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A REMARK ON INFINITELY REPEATED EXTENSIVE GAMES

by

Ariel Rubinstein*

and

Asher Wolinsky**

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*Department of Economics, Tel Aviv University, Tel Aviv, Israel and Department of Economics, Princeton University, Princeton, NJ 08544.

**Department of Economics, Northwestern University, Evanston, IL 60208.

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Abstract

Let Γ be a game in extensive form and G be its reduced normal form game. Let $\Gamma^\infty(\delta)$ and $G^\infty(\delta)$ be the infinitely repeated game versions of Γ and G respectively, with common discount factor δ. This note points out that the set of SPE payoff vectors of $\Gamma^\infty(\delta)$ might be different from that of $G_\infty(\delta)$, even when δ is arbitrarily close to 1. This difference can be substantial when G fails to satisfy the "dimensionality" condition (a-la Fudenberg and Maskin (1986) or Abreu, Dutta and Smith (1992)).

*Department of Economics, Tel Aviv University, Tel Aviv, Israel and Department of Economics, Princeton University, Princeton, NJ 08544.

**Department of Economics, Northwestern University, Evanston, Il 60208.

Introduction

Let Γ be a game in extensive form and G be its reduced normal form. Let $\Gamma^{\infty}(\delta)$ and $G^{\infty}(\delta)$ be the infinitely repeated game versions of Γ and G respectively with common discount factor δ. A play of $\Gamma^{\infty}(\delta)$ or $G^{\infty}(\delta)$ results in a sequence of one-shot payoff vectors, $(u^t)_{t=0,1,2,...}$. A feasible payoff vector of $\Gamma^{\infty}(\delta)$ or $G^{\infty}(\delta)$ is a vector whose i-th component is player i's average present value of a sequence of one-shot payoffs, i.e., $(\Sigma\delta^t u_i^{\,t})(1-\delta)$. Let $V[\Gamma^{\infty}(\delta)]$ and $V[G^{\infty}(\delta)]$ denote the sets of subgame perfect equilibrium (SPE) payoffs of these two games. Clearly $V[\Gamma^{\infty}(\delta)]\subseteq V[G^{\infty}(\delta)]$. Obviously, the inclusion can be strict. If, for example, Γ has a unique SPE but two Nash equilibria, then for small δ the set $V[\Gamma^{\infty}(\delta)]$ will include a unique point, while $V[G^{\infty}(\delta)]$ will include convex combinations of the Nash equilibrium payoff vectors. What happens for δ sufficiently large? The repeated game literature did not address this point, presumably because it seemed intuitively clear that, in the limit, as δ approaches 1, these sets coincide. In what follows we present three examples which contradict this intuition. The first two examples demonstrate that, when G does not satisfy the dimensionality condition of the repeated game literature, even the limiting set of SPE payoffs, $\cup_{\delta<1}V[\Gamma^{\infty}(\delta)]$, might be substantially different from $\cup_{\delta<1}V[G^{\infty}(\delta)]$. The third example shows that, when G satisfies that condition, the sets $\cup_{\delta<1}V[G^{\infty}(\delta)]$ and $\cup_{\delta<1}V[\Gamma^{\infty}(\delta)]$ may still differ with respect to vectors which are not strictly individually rational.

<u>Example 1</u>

Consider the extensive game Γ depicted below

```
  1--R--(1,1)
  |
  D
  |
  2--R--(2,2)
  |
  D
  |
(0,0)
```

The individually rational (minmax) level of each of the players is 1; player 1 can "punish" player 2 by playing R, and player 2 can enforce a payoff not higher than 1 by playing D. The normal form game, G, is

	R	D
R	1,1	1,1
D	2,2	0,0

Obviously, any convex combination of (1,1) and (2,2) which is the average present value of an arbitrary sequence of (1,1) and (2,2) is a SPE payoff vector of $G^{\infty}(\delta)$, since both (D,R) and (R,D) are Nash equilibria of G. However, for any $\delta<1$, the unique SPE of $\Gamma^{\infty}(\delta)$ is a repeated play of (D,R). To see it, let m be the infimum of a player's payoff over all SPE. Consider any SPE. If in his first move player 1 chooses D, the payoff to the players in the continuation cannot be less than $\delta m+(1-\delta)2$, since player 2 can guarantee that much by choosing R. Therefore, any SPE payoff y satisfies $y\geq\delta m+(1-\delta)2$ implying $m\geq\delta m+(1-\delta)2$ and hence $m=2$. ■

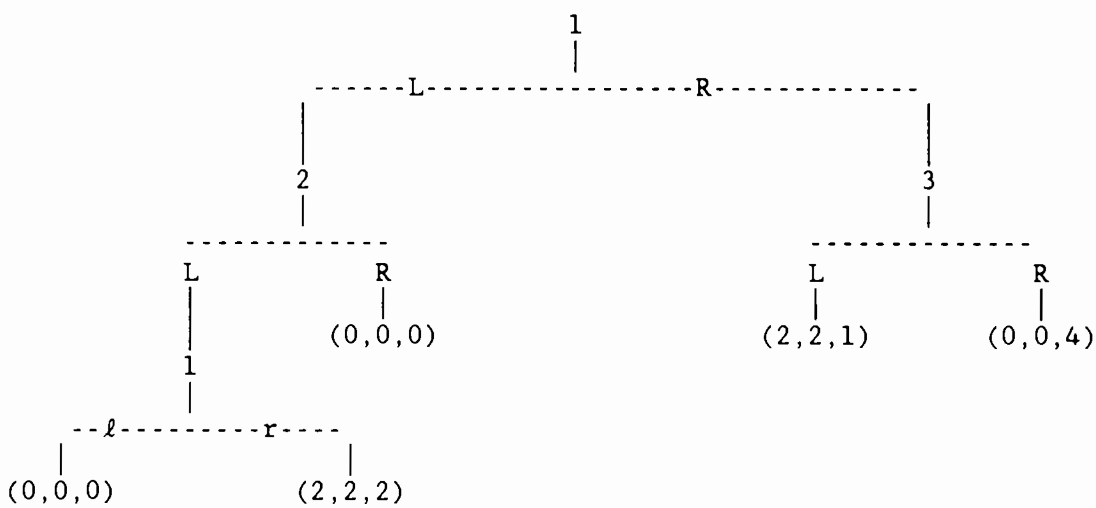
The basic difference between subgame perfection in $\Gamma^{\infty}(\delta)$ and in $G^{\infty}(\delta)$ is of course that, in the former, player 2's strategy has to be optimal after histories that end with player 1 choosing D, while in $G^{\infty}(\delta)$ player 2's strategy has to be optimal just after histories that end with 2's move.

The special feature of Γ that is responsible for the above observation is the one-dimensionality: all the payoff vectors lie on a single line. It is easy to see that, for the same game form without this property, the result will not hold. More generally, an n-player game is said to satisfy the dimensionality condition, if there are n feasible and strictly individually rational payoff vectors $(u(i))$ such that $u(j)_i > u(i)_i$ for all i and j. A certain version of the folk theorem says that, if G satisfies this dimensionality condition, then for every strictly feasible individually rational vector $u^\star$ there is $\delta^\star$ so that for all $\delta>\delta^\star$, $u^\star \in V[G^{\infty}(\delta)]$ (see Fudenberg and Maskin (1986) for the basic folk theorem with discounting and Abreu, Dutta and Smith (1992) for this particular version of it). It is easy to see that this theorem holds for extensive form games as well. That is, if Γ satisfies the dimensionality condition, then for every strictly feasible individually rational vector $u^\star$ there is $\delta^\star$ so that for all $\delta>\delta^\star$, $u^\star \in V[\Gamma^{\infty}(\delta)]$. The proof of this point essentially repeats the proof of the folk-theorem for G. The only change is that the choice of $\delta^\star$ should guarantee that deviations in unreached subgames of Γ are also deterred. (Thus, for each $u^\star$, the $\delta^\star$ for the folk theorem for Γ^{∞} is greater or equal than the corresponding $\delta^\star$ for G^{∞}.) The above example showed that for games which do not satisfy the dimensionality condition this result is not necessarily correct.

Observe that, for the point made here, the dimensionality condition matters already for a two players game. In contrast, Fudenberg and Maskin's theorem requires the dimensionality condition only when there are more than two players.

Example 2:

For two player games, the dimensionality condition fails only in one dimensional games as in example 1. For games with more than two players, the set of games that fail the dimensionality condition is of "higher dimension" and there are more interesting examples for the discrepancy between $V[\Gamma^{\infty}(\delta)]$ and $V[G^{\infty}(\delta)]$.



In this three players game each individually rational level is 0. The dimensionality condition is not satisfied (player 1's and 2's payoffs are fully correlated). Notice that, for δ large enough, $(2,2,1)\in V[G^{\infty}(\delta)]$. This vector is obtained by a constant play of $((R,\ell),R,L)$, which is supported by the threat to switch to a constant play of $((L,\ell), R, R))$ which is a Nash equilibrium of G. However, $V[\Gamma^{\infty}(\delta)]=\{(2,2,2)\}$. To see it, let m be the infimum of player 1's (and thus also player 2's) payoff over all SPE. We claim that $m=2$. To prove this, notice that, in any SPE, in any subgame that starts with player 1's choice between ℓ and r, player 1 cannot get less than $(1-\delta)2+\delta m$. From the equality between player 1's and 2's payoffs, it follows that player 2 cannot get less than $(1-\delta)2+\delta m$ in any subgame which starts with his move. Therefore, player 1 cannot get less than $(1-\delta)2+\delta m$ in any SPE. Since $(1-\delta)2+\delta m > m$ if $m < 2$, it must be that $m=2$. Thus, the only single period payoff vectors which may occur in any SPE are (2,2,2) or (2,2,1). It follows that a SPE cannot include any single period play in which the one-shot payoff is (2,2,1), since then player 3 could profitably deviate to the strategy that always plays R. ■

Example 3:

This example shows that, even when the dimensionality condition holds, there could be a point, which is individually rational, but not strictly individually rational, which is in $V[G^{\infty}(\delta)]$ for every δ but is never in $V[\Gamma^{\infty}(\delta)]$. Consider the following 3 player game:

```
  1------R-----2------R----3----R----(0,1,1)
  |D           |D          |D
(0,0,0)     (1,0,0)     (0,0,-1)
```

For sufficiently high δ, there are payoff vectors near (0.3, 0.7, 0.7) , (0.9, 0.1, 0.1) and (0.85, 0.12, 0.09) which are feasible and strictly individually rational and thus the dimensionality condition is satisfied. The payoff vector (1,0,0) is in $V[G^{\infty}(\delta)]$ for every δ since it is a G-Nash equilibrium payoff vector. However, it is not a payoff vector in any $V[\Gamma^{\infty}(\delta)]$. To see this, note that, in any SPE, player 3's payoff in a subgame starting with a move of player 3 must be strictly positive (he can guarantee it by choosing R at the origin of the subgame). Since player 2's payoff always weakly exceeds player 3's payoff, player 2 can guarantee a strictly positive payoff at a subgame starting with his move by choosing R. Therefore, there is no SPE in which the one-shot payoff vector (1,0,0) is received in all periods. But, this is the only sequence which can possibly yield the payoff vector (1,0,0) in Γ^{∞}. ■

Remark on the limit of the means Observe that, for repeated games with the limit of the means criterion, the SPE payoff vectors of Γ^{∞} and of G^{∞} coincide. Since in this case only infinite sequence of deviations may matter, the requirement that 2's strategy be optimal after histories that end with 1's move does not add any restriction and the Perfect-Folk theorem of Aumann and Shapley (1976) and Rubinstein (1977) holds.

Remark on the repeated extensive game with finite automata: A significant difference between the analysis of $\Gamma^{\infty}(\delta)$ and $G^{\infty}(\delta)$ has been pointed out in the context of the repeated game with finite automata (see Rubinstein (1986) and Abreu and Rubinstein (1988)). Piccione and Rubinstein (1991) show that the set of Nash equilibria of the machine game for $\Gamma^{\infty}(\delta)$, where Γ is an extensive game with perfect information, includes only repetitions of Nash equilibria of Γ, a much smaller set of the machine game equilibria of the corresponding $G^{\infty}(\delta)$.

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