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"Competition over Price and Service Rate when Demand is Stochastic: A Strategic Analysis"

by

Raymond Deneckere*

and

James Peck*

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*Department of Managerial Economics and Decision Sciences, J. L. Kellogg Graduate School of Management, Northwestern University, 2001 Sheridan Road, Evanston, Illinois 60208.

Abstract

We consider a two-stage game in which firms simultaneously select prices and capacities (or equivalently, outputs). Then, a random number of consumers attends the market, and each consumer selects a firm to visit. Consumers know all prices and quantities but not the realization of aggregate demand. The probability of being served at any firm depends on its capacity and the mixed strategy chosen by consumers. Consumers distribute themselves across firms so as to equalize the utility of each price-service pair. We show that there exists at most one equilibrium in which firms choose pure strategies, and characterize the "candidate" equilibrium. Consumers face a probability of being rationed, firms may have excess inventory, and the price remains above marginal cost. When there are many firms, the candidate is shown to be an equilibrium.

1. Introduction

In many industries, such as entertainment, transportation, and retailing, the possibility of stockouts and rationing is the rule rather than the exception. Prices often do not respond to short-run variability in demand. When demand is high, stores are willing to sell out their entire stock at one price and ration future customers, preferring not to raise the price as stocks become depleted. Our goal is to provide a formal strategic model of competition over price and service rate. This approach can develop testable implications for pricing patterns and service rates in markets, based upon the nature of the demand uncertainty and the technology for consumers to search across firms.

We consider a model in which firms, before knowing the state of demand, compete by simultaneously choosing a price and a level of output (or capacity). There is a nondegenerate distribution of consumers who will arrive at the market, knowing the prices and quantities chosen by each firm but not knowing the realization of aggregate demand. Consumers freely select a firm to visit, but consider visiting a second firm prohibitively expensive. If the ratio

$$s_i = (\text{total capacity of firm } i)/(\text{total number of consumers visiting firm } i)$$

is less than one, then some consumers visiting firm i are rationed, and s_i represents the probability of being serviced. Consumers optimally select a firm, based on the firm's price and service rate. Each firm's service rate is endogenously determined from consumer optimization, which equalizes the utility of each viable price-service pair offered in the market.

Having specified consumer behavior, we model the game played by the firms. For any aggregate distribution of consumers, we show that at most one Nash equilibrium exists in which firms choose pure strategies. Existence is guaranteed when the number of firms is sufficiently large. In equilibrium, consumers face a positive probability of being rationed, and the price charged by all firms remains above marginal cost, even as the number of firms approaches infinity.

In the literature on simultaneous competition in price and quantity, pure strategy equilibria often fail to exist. See Levitan and Shubik [1978] for a model in which demand is nonstochastic and consumers can costlessly move from firm to firm. The nonexistence problem holds in our context as well, where demand is nonstochastic, but consumers cannot move from firm to firm. The problem is that, when the lowest price is above marginal cost, firms have an incentive to undercut and serve the entire market. However, when the price equals marginal cost, then firms have an incentive to raise the price and reduce capacity. We show that the nonexistence result disappears with even the slightest amount of demand uncertainty, as the number of firms approaches infinity.

Carlton [1978] models the same type of competition we consider here, but in a non-strategic context. In his model, firms are assumed to earn zero expected profits. Competitive equilibrium is defined as a situation in which no firm can offer a price-service rate combination that offers consumers higher utility (than some fixed level offered by other firms) and earns positive expected profits. In our analysis, firms explicitly take into account the effects of their actions on the consumer utility offered by other firms, and the profits received by firms are determined endogenously. In view of the inability of the Bertrand model to deliver the competitive result, a formal game-theoretic justification for competitive equilibrium is important. Indeed, we demonstrate that, if firms were "utility takers", pure-strategy Nash equilibrium would never exist! On the other hand, we show that, as the number of firms approaches infinity, our Nash equilibrium converges to the outcome proposed by Carlton.

Our model is similar to that of Peters [1984], in that firms compete in prices and serve customers at the stated price until capacity is reached. Consumers can only visit a single firm, and mix over which firm to visit until the utility (as a function of price and service probability) offered by each firm is equated. In Peters' setting, firms in general must employ mixed strategies, although an approximate pure-strategy equilibrium exists when the number of firms approaches infinity. Our model differs along several dimensions, and we are able to derive stronger existence results. We assume inelastic demand, which avoids the problem of having to select a rationing rule. (See Kreps and Scheinkman [1983] and Davidson and Deneckere [1986] for a related model in which the choice of rationing rule makes a big difference.) We assume that there is aggregate risk, but no idiosyncratic risk; Peters [1984] considers the opposite case. In our model, firms must choose their output (equivalently, capacity) before the arrival of customers; in Peters' model, capacities are given, and consumers produce output to order as customers arrive (until capacity is reached).

The structure of our paper is as follows. In section 2, we set up the model and derive some technical properties of the service rate. In section 3, we prove that at most one equilibrium, in which all firms choose pure strategies, exists. We explicitly characterize the prices and capacities chosen in the candidate equilibrium. In section 4, we discuss the possible nonexistence of equilibrium, and prove that existence is guaranteed when the number of firms is sufficiently large. In section 5, we present an example with uniform demand uncertainty, and section 6 concludes the paper.

2. The Model

The economy is composed of n firms and a random number of consumers, drawn from a continuous distribution. Let the number of consumers per firm be denoted as $\tilde{a}$, where $\tilde{a}$ is a random variable with continuous density function $f(\tilde{a})$. The density function f is assumed to be strictly positive on the support $[\underline{a},\bar{a}]$, where $\bar{a}$ can be infinity. Let the expected number of consumers per firm be denoted as

$$\mu \equiv \int_{\underline{a}}^{\bar{a}} \tilde{a} f(\tilde{a}) d\tilde{a}.$$

It follows that the aggregate number of consumers is $n\tilde{a}$.[1]

We assume that all consumers are symmetric in the following sense. There are $n\bar{a}$ "potential" consumers.[2] When nature chooses the aggregate number of consumers at the market, $n\tilde{a}$, each of the potential consumers faces the same conditional probability of being selected. That is, we have

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[1]All of our results go through when we hold aggregate demand fixed as we vary n. The reason is that, when we scale aggregate demand by any factor (including 1/n), equilibrium capacities are scaled by the same factor. Prices and service rates are unchanged.

[2]Making the number of potential consumers coincide with the upper support of the density f was done to avoid introducing more notation. Any number of potential consumers greater than $n\bar{a}$ would work as well.

$$\text{Pr.(consumer } \alpha \text{ is at the market} | \tilde{a} = a) = a/\bar{a}.$$

Therefore, the unconditional probability of consumer α being at the market when $\tilde{a} = a$ is

$$\text{Pr.(consumer } \alpha \text{ is at the market } \underline{\text{and}}\ \tilde{a} = a) = af(a)/\bar{a},$$

and the unconditional probability of consumer α being at the market is

$$\text{Pr.(consumer } \alpha \text{ is at the market)} = \int_{\underline{a}}^{\bar{a}} \frac{\tilde{a}}{\bar{a}} f(\tilde{a}) d\tilde{a}.$$

From Bayes' rule, we have

$$\text{(2.1)} \qquad \text{Pr.}(\tilde{a} = a | \text{consumer } \alpha \text{ is at the market}) = \frac{af(a)}{\int_{\underline{a}}^{\bar{a}} \tilde{a} f(\tilde{a}) d\tilde{a}}.$$

It might seem unintuitive that one consumer out of a continuum receives any useful information from the fact that he or she is at the market. In other words, why is the conditional density in (2.1) different from the unconditional density, f? Consider the example of a rock concert that draws 10,000 fans with probability .5 or 50,000 fans with probability .5. If there are 50,000 potential concertgoers, a particular individual is part of all large crowds (50,000 fans) but only part of small crowds (10,000 fans) 20% of the time. Therefore, he or she arrives at a large crowd with probability .5, arrives at a small crowd with probability .1, and stays home with probability .4. Conditional on arriving, the probability of a small crowd is therefore 1/6, although the unconditional probability of a small crowd is 1/2.

The timing of actions is specified as follows. First, each firm i simultaneously chooses its capacity, k_i, and the price of its output, p_i. All firms face a constant marginal cost of capacity, c. Then nature chooses the set of consumers, and each consumer irrevocably chooses which firm to visit. Arriving consumers know the prices and capacities offered by all firms, and they know that they are at the market (hence, the conditional probability in equation (2.1) will be used), but they do not know the realization of aggregate demand.

The commodity is consumed in units of zero or one, with all consumers having the same reservation value, v. We assume that $c < v$ holds. If the number of consumers visiting a firm is less than the capacity, then all consumers are served. If there is excess demand, output is rationed anonymously, with all consumers at the firm receiving one unit with probability equal to the ratio of supply to demand.[3] If s is the probability of consuming the good (to be endogenously determined later), each consumer has the utility function, $U(p,s) = (v - p)s$. Firms are risk neutral, maximizing expected profits.

The service rate offered by firm i depends on firm i's capacity, the density f, and the choices consumers make about which firm to visit. Consumers choose a mixed strategy, $\tilde{q} = (\tilde{q}_1,\ldots,\tilde{q}_i,\ldots,\tilde{q}_n)$, where $\tilde{q}_i$ represents the probability that a consumer (at the market) visits firm i. Without loss of generality, we only consider Nash equilibria in which all consumers choose

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[3]We conjecture that allowing firms to choose any rationing rule would not affect the results.

the same strategy $\tilde{q}$.[4] In equilibrium, consumers will mix so as to equate the expected utility offered by each firm. A firm that slightly lowers its price, holding capacity constant, will not experience a jump in demand, because that would discontinuously lower the service rate. Instead, $\tilde{q}_i$ will increase slightly, which slightly lowers the service rate.

Before deriving the service rate, the following notation will be useful. Let $q_i \equiv \tilde{q}_i n$, and let $\hat{a}_i$ be a random variable representing the number of consumers arriving at firm i. The law of large numbers guarantees that we have

$$(2.2) \qquad \hat{a}_i = q_i \tilde{a}.$$

Definition 2.3: The service rate of firm i, conditional on a consumer visiting firm i, is given by

$$s(k_i/q_i) = E\left[\frac{\min(k_i, q_i\tilde{a})}{q_i(\tilde{a})} \,|\, \text{consumer } \alpha \text{ is at the market}\right].$$

Lemma 2.4: Let K_i be defined as $K_i = k_i/q_i$. Firm i's service rate is given by

$$(2.5) \qquad s(K_i) = K_i/\mu, \text{ for } K_i < \underline{a}$$

$$(2.6) \qquad s(K_i) = 1 - \frac{\int_{K_i}^{\bar{a}} (\tilde{a} - K_i) f(\tilde{a}) d\tilde{a}}{\mu}, \text{ for } \underline{a} \le K_i \le \bar{a}$$

[4]Because consumers are small, one can think of $\tilde{q}$ as a distribution of pure strategies chosen by different consumers. Under this interpretation, our uniqueness result refers to the strategies chosen by each firm and the distribution of actions chosen by consumers.

(2.7) $\qquad s(K_i) = 1, \text{ for } \bar{a} < K_i.$

Furthermore, s is increasing, concave, and continuously differentiable.

Proof: When $K_i < \underline{a}$ holds, we have $\min(k_i, q_i\tilde{a}) = k_i$. Definition (2.3) implies $s(K_i) = E[K_i/\tilde{a} | \text{consumer } \alpha \text{ is at the market}]$. From (2.1), we obtain:

$$s(K_i) = K_i \int_{\underline{a}}^{\bar{a}} 1/\tilde{a} \, \frac{\tilde{a}f(\tilde{a})d\tilde{a}}{\mu} \, d\tilde{a} = K_i/\mu.$$

When $\underline{a} \le K_i \le \bar{a}$ holds, definition (2.3) implies $s(K_i) = E[\min(K_i/\tilde{a}, 1) | \text{consumer}$ α is at the market$]$. From equation (2.1), we have

$$s(K_i) = \int_{\underline{a}}^{K_i} 1 \, \frac{\tilde{a}f(\tilde{a})d\tilde{a}}{\mu} + \int_{K_i}^{\bar{a}} K_i/\tilde{a} \, \frac{\tilde{a}f(\tilde{a})d\tilde{a}}{\mu}.$$

Therefore,

$$\begin{aligned} s(K_i) &= 1/\mu \left[\int_{\underline{a}}^{K_i} \tilde{a}f(\tilde{a})d\tilde{a} + \int_{K_i}^{\bar{a}} \tilde{a}f(\tilde{a})d\tilde{a} - \int_{K_i}^{\bar{a}} \tilde{a}f(\tilde{a})d\tilde{a} + \int_{K_i}^{\bar{a}} K_i f(\tilde{a})d\tilde{a}\right] \\ &= 1/\mu \left[\mu - \int_{K_i}^{\bar{a}} (\tilde{a} - K_i)f(\tilde{a})d\tilde{a}\right] \\ &= 1 - \frac{\int_{K_i}^{\bar{a}} (\tilde{a} - K_i)f(\tilde{a})d\tilde{a}}{\mu} \end{aligned}$$

When $K_i > \bar{a}$ holds, $s(K_i) = 1$ follows immediately from definition (2.3).

In the region $(\underline{a}, \bar{a})$, $s'(K_i)$ exists and is given by

(2.8) $\qquad s'(K_i) = 1/\mu \int_{K_i}^{\bar{a}} f(\tilde{a})d\tilde{a}.$

From equations (2.5), (2.7), and (2.8), it is trivial to verify that $s'(K_i)$ exists and is continuous at $K_i = \underline{a}$ and $K_i = \bar{a}$. For $K_i \in (\underline{a}, \bar{a})$ we have

$$(2.9) \qquad s''(K_i) = -(1/\mu)f(K_i).$$

Therefore, $s(K_i)$ is weakly concave for $K_i \leq \underline{a}$ and $K_i \geq \bar{a}$, and $s(K_i)$ is strictly concave for $K_i \in (\underline{a}, \bar{a})$. ∎

From Lemma (2.4) we conclude that the service rate offered by firm i to all of its customers is the ratio of the expected number of serviced customers to the expected number of arriving customers. The key step in the derivation of this result is to condition on a consumer arriving at the market.

3. Characterization of Equilibrium

In this section, we show that a Nash equilibrium in which all firms use pure strategies, if it exists, is unique. We characterize this "candidate" equilibrium and show how it may be computed. Our results are that (i) all firms choose the same price and capacity, (ii) there is a positive probability of rationing, (iii) there is a positive probability of unsold (wasted) goods, which keeps the price bounded above marginal cost, even as $n \to \infty$, and (iv) as $n \to \infty$, the equilibrium converges to that described by Carlton [1978].

Definition 3.1: Let k^* be defined as the unique solution to $s'(K) = c/(v\mu)$.

Observe that k^* is well defined, since $c < v$, $s'(a) = 1/\mu$ for $a \leq \underline{a}$, $s'(a) = 0$ for $a \geq \bar{a}$, and since s is continuously differentiable everywhere and strictly concave on $(\underline{a},\bar{a})$. Observe that $k^* \in (\underline{a},\bar{a})$.

Lemma 3.2: In an equilibrium in which firms use pure strategies, all active firms (with $k_i > 0$) receive positive expected profits.

Proof: Expected profits for firm i are given by

$$(3.3) \qquad \pi_i = p_i \mu q_i s(k_i/q_i) - ck_i.$$

Consumers mix so as to equate the expected utility offered by all active firms, so we have

$$(3.4) \qquad (v - p_i)s(k_i/q_i) = (v - p_j)s(k_j/q_j) \equiv U^*$$

for all active firms i and j.

Suppose we had an equilibrium in which $\pi_i = 0$ for some i (firms would stay inactive rather than allow $\pi_i < 0$). We know that reducing k_i leads consumers to reduce q_i and increase q_j, $j \neq i$. From expression (3.4), and the fact that p_j and k_j are unchanged, we know that U^* goes down. Holding p_i constant, $s(k_i/q_i)$ must also go down. Similarly, raising p_i, all else held constant, leads consumers to lower q_i and increase q_j, $j \neq i$. This raises $s(k_i/q_i)$.[5]

It follows that firm i could lower k_i to k_i' and raise p_i to p_i', such that firm i's service rate remains the same. That is, we have $s(k_i'/q_i') = s(k_i/q_i)$.

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[5]This argument relies on $K_j \leq \bar{a}$, but firm j cannot be optimizing with $K_j > \bar{a}$. Instead, firm j could reduce production without affecting sales.

Originally, profits are zero, which implies

$$(3.5) \qquad p_i\mu s(k_i/q_i) - c(k_i/q_i) = 0.$$

With the deviation to (k_i',p_i') firm i's profits are given by

$$\pi_i' = q_i'[p_i'\mu s(k_i/q_i) - c(k_i/q_i)].$$

Since p_i' and k_i' are small deviations, we know that $q_i' \neq 0$ holds. From $p_i' > p_i$ and equation (3.5), it follows that π_i' is positive, contradicting the supposition that $\pi_i = 0$. ■

<u>Lemma 3.6</u>: In any equilibrium in which all firms use pure strategies, all firms are active and receive positive profits.

<u>Proof</u>: First, observe that there must be at least one active firm. If no firms were active, any firm could charge a price of v and attract all consumers. As a function of K, this would yield profits of $v\mu s(K) - cK$, or

$$(3.7) \qquad v\mu K[s(K)/K - c/(v\mu)].$$

If $\underline{a} > 0$, expression (3.7) is positive at $K = \underline{a}$, because $s(\underline{a})/\underline{a} = 1/\mu$. Even if $\underline{a} = 0$, l'Hopital's rule implies $\lim_{K\to\underline{a}} s(K)/K = s'(\underline{a}) = 1/\mu$. Continuity then implies that expression (3.7) is positive for sufficiently small positive K.

Having established that there must be at least one active firm, suppose that firm i is active and firm j is inactive. Now consider the following profitable deviation for firm j. Choose $p_j' = p_i$ and $k_j' = \epsilon k_i$, were ϵ is a small positive scalar. When consumers optimize, taking the deviation into account, we must have $q_j' = \epsilon q_i'$, because consumers must be indifferent between choosing firms i and j.

The profits of firm j are given by

$$(3.8) \qquad \pi_j' = \epsilon q_i'[p_i \mu s(k_i/q_i') - c(k_i/q_i')].$$

For small enough ϵ, consumers do not change their mixing probabilities very much. Continuity implies π_j' is positive for some $\epsilon > 0$ whenever

$$(3.9) \qquad \epsilon q_i[p_i \mu s(k_i/q_i) - c(k_i/q_i)] > 0.$$

Inequality (3.9) is equivalent to $\epsilon\pi_i > 0$, which follows from Lemma (3.2). A profitable deviation for firm j is inconsistent with equilibrium; hence, all firms must be active in any pure-strategy equilibrium. ■

<u>Lemma 3.10</u>: In any equilibrium in which all firms choose pure strategies, we have $K_i = k_i/q_i = k^*$ for $i = 1,\ldots,n$.

<u>Proof</u>: From Lemma (3.6), all firms are active and earn positive profits. Suppose some firm, i, has $K_i \neq k^*$. We will show that firm i has a profitable deviation within the class of strategies that causes consumers to maintain their strategy, q. If U^* is the equilibrium utility offered to consumers, they will maintain their strategy, q, whenever (p_i',k_i') satisfies

$$(3.11) \qquad (v - p_i')s(k_i'/q_i) = U^*.$$

Consider a deviation such that $k_i' = q_i k^*$ and $p_i' = v - (U^*/s(k^*))$. Since equation (3.11) holds, it follows that consumers will continue to choose q_i. For now, we will allow the possibility that $p_i' < 0$. We have

$$(3.12) \qquad \pi_i' = q_i \left[\left[\frac{vs(k^*) - U^*}{s(k^*)}\right]\mu s(k^*) - ck^*\right].$$

Equation (3.12) can be simplified to

$$(3.13) \qquad \pi_i'/q_i = \mu vs(k^*) - ck^* - \mu U^*.$$

At the equilibrium strategy, firm i's profits are

$$(3.14) \qquad \pi_i/q_i = \mu vs(K_i) - cK_i - \mu U^*.$$

However, the right side of (3.13) is strictly greater than the right side of (3.14). To see this, consider the maximization problem

$$(3.15) \qquad \max_{0 \le K \le \infty} v\mu s(K) - cK.$$

Since $s'' \le 0$ everywhere and $s'' < 0$ for $K \in (\underline{a}, \bar{a})$, the first-order condition $s'(K) = c/(\mu v)$ gives rise to the unique optimum, k^* (see definition (3.1)). Therefore, the deviation (p_i', k_i') gives rise to higher profits than the equilibrium level. The supposition that $K_i \neq k^*$ is therefore false. ∎

The proof of Lemma (3.10) shows that, in any equilibrium with firms choosing pure strategies, all firms provide a service rate that maximizes gains from trade, given q. We can now prove the main result of this section.

Proposition 3.16: An equilibrium in which all firms choose pure strategies, if it exists, is unique. The equilibrium is given by

(3.17) $$k_i = k^* \text{ for } i = 1,\ldots,n,$$

$$p_i = \frac{ck^*n/(n-1)}{\mu s(k^*) + ck^*/(v(n-1))} \text{ for } i = 1,\ldots,n$$

and

$$q_i = 1 \text{ for } i = 1,\ldots n \text{ (along the equilibrium path).}$$

Proof: From Lemmas (3.6) and (3.10), we know that all firms are active and provide the same service rate, $s(k^*)$. It immediately follows that all firms choose the same price, which we denote by p. Now consider the optimization problem faced by firm i. Given the prices and capacities of other firms, firm i's price and capacity choice will determine q_i and q_j, $j \neq i$. Thus, a necessary condition for equilibrium is that firm i chooses a price, capacity, and induced q_i to solve

(3.18) $$\max\ q_i p_i \mu s(k_i/q_i) - ck_i$$

s.t.

$$(v - p_i)s(k_i/q_i) = (v - p)s(k_j/q_j)\ \forall\ j.$$

For any choice of q_i, k_j/q_j must be independent of j (in equilibrium, the constant is k^*), because all other firms are choosing the same price, p. Call this constant $\alpha(q_i)$. We therefore have

$$\alpha(q_i) = \textstyle\sum_{j\neq i} k_j / \sum_{j\neq i} q_j .$$

Since $\sum_{j\neq i} q_j + q_i = n$ holds, $\alpha(q_i)$ is equal to $(\sum_{j\neq i} k_j)/(n - q_i)$. Substituting the service rate offered by other firms, as a function of q_i, maximization problem (3.18) becomes

$$\text{(3.19)} \qquad \max_{p_i, k_i, q_i} \; q_i p_i \mu s(k_i/q_i) - c k_i$$

$$\text{s.t.} \quad (v - p_i) s(k_i/q_i) = (v - p) s(k_{-i}/(n - q_i))$$

where $k_{-i} \equiv \sum_{j\neq i} k_j$. Solving the constraint in problem (3.19) for p_i, and plugging it into the objective, we obtain the following unconstrained maximization problem :

$$\text{(3.20)} \quad \max_{k_i, q_i} \; q_i \left[v - (v - p) \frac{s(k_{-i})/(n - q_i)}{s(k_i/q_i)} \right] \mu s(k_i/q_i) - c k_i$$

Simplifying, we have

$$\text{(3.21)} \qquad \max_{k_i, q_i} \; q_i \mu s(k_i/q_i) v - (v - p) s\left(\frac{k_{-i}}{n - q_i} \right) q_i \mu - c k_i .$$

By Lemma (3.6), an equilibrium (if one exists) must involve firm i choosing $0 < q_i < n$, so the interior first order conditions are necessary. Taking the partial derivative with respect to k_i, we have

(3.22) $$s'(k_i/q_i) = c/(\mu v)$$

which implies $k_i/q_i = k^*$. Setting the partial derivative of expression (3.21) with respect to q_i equal to zero, we have

(3.23) $$0 = vs\left(\frac{k_i}{q_i}\right) - (v - p)s\left(\frac{k_{-i}}{n - q_i}\right) + q_i\left[v\, s'\left(\frac{k_i}{q_i}\right)\left(\frac{-k_i}{q_i^2}\right) + (v - p)s'\left(\frac{k_{-i}}{n - q_i}\right)\frac{k_{-i}}{(n - q_i)^2}\right].$$

From equation (3.22), and some straightforward calculations, we have

(3.24) $$k_{-i}/(n - q_i) = k^* = k_i/q_i.$$

Substituting (3.24) into (3.23) yields

(3.25) $$vs(k^*) = vs'(k^*)k^* + (v - p)s'(k^*)k^*(q_i/(n - q_i) + (v - p)s(k^*).$$

Since equation (3.25) must hold for all i, it follows that q_i is independent of i, so we have $q_i = 1$ for $i = 1,\ldots,n$. Therefore, equation (3.25) and $q_i = 1$ uniquely determine the price, p. Solving for p and substituting $s'(k^*) = c/(\mu v)$, the equilibrium price must equal the expression for p_i in (3.17). ■

From the properties of the function s derived in Section 2, $\underline{a} < k^* < \bar{a}$ must hold. (See the discussion following Definition (3.1).) Thus, all firms offer service rates strictly less than one at the candidate equilibrium.

There is a positive probability of rationing, when $\tilde{a} > k^*$, and there is a positive probability of unsold goods, when $\tilde{a} < k^*$. A positive probability of unsold goods means that the expected number of served customers, $\mu s(k^*)$, is strictly less than the capacity, k^*. Nonnegative profits implies $p_i \mu s(k^*) - ck^* \geq 0$, so we have $p_i \geq ck^*/\mu s(k^*) > c$. To compensate for the possibility of unsold goods, firms charge a price above marginal cost, even for large n. As n approaches infinity, the price approaches $ck^*/\mu s(k^*)$, and profits per firm approach zero. Thus, the candidate Nash equilibrium coincides in the limit with the solution proposed by Carlton [1978], who assumes that competition will drive profits to zero.

Observe that, in the limit, we have

$$\frac{(p-c)}{p} = \frac{k^* - \mu s(k^*)}{k^*} \tag{3.26}$$

Equation (3.26) states that the percentage markup over marginal cost equals the percentage of excess capacity.

4. Existence and Potential Nonexistence of Equilibrium

Before presenting the proposition that, for large n, an equilibrium exists in which all firms choose pure strategies, we want to emphasize the possibility that pure-strategy Nash equilibrium may fail to exist. In much of the literature on price-competition, existence problems arise because shading the price downward yields a discontinuous jump in a firm's demand. Carlton [1978] noticed that, in models where consumers must commit to a firm, the possibility of rationing eliminates this discontinuity; when a firm (offering a service rate less than one) marginally lowers its price, a jump in demand cannot occur, because the firm's service rate would deteriorate. Indeed, at the candidate equilibrium, no firm could improve by changing its price or by changing its capacity, holding the other variable constant.

What, then, may prevent an equilibrium from existing? The problem is that a firm might have an incentive to deviate from the candidate equilibrium (3.17) by cutting the price and expanding capacity simultaneously. This deviation allows the firm to maintain a service rate of $s(k^*)$, only slightly lower its profit margin (which remains positive), but greatly expand its customer base. Carlton [1978] assumes zero profits as a precondition for competitive equilibrium, so this undercut and expand strategy will not work.[6] However, when the number of firms is finite, there is no reason to think that profits will be zero. It will then be important to check whether a firm has an incentive to deviate from the strategy given in (3.17), cutting price and expanding to capture the entire market. If we are to justify competitive equilibrium as the limit of Nash equilibria, as n approaches infinity, it will be important to verify that no firm has an incentive to deviate near the limit (where profits are small but positive).

If we hold the utility offered by other firms constant at U^*, justified by the view that one small firm cannot affect the market utility level, then firm i's maximization problem is

$$\text{(4.1)} \qquad \max \pi_i = q_i[\mu s(K_i)p_i - cK_i]$$

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[6] Although the undercut and expand strategy will not work when firms are earning zero profits, there is still the problem that a firm could earn positive profits by raising the price and cutting capacity (keeping the service rate constant at $s(k^*)$).

s.t.

$$(v - p_i)s(K_i) = U^*.$$

Profits are linear in q_i, so a firm making profits (even small profits), would want to capture the market, by deviating to $k_i = n\ k^*$ and marking the price down by one cent. This deviation results in $K_i = k^*$ and $q_i = n$.

The assumption that firms are exact utility takers is self-negating, because a utility taker wants to capture the entire market, which multiplies its profits by n. Of course, a firm serving the entire market affects market utility. The reason that a Nash equilibrium (in which firms choose pure strategies) exists for large n is that firms take into account their small effect on market utility. Since the service rate offered by all firms is strictly less than one at the candidate equilibrium (3.17), when firm i deviates to capture it increases the service rates of all firms other than i. Firm i therefore raises the market utility, U^*, which requires a discrete reduction in p_i, or a discrete increase in firm i's service rate. When the number of firms is large, the profit margin, π/q, is close to zero. Therefore, a discrete drop in p_i (the required drop remains uniformly bounded away from zero for all n) will result in negative profits.[7] Similarly, a discrete increase in firm i's service rate will mean more unused capacity and negative profits.

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[7]Carlton assumes there is firm-idiosyncratic uncertainty but no aggregate uncertainty. By offering a capacity equal to market demand, and maintaining the same price, a firm can attract all customers and increase its profits more than n-fold. Thus, in Carlton's version of the model, a Nash equilibrium (with firms choosing pure strategies) will <u>never</u> exist. Idiosyncratic uncertainty creates increasing returns to scale through the law of large numbers.

For the degenerate case where aggregate demand is certain, the "candidate" equilibrium involves all firms offering a service rate of one. A deviation by one firm to capture the market does not increase the service rates offered by other firms, so firm i's profits are linear in q_i. Therefore, the price cannot be above marginal cost; but if the price equals marginal cost, a firm could make profits by raising the price and reducing capacity (maintaining a service rate of one while lowering the service rate of other firms). It follows that, without the presence of demand uncertainty, no equilibrium exists in which all firms use pure strategies.

The next proposition shows that, as long as there is some demand uncertainty, and n is sufficiently large, there exists an equilibrium in which all firms choose pure strategies. We show that attempts to gain significant market share above the candidate equilibrium level lead to negative profits, and that profits are quasi-concave over the domain where profits are positive. The first order conditions are therefore sufficient for profit maximization. The value of n that guarantees existence depends on the degree of uncertainty. When the density, f, approaches the degenerate case of certainty, the order of limits matters. For any f, there exists an n for which we have existence; however, for any n, there exists an f for which we have nonexistence. See section 5 for examples.

<u>Proposition 4.2</u>: For sufficiently large n, a Nash equilibrium in which all firms choose pure strategies exists. (From proposition (3.16), this equilibrium is unique and can be calculated from (3.17).)

<u>Proof</u>: We must check that, if all firms $j \neq i$ choose $k_j = k^*$ and $p_j = p^*$

(given in (3.17)), then firm i's best response is $p_i = p^*$ and $k_i = k^*$. Firm i's maximization problem can be posed as a choice of p_i, K_i, and q_i to maximize profits, subject to the constraint that q_i must be the optimal response chosen by consumers. We have:

$$(4.3) \qquad \max_{p,q,K} q[p\mu s(K) - cK]$$

$$\text{s.t.}$$

$$(v - p)s(K) = (v - p^*)s\left(\frac{k^*(n-1)}{n-q}\right), \; 0 \le q \le n.$$

The first constraint in maximization problem (4.3) is expressed as an equality because the firm would always raise the price rather than allow a strict inequality to hold.

From the proof of Lemma (3.10) it follows that, for any choice of q, $0 < q \le n$, the optimal K is k^*. Since firm i can ensure positive profits by going along with the candidate equilibrium, $q = 0$ cannot be optimal. Therefore, the solution to (4.3) must involve $K = k^*$.

Plugging $K = k^*$ into problem (4.3), and then solving the constraint for p as a function of q, we have the following unconstrained maximization problem, which is equivalent to (4.3).

$$(4.4) \qquad \max_{0 \le q \le n} \phi(q) \equiv q\left[v\mu s(k^*) - (v - p^*)s\left(\frac{k^*(n-1)}{n-q}\right)\mu - ck^*\right].$$

Let $\bar{k}$ denote $k^*(n - 1)/(n - q)$.

<u>Claim 1</u>: For all $\epsilon > 0$, there exists an n such that $\phi(q) > 0$ implies $\bar{k} - k^* < \epsilon$.

To see why Claim 1 is true, suppose there is an $\epsilon > 0$ for which $\bar{k} \geq k^* + \epsilon$ for all n. Then we have

$$(4.5) \qquad \phi(q) \leq q[v\mu s(k^*) - (v - p^*)\mu s(k^* + \epsilon) - ck^*].$$

Inequality (4.5) can be rewritten as

$$(4.6) \qquad \phi(q)/q \leq (v - p^*)\mu[s(k^*) - s(k^* + \epsilon)] + [\mu p^* s(k^*) - ck^*].$$

The second term in brackets on the right side of inequality (4.6) converges to zero as n approaches infinity. Also, $s'(k^*) = c/(v\mu)$ holds, so $s(k^*) - s(k^* + \epsilon)$ is bounded below zero, and the bound is independent of n. It follows that the right side of (4.6) becomes negative for large enough n, which contradicts the supposition and proves Claim 1.[8]

<u>Claim 2</u>: As n approaches infinity, $\phi(q)$ is strictly concave on an interval containing the set of q for which $\phi(q)$ is positive.

To prove claim 2, we differentiate the right side of (4.4), which yields

$$(4.7) \qquad \phi'(q) = v\mu s(k^*) - (v - p^*)\mu s(\bar{k}) - ck^* - q(v - p^*)\mu s'(\bar{k})\, k^*(n - 1)/(n - q)^2.$$

[8]Claim 1 proves that large deviations from the candidate equilibrium, including "taking the market," lead to negative profits (provided the number of firms is large).

From (4.7), we calculate the second derivative, keeping in mind that $\bar{k}$ depends on q. After simplifying, we have

$$(4.8) \qquad \phi''(q) = -\frac{(v - p^{*})\mu k^{*}(n - 1)n}{(n - q)^{3}} \left[2s'(\bar{k}) + s''(\bar{k})\bar{k}\frac{q}{n}\right].$$

Rewriting the definition of $\bar{k}$ as

$$(4.9) \qquad \frac{q}{n} = 1 - \frac{k^{*}}{\bar{k}}\frac{(n - 1)}{n},$$

and substituting equation (4.9) into (4.8) yields

$$(4.10) \qquad \phi''(q) = -\frac{(v - p^{*})\mu k^{*}(n - 1)n}{(n - q)^{3}} \left[2s'(\bar{k}) + s''(\bar{k})\left(\bar{k} - k^{*} + \frac{k^{*}}{n}\right)\right].$$

Using Claim 1 (and choosing the ϵ in Claim 1 to be $\epsilon'/2$), for all $\epsilon' > 0$ we can find an n such that $\phi(q) > 0$ implies

$$(4.11) \qquad \phi''(q) < -\frac{(v - p^{*})\mu k^{*}(n - 1)n}{(n - q)^{3}}\left[2s'(\bar{k}) + s''(\bar{k})\epsilon'\right].$$

The fact that the density f is bounded implies s" is bounded[9], so the term in brackets becomes positive as n approaches infinity. Thus, over the range of q for which profits are nonnegative, ϕ is strictly concave. Since ϕ is concave over the relevant range of q, there is a unique solution to maximization problem (4.4), which solves the first-order condition, $\phi'(q) = 0$. Substituting the formula for p^{*}, given in (3.17), into the right side of

[9]Instead of assuming that f is continuous everywhere, it is sufficient to assume that f is continuous on a neighborhood of k*. Therefore, f and s" are bounded on a neighborhood of k*. Thus, we can handle exponential distributions.

(4.7), it is evident that $q = 1$ is the unique value of q which satisfies $\phi'(q) = 0$. Thus, $p_i = p^*$, $K_i = k^*$, and $q_i = 1$ is the best response to the strategies of the other firms. For large enough n, the candidate equilibrium (3.17) is therefore an equilibrium. ■

5. An Example

In this section, we present some examples illustrating the existence of pure-strategy equilibrium for large n, and the possible nonexistence for small n. Let $\tilde{a}$ be uniformly distributed on the interval $[1 - \epsilon, 1 + \epsilon]$, for $\epsilon < 1$. Observe that an increase in ϵ produces a mean-preserving spread, with $\mu = 1$. From Lemma (2.4), we have

$$s(K) = \begin{cases} K & \text{for } K \le 1 - \epsilon \\ 1 - (1/4\epsilon)(1 + \epsilon - K)^2 & \text{for } 1 - \epsilon \le K \le 1 + \epsilon \\ 1 & \text{for } 1 + \epsilon \le K. \end{cases} \tag{5.1}$$

Consider the parameter values $\epsilon = .1$, $c = .1$, and $v = 1$. Definition (3.1) and straightforward algebra yield $k^* = 1.08$ and $s(k^*) = .999$.

From equation (4.10), we have

$$\text{sign } \phi''(q) = -\text{ sign}[2s'(\bar{k}) + s''(\bar{k})(\bar{k} - k^* + k^*/n)]. \tag{5.2}$$

(Recall that $\bar{k} \equiv k^*(n - 1)/(n - q)$.) For $.9 \le \bar{k} \le 1.1$, (5.1) implies $s'(\bar{k}) = 5.5 - 5\bar{k}$ and $s''(\bar{k}) = -5$. Equation (5.2) becomes

$$\text{sign } \phi''(q) = \text{sign } [15\bar{k} - 16.4 + 5.4/n]. \tag{5.3}$$

Since $\bar{k}$ is strictly increasing in q, $0 \le q \le n$, and the expression in brackets in (5.3) is linear in $\bar{k}$, ϕ can change convexity at most once. When $n \le 27$ holds, we see from equation (5.3) that ϕ is convex at $q = 1$, so there can be no equilibrium. When $n > 27$ holds, then there is a range of q (including $q = 1$) on which ϕ is concave. Above some cutoff value of q, ϕ becomes convex. Finally, above a second cutoff value of q, we have $\bar{k} \ge 1.1$, and ϕ is a linear function of q. Either ϕ is negative, or ϕ is an increasing linear function of q. Therefore, to check whether or not an equilibrium exists for $n > 30$, we need only compare $\phi(1)$ with $\phi(n)$.

For $n = 100$, we have $p^* = .10908$ and $\phi(1) = .000972$. However, by deviating to $k_i = 108$ and $p_i = .10819$, firm i captures the market ($q_i = n$) and maintains $K_i = 1.08$. Firm i's profits are given by $\phi(n) = .008109$. Therefore, no equilibrium with all firms choosing pure strategies can exist when $n = 100$.

For $n = 108$, we have $p^* = .109$ and $\phi(1) = .000899$. By deviating to $p_i = .108117$ and maintaining $K_i = 1.08$, firm i makes profits of $\phi(n) = .000910$. The deviation to steal the market just barely breaks the candidate equilibrium. For $n = 109$, we have $p^* = .109$ and $\phi(1) = .000891$. Since $\phi(n) = 0$ holds, and by the concavity-convexity of ϕ, we know that a unique pure-strategy equilibrium exists.

Figures 1, 2 and 3 graph profit (or zero when profit is negative) as a function of q ($0 \le q \le 15$)[10] and price ($.107 \le p \le .11$) when all other firms are choosing the equilibrium capacity, k^*, and the equilibrium price, p^*. Parameter values are $c = .1$, $v = 1$, and $\epsilon = .1$. For Figures 1, 2, and 3, the

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[10]Note that the constraint in (3.19) implies that the firm's capacity is an increasing function of q, i.e. $k = q\, s^{-1}([(v-p^*)/(v-p)]s[(n-1)k^*/(n-q)])$.

number of firms is 100, 108, and 110, respectively.

The example was chosen to illustrate the fact that pure-strategy Nash equilibrium might not exist, even when the number of firms is substantial. Nonetheless, equilibrium must eventually exist. The parameter values $c = .1$, $v = 1$, $\epsilon = .1$ do not appear to be extreme, yet they give rise to $s(k^*) = .999$. Since a deviation to capture the market only raises the service rate the firm must offer consumers by a factor of .001, this deviation will be attractive unless the number of firms is quite large and profit margins are quite small. Increasing c and ϵ greatly reduces the cutoff value for n at which pure-strategy equilibrium begins to exist.

6. Discussion

Here, as in Carlton [1978] and Peters [1984], there is an infinite cost of searching beyond the first firm. Unlike most other search papers, however, the supply-side of the market is known to all consumers before they must search out a firm. In any pure-strategy equilibrium, all firms choose the same price, and so our model predicts that no price dispersion will exist in equilibrium. This is a consequence of our assumption that all consumers are identical. However, if we allowed consumers to have different reservation prices, we would expect price dispersion to appear. Different firms would serve different market niches, with some firms offering high-valuation consumers a high price and high service rate, while other firms would offer low valuation consumers a low price and low service rate. This "demand-side" motivation for price dispersion has very different (and testable) implications from the traditional "supply-side" motivation. Low-price firms should be more likely to ration consumers than high-price firms, and high-valuation consumers should be more likely than low-valuation consumers to visit high-price firms.

Although we have been assuming that firms commit to a fixed price and ration any excess demand, this form of price competition could be selected even if the firm could choose any anonymous mechanism for making transactions. Peck [1992] compares the game, $\bar{\Gamma}$, where firms must choose prices and quantities to the game, Γ, where firms choose quantities and any anonymous transactions mechanism. (A transactions mechanism is a mapping from ordered lists of consumers visiting the firm to allocations of the commodity and money.) It is shown that any equilibrium of $\bar{\Gamma}$ is an equilibrium of Γ. Conversely, all equilibria of Γ involve firms choosing to fix prices and ration excess demand. The crucial assumptions are that consumers are identical, demand one unit of the commodity, and cannot commit to pay money unless they receive the good.

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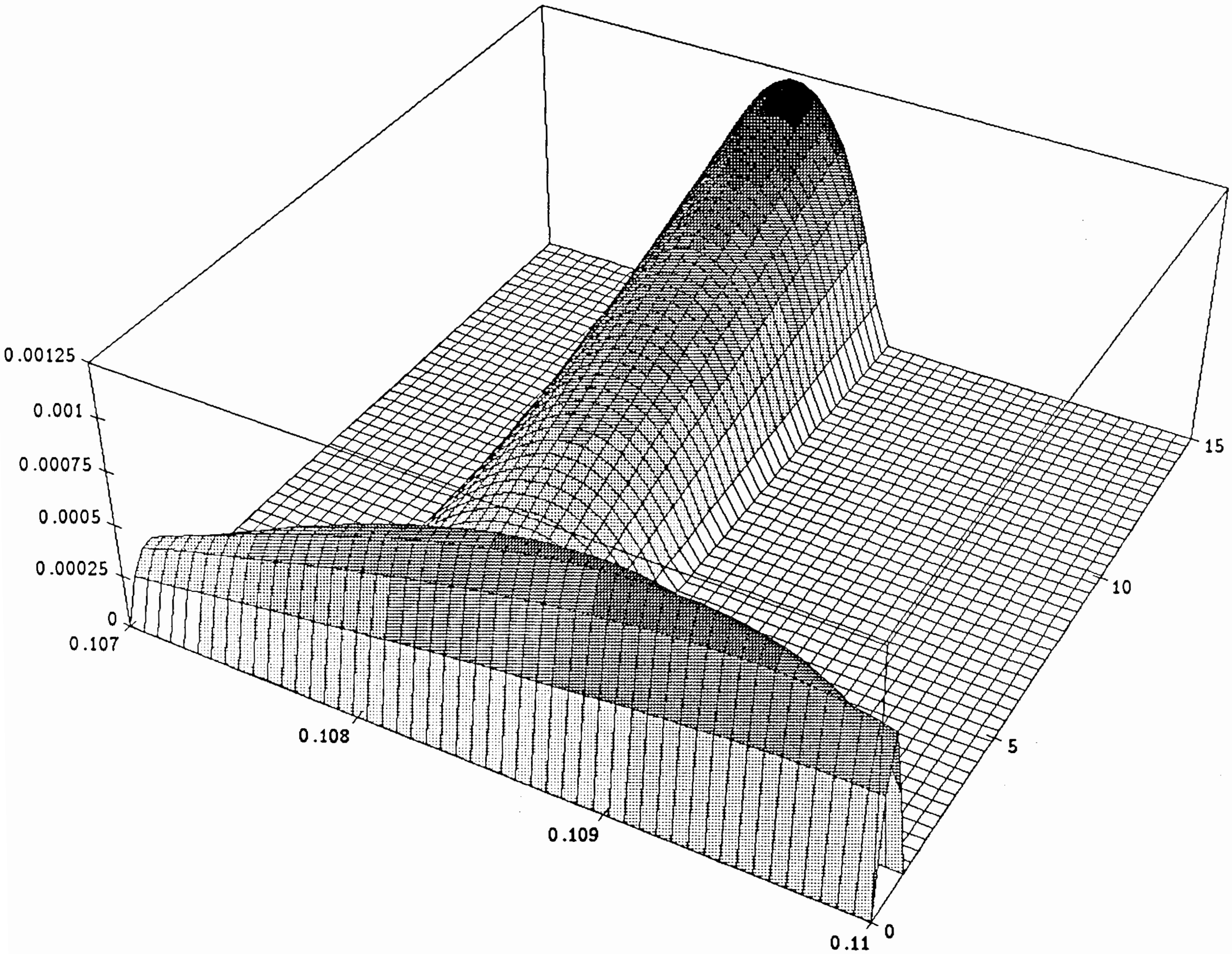


Figure 1

The maximum of profit and zero, as a function of the customer base q ($0 \leq q \leq 15$) and price ($.107 \leq p \leq .11$) when all other firms are choosing the equilibrium capacity, k^*, and the equilibrium price, p^*. Parameter values are $n = 100$, $c = .1$, $v = 1$, and $\epsilon = .1$.

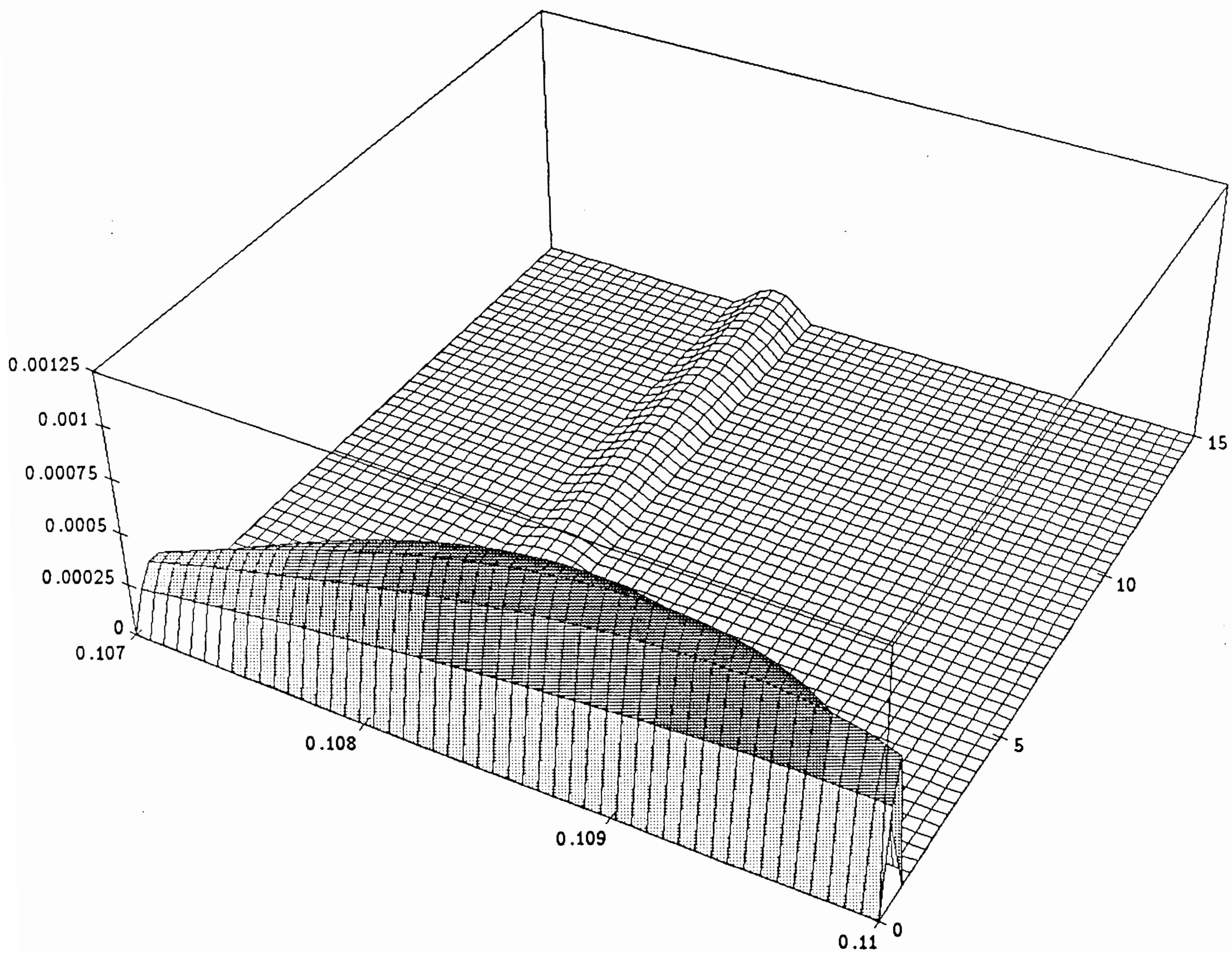
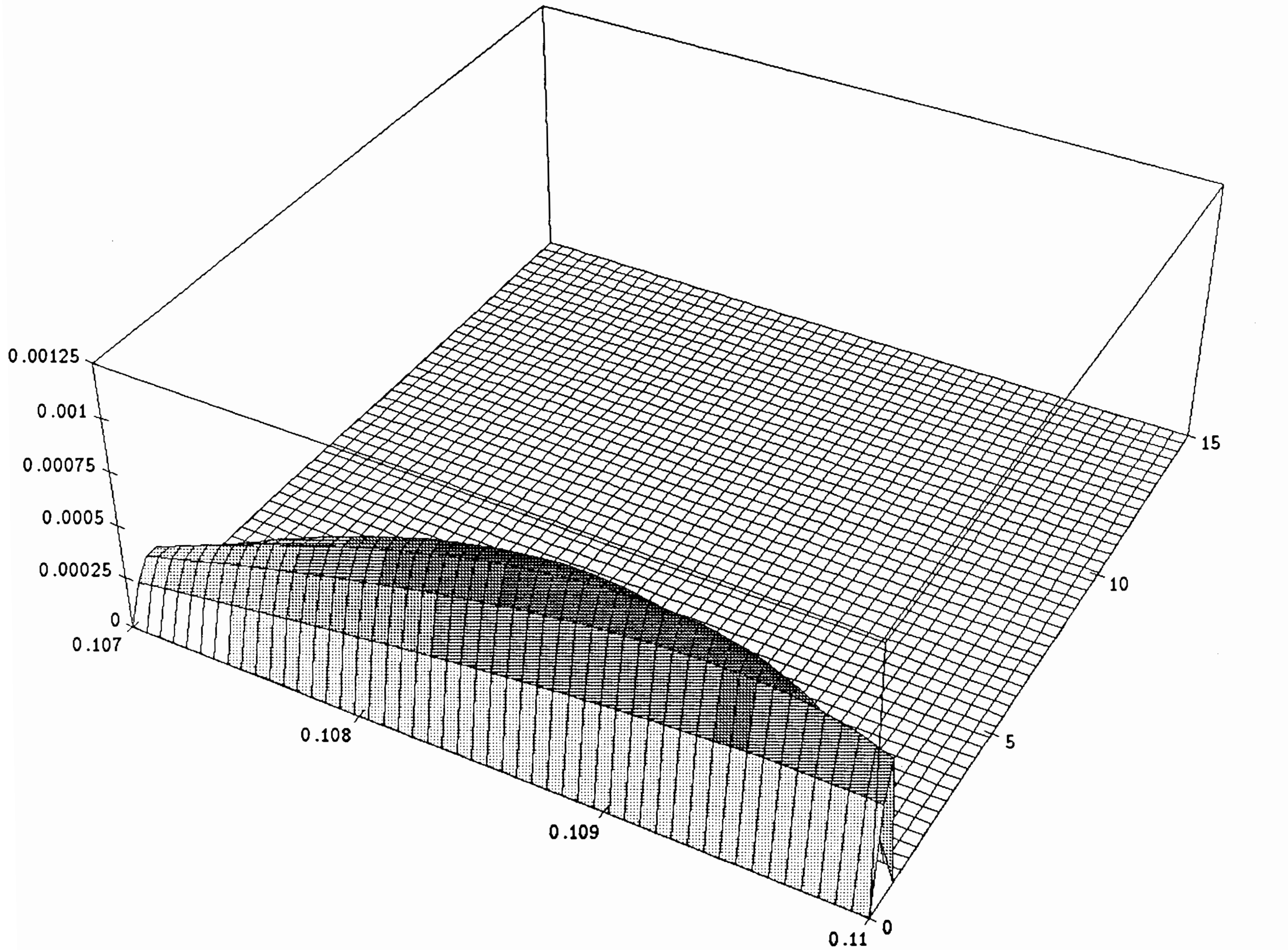


Figure 2

The maximum of profit and zero, as a function of the customer base q ($0 \le q \le 15$) and price ($.107 \le p \le .11$) when all other firms are choosing the equilibrium capacity, k^*, and the equilibrium price, p^*. Parameter values are $n = 108$, $c = .1$, $v = 1$, and $\epsilon = .1$.



<u>Figure 3</u>

The maximum of profit and zero, as a function of the customer base q ($0 \le q \le 15$) and price ($.107 \le p \le .11$) when all other firms are choosing the equilibrium capacity, k^*, and the equilibrium price, p^*. Parameter values are $n = 110$, $c = .1$, $v = 1$, and $\epsilon = .1$.

#990

"Competition over Price and Service Rate when Demand is Stochastic: a Strategic Analysis"

by

Raymond Deneckere*

and

James Peck*

May, 1992

*Department of Managerial Economics and Decision Sciences, J. L. Kellogg Graduate School of Management, Northwestern University, 2001 Sheridan Road, Evanston, Illinois 60208.

[P4-2]