\
&= ug(c|u) + \int_0^c \frac{d}{dc} U_{t+1}^{c-w} g(w|u)\, dw + U_{t+1}^0\, g(c|u) \\
&\quad - g(c|u)\, U_{t+1}^c + (1 - G(c|u)) \frac{d}{dc} U_{t+1}^c \\
&= (u - U_{t+1}^c) g(c|u) + \int_0^c \frac{d}{dc} U_{t+1}^{c-w} g(w|u)\, dw + (1 - G(c|u)) \frac{d}{dc} U_{t+1}^c
\end{aligned}$$

and

$$\begin{aligned}
&\frac{d^2}{dc^2}\left[uG(c|u) + \int_0^c U_{t+1}^{c-w} g(w|u)\,dw + (1-G(c|u))U_{t+1}^c\right] \\
= \;& (u - U_{t+1}^c)g'(c|u) - g(c|u)\frac{d}{dc}U_{t+1}^c + \int_0^c \frac{d^2}{dc^2}U_{t+1}^{c-w} g(w|u)\,dw \\
+ \;& \frac{d}{dw}U_{t+1}^w\big|_{w=0}\, g(c|u) - g(c|u)\frac{d}{dc}U_{t+1}^c + (1-G(c|u))\frac{d^2}{dc^2}U_{t+1}^c. \qquad (7)
\end{aligned}$$

The last term is non-positive by the concavity of U_{t+1}^c, the first term is non-positive because $u > U_{t+1}^c$ and because $G(c|u)$ has a non-increasing density by assumption. In addition, $g(c|u)\frac{d}{dc}U_{t+1}^c$ is non-negative, and hence (7) is bounded from above by

$$\int_0^c \frac{d^2}{dc^2}U_{t+1}^{c-w} g(w|u)\,dw + g(c|u)\left(\frac{d}{dw}U_{t+1}^w\big|_{w=0} - \frac{d}{dc}U_{t+1}^c\right).$$

But $\int_0^c \frac{d^2}{dc^2}U_{t+1}^{c-w} g(w|u)\,dw$ may be bounded from above by $g(c|u)\ \int_0^c \frac{d^2}{dc^2}U_{t+1}^{c-w}\,dw$ because of the decreasing density and because $\frac{d^2}{dc^2}U_{t+1}^{c-w} \leq 0$. Thus,

$$\begin{aligned}
&\frac{d^2}{dc^2}\left[uG(c|u) + \int_0^c U_{t+1}^{c-w} g(w|u)\,dw + (1-G(c|u))U_{t+1}^c\right] \\
\leq \;& g(c|u)\ \left[\int_0^c \frac{d^2}{dc^2}U_{t+1}^{c-w}\,dw + \frac{d}{dw}U_{t+1}^w\big|_{w=0} - \frac{d}{dc}U_{t+1}^c\right] \\
= \;& g(c|u)\ \left[\int_0^c \frac{d^2}{dw^2}U_{t+1}^{c-w}\,dw + \frac{d}{dw}U_{t+1}^w\big|_{w=0} - \frac{d}{dc}U_{t+1}^c\right] \;=\; 0. \qquad (8)
\end{aligned}$$

Case 2: $u \leq U_{t+1}^c$. Here $u = U_{t+1}^c - U_{t+1}^{c-w_t^c(u)}$. Consequently, the bracket term in (6) becomes

$$U_{t+1}^c - U_{t+1}^{c-w_t^c(u)} G(w_t^c(u)|u) + \int_0^{w_t^c(u)} U_{t+1}^{c-w} g(w|u)\,dw. \qquad (9)$$

Before computing its first and second derivatives, we differentiate the identity $u = U_{t+1}^c - U_{t+1}^{c-w_t^c(u)}$ to obtain an expression for $\frac{d}{dc}w_t^c(u)$ (derivative from the right if $u = U_{t+1}^c$):

$$0 \;=\; \frac{d}{dc}U_{t+1}^c - \frac{d}{dw}U_{t+1}^w\big|_{w=c-w_t^c(u)}\left(1 - \frac{d}{dc}w_t^c(u)\right).$$

Since indeed $\frac{d}{dw}U_{t+1}^w > 0$ in our setup with strictly positive densities, this implies

$$\frac{d}{dc}w_t^c(u) \;=\; \frac{\frac{d}{dw}U_{t+1}^w\big|_{w=c-w_t^c(u)} - \frac{d}{dc}U_{t+1}^c}{\frac{d}{dw}U_{t+1}^w\big|_{w=c-w_t^c(u)}}. \qquad (10)$$

By concavity of U_{t+1}^c, its derivative is non-increasing and hence the identity (10) yields in particular $\frac{d}{dc}w_t^c(u) \geq 0$. We now compute the derivatives of (9):

$$\begin{aligned}
&\frac{d}{dc}\left[U_{t+1}^c - U_{t+1}^{c-w_t^c(u)} G(w_t^c(u)|u) + \int_0^{w_t^c(u)} U_{t+1}^{c-w} g(w|u)\, dw\right] \\
&= \frac{d}{dc}U_{t+1}^c - \frac{d}{dw}U_{t+1}^w\Big|_{w=c-w_t^c(u)}\left(1 - \frac{d}{dc}w_t^c(u)\right) G(w_t^c(u)|u) - U_{t+1}^{c-w_t^c(u)} g(w_t^c(u)|u)\frac{d}{dc}w_t^c(u) \\
&+ U_{t+1}^{c-w_t^c(u)} g(w_t^c(u)|u)\frac{d}{dc}w_t^c(u) + \int_0^{w_t^c(u)} \frac{d}{dc}U_{t+1}^{c-w} g(w|u)\, dw \\
&\overset{(10)}{=} \frac{d}{dc}U_{t+1}^c - \frac{d}{dc}U_{t+1}^c\, G(w_t^c(u)|u) + \int_0^{w_t^c(u)} \frac{d}{dc}U_{t+1}^{c-w} g(w|u)\, dw \\
&= \frac{d}{dc}U_{t+1}^c(1 - G(w_t^c(u)|u)) + \int_0^{w_t^c(u)} \frac{d}{dc}U_{t+1}^{c-w} g(w|u)\, dw.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\frac{d^2}{dc^2}\left[U_{t+1}^c - U_{t+1}^{c-w_t^c(u)} G(w_t^c(u)|u) + \int_0^{w_t^c(u)} U_{t+1}^{c-w} g(w|u)\, dw\right] \\
&= \frac{d^2}{dc^2}U_{t+1}^c(1 - G(w_t^c(u)|u)) - \frac{d}{dc}U_{t+1}^c\, g(w_t^c(u)|u)\frac{d}{dc}w_t^c(u) \\
&+ \frac{d}{dw}U_{t+1}^w\Big|_{w=c-w_t^c(u)} g(w_t^c(u)|u)\frac{d}{dc}w_t^c(u) + \int_0^{w_t^c(u)} \frac{d^2}{dc^2}U_{t+1}^{c-w} g(w|u)\, dw \\
&\leq g(w_t^c(u)|u)\frac{d}{dc}w_t^c(u)\left(\frac{d}{dw}U_{t+1}^w\Big|_{w=c-w_t^c(u)} - \frac{d}{dc}U_{t+1}^c\right) + \int_0^{w_t^c(u)} \frac{d^2}{dw^2}U_{t+1}^{c-w} g(w|u)\, dw.
\end{aligned}$$

For the final inequality we used concavity of U_{t+1}^c, as well as $\frac{d^2}{dc^2}U_{t+1}^{c-w} = \frac{d^2}{dw^2}U_{t+1}^{c-w}$. Noting that (10) implies that $\frac{d}{dc}w_t^c(u) \leq 1$ and once more using concavity of U_{t+1}^c, we may bound the first term from above. Since $g(w|u)$ is non-increasing in w, we can also bound the second term to obtain:

$$\begin{aligned}
&\frac{d^2}{dc^2}\left[U_{t+1}^c - U_{t+1}^{c-w_t^c(u)} G(w_t^c(u)|u) + \int_0^{w_t^c(u)} U_{t+1}^{c-w} g(w|u)\, dw\right] \qquad (11) \\
&\leq g(w_t^c(u)|u)\left(\frac{d}{dw}U_{t+1}^w\Big|_{w=c-w_t^c(u)} - \frac{d}{dc}U_{t+1}^c + \int_0^{w_t^c(u)} \frac{d^2}{dw^2}U_{t+1}^{c-w}\, dw\right) = 0.
\end{aligned}$$

Taken together, (8) and (11) establish concavity of the integrand in (6) with respect to c. This implies that U_t^c is concave. Having a second look at the computations just done reveals that the integrand in (6) has a kink in the second derivative at $u = U_{t+1}^c$. However, this event has measure zero for any given c, so that we also get that U_t^c is twice continuously differentiable. This completes the induction step. ■

Proof of Theorem 3. The main idea of the proof is to translate the problem of setting revenue-maximizing prices when w is observable into the problem of maximizing welfare with respect to virtual values (rather than the values themselves), and then to use Lemma 1.

To begin with, note that there is a dual way to describe the policy that maximizes expected welfare under complete information. In the proof of Lemma 1, we characterized it by optimal weight thresholds $w_t^c(u)$. Alternatively, given any requested quantity w, (not greater than the remaining c) we may set a valuation per unit threshold $v_t^c(w)$. Requests above this valuation are accepted, those below are not. Optimal such thresholds are characterized by the Bellman-type condition:

$$w\, v_t^c(w) \;=\; U_{t+1}^c - U_{t+1}^{c-w}. \tag{12}$$

Thus, one way of writing the optimal expected welfare under complete information is:

$$\begin{aligned} U_t^c \;&=\; \int_0^c w \int_{v_t^c(w)}^\infty v f(v|w)\, dv\, \bar{f}_w(w)\, dw \\ &+ \int_0^c \left[(1 - F(v_t^c(w)|w))U_{t+1}^{c-w} \;+\; F(v_t^c(w)|w)U_{t+1}^c\right] \bar{f}_w(w)\, dw. \end{aligned} \tag{13}$$

In contrast, the optimal expected revenue with complete information about w but incomplete information about v satisfies:

$$\begin{aligned} R^*(c, T+1-t) \;&=\; \int_0^c w\, p_t^c(w)\, (1 - F(p_t^c(w)|w))\, \bar{f}_w(w)\, dw \\ &+ \int_0^c [(1 - F(p_t^c(w)|w))R^*(c-w, T-t) \;+\; F(p_t^c(w)|w)R^*(c, T-t)]\; \bar{f}_w(w)\, dw, \end{aligned} \tag{14}$$

where $p_t^c(w)$ are the per-unit prices from (1). We rephrase this in terms of $\hat{F}$, whose definition required monotonicity of virtual values. Setting $\hat{v}_t^c(w) := \hat{v}(p_t^c(w), w)$ we have on the one hand:

$$F(p_t^c(w)|w) \;=\; \hat{F}(\hat{v}_t^c(w)|w).$$

On the other hand:

$$\begin{aligned} p_t^c(w)\, (1 - F(p_t^c(w)|w)) \;&=\; \int_{p_t^c(w)}^\infty [v\, f(v|w) - (1 - F(v|w))]\; dv \\ &=\; \int_{p_t^c(w)}^\infty \hat{v}(v, w)\, \hat{f}(\hat{v}(v, w)|w) \frac{d}{dv}\hat{v}(v, w)\, dv \\ &=\; \int_{\hat{v}_t^c(w)}^\infty \hat{v}\, \hat{f}(\hat{v}|w)\, d\hat{v}. \end{aligned}$$

Plugging this and the identities for the marginal densities in w into (14), we obtain:

$$\begin{aligned} R^*(c, T+1-t) \;&=\; \int_0^\infty w \int_{\hat{v}_t^c(w)}^\infty \hat{v} \hat{f}(\hat{v}|w)\, d\hat{v}\, \bar{\hat{f}}_w(w)\, dw \\ &+ \int_0^\infty \left[(1 - \hat{F}(\hat{v}_t^c(w)|w))R^*(c-w, T-t) \;+\; \hat{F}(\hat{v}_t^c(w)|w)R^*(c, T-t)\right] \bar{\hat{f}}_w(w)\, dw. \end{aligned}$$

Comparing this with (13), it follows that maximizing expected revenue when w is observable is equivalent to maximizing expected welfare with respect to the distribution of weight and conditional virtual valuation (note the identical zero boundary values at $T+1$). Invoking Lemma 1 applied to $\hat{G}$, we see that $R^*(c, T+1-t)$ is concave with respect to c for all t (note that the fact that the support of virtual valuations contains also negative numbers does not matter for the argument of Lemma 1). ■

Proof of Proposition 2. The Proposition is an immediate consequence of the characterization (4) of optimal solutions for the w-subproblems given $\text{Cap}(w)$, and of a straightforward variational argument ensuring that marginal revenues from marginal increase of $\text{Cap}(w)$ must be constant almost surely in w. ■

Proof of Theorem 4. We need to distinguish two cases:

Case 1: Assume that $C > T\int_0^\infty \lambda^{w,*} w \bar{f}_w(w)\, dw$. In this case, $\beta(C,T) = 0$ and $R^d(C,T) = T\int_0^\infty r^w(\lambda^{w,*}) w \bar{f}_w(w)\, dw$. We also know that $R^*(C,T) \le R^*(+\infty, T)$, where $R^*(+\infty, T)$ denotes the optimal expected revenue from a stochastic problem without any capacity constraint. But, for such a problem, the optimal Markovian policy maximizes at each period the instantaneous expected revenue upon observing w_t, $w_t\, r^{w_t}(\lambda)$. That is, the optimal policy sets $\lambda_t^{w_t}[+\infty] = \lambda^{w,*}$. Thus,

$$R^*(C,T) \le R^*(+\infty, T) \;=\; T\int_0^\infty w\, r^w(\lambda^{w,*}) \bar{f}_w(w)\, dw \;=\; R^d(C,T).$$

Case 2: Assume now that $C \le T\int_0^\infty \lambda^{w,*} w \bar{f}_w(w)\, dw$. For $\mu \ge 0$, consider the unconstrained maximization problem

$$\max_{\text{Cap}(\cdot)} \left[\int_0^\infty r^w\left(\frac{\text{Cap}(w)}{Tw\bar{f}_w(w)}\right) Tw\bar{f}_w(w)\, dw \;+\; \mu\left(C - \int_0^\infty \text{Cap}(w)\, dw\right)\right]$$

The Euler-Lagrange equation is $\left(\frac{d}{d\lambda} r^w\right)\left(\frac{\text{Cap}(w)}{Tw\bar{f}_w(w)}\right) \;=\; \mu$. Hence, if we write $R^d(C,T,\mu)$ for the optimal value of the above problem, and if we let $\mu = \beta(C,T)$ where $\beta(C,T)$ is the constant from Proposition 2, then the solution equals the one of the constrained deterministic problem. In particular $\int_0^\infty \text{Cap}(w)\, dw \;=\; C$, and $R^d(C,T,\beta(C,T)) \;=\; R^d(C,T)$.

Recall that for the stochastic problem, and for any Markovian policy α we have

$$R_\alpha(C,T) = E_\alpha\left[\sum_{t=1}^{T} w_t\, p^{w_t}(\lambda_t^{w_t}[c_t])\, I_{\{v_t \ge p^{w_t}(\lambda_t^{w_t}[c_t])\}}\right],$$

and define

$$R_\alpha(C,T,\beta(C,T)) \;=\; R_\alpha(C,T) \;+\; \beta(C,T)\left(C - E_\alpha\left[\sum_{t=1}^{T} w_t\, I_{\{v_t \ge p^{w_t}(\lambda_t^{w_t}[c_t])\}}\right]\right).$$

Since for any policy α that is admissible in the original problem, it holds that

$$\sum_{t=1}^{T} w_t\, I_{\{v_t \ge p^{w_t}(\lambda_t^{w_t}[c_t])\}} \;\le\; C \quad a.s.,$$

we have $R_\alpha(C,T) \le R_\alpha(C,T,\beta(C,T))$. We will show below that, for arbitrary α (which satisfies the capacity constraint or not), it holds that:

$$R_\alpha(C,T,\beta(C,T)) \le R^d(C,T,\beta(C,T)). \tag{15}$$

This yields for any α that is admissible in the original problem:

$$R_\alpha(C,T) \le R_\alpha(C,T,\beta(C,T)) \le R^d(C,T,\beta(C,T)) = R^d(C,T).$$

Taking the supremum over α concludes then the proof for the second case.

It remains to prove (15). The argument uses the filtration $\{\mathcal{F}_t\}_{t=1}^T$ of σ - algebras containing information prior to time t (in particular the value of c_t) and in addition the currently observed w_t.

$$\begin{aligned}
R_\alpha(C,T,\beta(C,T)) &= E_\alpha\left[\sum_{t=1}^T w_t\,(p^{w_t}(\lambda_t^{w_t}[c_t]) - \beta(C,T))\; I_{\{v_t \ge p^{w_t}(\lambda_t^{w_t}[c_t])\}}\right] + \beta(C,T)\,C \\
&= E_\alpha\left[\sum_{t=1}^T E_\alpha\left[w_t\,(p^{w_t}(\lambda_t^{w_t}[c_t]) - \beta(C,T))\; I_{\{v_t \ge p^{w_t}(\lambda_t^{w_t}[c_t])\}} | \mathcal{F}_t\right]\right] + \beta(C,T)\,C \\
&= E_\alpha\left[\sum_{t=1}^T w_t\,(p^{w_t}(\lambda_t^{w_t}[c_t]) - \beta(C,T))\, E_\alpha\left[I_{\{v_t \ge p^{w_t}(\lambda_t^{w_t}[c_t])\}} | \mathcal{F}_t\right]\right] + \beta(C,T)\,C \\
&= E_\alpha\left[\sum_{t=1}^T w_t\,(r^{w_t}(\lambda_t^{w_t}[c_t]) - \beta(C,T)\lambda_t^{w_t}[c_t])\right] + \beta(C,T)\,C \\
&\le E_\alpha\left[\sum_{t=1}^T w_t\left(r^{w_t}(\lambda^{w_t,d}) - \beta(C,T)\lambda^{w_t,d}\right)\right] + \beta(C,T)\,C \\
&= E_{(w_t)_{t=1}^T}\left[\sum_{t=1}^T w_t\left(r^{w_t}(\lambda^{w_t,d}) - \beta(C,T)\lambda^{w_t,d}\right)\right] + \beta(C,T)\,C \\
&= T\int_0^\infty (r^w(\lambda^{w,d}) - \beta(C,T)\lambda^{w,d}) w \bar{f}_w(w)\, dw + \beta(C,T)\,C = R^d(C,T,\beta(C,T)).
\end{aligned}$$

For the inequality, we have used that $\lambda^{w,d}$ maximizes $r^w(\lambda) - \beta(C,T)\lambda$. ■

For the proof of Theorem 5, we first need a Lemma:

Lemma 2 *Let $R^{TI}(C,T)$ be the revenue obtained form the stationary policy α_{TI}. Let $(\widetilde{w}_t, \widetilde{v}_t)_{t=1}^T$ be an independent copy of the process $(w_t, v_t)_{t=1}^T$. It holds that:*

1.

$$R^{TI}(C,T) = E_{(w_t)_{t=1}^T}\left[\sum_{t=1}^T r^{w_t}(\lambda^{w_t,d})\, w_t \left(1 - P\left[\sum_{s=1}^{t-1} \widetilde{w}_s I_{\{\widetilde{v}_s \ge p^{\widetilde{w}_s,d}\}} > C - w_t\right]\right)\right]. \tag{16}$$

2.

$$\frac{R^{TI}(C,T)}{R^d(C,T)} \geq 1 \tag{17}$$
$$- \frac{\sum_{t=1}^T \int_0^\infty r^w(\lambda^{w,d})\, w \left(\min\left(1, \frac{(t-1)\sigma_d^2}{((T-t+1)\mu_d - w)^2}\right) I_{\{w\leq (T-t+1)\mu_d\}} + I_{\{w>(T-t+1)\mu_d\}}\right) \bar{f}_w(w)\, dw}{T\int_0^\infty r^w(\lambda^{w,d}) w \bar{f}_w(w)\, dw}$$

where $\mu_d := \frac{\min(C, T\int_0^\infty \lambda^{w,*} w \bar{f}_w(w)\, dw)}{T}$ *, and where* $\sigma_d^2 := E[w^2 I_{\{v\geq p^{w,d}\}}] - \mu_d^2 = \int_0^\infty w^2\lambda^{w,d}\bar{f}_w(w)\, dw - \mu_d^2$.

Proof. 1. $R^{TI}(C,T)$ may be written as:

$$R^{TI}(C,T) = E_{(w_t,v_t)_{t=1}^T}\left[\sum_{t=1}^T p^{w_t,d}\, w_t\, I_{\{v_t\geq p^{w_t,d}\}}\, I_{\{\sum_{s=1}^{t-1} w_s\, I_{\{v_s\geq p^{w_s,d}\}} \leq C - w_t\}}\right]$$
$$= E_{(w_t)_{t=1}^T}\left[\sum_{t=1}^T r^{w_t}(\lambda^{w_t,d})\, w_t\right] - E_{(w_t,v_t)_{t=1}^T}\left[\sum_{t=1}^T p^{w_t,d}\, w_t\, I_{\{v_t\geq p^{w_t,d}\}}\, I_{\{\sum_{s=1}^{t-1} w_s\, I_{\{v_s\geq p^{w_s,d}\}} > C - w_t\}}\right]$$

In order to simplify the second term, we use the fact that v_t and $(w_s, v_s)_{s=1}^{t-1}$ are independent conditional on w_t:

$$E_{(w_t,v_t)_{t=1}^T}\left[\sum_{t=1}^T p^{w_t,d}\, w_t\, I_{\{v_t\geq p^{w_t,d}\}}\, I_{\{\sum_{s=1}^{t-1} w_s\, I_{\{v_s\geq p^{w_s,d}\}} > C - w_t\}}\right]$$
$$= E_{(w_t,v_t)_{t=1}^T}\left[\sum_{t=1}^T E\left[p^{w_t,d}\, w_t\, I_{\{v_t\geq p^{w_t,d}\}}\, I_{\{\sum_{s=1}^{t-1} w_s\, I_{\{v_s\geq p^{w_s,d}\}} > C - w_t\}} \mid w_t\right]\right]$$
$$= E_{(w_t,v_t)_{t=1}^T}\left[\sum_{t=1}^T p^{w_t,d}\, w_t\, E\left[I_{\{v_t\geq p^{w_t,d}\}}|w_t\right]\, E\left[I_{\{\sum_{s=1}^{t-1} w_s\, I_{\{v_s\geq p^{w_s,d}\}} > C - w_t\}} \mid w_t\right]\right]$$
$$= E_{(w_t,v_t)_{t=1}^T}\left[\sum_{t=1}^T p^{w_t,d}\, w_t\, \lambda^{w_t,d}\, P\left[\sum_{s=1}^{t-1} \widetilde{w}_s\, I_{\{\widetilde{v}_s\geq p^{\widetilde{w}_s,d}\}} > C - w_t\right]\right]$$
$$= E_{(w_t)_{t=1}^T}\left[\sum_{t=1}^T r^{w_t}(\lambda^{w_t,d})\, w_t\, P\left[\sum_{s=1}^{t-1} \widetilde{w}_s\, I_{\{\widetilde{v}_s\geq p^{\widetilde{w}_s,d}\}} > C - w_t\right]\right].$$

This establishes equation (16).

2. Recall that $R^d(C,T) = T\int_0^\infty r^w(\lambda^{w,d}) w\bar{f}_w(w)\, dw$. Observe furthermore that $\lambda^{w,d}$ depends on C and T only through the ratio $\frac{C^{\text{eff}}}{T}$, where $C^{\text{eff}} = \min(C, T\int_0^\infty \lambda^{w,*} w\bar{f}_w(w)\, dw)$, via $E[wI_{\{v\geq p^{w,d}\}}] = \int_0^\infty w\lambda^{w,d}\bar{f}_w(w)\, dw = \frac{C^{\text{eff}}}{T} = \mu_d$. Observe first that

$$P\left[\sum_{s=1}^{t-1} \widetilde{w}_s\, I_{\{\widetilde{v}_s\geq p^{\widetilde{w}_s,d}\}} > C - w_t\right] \leq P\left[\sum_{s=1}^{t-1} \widetilde{w}_s\, I_{\{\widetilde{v}_s\geq p^{\widetilde{w}_s,d}\}} > T\mu_d - w_t\right]$$
$$= P\left[\sum_{s=1}^{t-1} \widetilde{w}_s\, I_{\{\widetilde{v}_s\geq p^{\widetilde{w}_s,d}\}} - (t-1)\mu_d > (T-t+1)\mu_d - w_t\right].$$

We trivially bound the last expression by 1 if $(T-t+1)\mu_d - w_t \leq 0$, and otherwise use Chebychev's inequality to deduce

$$\begin{aligned}
&P\left[\sum_{s=1}^{t-1} \widetilde{w}_s\, I_{\{\widetilde{v}_s \geq p^{\widetilde{w}_s,d}\}} \;-\; (t-1)\mu_d \;>\; (T-t+1)\mu_d - w_t\right] \\
&\quad\leq\; P\left[\left(\sum_{s=1}^{t-1} \widetilde{w}_s\, I_{\{\widetilde{v}_s \geq p^{\widetilde{w}_s,d}\}} \;-\; (t-1)\mu_d\right)^2 \;>\; ((T-t+1)\mu_d - w_t)^2\right] \\
&\quad\leq\; \frac{E\left[\left(\sum_{s=1}^{t-1} \widetilde{w}_s\, I_{\{\widetilde{v}_s \geq p^{\widetilde{w}_s,d}\}} \;-\; (t-1)\mu_d\right)^2\right]}{((T-t+1)\mu_d - w_t)^2} \;=\; \frac{(t-1)\sigma_d^2}{((T-t+1)\mu_d - w_t)^2}.
\end{aligned}$$

As we are bounding a probability, we can replace this estimate by the trivial bound 1 again whenever this is better, i.e. if w_t is smaller than but close to $(T-t+1)\mu_d$. Thus,

$$\begin{aligned}
&E_{(w_t,v_t)_{t=1}^T}\left[\sum_{t=1}^{T} p^{w_t,d}\, w_t\, I_{\{v_t \geq p^{w_t,d}\}}\, I_{\{\sum_{s=1}^{t-1} w_s\, I_{\{v_s \geq p^{w_s,d}\}} > C - w_t\}}\right] \\
&\leq\; \sum_{t=1}^{T} \int_0^\infty r^w(\lambda^{w,d})\, w\left(\min\left(1, \frac{(t-1)\sigma_d^2}{((T-t+1)\mu_d - w)^2}\right) I_{\{w \leq (T-t+1)\mu_d\}} + I_{\{w > (T-t+1)\mu_d\}}\right) \bar{f}_w(w)\, dw.
\end{aligned}$$

Finally, dividing by $R^d(C,T)$ yields the desired estimate. ■

Proof of Theorem 5. **1.** The starting point is the estimate from (17). Note that $r^w(\lambda^{w,d}) w \bar{f}_w(w)$ is an integrable upper bound for

$$r^w(\lambda^{w,d})\, w\left(\min\left(1, \frac{(t-1)\sigma_d^2}{((T-t+1)\mu_d - w)^2}\right) I_{\{w \leq (T-t+1)\mu_d\}} + I_{\{w > (T-t+1)\mu_d\}}\right) \bar{f}_w(w).$$

Moreover, for fixed w, for arbitrary $\eta \in (0,1)$ and for $t \leq \eta T$ we have $w < (1-\eta)T\mu_d$ eventually as $T, C \to \infty$, $\frac{C}{T} = \text{const}$. Moreover,

$$\frac{(t-1)\sigma_d^2}{((T-t+1)\mu_d - w)^2} \;\leq\; \frac{\eta T \sigma_d^2}{((1-\eta)T\mu_d - w)^2} \;\to\; 0, \text{ as } T \to \infty.$$

The Dominated Convergence Theorem implies then that

$$\int_0^\infty r^w(\lambda^{w,d})\, w\left(\min\left(1, \frac{(t-1)\sigma_d^2}{((T-t+1)\mu_d - w)^2}\right) I_{\{w \leq (T-t+1)\mu_d\}} + I_{\{w > (T-t+1)\mu_d\}}\right) \bar{f}_w(w)\, dw \;\to\; 0,$$

in the considered limit, for arbitrary $\eta \in (0,1)$ and for $t \leq \eta T$. Consequently, also the term that is subtracted in the estimate (17) converges to zero.

2. If w and v are independent, all the $\lambda^{w,d}$ for different w coincide, as do the $\lambda^{w,*}$. Call them λ^d and λ^*, respectively. We have then

$$\begin{aligned}
R^{TI}(C,T) &= p(\lambda^d) E\left[\min\left(C, \sum_{t=1}^{T} w_t\, I_{\{v_t \geq p(\lambda^d)\}}\right)\right] \\
&= p(\lambda^d) E\left[\sum_{t=1}^{T} w_t\, I_{\{v_t \geq p(\lambda^d)\}} \;-\; \left(\sum_{t=1}^{T} w_t\, I_{\{v_t \geq p(\lambda^d)\}} \;-\; C\right)^+\right]
\end{aligned}$$

We use now the following estimate for $E[(X-k)^+]$, where X is a random variable with mean m and variance σ^2 and where k is a constant:

$$E[(X-k)^+] \leq \frac{\sqrt{\sigma^2 + (k-m)^2} - (k-m)}{2}.$$

Note that by independence

$$\begin{aligned} E\left[\sum_{t=1}^{T} w_t\, I_{\{v_t \geq p(\lambda^d)\}}\right] &= E[w]T\lambda^d; \\ Var\left[\sum_{t=1}^{T} w_t\, I_{\{v_t \geq p(\lambda^d)\}}\right] &= T\left(E[(w\, I_{\{v \geq p(\lambda^d)\}})^2] - E[w]^2(\lambda^d)^2\right) \\ &= T\left(E[w^2]\lambda^d - E[w]^2(\lambda^d)^2\right) \end{aligned}$$

If $\lambda^* T E[w] > C$ and hence if $\lambda^d = \frac{C}{TE[w]}$ this yields:

$$R^{CP}(C,T) \geq R^d(C,T) - p(\lambda^d)\frac{\sqrt{TE[w^2]\lambda^d}}{2} = R^d(C,T)\left(1 - \frac{\sqrt{E[w^2]/E[w]}}{2\sqrt{C}}\right).$$

If $\lambda^* T E[w] \leq C$ and hence if $\lambda^d = \lambda^*$, then $C \geq E\left[\sum_{t=1}^{T} w_t\, I_{\{v_t \geq p(\lambda^d)\}}\right]$, so that $E\left[\left(\sum_{t=1}^{T} w_t\, I_{\{v_t \geq p(\lambda^d)\}} - C\right)^+\right] \leq \frac{\sqrt{\sigma^2}}{2}$. Thus,

$$R^{TI}(C,T) \geq R^d(C,T) - p(\lambda^*)\frac{\sqrt{\lambda^* T E(w^2)}}{2} = R^d(C,T)\left(1 - \frac{\sqrt{E[w^2]/E[w]}}{2\sqrt{\lambda^* E(w) T}}\right).$$

Hence, we can conclude that:

$$\frac{R^{TI}(C,T)}{R^d(C,T)} \geq \left(1 - \frac{\sqrt{E[w^2]/E[w]}}{2\sqrt{\min(C, T\lambda^* E[w])}}\right).$$

■

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