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ADDITIVE REPRESENTATIONS OF NON-ADDITIVE MEASURES AND THE CHOQUET INTEGRAL*

by

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and

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Abstract

This paper studies some new properties of set functions (and, in particular, "non-additive probabilities" or "capacities") and the Choquet integral with respect to such functions, in the case of a finite domain.

We use an isomorphism between non-additive measures on the original space (of states of the world) and additive ones on a larger space (of events), and embed the space of real-valued functions on the former in the corresponding space on the latter. This embedding gives rise to the following results:

-- the Choquet integral with respect to any totally monotone capacity is an average over minima of the integrand;

-- the Choquet integral with respect to any capacity is the difference between minima of regular integrals over sets of additive measures;

-- under fairly general conditions one may define a "Radon-Nikodym derivative" of one capacity with respect to another;

-- the "optimistic" pseudo-Bayesian update of a non-additive measure follows from the Bayesian update of the corresponding additive measure on the larger space.

We also discuss the interpretation of these results and the new light they shed on the theory of expected utility maximization with respect to non-additive measures.

1. Introduction

The representation of beliefs by real-valued set functions which do not necessarily satisfy additivity dates back to Dempster (1967, 1968) and Shafer (1976) at the latest. Their theory is not directly related to decision making under uncertainty, nor is their concept of "probability" derived from preferences. Rather, they assume that "weight of evidence" for events is a primitive, and study the "belief functions" which are generated by summation of such weights. Belief functions are a special class of "non-additive measures" or "capacities," characterized by a condition called "total monotonicity."

In modern economic and decision theory, on the other hand, terms such as "utility" and "probability" are defined via preferences which, at least in principle, are supposed to be observable. Von Neumann and Morgenstern (1944) have defined a "utility" by preferences over lotteries, with given probabilities. Building upon works of Ramsey (1931) and de Finetti (1937), Savage (1954) provided a simultaneous derivation of "utility" and "probability" from preferences over objects ("act") which did not presuppose either of these (potentially-metaphysical) concepts. In the same vein, Anscombe and Aumann (1963) provided a similar axiomatization of "subjective" probability (assuming that probabilities on an auxiliary space are given). Apart from bestowing "cognitive significance" upon the term "probability," the advantage of the axiomatic approach is that this term is derived together with a procedure to use it. While abstract set functions which represent "beliefs" do not, in and of themselves, prescribe a way to make decisions in face of uncertainty, the "probability" measure derived by Savage (1954) and Anscombe-

Aumann (1963) are to be used in a very specific way, namely in the maximization of expected utility.

However, the expected utility paradigm, which is still the dominant approach in economic, decision and game theory, was subjected to empirical refutations as a descriptive theory, and sometimes also to theoretical attacks as a normative one. Among the most famous experiments, mind-experiments and "paradoxes" are Allais (1953), Ellsberg (1961) and Kahnemann-Tverskey (1979). (See Machina (1987) for a survey and references.) While Allais (1953) and Kahnemann-Tverskey (1979) do not necessarily undermine the concept of probability per se, Ellsberg's (1961) findings are incompatible with the very notion of an (additive) probability measure as representing beliefs. That is to say, neither expected utility maximization nor any other reasonable procedure which relies only on the distributions induced by an additive probability could account for observed choices.

Although his original motivation was somewhat different, Schmeidler (1982, 1986, 1989) suggested a generalization of expected utility which could accommodate Ellsberg's evidence. He provided an axiomatization derivation of both utilities and not-necessarily-additive probabilities, such that a decision maker's preferences are equivalent to expected utility maximization, where expectation with respect to a non-additive measure is computed by the Choquet integral (Choquet (1953-4)). Conceptually similar but mathematically different, Gilboa (1987), Wakker (1989) Sarin-Wakker (1990) and Fishburn (1988) provided additional such axiomatizations for other frameworks, the latter allowing for intransitive preferences as well.

In a different model, Gilboa and Schmeidler (1989) characterized preferences which may be represented by a utility function and a set of additive measures, in the sense that preferences obey maximization of the minimal expected utility over all measures in the given set. (See also Bewley (1986) who deals with a set of probabilities with partially ordered preferences.) These preferences can also be represented by the non-additive model (with maximization of the Choquet integral) in case the set of measures is the core of a convex non-additive measure. (The definitions will be given below. At this point let us only mention that convex non-additive probabilities correspond to uncertainty aversion and that belief functions are, in particular, convex.)

The non-additive expected utility theory in general, and with convex probabilities in particular, has been recently applied to a variety of problems. In economics and finance, such applications include Dow and Werlang (1990, 1992), Dow, Madrigal and Werlang (1989), Simonson and Werlang (1990), and Yoo (1990, 1991). The same mathematical structure, though differently interpreted, is also used in other fields of decision theory. (See, for instance, Gilboa (1989a) for multi-stage decisions and Ben-Porath and Gilboa (1991) for the measurement of inequality.)

However, the interest in non-additive measures and the Choquet integral stems from other applications as well. The theory of transferable-utility cooperative games deals with non-additive set functions, and, at times, also with integration with respect to them. (See Rosenmuller (1971, 1972).) In artificial intelligence, belief functions have been used to represent uncertainty (see Dubois and Prade (1986, 1991) Halpern and Fagin (1989), Dubois, Prade and Ramer (1991), and others.) Following Dempster, belief functions are also used in statistics in the absence of an additive prior. (See Huber and Strassen (1973), Huber (1973), Walley and Fine (1982), Walley

(1987), Wasserman (1990) and Wasserman and Kadane (1990).) Additional studies of mathematical properties of the Choquet integral include Denneberg (1990) and Dyckerhoff and Mosler (1990).

In this paper we present some results, which shed new light on Choquet integration and suggest ways to re-interpret various models using it. Although the results are interpretable in any domain, we will adhere to decisions under uncertainty for the most part. The counterpart interpretations in other domains will be obvious.

The mathematical cornerstone of this paper is a well-known theorem in cooperative game theory, according to which the space of all non-additive measures ("games") is spanned by a natural linear basis (of "unanimity games"). This result may be viewed as suggesting an isomorphism between non-additive set functions on the original space (of states of the world) and additive one on a larger space (of all events). (Similarly, one may use the basis of the "duals" of the unanimity games, which would lead to "dual" results of those presented here.)

Using this result we show that the Choquet integral with respect to any non-additive set function v is simply some linear combination of the minima of the integrand (over various events). Furthermore, if v is a belief function, this linear combination reduces to a weighted average. Thus, for such probabilities v, the integral is both mean of minima (over events) and, since they are also convex, minimum of means (where the minimum is taken over additive measures in the core). (Related results, with a somewhat different interpretation, were obtained in Jaffray (1989) and Wasserman (1990).)

We then provide an interpretation of this result, according to which the space of real-valued functions on the original space is embedded in the corresponding one on the larger space of events in an integral-preserving way. This view of the Choquet integral may accommodate models in which the space of states of the world may be misspecified, in which case the non-additivity of the measure is due to possibly missing states, which are accounted for in the larger space.

Reinterpreting the "mean of minima" result in the context of social choice, one finds that any social welfare function which satisfies Schmeidler's (1989) axioms is a linear combination of coalitional "utility levels," where the utility level of a coalition is simply the minimal utility of each of its members. Again, if the underlying non-additive measure is totally monotone, this linear combination reduces (up to a positive constant) to a weighted average. (See also Remark 5.3 in the sequel.)

Another result which follows from the linear combination representation is that any non-additive measure is the difference of two totally monotone set functions (i.e., belief functions multiplied by a non-negative constant). It then follows that for every set function v there are sets of additive measures C^+ and C^-, such that the Choquet integral with respect to v equals the difference of two minima: one of all integrals with respect to measures in C^+, the other--with respect to C^-. Thus, while minima of integrals is only a subset of the functionals described by Choquet integration, differences of such minima exhaust all these functionals. We later discuss further interpretations of this result.

Next we proceed to deal with the question of Radon-Nikodym derivative of one set-function with respect to another. While straightforward generalizations of the theorem for the finite case do not seem to hold, it appears that the (larger) space of events is the appropriate one for such a generalization.

Finally, we address the question of updating a non-additive probability measure. Recent studies of this problem include Fagin and Halpern (1989), Jaffray (1990) and Gilboa-Schmeidler (1991). The latter axiomatize the Dempster-Shafer update rule (see Dempster (1968)) as a "pessimistic" one, as well as a corresponding "optimistic" one (also used in Gilboa (1989b).)

The isomorphism between non-additive measures and additive ones (on the space of events) suggests a new look at the updating problem: since there is little disagreement regarding the way additive measures should be updated, one may update the additive measure and project it back to the original space, to obtain an updated, possibly non-additive, measure on it. This construction leads to the "optimistic" update. Using a similar update with the dual space gives rise to the "pessimistic" one. (See Dubois-Prade (1991), and Lipman (1992).)

In this paper we restrict ourselves to the case of a finite space of states of the world. All the mathematical results we present range from immediate to simple. Indeed, practically all of them have natural counterparts in the general case, where the mathematics is considerably more complicated. In order to highlight the conceptual issues, we chose to focus here only on the finite case, and deal with the general one in Gilboa-Schmeidler (1992).

The rest of the paper is organized as follows. Section 2 presents notations and definitions, and Section 3 is devoted to quoting some known results. From Section 4 on we present our new results and discuss their interpretation.

2. Notations and Definitions

Let Ω be a nonempty set of states of the world and let Σ be an algebra of events defined on it. We will assume w.l.o.g that $\Sigma = 2^{\Omega}$.

A function $v: \Sigma \to \mathbb{R}$ with $v(\emptyset) = 0$ is called a non-additive signed measure or a capacity. The space of all capacities will be denoted by V and will be considered as a linear space (over $\mathbb{R}$) with the natural (pointwise) operations.

For $v \in V$ we will use the following definitions:

(1) v is monotone if $A \subsetneq B$ implies $v(A) \leq v(B)$ for all $A,B \in \Sigma$.

(2) v is normalized if $v(\Sigma) = 1$.

(3) v is additive if $v(A \cup B) = v(A) + v(B)$ for all $A,B \in \Sigma$ with $A \cap B = \emptyset$. An additive v is also called a signed measure.

(4) v is convex if for every $A,B \in \Sigma$, $v(A \cup B) + v(A \cap B) \geq v(A) + v(B)$. It is superadditive if the above holds for all $A,B \in \Sigma$ with $A \cap B = \emptyset$.

(5) v is nonnegative if $v(A) \geq 0$ for all $A \in \Sigma$.

(6) v is totally monotone if it is nonnegative and, for every $A_1, \ldots A_n \in \Sigma$, $v(\cup_{i=1}^{n} A_i) \geq \sum_{\{I | \phi \neq I \subseteq \{1, \ldots, n\}\}} (-1)^{|I|+1} v(\cap_{i \in I} A_i)$.

(7) v is a measure if it is nonnegative and additive.

(8) v is a belief function if it is normalized and totally monotone.

Observe that additive capacities are totally monotone, totally monotone are convex and convex are superadditive.

We denote the space of real-valued functions on Ω (or random variables) by $F = \{f | f: \Omega \to \mathbb{R}\} = \Omega^{\mathbb{R}}$.

For $v \in V$ and $f \in F$, the Choquet integral of f w.r.t. (with respect to) v is defined to be

$$\int f dv = \int_0^{\infty} v(\{\omega | f(\omega) \geq t\}) dt + \int_{-\infty}^{0} [v(\{\omega | f(\omega) \geq t\}) - v(\Omega)] dt.$$

Note that it is always well-defined. Also, observe that this definition coincides with the standard one if v is additive.

For $v \in V$ we define the core to be

$$\text{Core}(v) = \{p | \text{(i)} \quad p \text{ is a measure;} \\ \text{(ii)} \quad p(A) \geq v(A),\ \forall\ A \in \Sigma; \\ \text{(iii)} \ p(\Omega) = v(\Omega)\}.$$

Note that we allow for a measure to be identically zero. For instance, if $v = 0$, $\text{Core}(v) = \{v\}$.

It will be useful to denote $\Sigma' = \Sigma \setminus \{\emptyset\}$.

For $T \in \Sigma'$, define the unanimity game on T ("elementary belief function" in Jaffray (1989).) to be the capacity $u_T \in V$ defined by

$$u_T(A) = \begin{cases} 1 & A \supseteq T \\ 0 & \textit{otherwise.} \end{cases}$$

3. Some Known Results

The following results will be used in the sequel.

Theorem 3.1 **(Shapley, 1965)**: Every convex nonnegative game has a nonempty core.

Theorem 3.2 (Rosenmuller (1971, 1972). See also Schmeidler (1984, 1986)): A monotone game v is convex if and only if

(i) $\text{Core}(v) \neq \emptyset$;

(ii) for every $f \in F$,

$$\int f dv = \min_{p \in Core(v)} \int f dp.$$

Next is the canonical representation theorem which will drive basically all the following results:

Theorem 3.3: The set $\{u_T\}_{T \in \Sigma'}$ is a linear basis for V. The unique coefficients $\{\alpha_T^v\}_{T \in \Sigma'}$ satisfying

$$v = \sum_{T \in \Sigma'} \alpha_T^v u_T$$

are given by

$$\alpha_T^v = \sum_{S \subseteq T} (-1)^{|T|-|S|} v(S) =$$
$$= v(T) - \sum_{\{I | \phi \neq I \subseteq \{1,\ldots,n\}\}} (-1)^{|I|+1} v(\cap_{i \in I} T_i)$$

where $T_i = T \backslash \{\omega_i\}$ and $T = \{\omega_1, \ldots, \omega_n\}$.

In the sequel, $\{\alpha_T^v\}$ will refer to the above coefficients.

Theorem 3.4: For every $v \in V$, v is totally monotone iff $\alpha_T^v \geq 0$ for all $T \in \Sigma'$.

Theorem 3.4 is due to Dempster (1967) and Shafer (1976). Both Theorems 3.3 and 3.4 are generalized in Gilboa-Lehrer (1991) to real-valued functions defined on arbitrary finite lattices.

4. The Choquet Integral: Min of Means and Mean of Mins.

We first have:

Observation 4.1: The Choquet integral is linear in the game v. That is, for all $v,w \in V$, $\alpha,\beta \in \mathbf{R}$ and $f \in F$,

$$\int fd(\alpha v + \beta w) = \alpha \int fdv + \beta\int fdw$$

It is also useful to state the following lemma:

Lemma 4.2: For $f \in F$ and $T \in \Sigma'$,

$$\int fdu_T = \min\{f(\omega)|\omega \in T\}$$

Proof: Immediate. (Also appears in Rosenmuller (1971).)

We can now prove:

Theorem 4.3 (A version of this result appears in Wasserman (1990). See also Smets (1981) and Dubois-Prade (1985).): For every $v \in V$ and $f \in F$,

$$\int fdv = \sum_{T\in \Sigma'} \alpha_T^v[\min_{\omega\in T} f(\omega)].$$

Proof: By the auxiliary results above,

$$\int fdv = \int fd(\textstyle\sum_{T\in\Sigma'} \alpha_T^v u_T) =$$

$$= \sum_{T\in\Sigma'} \alpha_T^v(\int f du_T) =$$

$$= \sum_{T\in\Sigma'} \alpha_T^v[\min_{\omega\in T} f(\omega)]. \quad \blacksquare$$

Recall that if v is totally monotone, $\alpha_T^v \geq 0$ for all $t \in \Sigma'$. If, furthermore, v is normalized, i.e., it is a belief function, then

$$\sum_{T\in\Sigma'} \alpha_T^v = v(\Omega) = 1,$$

which implies that the Choquet integral of a function of w.r.t. v can be represented as a (weighted) average over all minima of the function f, i.e., over its minima on all non-empty events.

In one extreme case v is additive, which is easily seen to be equivalent to

$$\alpha_T^v = 0 \text{ for all } T \text{ with } |T| \geq 2.$$

In this case, indeed, the integral of f w.r.t. v is an average of the values of f--or, if you will, of the minima of f over singletons.

Another extreme case is where $v = u_\Omega$, and the integral of f w.r.t. v is simply the minimum f obtains over all of Ω.

While both these extreme cases were known to be special cases of the Choquet integral, Theorem 4.3 shows that <u>any</u> Choquet integral (to be precise, the integral w.r.t. <u>any</u> game v) is no more than some average over minima. (Where "average" has its usual meaning if v is normalized and totally monotone.)

On the other hand, let us recall that totally monotone capacities are, in particular, convex (though the converse is false). Hence, applying Theorem 3.2, the Choquet integral of some f w.r.t. to a totally monotone v may be also represented as the minimum of all integrals of f w.r.t. measures in a certain set (the core of v). If v is also normalized, each of these measures p is simply some "weight" vector, and the integral of f w.r.t. p is a p-average over f's values.

To sum, if v is a belief function, the Choquet integral w.r.t. v is both a minimum of averages and an average of minima:

Corollary 4.4: Assume that v is a belief function. Then for every $f \in F$

$$\int f dv = \sum_{T\in\Sigma'} \alpha_T^v[\min_{\omega\in T} f(\omega)] = \min_{p\in \mathrm{Core}(v)} \sum_{\omega\in\Omega} p(\{\omega\})f(\omega).$$

5. Completion of a Misspecified Model

A few more words on the interpretation of Theorem 4.3 may be in order. The approach of Dempster (1967, 1968), and Shafer (1976), is, roughly, the following: evidence supporting our belief in certain events is usually not well-specified enough to be given as a distribution over states of the world. Rather, they assume there is a function $m: 2^\Omega \to [0,1]$ with $m(\emptyset) = 0$ and $\sum_{T\subseteq\Omega} m(T) = 1$ such that m(T) is the "direct evidence" for T, which cannot be further specified in terms of subsets of T. The belief in an event T is given by

$$v(T) = \sum_{S\subseteq T} m(S).$$

(Obviously, in our terms $m(S) = \alpha_S^v$.)

One of the reasons one gets direct evidence for T but not for any subset thereof may be model misspecifications, i.e., that the states of the world included in the model do not exhaust the "actual" ones.

Thus, one is led to an extended model, $\tilde{\Omega}$, in which for every $T \in \Sigma'$ there is $\tilde{\omega}_T \in \tilde{\Omega}$, and the belief function v corresponds to the measure μ_v on $\tilde{\Omega}$ given by $\mu_v(\tilde{\omega}_T) = \alpha_T^v$. A function $f: \Omega \to \mathbb{R}$ may be naturally "extended" to $\tilde{\Omega}$ by $\tilde{f}(\tilde{\omega}_T) = \min_{\omega \in T} f(\omega)$. Then, indeed, Theorem 4.3 may be written as

$$\int_{\Omega} f dv = \int_{\tilde{\Omega}} \tilde{f} d\mu_v.$$

In other words, the non-additivity of the "probability" v may be explained by "omitted" states of the world. If those were introduced into the model explicitly, the non-additivity would disappear.

Note that restricting the extension $f \mapsto \tilde{f}$ to indicator functions yields a natural embedding of events in Ω into events in $\tilde{\Omega}$. That is to say, for $T \subseteq \Omega$, $T \neq \emptyset$, there corresponds

$$\tilde{T} = \{S \in \Sigma' \mid S \subseteq T\} \subseteq \tilde{\Omega} = \Sigma'$$

and we have

$$v(T) = \mu_v(\tilde{T}).$$

Furthermore, notice that the function $\psi: v \mapsto \mu_v$ is linear, continuous, and together with $\phi: f \mapsto \tilde{f}$, preserves integral values for all $f \in F$.

We therefore conclude that every decision model with a non-additive measure v on Ω may be embedded in a model with an additive measure μ_v on Σ^*.

Obviously, v and μ_v contain precisely the same information and both require $(2^{|\Omega|} - 1)$ real numbers for their specification. Yet, the extended model is more "wasteful": it includes many functions (in $\mathbf{R}^{\Sigma'}$) which do not correspond to any function in the original space F. Put differently, the objects of choice in the extended model, $\{\tilde{f} | f \in F\}$, are each fully characterized by $|\Omega|$ values (i.e., by $\tilde{f}(\{\omega\})$ for $\omega \in \Omega$), though they are points in a $(2^{|\Omega|} - 1)$-dimensional space.

Looked at from the opposite direction, then, a non-additive measure on Ω can be viewed as a concise way to represent beliefs and preferences on a larger space. Instead of $(2^{|\Omega|} - 1)$ dimensions with an additive μ_v, one may use $|\Omega|$ dimensions at the expense of additivity (of v). Obviously, this more concise representation may hold only if all the available acts $\tilde{f}: \Sigma \to \mathbf{R}$ satisfy

$$\tilde{f}(T) = \min_{\omega \in T} \tilde{f}(\{\omega\})$$

which may be interpreted as an uncertainty-averse assessment of $\tilde{f}$ where it is not fully specified.

Remark 5.1: One may wonder whether minima play here a special role (as opposed to, say, maxima) and if so, why. Indeed, when one considers belief functions, the maximization of Choquet-expected utility is uncertainty averse, and the minima capture this intuition. (Both in the "mean-of-min" and in the "min-of-mean" theorems.) However, in general Choquet expected utility may be uncertainty-seeking as well. Furthermore, all the theory (or theories) with minima have natural dual theories with maxima.

For instance, if one defines, for $T \in \Sigma$, $T \neq \Omega$,

$$w_T(S) = \begin{cases} 1 & S \not\subseteq T \\ 0 & S \subseteq T \end{cases}$$

It is easy to check that $\{w_T\}_{T \neq \Omega}$ is also a linear basis for V. (w_T is the "dual" of u_{T^c} in the sense

$$w_T(S) = 1 - u_{T^c}(S^c) \ \forall\, S \in \Sigma.)$$

It is also immediate that for all $f \in F$,

$$\int f dw_T = \max_{\omega \in T^c} f(\omega)$$

and therefore, if $v = \sum_{T \neq \Omega} \beta_T^v w_T$,

$$\int f dv = \sum_{T \neq \Omega} \beta_T^v \max_{\omega \in T^c} f(\omega).$$

(For additional duality results, see Gilboa (1989c).)

However, the set of capacities v for which $\beta_T^v \geq 0$ is different from (and the dual of) the set of totally monotone capacities.

<u>Remark 5.2</u>: The notion of a "misspecified model" above is rather vague. It may be formalized as follows: assume we are given a set of propositions P endowed with two binary relations on it, $I, IN \in P^2$, to be interpreted as "implies" and "implies not," and a function $v: P \to [0,1]$ measuring "belief." A pair (Ω, ψ) where $\psi: P \to 2^{\Omega} \setminus \{\emptyset\}$ is a <u>model</u> for (P,I,IN) if

(i) $\forall\, p,q \in P$, $p \ I \ q \Rightarrow \psi(p) \subseteq \psi(q)$;

(ii) $\forall\, p,q \in P$, $p \ IN \ q \Rightarrow \psi(p) \cap \psi(q) = \emptyset$.

2^{Ω}, which will also be denoted by v.

We call a model (Ω,ψ) complete w.r.t. v if v can be extended to a measure on Ω. If it cannot, (Ω,ψ) is incomplete w.r.t. the beliefs v. However, if range $(\psi) = \Sigma\backslash\{\emptyset\}$ and v happens to be totally monotone on Ω, one may consider the completion of (Ω,ψ) which is (Σ',ϕ) defined by

$$\Sigma' = 2^{\Omega}\backslash\{\emptyset\}$$

$$\phi(p) = \tilde{\psi}(p) = \{A \in \Sigma' \mid A \subseteq \psi(p)\}.$$

Notice that for all Ω the map $B \mapsto \tilde{B} = \{A \in \Sigma' \mid A \subseteq B\}$ satisfies

(i) $A \subsetneq B \Rightarrow \tilde{A} \subsetneq \tilde{B}$;

(ii) $A \cap B = \emptyset \Rightarrow \tilde{A} \cap \tilde{B} = \emptyset$.

Hence, if (Ω,ψ) is a model for (P,I,IN), so is (Σ',ϕ). However, (Σ',ϕ) is a complete model w.r.t. v (extended by μ_v) even if (Ω,ψ) is not.

Remark 5.3: The interpretation of Theorem 4.3 in the context of social choice may also be of interest. Consider a social welfare function, or, more generally, a social preference order, satisfying the axioms of Schmeidler (1989). (From a conceptual viewpoint, the most important of these is "comonotonic independence.") Such a function is representable by a Choquet integral of some utility function with respect to a non-additive measure (or a "game") v.

As in the context of uncertainty, two well-known special cases of such functionals are the utilitarian function (if v is additive) and the egalitarian one (if $v = u_{\Omega}$). And here again we find that every such functional is a linear combination of "utility levels" of all coalitions, functional is a linear combination of "utility levels" of all coalitions, where these are defined in the egalitarian spirit: the "utility level" of a coalition is the minimal utility of its members. Thus, this class of social welfare functions can be thought of as "utilitarian" with respect to coalitions and "egalitarian" with respect to individuals within coalitions.

Similarly, one may define a "completion" of a society Ω to be the "society" Σ' whose members are (nonempty) coalitions of Ω. A utilitarian social preference order in the "society of coalitions" corresponds to a possibly non-utilitarian (but Choquet-representable) preference order in the society of individuals.

Another special case of these functionals was studied by Ben-Porath and Gilboa (1991). They axiomatize--under the assumption of a linear utility function of income--the social welfare functions which are a linear combination of total income and (a version of) the Gini index for the measurement of inequality. It turns out that these are precisely the Choquet integrals with respect to a symmetric non-additive measure v, whose coefficients satisfy

$$\alpha_T^v = 0 \text{ if } |T| \geq 3.$$

In other words, while utilitarian functions are concentrated only on singleton coalitions, these functions are concentrated on singletons and pairs. Thus they can also represent envy-aversion: the introduction of terms such as

$$\tilde{f}(\{i,j\}) = \min\{f(i),f(j)\}, \ i,j \in \Omega$$

reduces the overall impact of a "gift" given to (f-)rich individuals; the coefficient of such a term,

$$\mu_v(\{i,j\}) = \alpha^v_{\{i,j\}}$$

may therefore be interpreted as the relative weight put in the social welfare function on i's envy in j (or vice versa).

It should be noted, however, that such functionals, while obviously nonlinear in the payoff distribution f, are, in a sense, "linear in envy." Consider, for instance, a society $\Omega = \{1,2,3\}$ and a payoff (or income) distribution (0,1,1). According to a non-additive measure v with

$$\alpha^v_T = 0 \text{ for } |T| \geq 3$$

individual 1's envy in 2 and 3 is the "sum" of his envy in each of them. More generally, one may suspect the individual 1 in this example will feel even "greater" envy than this "sum." After all, s/he may justly claim that "Everyone is better off than I am." Thus, more general functionals may capture the fact that envy itself is not always linear.

Choquet integration with respect to general (say, totally monotone) non-additive measures represents a rich enough class of preferences to reflect nonlinear envy. On the other hand, every totally monotone v induces a functional which is some average of payoffs and of the "envy level" in various coalitions. (Note that, in general, these functionals may not be symmetric. For instance, one may only envy one's neighbors, in which case not all pairs (triple, quadruples) would be equally weighed in the social welfare function.)

6. The Choquet Integral: Min Minus Min

As stated by Theorem 3.2, if a game v is convex, the Choquet integral of a function f w.r.t. v is the minimum of integrals of f w.r.t. additive set functions. However, when v is not convex such a result cannot be proven. (The fact that Core (v) will not do as a set of measures follows from Theorem 3.2 itself. Gilboa-Schmeidler (1989) prove that other sets cannot serve this purpose either.)

In many applications, the min-of-mean representation is both useful and intuitive. For instance, maximizing the Choquet integral w.r.t. a non-additive set function is typically less palatable to economists than maximizing the minimal (regular) integral w.r.t. a set of (regular) measures. In this subsection we provide an extension to Theorem 3.2, which will represent any Choquet integral (even w.r.t. a non-convex v) in a more intuitive way.

We start with:

<u>Lemma 6.1</u>: For every $v \in V$ there exist totally monotone $v^+, v^- \in V$ such that

$$v = v^+ - v^-.$$

<u>Proof</u>: Given $v \in V$, consider the coefficients $\{\alpha_T^v\}_{T\in\Sigma'}$. Define $\Sigma^+ = \{T \in \Sigma' \mid \alpha_T^v \geq 0\}$, and

$$v^+ = \sum_{T\in\Sigma^+} \alpha_T^v u_T$$
$$v^- = \sum_{t\in\Sigma'\setminus\Sigma^+} (-\alpha_T^v) u_T$$

It is obvious that $v = v^+ - v^-$, and that (by Theorem 4.4) both v^+ and v^-

are totally monotone. ■

Next we have

Theorem 6.2: For every $v \in V$ there exist two non-empty sets of measures on Σ, C^+ and C^-, such that for all $f \in F$

$$\int f dv = \min_{p \in C^+} \int f dp - \min_{p \in C^-} \int f dp.$$

Proof: Given v, let v^+ and v^- be the totally monotone capacities provided by Theorem 6.1. By Observation 4.1,

$$\int f dv = \int f dv^+ - \int f dv^-$$

for all $f \in F$. Defining

$$C^+ = \mathrm{Core}(v^+),\ C^- = \mathrm{Core}(v^-)$$

and using Theorem 3.2 (twice) completes the proof. ■

We note that neither v^+, v^- in Theorem 6.1 nor C^+, C^- in Theorem 6.2 are unique. Furthermore, the representation of Theorem 4.5 may also hold with sets C^+, C^- which are not the cores of totally monotone capacities. For instance, consider, for $\Omega = \{1,2,3\}$, the capacity

$$v = u_{\{1,2\}} + u_{\{1,3\}} + u_{\{2,3\}} - u_{\{1,2,3\}}$$

which is convex but not totally monotone. Then, by Theorem 3.2, $\int\bullet dv$ has one representation with

$$C^{+} = \mathrm{Core}(v), \quad C^{-} = \{0\}$$

which differs from that obtained in the proof of Theorem 6.2.

However, it is easy to see that v^{+} and v^{-} in Theorem 6.1 are the unique totally monotone capacities solving

$$\begin{aligned} &\mathrm{Min}\ v^{+}(\Omega) + v^{-}(\Omega) \\ &\text{s.t.}\quad v^{+} - v^{-} = v. \end{aligned}$$

In other words, if one defines the <u>norm</u> of $v \in V$ to be

$$\|v\| = \textstyle\sum_{T\in\Sigma'} |\alpha_T^v|,$$

then we have:

<u>Theorem 6.3</u>: For every $v \in V$ there are unique totally monotone $v^{+}, v^{-} \in V$ such that

$$v = v^{+} - v^{-}$$

and

$$\|v\| = \|v^{+}\| + \|v^{-}\|.$$

<u>Proof</u>: Immediate.

Similarly, one may make C^+ and C^- in Theorem 6.2 unique by further requiring that they be cores of totally monotone capacities, and that for all $p^+ \in C^+$, $p^- \in C^-$,

$$\|v\| = \|p^+\| + \|p^-\|.$$

(Notice that $\|v\| = v(\Omega)$ if and only if v is totally monotone, and, in particular

$$\|p\| = p(\Omega)$$

for every measure p.)

The results above reveal yet another facet of maximization of the Choquet integral. Suppose, for simplicity, that $f(\omega) \in [0,1]$ for all $\omega \in \Omega$ and all possible acts $f \in F$. (Recall that the utility function derived in Savage (1954) is bounded. See Fishburn (1970).) Then the maximization of $\int fdv$ over F is equivalent to maximization of

$$\min_{p\in C^+} \int fdp - \min_{p\in C^-} \int fdp = \min_{p\in C^+} \int fdp + \max_{p\in C^-} \int (-f)\,dp$$

or of

$$\min_{p\in C^+} \int fdp + \max_{p\in C^-} \int (1 - f)\,dp.$$

Considering $f(\omega)$ as payoff, $(1 - f(\omega))$ may be taken to be a measure of dissatisfaction with or disappointment from the outcome $f(\omega)$. Thus the Choquet integral is (up to a shift of size $v^-(\Omega)$) the minimal expected payoff (according to C^+) plus the maximal expected disappointment (according to C^-). Also note that changing the relative weights of "min" and "max" above reduces to multiplying all measures in C^+ (or in C^-, or in both) by some non-negative constant, and therefore remains a Choquet integral with respect to a (possibly) different $v \in V$.

In other words, the class of decision rules

$$\{\mathrm{Max} \int f dv | v \in V\}$$

is precisely equal to

$$\left\{Max\left[\alpha \,\mathbf{min}_{p\in C^+} \int f dp + (1 - \alpha)\,\mathbf{max}_{p\in C^-} \int (1 - f)\, dp\right] | C^+, C^-,\right.$$

non-empty closed sets of probability measures and $\alpha \in [0,1]\}$,

which may be thought of as a variation on Hurwicz's α-maxmin decision rule.

7. Radon-Nikodym Theorem for Non-Additive Measures.

For many applications, a version of the Radon-Nikodym theorem for not-necessarily-additive set functions would be useful. However, one cannot hope for such a theorem in a direct translation from the additive case: consider the claim that for some $v \in V$, for all $w \in V$ there is an $f^w \in F$ such that

$$w(S) = \int_S f^w \, dv$$

(where the integral restricted to S is appropriately defined). But if $|\Omega| = n$, the dimension of V is $(2^n - 1)$ while that of F is n, and one can hardly expect such a result to hold.

However, the canonical decomposition theorem, which basically represents non-additive set functions on Ω as additive ones on Σ', suggests a natural way to extend the Radon-Nikodym theorem and define the derivative of one set-function w.r.t. another.

We first need the following definition: for $w, v \in V$, w is said to be absolutely continuous w.r.t. v if for every $A_1, \ldots, A_n \in \Sigma$,

$$v(\cup_{i=1}^{n} A_i) = \sum_{\{I \mid \phi \neq I \subseteq \{1,\ldots,n\}\}} (-1)^{|I|+1} v(\cap_{i \in I} A_i)$$

implies

$$w(\cup_{i=1}^{n} A_i) = \sum_{\{I \mid \phi \neq I \subseteq \{1,\ldots,n\}\}} (-1)^{|I|+1} w(\cap_{i \in I} A_i).$$

Observation 7.1: If w is absolutely continuous w.r.t. v, then for all $T \in \Sigma'$ $\alpha_T^v = 0$ implies $\alpha_T^w = 0$.

Remark 7.2: Note that the converse does not hold. Let $\Omega = \{1,2,3\}$ and define v by

$$\alpha_T^v = 1, \; T \neq \Omega$$
$$\alpha_\Omega^v = -2.$$

Then for all $w \in V$, ($\alpha_T^v = 0$ implies $\alpha_T^w = 0$) trivially holds. However, for $A_1 = \{1,2\}$, $A_2 = \{3\}$ we have

$$v(\{1,2,3\}) = v(\{1,2\}) + v(\{3\})$$

which certainly does not imply the same equality for all $w \in V$. ∎

We can now state and prove

<u>Theorem 7.3</u>: Suppose Σ is finite. If $v, w \in V$ are such that w is absolutely continuous w.r.t. v, then there exists a function $g\colon \Sigma' \to \mathbf{R}$ such that for all $f \in F$,

$$\int f dw = \sum_{T\in\Sigma'} \alpha_T^v \, g(T) \, \min_{\omega\in T} f(\omega)$$

and, in particular, for every $A \in \Sigma$,

$$w(A) = \sum_{T\subseteq A;\, T\in\Sigma'} \alpha_T^v g(T).$$

The function g will be called the <u>derivative of w w.r.t. v</u>.

<u>Proof</u>: For all $T \in \Sigma'$ such that $\alpha_T^v \neq 0$, define $g(T) = \alpha_T^w/\alpha_T^v$, and define $g(T)$ arbitrarily otherwise. The result then follows from Theorem 4.3. ∎

In general, one may measure the non-additivity of $v \in V$ by

$$\sum_{\{T\in\Sigma' \mid |T|>1\}} |\alpha_T^v|$$

Given $v, w \in V$, assume that g is the Radon-Nikodym derivative of w w.r.t. v. Then $|g(T)|$ is a measure of the relative non-additivity of w and v at T, and $g(T)$ is a measure of their relative uncertainty aversion. For instance, if $g(T) > 1$ and v is totally monotonic, we may say that w is <u>more uncertainty averse than</u> v <u>at</u> T. We conjecture that this is the "appropriate" way to measure uncertainty aversion, in a way that will be the equivalent of the measurement of risk aversion in classical economic theory.

8. Pseudo-Bayesian Updates

The problem of updating beliefs is central to statistics, as well as to applications in artificial intelligence, economic theory and so forth.

Based on "Dempster rule of Combination" for belief functions in general, Dempster (1968) (see also Shafer (1976)) suggested that, given an event $A \in \Sigma'$, the new belief function $v_A \in V$ should be

$$v_A(B) = \frac{v((A \cap B) \cup A^c) - v(A^c)}{1 - v(A^c)}.$$

Gilboa (1989b) used, without any axiomatic derivation, the more straightforward adaptation of Bayes rule:

$$v_A(B) = v(A \cap B)/v(A).$$

Gilboa and Schmeidler (1991) approached the update problem axiomatically. From axioms on preference orders parameterized by events (the event assumed to be known) they derived pseudo-Bayesian update rules. Two of them, which correspond to a "pessimistic" and an "optimistic" interpretations, ended up being the two update rules given above. Furthermore, the pessimistic one, which coincides with the Dempster-Shafer rule, also has an interpretation of a maximum likelihood rule. Given that an event A has occurred, out of all Core(v), only those measures p which a priori maximized p(A) are retained in the set of measures, and they are each updated according to Bayes' rule.

The canonical representation theorem sheds a new light on the update problem: since for each $v \in V$ there corresponds an additive μ_v on Σ', one may try to update μ_v according to Bayes' rule, and then "project" the updated μ_v back to Ω.

However, it is not entirely clear how one translates the fact that "A has occurred" from Ω to Σ'. To formulate this issue, first define an update rule to be a function U: $V \times \Sigma' \to V$ with the interpretation that U(v,A) represents the updated beliefs given that $A \in \Sigma'$ has occurred and $v \in V$ represents the original beliefs.

For every translation function τ: $\Sigma' \to 2^{\Sigma'}$ one may define an update rule U^τ: $V \times \Sigma' \to V$ as follows: given $v \in V$ and $A \in \Sigma'$, update μ_V on Σ' as if $\tau(A)$ has occurred, and define $U^\tau(v,A)$ by the updated μ_V. That is,

$$U^\tau(v, A) = \sum_{S \in \Sigma'} (\alpha_S^v)' u_S$$

where

$$(\alpha_S^v)' = \begin{cases} 0 & S \notin \tau(A) \\ \dfrac{\alpha_S^v}{\sum_{T \in \tau(A)} \alpha_T^v} & S \in \tau(A) \end{cases}.$$

In particular, define $\tau^0(A) = \tilde{A} = \{B \in \Sigma' \mid B \subseteq A\}$.

Theorem 8.1:[1] (See Dubois-Prade (1991) for closely related results.) The translation function τ^0 gives rise to the optimistic update rule. That is, for all $v \in V$, $A \in \Sigma'$ and $B \in \Sigma$,

$$U^{\tau^0}(v, A)(B) = \frac{v(A \cap B)}{v(A)}$$

whenever these are well-defined. Moreover, both sides of the equality are well defined for the same events $A \in \Sigma'$.

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[1]We are grateful to Bart Lipman for pointing out an error in a previous version of this theorem.

<u>Proof</u>: Notice that

$$(\alpha_S^v)' = \begin{cases} 0 & S \not\subseteq A \\ \dfrac{\alpha_S^v}{\sum\limits_{T \in \tau(A)} \alpha_T^v} & S \subseteq A \end{cases}$$

if $v(A) = \sum_{T \subseteq A} \alpha_T^v > 0$. Then

$$U^{\tau c}(v, A)(B) = \sum_{T \subseteq B} (\alpha_S^v)' = \sum_{S \subseteq B \cap A} (\alpha_S^v)' = \frac{v(A \cap B)}{v(A)}.$$

Using the dual base $\{w_T\}_{T \in \Sigma'}$, one may obtain a similar derivation of the Dempster-Shafter ("pessimistic") update rule. (See Dubois-Prade (1991), and Lipman (1992).)

9. <u>Conclusion</u>

This paper suggests reinterpreting a known theorem from the theory of cooperative games as an embedding of not-necessarily-additive probabilities in additive ones (on a larger space). ˉince both the embedding function and the Choquet integral are linear, a variety of additional results follows, which shed new light on Choquet-expected utility theory. Furthermore, results and tools regarding additive measures (such as the Radon-Nikodym theorem and Bayes' update) may be naturally extended to non-additive ones using the isomorphism between these spaces. We trust that additional results may be derived in a similar manner, and therefore propose this isomorphism as a basic tool for the analysis of non-additive measures and Choquet integration with respect to them.

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