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A SOLUTION TO THE ENVY-FREE SELECTION PROBLEM IN ECONOMIES WITH INDIVISIBLE GOODS *

by

Enriqueta Aragones**

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**Department of Managerial Economics and Decision Sciences, J. L. Kellogg Graduate School of Management, Northwestern University, Evanston, Illinois 60208.

ABSTRACT

We consider the problem of selecting envy-free allocations in economies with indivisible objects and quasi-linear utility functions. We study the set of envy-free allocations for these economies and characterize the minimal amount of money necessary for its nonemptiness when negative distributions of money are not allowed. We also find that, when this is precisely the available amount of money, there is a unique way to combine objects and money such that these bundles may form an envy-free allocation. Based on this property, we propose a solution that selects a unique utility profile for any economy. When there is more money than is needed to solve the envy-free problem, our solution allocates it equally and we retain the uniqueness of the utility profile. Among the properties satisfied by this solution we find that it is invariant with respect to shifts in the utility scale of any agent, not obviously manipulable and can be computed by a polynomial-bounded algorithm. We show that when some agents leave the economy, the set of envy-free allocations for the new economy may offer new possible combinations of objects and money. Based on this, we argue that one should not expect a solution to this selection problem to satisfy any property related to consistency, as has been suggested in the literature.

1. INTRODUCTION.

Envy-free allocations, as defined by Foley (1967) are allocations for which every agent prefers his own bundle to those assigned to other agents. It is well known that they do not always exist when there are indivisible goods to be allocated among the agents. For instance, consider a two person economy with two indivisible goods to be allocated and such that one of the goods is preferred to the other one by both agents. Obviously, in any allocation one of the agents does not get the most preferred good and will envy the other agent. In order to guarantee the existence of envy-free allocations for these economies, there has to exist an infinitely divisible good (that we may think of as money) to compensate agents when the distribution of the indivisible ones generates envy among them.

The availability of the infinitely divisible good (money) guarantees the existence of envy-free allocations if we do not have any restriction on the distribution of money (Svensson (1983) and Alkan, Demange and Gale (1991)). Sometimes, we may want the money allocations to be nonnegative, since the allocation of negative quantities of money requires additional assumptions on the original wealth of the agents. In this case, envy-free allocations only exist when the quantity of money available in the economy is large enough (Maskin (1987) and Alkan, Demange and Gale (1991)). In this paper we study both cases.

The economies we deal with are defined by a set of objects, a quantity of money and a set of agents with preferences defined on objects and money. The agents are going to receive an object and an amount of money in addition to the wealth they may already have. A classical example is the allocation of an inheritance. Another typical example is the allocation of jobs and money compensations, where money comes from taxes paid by the agents. In this case we may have special interest in solutions that choose allocations using the minimal possible amount of money. We address here the problem of deciding in which way the goods and the money should be allocated among the agents if all agents are supposed to have the same property rights over the objects and the money (see Moulin (1990)). First we are interested in allocations that do not generate any envy, but for these economies the set of envy-free allocations may be quite large. Thus we are faced with a selection problem. Since a solution to this problem that chooses many different allocations is not very useful, the goal of this paper is to find a solution that gives a precise recommendation of how to allocate objects (indivisible goods) in a way that is envy-free and normatively justified.

This problem has been addressed by Alkan, Demange and Gale (1991) and Tadenuma and Thomson (1991.c) who proposed solution based on intuitive considerations of fairness, and by Tadenuma and Thomson (1991.a) whose solutions were defined axiomatically.

Traditionally, an allocation is called "fair" if it is both Pareto efficient and envy-free. It has been proved (Svensson(1983) and Alkan, Demange and Gale (1991)) that for economies with indivisible goods and money, given continuous utility functions which are unbounded in money, the set of Pareto efficient allocations contains all envy-free allocations. Therefore, the set of "fair" allocations for these economies does not satisfy our requirement of a precise recommendation.

We use the model suggested by Tadenuma and Thomson (1991.a and 1991.c) with the additional assumption that the utility functions are quasi-linear. A few words on this condition are in order. We find that envy-freeness is a relevant condition on the "desired" allocations only as long as there exists some homogeneity among the agents. For example, if we were to allocate an inheritance among a set of agents of which some were very wealthy while others were extremely poor, the envy that the rich may experience would hardly be a convincing argument to rule out certain allocations. Thus, the focus on the concept of envy seems to presuppose some comparability or homogeneity of the agents. In our model the homogeneity among agents is represented by the assumption of identical and constant marginal utility of money, i.e. the utility functions are quasi-linear in money (additively separable and linear in money). This special case is mentioned in Tadenuma and Thomson (1991.a) and (1991.c).

Given this notion of homogeneity, a natural suggestion for a solution is to equate the utility of all agents. This criterion seems to make sense since our utility functions allow interpersonal utility comparisons. We call it the egalitarian criterion, and we find that in our economies egalitarian allocations always exist but the set of envy-free allocations satisfying this property may be empty. We characterize the class of economies for which this set is nonempty. For the economies in this class, the egalitarian allocations satisfying envy-freeness are almost unique, and coincide with the solution proposed by Alkan, Demange and Gale (1991). But for the economies out of this class this criterion does not provide a solution.

However, in many problems the utility functions of the agents may not be available. Since the preferences are defined on bundles composed by an object and money, we will only be able to learn the differences between the values that each agent assigns to each pair of objects. Typically, we will not have an object that is necessarily "as valuable" to all agents, even if we include a "nothing" object. For instance, in a job-allocation problem we cannot always assume that all agents value leisure in the same way. Thus we will assume that the utility functions of the agents are given up to a shift by a constant. This renders the egalitarian criterion devoid of content, since it again becomes meaningless to state that "agent i is as well-off as agent j". It is crucial to note two points: first, the assumption of quasi-linear utilities does not imply interpersonal comparisons of absolute utility levels: only the differences between such levels are comparable. Second, the notions of envy and envy-freeness (as well as Pareto efficiency, of course) are well-defined even if the utilities are given up to a numerical shift.

We approach the selection problem by applying a pseudo-egalitarian criterion. First, we construct an envy-free allocation that allocates nonnegative quantities of money to the agents and uses the minimal amount of money. We find that this allocation is "almost unique" in two ways: first, there is only one way of combining objects and money to have envy-freeness; second, all agents are indifferent among all envy-free allocations (Tadenuma and Thomson (1989) refer to this property as "single-valuedness up to indifferent permutations"). When there is more than one envy-free allocation, the same bundles are given to different agents. This implies that when we have just enough money to guarantee the existence of envy-free allocations the envy-free solution gives as precise a recommendation to the allocation problem as one could hope for (obviously, ties can always be broken arbitrarily or randomly). When money is given exogenously and exceeds the amount needed to solve the envy-free problem, we allocate it equally and we retain the egalitarian and envy-free properties as well as the uniqueness of the utility profile. Therefore, this allocation satisfies the requirements described at the beginning and is proposed here as a solution to the selection problem. We describe a procedure to calculate the set of allocations selected by our solution. The computation of these allocations can be done by an algorithm of a polynomial time complexity.

Comparing the allocations chosen by our solution for two economies which only differ in the total amount of money, we find that the solution chooses an allocation that distributes the difference between the two total amounts of money equally among the agents. We name this property monotonicity with respect to money. It implies the properties of "money-monotonicity" and "translation invariance" defined by Tadenuma and Thomson (1989). Furthermore, our solution depends only on the differences between the values every agent assigns to each pair of objects. Hence, it is invariant with respect to shifts in the utility scale of any agent.

We note that most solution concepts proposed in the literature fail to choose a unique (or almost unique) allocation. An exception is Alkan, Demange and Gale (1991) who proposed a solution that also chooses a unique utility profile, and is based on the maximin criterion, where the worst off person is as well off as possible. In order to find the allocation chosen by this solution we need to know not only the differences between the values every agent assigns to each pair of objects but also the differences of these values for all pairs of agents and objects.

Furthermore, an implementation problem may also arise here: assume (rather realistically) that the utilities are not commonly known and have to be reported by the agents. Knowing that they are to get the allocation chosen by this solution, some agents may have an incentive to misreport utility values. To be precise, for every agent truth-telling (assuming well-defined) is dominated by reporting a utility function which is shifted down by an (additive) constant. While our solution also fails to be nonmanipulable (see Tadenuma and Thomson (1991.b), who prove that such do not exist), it is, at least, robust with respect to these dominant manipulation strategies. Not knowing what the utilities of the others really are, other manipulations may or may not be profitable. Hence, even though truth-telling is by no means a dominant strategy (given our solution), it is at least an undominated one. We say that a solution is "not obviously manipulable" when it satisfies this property.

We also find that our solution does not satisfy the axioms related to consistency required by Tadenuma and Thomson (1991.a). In studying the reasons we find that the number of envy-free allocations increases when the economy becomes smaller. Intuitively this means that when there are less agents and objects they may envy each other as much as before, but no more. We conclude that we should not expect an envy-oriented solution to the selection problem to be invariant with respect to the number of objects or agents.

The rest of the paper is organized as follows. Section 2 describes the model and defines the properties for allocations that we use in the paper. Section 3 studies the set of envy-free allocations when there are no restrictions on the distribution of money and when nonnegative allocations of money are required. Section 4 analyzes the set of egalitarian allocations. Section 5 describes the recommended solution to the selection problem and its properties. Section 6 includes some comments on consistency and its relevance to the envy-free problem. Finally, section 7 describes how our solution can be applied to economies in which the number of agents and objects differ.

2. MODEL AND DEFINITIONS

An economy is represented by an ordered pair $e = (F, M)$, where M is a real number representing the amount of an infinitely divisible good, which we call money. F describes the fundamentals of economy e and is given by an ordered triple $F = (Q, A, u_Q)$, where $Q = \{1, 2, ..., n\}$ is a finite set of agents, $A = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ is a finite set of objects and the component u_Q is constructed thus:

Each agent $i \in Q$ is endowed with a preference relation defined on the product of A and the real line, which is assumed to admit a numerical representation by a quasi-linear utility function:

$$U_i(\alpha_j, x) = u_i(\alpha_j) + x$$

This function is interpreted as the utility that agent $i \in Q$ derives when he receives an object $\alpha_j \in A$ and an amount of money $x \in \Re$. The symbol u_Q, in the economy description, stands for a list of n nonnegative vectors, one for each agent $i \in Q$.

$$u_Q = \{ [u_i(\alpha_1), u_i(\alpha_2), \ldots, u_i(\alpha_n)] \}_{i \in Q}$$

Each component of the vector $[u_i(\alpha_1), u_i(\alpha_2), \ldots , u_i(\alpha_n)]$ denotes the utility levels that agent i derives from receiving each one of the objects in A.

We will denote by $\mathcal{E}$ the class of such economies.

Given an economy $e = (F, M) \in \mathcal{E}$, we define an <u>allocation</u> for this economy to be a pair $z=(\sigma,m)$, where σ is a bijection, $\sigma:Q \to A$, assigning to each agent $i \in Q$ an element $\alpha_i \in A$, and where $m = \{m_1, m_2, \ldots, m_n\}$ is a vector in R^n and m_i is to be thought of as the amount of money agent i receives.

We say that an allocation $z = (\sigma, m)$ is <u>feasible</u> when the total amount of money is distributed among the agents, i.e.:

$$\sum_{i=1}^{n} m_i = M$$

$Z(e)$ will denote the set of feasible allocations for the economy $e \in \mathcal{E}$.

In some cases it may be of little interest to consider allocations in which agents receive negative quantities of money, unless we make additional assumptions (e.g., the agents hold positive quantities of money). Therefore, we also study the case in which agents can only receive nonnegative quantities of money. Obviously, only economies with a nonnegative total quantity of money M are of interest in this case. The sub-class of $\mathcal{E}$ containing these economies will be called $\mathcal{E}_+$, i.e.:

$$\mathcal{E}_+ = \{e = (F, M) \in \mathcal{E} \,/\, M \geq 0\}.$$

Correspondingly, we define a modified concept of feasibility: an allocation $z = (\sigma,m) \in Z(e)$, where $e = (F, M) \in \mathcal{E}_+$, is called feasible with nonnegative transfers if it is feasible and $m \geq 0$.

$Z_+(e)$ will denote the set of feasible allocations with nonnegative transfers for the economy $e \in \mathcal{E}_+$. Naturally, for each economy $e \in \mathcal{E}_+$, $Z_+(e) \subsetneq Z(e)$.

For each economy $e = (F, M) \in \mathcal{E}$ there are n! different ways to distribute the objects among the agents, which we will generically denote by σ, σ', ... and for each σ there are infinitely many ways to divide the total amount of money M among the agents. Therefore, the set Z(e) contains infinitely many allocations. The following properties may aid in the selection of a normatively appealing allocation.

Definition 1:

An allocation $z = (\sigma,m) \in Z(e)$, where $e = (F, M) \in \mathcal{E}$, is called Pareto efficient if there is no feasible allocation $z'=(\sigma',m') \in Z(e)$ such that

$$u_i(\sigma'(i)) + m_i' \geq u_i(\sigma(i)) + m_i \qquad \text{for all } i \in Q, \text{ and}$$

$$u_j(\sigma'(j)) + m_j' > u_j(\sigma(j)) + m_j \qquad \text{for some } j \in Q.$$

P(e) will denote the set of Pareto efficient allocations for $e \in \mathcal{E}$.

Definition 2:

(Foley (1967)). An allocation $z = (\sigma,m) \in Z(e)$, where $e = (F, M) \in \mathcal{E}$, is called envy-free if

$$u_i(\sigma(i)) + m_i \geq u_i(\sigma(j)) + m_j \qquad \text{for all } i, j \in Q.$$

E(e) will denote the set of envy-free allocations for the economy $e \in \mathcal{E}$.

Definition 3:

An allocation $z = (\sigma, m) \in Z(e)$, where $e = (F, M) \in \mathcal{E}$, is called egalitarian if

$$u_i(\sigma(i)) + m_i = u_j(\sigma(j)) + m_j \qquad \text{for all } i, j \in Q.$$

$I(e)$ will denote the set of egalitarian allocations for $e \in \mathcal{E}$.

Definition 4:

An allocation $z = (\sigma, m) \in Z(e)$, where $e = (F, M) \in \mathcal{E}$, is called α-efficient if it is optimal with respect to the utilitarian criterion, i.e.,

$$\sum_{i=1}^{n} u_i(\sigma(i)) \geq \sum_{i=1}^{n} u_i(\sigma'(i)) \qquad \text{for every bijection } \sigma': Q \to A.$$

$\alpha P(e)$ will denote the set of α-efficient allocations for $e \in \mathcal{E}$. Since the requirement is only on the distribution of objects, we shall say that σ is α-efficient if this condition is satisfied.

The intersection of $P(e)$, $I(e)$, $E(e)$ and $\alpha P(e)$ with $Z_+(e)$ are denoted by $P_+(e)$, $E_+(e)$, $I_+(e)$ and $\alpha P_+(e)$. These are the sets of Pareto efficient, envy-free, egalitarian and α-efficient allocations again which are also feasible with nonnegative transfers.

3. ENVY-FREE ALLOCATIONS

In order to analyze the set of envy-free allocations for our economies we give a characterization for Pareto efficient allocations and we derive the already known results of existence and efficiency of envy-free allocations for our economies. We then proceed to show how the nonnegativity requirement on the distribution of money affects these results. The proof of existence of envy-free allocations for our economies reveals the special structure of the set $E(e)$, and will also suggest a natural solution to the selection problem. We also include a very simple proof for efficiency. See Svensson (1983) and Alkan, Demange and Gale (1988) for more general proofs.

Since for each economy $e = (F, M) \in \mathcal{E}$, there is a finite number (n!) of different distributions of money among agents, there will always exist at least one satisfying α-efficiency. Therefore, the set $\alpha P(e)$ is never empty. And this property characterizes the set of Pareto efficient allocations for the class of economies we are interested in.

PROPOSITION 1

For each $e \in \mathcal{E}$, $\alpha P(e) = P(e)$.

Proof:

First, we show that $\alpha P(e) \subseteq P(e)$, for each $e \in \mathcal{E}$.

Let $z = (\sigma, m)$ be an α-efficient allocation. Suppose that there exists $z' = (\sigma', m')$ such that

$$u_i(\sigma'(i)) + m_i' \geq u_i(\sigma(i)) + m_i$$

for all $i \in Q$, with at least one strict inequality. Summing up these inequalities for all $i \in Q$, we have:

$$\sum_{i=1}^{n} u_i(\sigma'(i)) + M > \sum_{i=1}^{n} u_i(\sigma(i)) + M$$

Thus

$$\sum_{i=1}^{n} u_i(\sigma'(i)) > \sum_{i=1}^{n} u_i(\sigma(i))$$

So we must have $z = (\sigma, m) \notin \alpha P(e)$ which contradicts our initial assumption. Therefore, $z \in \alpha P(e)$ implies $z \in P(e)$.

To show that $P(e) \subseteq \alpha P(e)$ for each $e \in \mathcal{E}$, assume that there exists $z = (\sigma, m) \in P(e)$ such that $z = (\sigma, m) \notin \alpha P(e)$. This implies that there exists a bijection σ': $Q \rightarrow A$ such that:

$$\sum_{i=1}^{n} u_i(\sigma'(i)) > \sum_{i=1}^{n} u_i(\sigma(i)).$$

Using the distribution σ' we construct a new allocation z' = (σ', m') such that:

$$m_i' = m_i + u_i(\sigma(i)) - u_i(\sigma'(i))$$

This allocation is at least as preferred as z = (σ, m) for all the agents. But, it is not feasible since:

$$\sum_{i=1}^{n} m_i' = \sum_{i=1}^{n} m_i + \sum_{i=1}^{n} [u_i(\sigma(i)) - u_i(\sigma'(i))] < \sum_{i=1}^{n} m_i = M.$$

From z' it is easy to construct a feasible allocation, for instance, assigning the residual money to one agent $j \in Q$:

$$m_j'' = m_j' + M - \sum_{i=1}^{n} m_i'$$

$$m_i'' = m_i' \quad \text{for each } i \in Q \text{ with } i \neq j.$$

Now we have a new allocation z'' = (σ', m'') which Pareto dominates z = (σ, m) and so $z \notin P(e)$.•

As an implication of this result, the Pareto efficient allocations are characterized only by a condition on the fundamentals of the economy. Since for each economy $e \in \mathcal{E}$ there is always a σ satisfying α-efficiency, we shall always have infinitely many Pareto efficient allocations for each economy: $\{(\sigma, m)\text{: } \sigma \text{ is } \alpha\text{-efficient and } \sum_{i=1}^{n} m_i = M\}$. To show the existence of envy-free allocations for any economy in $\mathcal{E}$ we need some additional notation.

<u>Notation</u>: Let k_{ij}^{σ} denote the extent to which agent i envies agent j given the distribution of objects among agents σ, i.e.,

$$k^{\sigma}_{ij} = u_i(\sigma(j)) - u_i(\sigma(i)).$$

(Note that this expression may, of course, be negative.)

Observe that $z = (\sigma,m) \in \alpha P(e)$ if and only if

$$\sum_{i=1}^{n} k^{\sigma}_{i\sigma^{-1}(\sigma'(i))} \leq 0 \quad \text{for every bijection } \sigma':Q \rightarrow A.$$

And $z = (\sigma,m) \in E(e)$ if and only if

$$m_i - m_j \geq k^{\sigma}_{ij} \quad \text{for all } i, j \in Q.$$

Given $F = (Q, A, u_Q)$ and a distribution of objects $\sigma:Q \rightarrow A$, we construct a directed graph $G_\sigma = (Q, Q^2)$, where every agent is represented by a node and every two agents are connected by an arc. We define the weight of arc (i, j) to be k^{σ}_{ij}. The total weight of a directed path $(r, T) = [r(1), ..., r(T)]$, with $r(t) \in Q$ for each $t = 1, ..., T$, is given by

$$w(r,T) = \sum_{t=1}^{T-1} k^{\sigma}_{r(t)r(t+1)}$$

A cycle is a directed path (r, T) with $r(1) = r(T)$.

<u>LEMMA 1</u>

An allocation $z = (\sigma, m)$ is α-efficient if and only if every cycle in G_σ has a nonpositive total weight.

<u>Proof</u>:

Given σ and $z = (\sigma, m)$, z is α-efficient if and only if:

$$\sum_{i=1}^{n} k^{\sigma}_{i\sigma^{-1}(\sigma'(i))} \leq 0 \quad \text{for every bijection } \sigma':Q \rightarrow A.$$

Given σ', we can define a set of cycles in the graph G_σ. Notice that every disjoint set of cycles defines a σ'. Hence, σ satisfies α-efficiency if and only if the sum of the total weights of these cycles is nonpositive. We want to prove that this is true if and only if the weight of every cycle in the graph is nonpositive. However, the "if" is immediate and the "only if" is simple: Suppose one of this cycles has positive total weight, i.e., for some $i \in Q$

$$\sum_{t=1}^{T-1} k^{\sigma}_{r(t)r(t+1)} > 0, \text{ where } r(1) = r(T) = i$$

Supplementing r by (n - T) trivial cycles $\{(i, i)\}_{i \notin \{r(j)\}_{j=i}^{T}}$ yields a contradiction. Hence σ is α-efficient if and only if all cycles are nonpositive.•

LEMMA 2

If σ satisfies α-efficiency, then for each $i \in Q$

$$w(r_i, T_i) = \max \sum_{t=1}^{T-1} k^{\sigma}_{r(t)r(t+1)}$$

$$\text{s.t.} \quad r(1) = i$$

$$(r, T) \text{ is any path in } G_\sigma$$

has a finite and nonnegative solution.

Proof:

First, we show that for every path (r, T) there is another path (r', T') with $T' \leq n$ such that $w(r', T') \geq w(r, T)$. If $T > n$ then (r, T) must contain a nontrivial cycle, because the graph has only n nodes. For each cycle there exist t* and t** such that $t^* < t^{**} \leq T$ and $r(t^*) = r(t^{**})$. Therefore, the total weight of (r, T) can be decomposed as follows:

$$w(r, T) = \sum_{t=1}^{T-1} k^{\sigma}_{r(t)r(t+1)} = \sum_{t=1}^{t^*-1} k^{\sigma}_{(t)r(t+1)} + \sum_{t=t^*}^{t^{**}-1} k^{\sigma}_{r(t)r(t+1)}$$

$$+ \sum_{t=t^{**}}^{T-1} k^{\sigma}_{r(t)r(t+1)}$$

Since $[r(t^*), r(t^*+1), \ldots, r^*(t^{**})]$ is a cycle, α-efficiency implies that its total weight must be non positive. Then

$$w(r, T) = \sum_{t=1}^{T-1} k^{\sigma}_{r(t)r(t+1)} \leq \sum_{t=1}^{t^*-1} k^{\sigma}_{r(t)r(t+1)} + \sum_{t=t^{**}}^{T-1} k^{\sigma}_{r(t)r(t+1)}$$

Following this reasoning for all cycles, we shall find a path (r', T') with T'<n contained in the original path (r, T). Therefore for each $i \in Q$ there is a path (r_i, T_i) with $T_i < n$, which satisfies:

$$w(r_i, T_i) = \max \sum_{t=1}^{T-1} k^{\sigma}_{r(t)r(t+1)} \quad = \quad \max \sum_{t=1}^{T-1} k^{\sigma}_{r(t)r(t+1)}$$

$$\begin{array}{llll} \text{s.t.} & r(1) = i & \text{s.t.} & r(1) = i \\ & (r, T) \text{ is any path in } G_{\sigma} & & (r, T) \text{ is any path in } G_{\sigma} \\ & & & \text{with } T \leq n \end{array}$$

Since the solution for the right hand side problem exists and is finite, we also have a finite solution to our problem. This solution must be nonnegative, since loops (r, T), with w(r, T) = 0, are also in the feasible set.•

To prove the existence of envy-free allocations we start by showing that, given the fundamentals of an economy, for every α-efficient distribution of objects among agents we can find a distribution of money such that they form an envy-free allocation for the economy defined by these fundamentals and the amount of money given by the total amount distributed.

PROPOSITION 2

Given $F = (Q, A, u_Q)$, for each bijection $\sigma: Q \rightarrow A$ that satisfies α-efficiency there exists $m^* \in R^n$ such that $z = (\sigma, m^*) \in E((F, M^*))$ where $M^* = \sum_{i=1}^{n} m_i^*$.

Proof:

Given an α-efficient σ, by Lemma 2, for each $i \in Q$ there is a path (r_i, T_i) solving the maximization problem. Define m_i^* by:

$$m_i^* = w(r_i, T_i) = \max \sum_{t=1}^{T_i-1} k^{\sigma}_{r_i(t)r_i(t+1)} = \max \sum_{t=1}^{T-1} k^{\sigma}_{r(t)r(t+1)}$$

$$\text{s.t. } r(1) = i$$

$$(r, T) \text{ is any path in } G_{\sigma}$$

Let $m^* = (m_1^*, ..., m_n^*)$. To show that $(\sigma, m^*) \in E((F, M^*))$ we need to prove that for all $i, j \in Q$ $m_i^* - m_j^* \geq k^{\sigma}_{ij}$. Suppose, to the contrary, that $m_i^* - m_j^* < k^{\sigma}_{ij}$ for some $i, j \in Q$, then we must have:

$$m_i^* - m_j^* = \sum_{t=1}^{T_i-1} k^{\sigma}_{r_i(t)r_i(t+1)} - \sum_{t=1}^{T_j-1} k^{\sigma}_{r_j(t)r_j(t+1)} < k^{\sigma}_{ij}$$

which implies:

$$\sum_{t=1}^{T_i-1} k^{\sigma}_{r_i(t)r_i(t+1)} < k^{\sigma}_{ij} + \sum_{t=1}^{T_j-1} k^{\sigma}_{r_j(t)r_j(t+1)}$$

Since the right hand side is the total weight of a path starting at node i, i.e., [i, $r_j(1)$, ..., $r_j(T_j)$], this inequality contradicts the fact that the path (r_i, T_i) is a solution to the maximization problem for agent i. Therefore, we must always have $m_i^* - m_j^* \geq k^{\sigma}_{ij}$, which proves that $z = (\sigma, m^*) \in E(e^*)$.•

A correspondence $\Phi: e \to Z(e)$ (for $e \in \mathcal{E}$) satisfies monotonicity with respect to money if given two economies that only differ in the total amount of money, for each allocation chosen by the correspondence for one of them, it will choose an allocation for the other that only differs in the distribution of money and in which agents share equally the difference between the two amounts of money,.

Definition 5:

A correspondence $\Phi: e \to Z(e)$ (for $e \in \mathcal{E}$) satisfies monotonicity with respect to money if given an allocation $z = (\sigma, m) \in \Phi((F, M))$ for every M' the allocation $z' = (\sigma, m') \in \Phi((F, M'))$ where $m_i' = m_i + (M' - M) / n$ for all $i \in Q$.

The next proposition shows that the set of envy-free allocations satisfies monotonicity with respect to money.

PROPOSITION 3

E(.) satisfies monotonicity with respect to money.

Proof:

$z = (\sigma, m) \in E((F, M))$ if and only if

$$m_i - m_j \geq k^{\sigma}_{ij} \text{ for all } i, j \in Q \quad \text{and} \quad \sum_{i=1}^{n} m_i = M .$$

By construction of z', we have: $m_i' - m_j' = m_i - m_j$, therefore

$$m_i' - m_j' \geq k^{\sigma}_{ij} \text{ for all } i, j \in Q \quad \text{and} \quad \sum_{i=1}^{n} m_i' = \sum_{i=1}^{n} m_i + M' - M = M'$$

which is the definition of $z'=(\sigma,m') \in E((F, M'))$.•

Propositions 2 and 3 give the existence of envy free allocations for any economy $e = (F, M) \in \varepsilon$. We will now show they are also efficient.

PROPOSITION 4 (Svensson (1983))

For each $e \in \varepsilon$, $E(e) \subseteq \alpha P(e) = P(e)$.

Proof:

Suppose that there exists $z = (\sigma, m) \in E(e)$ and $z = (\sigma, m) \notin P(e) = \alpha P(e)$. Then, there must exists $\sigma':Q \to A$ such that

$$\sum_{i=1}^{n} k^{\sigma}_{i\sigma^{-1}(\sigma'(i))} > 0$$

σ' defines at least one cycle $[r(1), r(2), ..., r(T)=r(1)]$ with positive total weight:

$$\sum_{t=1}^{T-1} k^{\sigma}_{r(t)r(t+1)} > 0.$$

Since $z = (\sigma, m) \in E(e)$, we must have that $m_i - m_j \geq k^{\sigma}_{ij}$ for all $i, j \in Q$. Therefore,

$$m_{r(1)} \geq m_{r(2)} + k^{\sigma}_{r(1)r(2)} \geq m_{r(T)} + k^{\sigma}_{r(T-1)r(T)} + \ldots + k^{\sigma}_{r(1)r(2)}$$

$$= m_{r(1)} + \sum_{t=1}^{T-1} k^{\sigma}_{r(t)r(t+1)} > m_{r(1)}.$$

Which is a contradiction.•

When we restrict our attention to the set of feasible allocations with nonnegative transfers, we still have existence for α-efficient allocations but this property no longer characterizes the Pareto efficient allocations. The result we have here is the following:

PROPOSITION 5

For each $e \in \mathcal{E}_+$, $\alpha P_+(e) \subseteq P_+(e)$.

The first part of the proof for Proposition 1, in which we show that $\alpha P(e) \subseteq P(e)$ also applies here. Observe that following the reasoning of the proof of Proposition 4 we can also prove the next Proposition.

PROPOSITION 6

For each $e \in \mathcal{E}_+$,$E_+(e) \subseteq \alpha P_+(e)$.

Hence, the distributions of objects that can generate envy-free allocations must satisfy α-efficiency. The following example shows that there may exist Pareto efficient allocations that are no longer α-efficient.

Example 1:

Let $e = (F, M) \in \mathcal{E}$ be such that: $Q = \{1, 2\}$, $A = \{\alpha_1, \alpha_2\}$, $u_Q = \{ [u_1(\alpha_1) = 10, u_1(\alpha_2) = 1] , [u_2(\alpha_1) = 6, u_2(\alpha_2) = 5] \}$ and $M = 10$.

In this case the α-efficient allocations with nonnegative transfers are such that agent 1 receives object α_1 and agent 2 receives object α_2 because:

$$u_1(\alpha_1) + u_2(\alpha_2) = 10 + 5 \geq 1 + 6 = u_1(\alpha_2) + u_2(\alpha_1).$$

However, we can easily check that the allocation $z = [\ (\alpha_2, 0), (\alpha_1, 10)\ ]$ which is not α-efficient, is Pareto efficient since there is no feasible allocation with nonnegative transfers that dominates it.•

Furthermore when we select allocations in the set $Z_+(e)$ the set of envy-free allocations may be empty:

Example 2:

Let $e = (F, M) \in \mathcal{E}_+$ be such that: $Q = \{1, 2\}$, $A = \{\alpha_1, \alpha_2\}$, $u_Q = \{\ [u_1(\alpha_1) = 10, u_1(\alpha_2) = 2], [u_2(\alpha_1) = 9, u_2(\alpha_2) = 4]\ \}$ and $M = 3$.

α-efficient allocations for this economy assign the object α_1 to agent 1 and the object α_2 to agent 2, since:

$$u_1(\alpha_1) + u_2(\alpha_2) = 10 + 4 > 9 + 2 = u_1(\alpha_2) + u_2(\alpha_1).$$

Given the α-efficient distribution of objects, the distribution of money among the agents in an envy-free allocation must be such that:

$$k_{12} \leq m_1 - m_2 \leq -k_{21},$$

$$m_1 + m_2 = M,$$

$$m_1 \geq 0, \; m_2 \geq 0$$

where:

$$k_{12} = u_1(\alpha_2) - u_1(\alpha_1) = -8$$

$$k_{21} = u_2(\alpha_1) - u_2(\alpha_2) = 5$$

$$M = 3.$$

It is easy to see that there are no real numbers satisfying the last conditions. Therefore, for this economy the set of envy free allocations with nonnegative transfers is empty.•

The existence of envy-free allocations for our economies can only be guaranteed when the total quantity of money in the economy is large enough. Maskin (1987) and Alkan, Demange and Gale (1991) give sufficient conditions on the total quantity for a more general class of economies. For our particular model we have a necessary and sufficient condition on the total quantity of money for envy-free allocations with nonnegative transfers to exist.

THEOREM 1

Given $F = (Q, A, u_Q)$ there exists a nonnegative number $M_E(F)$ such that the set $E_+((F, M))$ is non-empty if and only if $M \geq M_E(F)$. Moreover, for every α-efficient σ there exists $m^\sigma \geq 0$ such that $(\sigma, m^\sigma) \in E((F, M_E(F)))$.

Proof:

The strategy of the proof is as follows: First, we show that for every α-efficient distribution of objects σ there is a number $M_E^\sigma(F)$ such that there exists $z = (\sigma, m) \in E_+((F, M_E^\sigma(F)))$ if and only if $M \geq M_E^\sigma(F)$. Next, we will show that $M_E^\sigma(F)$ is independent of σ.

Given $F = (Q, A, u_Q)$, fix an α-efficient bijection $\sigma: Q \to A$ and consider an allocation $z = (\sigma, m^*)$, where for each $i \in Q$

$$m_i^* = w(r_i, T_i) = \max \sum_{t=1}^{T-1} k^\sigma_{r(t)r(t+1)}$$

$$\text{s.t.} \quad r(1) = i$$

$$(r, T) \text{ is any path in } G_\sigma$$

By Lemma 2 we know that this maximization problem has a finite and nonnegative solution. First, we show that $z = (\sigma, m^*)$ is an envy-free allocation for the economy $e=(F, M^*)$ where $M^* = \sum_{i=1}^{n} m_i^*$. By Lemma 2 we know that $m_i^* \geq 0$ for all $i \in Q$. Suppose we had:

$$m_i^* - m_j^* = \sum_{t=1}^{T_i-1} k^{\sigma}_{r_i(t)r_i(t+1)} - \sum_{t=1}^{T_j-1} k^{\sigma}_{r_j(t)r_j(t+1)} < k^{\sigma}_{ij}.$$

This inequality implies that there is a path $[i, r_j(1), ..., r(T_j)]$ starting at i with a higher total weight than (r_i, T_i), which is a contradiction. This yields:

$$m_i^* - m_j^* = \sum_{t=1}^{T_i-1} k^{\sigma}_{r_i(t)r_i(t+1)} - \sum_{t=1}^{T_j-1} k^{\sigma}_{r_j(t)r_j(t+1)} \geq k^{\sigma}_{ij} \text{ for all } i, j \in Q.$$

Therefore $z = (\sigma, m^*) \in E_+((F, M))$. Now, we show that M^* is the minimal quantity of money needed to generate envy-free allocations from σ. Suppose $m' = (m_1', ..., m_n') \geq 0$ is another distribution of money such that $\sum_{i=1}^{n} m_i' = M' < M^*$ and $z = (\sigma, m') \in E_+((F, M))$. Then, for some $i \in Q$ we must have $m_i' < m_i^*$. And, since (σ, m') is envy-free, $m_h' - m_j' \geq k^{\sigma}_{hj}$ must hold for all $h, j \in Q$. This implies:

$$m_i' < m_i^* = \sum_{t=1}^{T_i-1} k^{\sigma}_{r_i(t)r_i(t+1)} \leq m'_{r_i(1)} - m'_{r_i(2)} + m'_{r_i(2)} - m'_{r_i(3)} + ... +$$

$$m'_{r_i(T_i-1)} - m'_{r_i(T_i)} = m'_{r_i(1)} - m'_{r_i(T_i)} = m_i' - m'_{r_i(T_i)}$$

which, in turn, implies that $m'_{r_i(T_i)} < 0$ and therefore $z = (\sigma, m')$ is not feasible.

Finally, note that if $M > M^*$, by Proposition 3, there is $m > m^*$ such that $z = (\sigma, m) \in E_+((F, M))$. Therefore, $z = (\sigma, m)$ is envy-free if and only if $M \geq M^*$.

Thus, for each σ that satisfies α-efficiency we can find the minimal quantity of money needed to generate envy-free allocations. We denote it by $M_E^{\sigma}(F)$. We now wish to show that it is, in fact, independent of σ. That is, that there is $M_E(F)$ such that, for all σ, either $z = (\sigma, m)$ is not an envy-free allocation for any m, or else $M_E^{\sigma}(F) = M_E(F)$. We will do so by proving that envy-free allocations may be permuted and still be envy-free. Let σ and σ' be two α-efficient allocations of objects and let $z = (\sigma, m) \in E_+((F, M_E^{\sigma}(F)))$. Consider the allocation $z' = (\sigma', m')$ where $m_i' = m_{\sigma^{-1}(\sigma'(i))}$ for all $i \in Q$ and suppose z' is not envy-free. Then, there exist $i, h \in Q$ such that

$$u_i(\sigma'(i)) + m_i' < u_i(\sigma'(h)) + m_h'.$$

Denoting $j = \sigma^{-1}(\sigma'(h))$ and by definition of z'

$$u_i(\sigma'(i)) + m_i' = u_i(\sigma'(i)) + m_{\sigma^{-1}(\sigma'(i))} < u_i(\sigma(j)) + m_j = u_i(\sigma'(h)) + m_h'.$$

Rearranging terms and using the definition of k_{ij}^{σ} we have:

$$m_{\sigma^{-1}(\sigma'(i))} - m_j < u_i(\sigma(j)) - u_i(\sigma'(i)) = u_i(\sigma(j)) - u_i(\sigma(i)) + u_i(\sigma(i)) - u_i(\sigma'(i))$$

$$= k_{ij}^{\sigma} - k_{i\sigma^{-1}(\sigma'(i))}^{\sigma}$$

Because both σ and σ' are α-efficient $\sum_{h=1}^{n} k_{h\sigma^{-1}(\sigma'(h))}^{\sigma} = 0$, and we can write:

$$k_{ij}^{\sigma} - k_{i\sigma^{-1}(\sigma'(i))}^{\sigma} \quad = k_{ij}^{\sigma} + \sum_{h=1;h\neq i}^{n} k_{h\sigma^{-1}(\sigma'(h))}^{\sigma}$$

Since z = (σ, m) is envy-free we have:

$$k_{ij}^{\sigma} + \sum_{h=1;h\neq i}^{n} k_{h\sigma^{-1}(\sigma'(h))}^{\sigma} \leq k_{ij}^{\sigma} + \sum_{h=1;h\neq i}^{n} m_h - m_{\sigma^{-1}(\sigma'(h))}$$

$$= k_{ij}^{\sigma} + M_E^{\sigma}(F) - m_i - M_E^{\sigma}(F) + m_{\sigma^{-1}(\sigma'(i))}$$

$$= k_{ij}^{\sigma} - m_i + m_{\sigma^{-1}(\sigma'(i))}$$

We get $m_{\sigma^{-1}(\sigma'(i))} - m_j < k_{ij}^{\sigma} - m_i + m_{\sigma^{-1}(\sigma'(i))}$ which implies that $m_i - m_j < k_{ij}^{\sigma}$ and contradicts the fact that z is envy-free. Therefore, we have that z' = (σ', m') ∈ $E_+((F, M_E^{\sigma}))$ and by definition of $M_E^{\sigma'}$ we must have $M_E^{\sigma} \geq M_E^{\sigma'}$. But following the same reasoning for σ' we would get $M_E^{\sigma} \leq M_E^{\sigma'}$, hence $M_E^{\sigma} = M_E^{\sigma'}$. This completes the proof of the theorem.•

4. EGALITARIAN ALLOCATIONS

In this section we study the existence of egalitarian allocations. According to these allocations all agents achieve the same utility level. First, we do it for the general case, without any restrictions on the amounts of money, and then for allocations with nonnegative transfers.

PROPOSITION 7

Given an economy $e \in \mathcal{E}$, an allocation $z = (\sigma, m)$ is egalitarian if and only if

$$m_j = \frac{M + \sum_{i=1}^{n}[u_i(\sigma(i)) - u_j(\sigma(j))]}{n} \quad \text{for all } j \in Q$$

Proof:

To see that z is egalitarian note that for all $j \in Q$

$$u_j(\sigma(j)) + m_j = \frac{M + \sum_{i=1}^{n} u_i(\sigma(i))}{n}$$

On the other hand, if $z \in I(e)$ we know that $u_i(\sigma(i)) + m_i = K$ for all $i \in Q$ and some $K \in \Re$, therefore

$$K = \frac{M + \sum_{i=1}^{n} u_i(\sigma(i))}{n}$$

and $m_i = K - u_i(\sigma(i))$ for all $i \in Q$ which is the desired conclusion.•

Since for each economy there always exists a distribution of objects among the agents which generates efficient allocations, from the last result we can conclude that for each economy there is always an efficient and egalitarian allocation. All one has to do is to choose an α-efficient distribution of objects and the distribution of money given by the last proposition. When we consider nonnegative transfers of money, however, egalitarian allocations may fail to exist if the total quantity of money in the economy is not large enough. The next result gives necessary and sufficient conditions on the quantity of money for the set $I_+((F, M))$ to be nonempty.

THEOREM 2

Given $F = (Q, A, u_Q)$ there exists a nonnegative number $M_I(F)$ such that the set $I_+((F, M))$ is non-empty if and only if $M \geq M_I(F)$.

Proof:

By Proposition 7, given the distribution of objects σ, $z = (\sigma,m) \in I_+((F, M))$ if and only if the amounts of money received by the agents satisfy:

$$m_j = \frac{M + \sum_{i=1}^{n}[u_i(\sigma(i)) - u_j(\sigma(j))]}{n} \geq 0 \qquad \text{for all } j \in Q$$

Therefore, a necessary and sufficient condition for the existence of $z = (\sigma, m)$ in $I_+((F, M))$ is

$$M \geq \sum_{i=1}^{n}[u_j(\sigma(j)) - u_i(\sigma(i))] = n\, u_j(\sigma(j)) - \sum_{i=1}^{n} u_i(\sigma(i)) \qquad \text{for all } j \in Q$$

and defining

$$M_I^{\sigma}(F) = n \max_{j\in Q} \{u_j(\sigma(j))\} - \sum_{i=1}^{n} u_i(\sigma(i))$$

we conclude that such allocations exist if and only if $M \geq M_I^{\sigma}(F)$. Since for each economy there is a finite number of distributions of objects, we can always find the minimum of these total quantities:

$$M_I(F) = \min \{ M_I^{\sigma} / \sigma{:}Q \to A \text{ is a bijection} \}. \bullet$$

5.SOLUTIONS.

The goal of this section is to define the best allocation for each economy following the criteria described in the introduction. The set of feasible allocations we have under consideration will be Z(e). However, all the following results apply when transfers of money are required to be nonnegative if a condition for the existence of the suitable allocations is satisfied.

Definition 6:

A solution is defined as a correspondence Φ which assigns to each economy $e \in \mathcal{E}$ a non-empty subset $\Phi(e)$ of $Z(e)$.

We want a solution to choose allocations satisfying some properties in the feasible set and we would like it to give precise recommendations regarding the distribution of objects and money among the agents. We have seen that the Pareto solution, the Envy-free solution and the Egalitarian solution choose a large number of allocations.In particular, we want a solution to choose from the set of envy-free allocations. Given the fact that $E(e) \subseteq P(e)$, it is natural to suggest $I(e) \cap E(e) \subseteq P(e)$ as a solution. But allocations that are both envy-free and egalitarian do not exist in general. (See Thomson (1990).) The following proposition characterizes the subclass of economies for which these allocations do exist. The necessary and sufficient condition for the set $I(e) \cap E(e)$ to be nonempty requires the existence of a distribution of objects such that each agent values the object received from it more than any other agent. This sub-class will be denoted ξ.

Definition 7:

$e \in \xi$ if there exist a σ such that $u_i(\sigma(i)) \geq u_j(\sigma(i))$ for all $i, j \in Q$.

PROPOSITION 8

For each $e \in \mathcal{E}$, there exists $z = (\sigma, m) \in I(e) \cap E(e)$ if and only if $e \in \xi$.

Proof:

Let us begin with the "only if" part. Let $z = (\sigma, m) \in E(e) \cap I(e)$. Since $z \in I(e)$ it must satisfy:

$$u_i(\sigma(i)) + m_i = u_j(\sigma(j)) + m_j, \text{ for all } i,j \in Q.$$

Since $z \in E(e)$, it must also satisfy:

$$m_i - m_j \leq u_j(\sigma(j)) - u_j(\sigma(i)), \text{ for all } i,j \in Q.$$

But since $m_i - m_j = u_j(\sigma(j)) - u_i(\sigma(i))$, we must have

$$u_j(\sigma(j)) - u_i(\sigma(i)) \leq u_j(\sigma(j)) - u_j(\sigma(i)), \text{ for all } i,j \in Q.$$

And this is true if and only if:

$$u_i(\sigma(i)) \geq u_j(\sigma(i)) \text{ for all } i,j \in Q.$$

As for the "if" part, assuming $e \in \xi$, let σ be such that $u_i(\sigma(i)) \geq u_j(\sigma(i))$ for all $i,j \in Q$. By Proposition 7 there exists $m = (m_1, ..., m_n)$ such that $z = (\sigma, m) \in I(e)$ and $m_j = \dfrac{M + \sum_{i=1}^{n}[u_i(\sigma(i)) - u_j(\sigma(j))]}{n}$ for all $j \in Q$. To see that z is an envy allocation we have to prove that $m_h - m_j \geq k^{\sigma}_{hj}$ holds for all $i, j \in Q$:

$$m_h - m_j = \frac{M + \sum_{i=1}^{n}[u_i(\sigma(i)) - u_h(\sigma(h))]}{n} - \frac{M + \sum_{i=1}^{n}[u_i(\sigma(i)) - u_j(\sigma(j))]}{n}$$

$$= u_j(\sigma(j)) - u_h(\sigma(h)) \geq u_h(\sigma(j)) - u_h(\sigma(h)) = k^{\sigma}_{hj}$$

Where the inequality follows because $u_i(\sigma(i)) \geq u_j(\sigma(i))$ for all $i,j \in Q$. Therefore, we have $z \in I(e) \cap E(e)$.•

When feasibility requires nonnegative transfers we need an additional condition to guarantee that the set $I_+(e) \cap E_+(e)$ is non-empty.

PROPOSITION 9

Given $e = (F, M) \in \mathcal{E}_+$, the set $I_+(e) \cap E_+(e)$ is non-empty if and only if there exists $\sigma^*: Q \rightarrow A$ such that

(a) $u_i(\sigma^*(i)) \geq u_j(\sigma^*(i))$ for all $i, j \in Q$.

(b) $M \geq M^{\sigma^*}_{I}(F)$

Proof:

Starting with the "if" part, assume such σ^* does exist. By condition (a) and Proposition 8 we have that there exists $z = (\sigma^*, m) \in I(e) \cap E(e)$. By condition (b) the distribution of money for an egalitarian allocation generated by σ^* is nonnegative. Therefore, $z = (\sigma^*, m) \in I_+(e) \cap E_+(e)$. On the other hand, if there exists $z = (\sigma, m) \in I_+(e) \cap E_+(e)$ then $z = (\sigma, m) \in I(e) \cap E(e)$ and by Proposition 8 we must have $e \in \xi$, from which condition (a) follows. Since $z = (\sigma, m) \in I_+(e)$ we have that $z \in I(e)$ and $m \geq 0$ and condition (b) follows from the definition of $M_I^{\sigma}(F)$.•

The correspondence $\Phi(e) = I(e) \cap E(e)$ could have been a good candidate for a solution, but is has two flaws: (i) it fails to give any recommendation at all for a subset of economies, and (ii) like the set I(e) itself, it is not invariant with respect to shifts in the utility function of an agent, thus rendering "underestimated" self-reports a dominant strategy. Following the criteria described in the introduction we propose a solution which may be applied to all economies in our class, will always choose a unique allocation from E(e) and is invariant with respect to utility-scale shifts. This solution is defined as follows:

$$\Phi^*(F, M) = \{ (\sigma, m) \in E((F, M)):$$

$$m_i = m_i^* + (M - M_E(F)) / n \text{ and } (\sigma, m^*) \in E((F, M_E(F)))\}$$

It allocates the money as follows: given any α-efficient allocation each agent will receive the unique amount of money indicated by the envy-free allocation for $(F, M_E(F))$ plus an equal share of the amount of money left over, i.e. it allocates the minimal amount of money to generate an envy-free allocation and when there is no more envy applies only the egalitarian criterion to allocate the rest of the money. Since envy-free allocations satisfy monotonicity with respect to money the allocations chosen by this solution are envy-free.

This solution clearly chooses at least one allocation for each economy, since Theorem 1 shows the existence of $M_E(F)$ for each one. Furthermore, it chooses a unique distribution of money for each α-efficient distribution of objects. Finally, it has the property that all agents are indifferent among all allocations chosen by Φ^*.

THEOREM 3

a) σ satisfies α-efficiency if and only if there is a vector m^σ such that $z = (\sigma, m^\sigma) \in \Phi^*((F, M))$. Furthermore, in this case the vector m^σ is unique.

b) For any $z = (\sigma, m^\sigma)$, $z' = (\sigma', m^{\sigma'}) \in \Phi^*((F, M))$ we have

$$u_i(\sigma(i)) + m_i^\sigma = u_i(\sigma'(i)) + m_i^{\sigma'} \text{ for all } i \in Q.$$

Proof:

To prove part (a), let us begin with the "only if" part. Let σ be α-efficient. Since Φ^* satisfies monotonicity with respect to money we can restrict our attention to the economy (F, M_E). In the proof of Theorem 1 we have constructed an envy-free allocation $z = (\sigma, m^*)$ for the economy (F, M_E) for which m_i^* is defined as the minimal quantity of money that we have to give agent i, given that he receives object $\sigma(i)$, in any envy-free allocation. For any other distribution of money m' summing up to M_E we must have $m_i^* > m_i'$ for some $i \in Q$. Therefore, it cannot be envy-free. As for the "if" part, since $z = (\sigma, m^\sigma) \in \Phi^*((F, M))$ we also have that $z = (\sigma, m^\sigma) \in E((F, M))$ and by Proposition 4 σ must be α-efficient. This proves part (a).

To prove part (b), and again from the proof of Theorem 1, we know that if $z = (\sigma, m) \in E_+((F, M_E)) = \Phi^*((F, M_E))$ and σ' is another α-efficient distribution of objects and $m_i' = m_{\sigma^{-1}(\sigma'(i))}$ then $z' = (\sigma', m') \in E_+((F, M_E)) = \Phi^*((F, M_E))$, i.e. z' allocates exactly the same combinations of objects and money as z but to different agents. Since z is an envy-free allocation which uses the smallest amount of money, for each $i \in Q$ we must have that $u_i(\sigma(i)) + m_i = \max_{j \in Q} \{u_i(\sigma(j)) + m_j\}$. By the same reason $u_i(\sigma'(i)) + m_i = \max_{j \in Q} \{u_i(\sigma'(j)) + m_{\sigma^{-1}(\sigma'(j))}\}$., and by construction of z' these quantities are the same. this proves part b) for (F, M_E). For any (F, M) the solution only adds the same amount of money to every agent, and therefore the result holds.•

By construction, the differences between the values that every agent assigns to each pair of objects is all the information we need in order to find the allocation chosen by our solution. Hence, our solution is invariant to shifts in the utility scale of any agent.

Definition 8:

u(.) and u'(.) are equivalent if $u(\alpha) = u'(\alpha) + b$ for all $\alpha \in A$ for some $b \in \Re$.

It is easy to see that our solution also satisfies monotonicity with respect to money.

PROPOSITION 10

a) $\Phi^*((F, M))$ is invariant with respect to shifts in the utility scale of any agent: for all $F = (Q, A, u_Q)$ and $F' = (Q, A, u'_Q)$, with u_i and u_i' equivalent for all $i \in Q$, we have that $\Phi^*((F, M)) = \Phi^*((F', M))$.

b) $\Phi^*((F, M))$ satisfies monotonicity with respect to money.

Proof:

To prove part (a) first we have to show that the α-efficient distributions of objects are the same for both economies. This is clear from the definition of α-efficiency: in the economy (F', M) we have that σ is α-efficient if and only if

$$\sum_{i=1}^{n} u'_i(\sigma(i)) = \sum_{i=1}^{n} u_i(\sigma(i)) + \sum_{i=1}^{n} b_i \geq \sum_{i=1}^{n} u_i(\sigma'(i)) + \sum_{i=1}^{n} b_i$$

$$= \sum_{i=1}^{n} u'_i(\sigma'(i)) \qquad \text{for all } \sigma':Q \rightarrow A.$$

This holds if and only if $\sum_{i=1}^{n} u_i(\sigma(i)) \geq \sum_{i=1}^{n} u_i(\sigma'(i))$ for all $\sigma':Q \rightarrow A$, which implies that σ is an α-efficient distribution of objects for economy (F, M).

The distribution of money chosen by Φ^* for any α-efficient distribution of objects for the economy (F, $M_E(F)$) is given by Theorem 1 as follows:

$$m_i^* = w(r_i,T_i) = \max \sum_{t=1}^{T-1} k^{\sigma}_{r(t)r(t+1)}$$

$$\text{s.t.} \quad r(1) = i$$

$$(r, T) \text{ is any path in } G_\sigma$$

Since it only depends on $k^{\sigma}_{ij} = u_i(\sigma(j)) - u_i(\sigma(i))$ for $i, j \in Q$, we have that

$$k'^{\sigma}_{ij} = u'_i(\sigma(j)) - u'_i(\sigma(i)) = u_i(\sigma(j)) - u_i(\sigma(i)) \text{ for } i, j \in Q.$$

Thus, the distribution of money chosen by Φ^* for the economy (F', $M_E(F')$) is the same and $M_E(F) = M_E(F')$. Hence, for the economies (F, M) and (F', M), after the equal distribution of the money left (i.e., M - $M_E(F)$ for both of them), the solution will still choose the same allocations. This proves part (a).

To prove part (b), suppose $z = (\sigma, m) \in \Phi^* ((F, M)), (\sigma, m^*) \in \Phi^* ((F, M_E(F)))$ where $M' \in \Re$ and consider

$$\begin{aligned} m_i' &= m_i + (M' - M) / n \\ &= m_i^* + [(M - M_E(F)) / n] + [(M' - M) / n] \\ &= m_i^* + (M' - M_E(F)) / n. \end{aligned}$$

Clearly $z' = (\sigma, m') \in \Phi^* ((F, M'))$.•

An important property of our solution is that if the utility functions of the agents are to be reported then truth-telling is an undominated strategy. We say that a solution concept is "not obviously manipulable" when there does not exist an obvious way to improve the outcome that an agent will receive by misreporting his utility function, given that he does not know the utility functions reported by the rest of the agents. I.e., for any utility function that agent i may report there exist a utility profile for the rest of the agents such that agent i is better off by reporting his "true" utility function. It is shown in the next theorem that our solution is "not obviously manipulable".

Definition 9:

A solution Φ is "not obviously manipulable" if given Q, A, M, for any agent $i \in Q$ and any u_i and u_i' which are not equivalent, we can find $\{u_j\}_{j \neq i}$ such that $U_i(\Phi_i(F, M)) > U_i(\Phi_i(F', M))$ where $U_i(.) = u_i(.) + m$ is interpreted as the "true" utility function of agent i, $F = (Q, A, u_Q)$ where $u_Q = \{u_1, u_2, ..., u_n\}$ and $F' = (Q, A, u_Q')$ where $u_Q' = \{u_1, ..., u_{i-1}, u_i', u_{i+1}, ..., u_n\}$.

THEOREM 4

Φ^* is "not obviously manipulable".

Proof:

Let $u_1 = \{u_1(\alpha_1), u_1(\alpha_2), ..., u_1(\alpha_n)\}$ be interpreted as the "true" utility for agent 1 and let $u_1' = \{u_1'(\alpha_1), u_1'(\alpha_2), ..., u_1'(\alpha_n)\}$ be the utility reported by agent 1. Suppose that u_1 and u_1' are not equivalent. For any such u_1' we have to find a profile of utilities for agents 2 to n, $\{u_2, u_3, ..., u_n\}$, such that if the other agents report $\{u_2, u_3, ..., u_n\}$ agent 1 will be strictly better off reporting u_1 rather than u_1'.

Suppose that the objects are indexed such that $u_1(\alpha_j) \geq u_1(\alpha_{j+1})$ for all $1 \leq j < n$. By proposition 10 we know that $\Phi^*((F, M))$ is invariant with respect to shifts in the utility scale of any agent, therefore we can assume that $u_1'(\alpha_1) = u_1(\alpha_1)$. Then, we will have that either $u_1'(\alpha) > u_1'(\alpha_1)$ for some $\alpha \in A$ or $u_1'(\alpha) \leq u_1'(\alpha_1)$ for all $\alpha \in A$. These two cases are considered separately.

Case 1: Suppose that $\max_{j\in Q} \{u_1'(\alpha_j)\} = u_1'(\alpha_{j^*}) > u_1'(\alpha_1)$. Consider the following utility profile for agents 2 to n:

$$u_i(\alpha_j) = 0 \text{ for all } j \in Q \text{ and for all } 1 < i \leq n$$

First we compute the allocation given by Φ^* if agent 1 reports his "true" utility. Notice that $u_1(\alpha_1) \geq u_1(\alpha_j)$ for all $j \in Q$ implies that every α-efficient distribution of objects satisfies $\sigma(1) = \alpha_1$ for the economy given by e = (F, M) and the minimal amount of money for an envy-free allocation to exist is $M_E(F) = 0$, since every agent gets his most preferred object. Therefore, $\Phi^*(F, M)$ assigns to agent 1:

$$\Phi^*_1(F, M) = (\alpha_1, (M - M_E(F))/n) = (\alpha_1, M/n)$$

Which corresponds to the following utility level:

$$U_1(\Phi^*_1(F, M)) = u_1(\alpha_1) + M/n.$$

Similarly, we compute the allocation given by Φ^* if agent 1 reports u_1'. In this case every α-efficient distribution of objects satisfies $\sigma(1) = \alpha_{j^*}$, $\sigma(i) = \alpha_i$ and the minimal amount of money for an envy-free allocation to exist is again $M_E(F') = 0$. Therefore, $\Phi^*(F', M)$ assigns to agent 1:

$$\Phi^*_1(F', M) = (\alpha_{j^*}, (M - M_E(F'))/n) = (\alpha_{j^*}, M/n)$$

Which corresponds to the following utility level:

$$U_1(\Phi^*_1(F', M)) = u_1(\alpha_{j^*}) + M/n.$$

Then, $U_1(\Phi^*_1(F, M)) - U_1(\Phi^*_1(F', M)) = u_1(\alpha_1) - u_1(\alpha_{j^*}) > 0$. So agent 1 may be worse off by reporting $u_1'(\alpha_j) > u_1(\alpha_1)$ for any $j \in Q$.

Case 2: Suppose $u_1'(\alpha_1) \geq u_1'(\alpha_j)$ for all $j \in Q$. Let α_{k+1}, with $1 \leq k < n$, be the object with the lowest index with misreported utility, i.e., u_1' is such that

$$u_1'(\alpha_j) = u_1(\alpha_j) \text{ for all } j \leq k,$$

$$u_1'(\alpha_{k+1}) = u_1(\alpha_k) - b \neq u_1(\alpha_{k+1}) = u_1(\alpha_k) - a,\ a \geq 0.$$

Here we consider two different cases, depending on whether b is greater or smaller than a.

Case 2.1: Assume that $b > a$ and consider the following utility profile for agents 2 to n.

$$\begin{array}{ll} \text{if } 1 < i \leq k: & u_i(\alpha_{i-1}) = L \\ & u_i(\alpha) = 0 \text{ for all } \alpha \neq \alpha_{i-1} \\ \text{if } i = k+1: & u_i(\alpha_{i-1}) = \delta \\ & u_i(\alpha) = 0 \text{ for all } \alpha \neq \alpha_{i-1} \\ \text{if } k+1 < i \leq n: & u_i(\alpha_i) = L \\ & u_i(\alpha) = 0 \text{ for all } \alpha \neq \alpha_i \end{array}$$

where $a < \delta < b$ and $L > \max \{ \sum_{i=1}^{n} u_1(\alpha_i) + \delta,\ \sum_{i=1}^{n} u_1'(\alpha_i) + \delta\}$. First we compute the allocation given by Φ^* if agent 1 reports his "true" utility. Notice that $\delta >$ a and $L > \max \{ \sum_{i=1}^{n} u_1(\alpha_i) + \delta,\ \sum_{i=1}^{n} u_1'(\alpha_i) + \delta\}$ imply that the bijection σ given by: $\sigma(1) = \alpha_{k+1}$, $\sigma(i) = \alpha_{i-1}$ if $1 < i \leq k+1$, $\sigma(i) = \alpha_i$ if $k+1 < i \leq n$, is the only α-efficient distribution of objects for e = (F, M). The amount of money that Φ^* assigns to agent 1 depends on the value of the parameters in the following way:

Case 2.1.1: If $\delta < u_1(\alpha_1) - u_1(\alpha_k) + a$ the minimal amount of money for an envy-free allocation to exist is $M_E(F) = 2\,(u_1(\alpha_1) - u_1(\alpha_k) + a) - \delta$, since we have to compensate agent 1 by $u_1(\alpha_1) - u_1(\alpha_k) + a$, because he does not get his most preferred object, and the envy of agent k+1 with respect to agent 1 by $u_1(\alpha_1) - u_1(\alpha_k) + a - \delta$. Therefore, $\Phi^*(F, M)$ assigns to agent 1:

$$\Phi^*_1(F, M) = (\alpha_{k+1}, (u_1(\alpha_1) - u_1(\alpha_k) + a + (M - M_E(F))/n)$$

$$= (\alpha_{k+1}, (1/n)[M + \delta + (n-2)(u_1(\alpha_1) - u_1(\alpha_k) + a)])$$

Which corresponds to the following utility level:

$$U_1(\Phi^*_1(F, M)) = u_1(\alpha_{k+1}) + (1/n)[M + \delta + (n-2)(u_1(\alpha_1) - u_1(\alpha_k) + a)]$$

$$= u_1(\alpha_k) - a + (1/n)[M + \delta + (n-2)(u_1(\alpha_1) - u_1(\alpha_k) + a)]$$

$$= (1/n)[M + \delta + (n-2)\, u_1(\alpha_1) - 2\,(a - u_1(\alpha_k))].$$

Case 2.1.2: If $u_1(\alpha_1) - u_1(\alpha_k) + a < \delta < b$ the minimal amount of money for an envy-free allocation to exist is $M_E(F) = u_1(\alpha_1) - u_1(\alpha_k) + a$, since we only have to compensate agent 1 because he does not get his most preferred object. Therefore, $\Phi^*(F, M)$ assigns to agent 1:

$$\Phi^*_1(F, M) = (\alpha_{k+1}, (u_1(\alpha_1) - u_1(\alpha_k) + a + (M - M_E(F))/n)$$

$$= (\alpha_{k+1}, (1/n)[M + (n-1)(u_1(\alpha_1) - u_1(\alpha_k) + a)])$$

Which corresponds to the following utility level:

$$U_1(\Phi^*_1(F, M)) = u_1(\alpha_{k+1}) + (1/n)[M + (n-1)(u_1(\alpha_1) - u_1(\alpha_k) + a)]$$

$$= u_1(\alpha_k) - a + (1/n)[M + (n-1)(u_1(\alpha_1) - u_1(\alpha_k) + a)]$$

$$= (1/n)[M + (n-1)\, u_1(\alpha_1) - (a - u_1(\alpha_k))].$$

Now we compute the allocation given by Φ^* if agent 1 reports u_1'. Notice that $a < \delta < b$ and $L > \max\{\sum_{i=1}^{n} u_1(\alpha_i) + \delta, \sum_{i=1}^{n} u_1'(\alpha_i) + \delta\}$ imply that the bijection σ given by: $\sigma(1) = \alpha_k$, $\sigma(i) = \alpha_{i-1}$ if $1 < i < k+1$, $\sigma(i) = \alpha_i$ if $k+1 \le i \le n$, is the only α-efficient distribution of objects for e = (F', M). And the minimal amount of money for an envy-free allocation to exist is $M_E(F') = 2\,(u_1(\alpha_1) - u_1(\alpha_k)) + \delta$, since we have to compensate agent 1 by $u_1(\alpha_1) - u_1(\alpha_k)$, because he does not get his most preferred object, and the envy of agent k+1 with respect to agent 1 by $\delta + u_1(\alpha_1) - u_1(\alpha_k)$. Therefore, $\Phi^*(F', M)$ assigns to agent 1:

$$\Phi^*_1(F', M) \quad = (\alpha_k, (u_1(\alpha_1) - u_1(\alpha_k) + (M - M_E(F'))/n\ )$$

$$= (\alpha_k, (1/n)[M - \delta + (n-2)(u_1(\alpha_1) - u_1(\alpha_k))]\ )$$

Which corresponds to the following utility level:

$$U_1(\Phi^*_1(F', M)) \quad = u_1(\alpha_k) + (1/n)\,[M - \delta + (n-2)(u_1(\alpha_1) - u_1(\alpha_k))]$$

$$= (1/n)\,[\,M - \delta + (n-2)\,u_1(\alpha_1) + 2\,u_1(\alpha_k)].$$

Then, if case 2.1.1 applies $U_1(\Phi^*_1(F, M)) - U_1(\Phi^*_1(F', M)) = (1/n)\,2\,(\delta - a) > 0$ since $\delta > a$ and if case 2.1.2 applies $U_1(\Phi^*_1(F, M)) - U_1(\Phi^*_1(F', M)) = (1/n)\,(\delta - a + u_1(\alpha_1) - u_1(\alpha_k)) > 0$ since $\delta > a$ and $u_1(\alpha_1) > u_1(\alpha_k)$. So agent 1 may be worse off by reporting $u_1'(\alpha_k) - u_1'(\alpha_{k+1}) = b > a = u_1(\alpha_k) - u_1(\alpha_{k+1})$.

Case 2.2: Suppose that $b < a$ and consider the same utility profile for agents 2 to n as in case 2.1, with $\delta > a$ and $L > \max\{\sum_{i=1}^{n} u_1(\alpha_i) + \delta,\ \sum_{i=1}^{n} u_1'(\alpha_i) + \delta\}$. First we compute the allocation given by Φ^* if agent 1 reports his "true" utility. Notice that $\delta > a$ and $L > \max\{\sum_{i=1}^{n} u_1(\alpha_i) + \delta,\ \sum_{i=1}^{n} u_1'(\alpha_i) + \delta\}$ imply that the bijection σ given by: $\sigma(1) = \alpha_{k+1}$, $\sigma(i) = \alpha_{i-1}$ if $1 < i \leq k+1$, $\sigma(i) = \alpha_i$ if $k+1 < i \leq n$, is the only α-efficient distribution of objects for $e = (F, M)$. And the minimal amount of money for an envy-free allocation to exist is $M_E(F) = u_1(\alpha_1) - u_1(\alpha_k) + a$, since we only have to compensate agent 1 because he is the only one who does not get his most preferred object. Therefore, $\Phi^*(F, M)$ assigns to agent 1:

$$\Phi^*_1(F, M) \quad = (\alpha_{k+1}, (u_1(\alpha_1) - u_1(\alpha_k) + a + (M - M_E(F))/n\ )$$

$$= (\alpha_{k+1}, (1/n)[M + (n-1)(u_1(\alpha_1) - u_1(\alpha_k) + a)]\ )$$

Which corresponds to the following utility level:

$$U_1(\Phi^*_1(F, M)) \quad = u_1(\alpha_{k+1}) + (1/n)\,[M + (n-1)(u_1(\alpha_1) - u_1(\alpha_k) + a)]$$

$$= u_1(\alpha_k) - a + (1/n)\,[M + (n-1)(u_1(\alpha_1) - u_1(\alpha_k) + a)]$$

$$= (1/n)\,[\,M + (n-1)\,u_1(\alpha_1) - (a - u_1(\alpha_k))].$$

Now we compute the allocation given by Φ^* if agent 1 reports u_1'. Notice that $b < \delta$ and $L > \max\{\sum_{i=1}^{n} u_1(\alpha_i) + \delta,\ \sum_{i=1}^{n} u_1'(\alpha_i) + \delta\}$ imply that the same bijection σ (that is: $\sigma(1) = \alpha_{k+1}$, $\sigma(i) = \alpha_{i-1}$ if $1 < i \leq k+1$, $\sigma(i) = \alpha_i$ if $k+1 < i \leq n$) is the only α-efficient distribution of objects for $e = (F', M)$. Here again, the amount of money that Φ^* assigns to agent 1 depends on the value of the parameters:

Case 2.2.1: If $\delta > u_1(\alpha_1) - u_1(\alpha_k) + b$ the minimal amount of money for an envy-free allocation to exist is $M_E(F') = u_1(\alpha_1) - u_1(\alpha_k) + b$, since we only have to compensate agent 1 because he is the only one who does not get his most preferred object. Therefore, $\Phi^*(F', M)$ assigns to agent 1:

$$\begin{aligned}\Phi^*_1(F', M) \quad &= (\alpha_{k+1}, (u_1(\alpha_1) - u_1(\alpha_k) + b + (M - M_E(F'))/n\,) \\ &= (\alpha_{k+1}, (1/n)[M + (n-1)(u_1(\alpha_1) - u_1(\alpha_k) + b)]\,)\end{aligned}$$

Which corresponds to the following utility level:

$$\begin{aligned}U_1(\Phi^*_1(F', M)) \quad &= u_1(\alpha_{k+1}) + (1/n)\,[M + (n-1)(u_1(\alpha_1) - u_1(\alpha_k) + b)] \\ &= u_1(\alpha_k) - a + (1/n)\,[M + (n-1)(u_1(\alpha_1) - u_1(\alpha_k) + b)] \\ &= (1/n)\,[\,M + (n-1)\,u_1(\alpha_1) + u_1(\alpha_k) + (n-1)\,b - n\,a].\end{aligned}$$

Case 2.2.2: If $a < \delta < u_1(\alpha_1) - u_1(\alpha_k) + b$ the minimal amount of money for an envy-free allocation to exist is $M_E(F') = 2\,(u_1(\alpha_1) - u_1(\alpha_k) + b) - \delta$, since we have to compensate agent 1 by $u_1(\alpha_1) - u_1(\alpha_k) + b$ because he does not get his most preferred object and the envy of agent k+1 with respect to agent 1 by $u_1(\alpha_1) - u_1(\alpha_k) + b - \delta$. Therefore, $\Phi^*(F', M)$ assigns to agent 1:

$$\begin{aligned}\Phi^*_1(F', M) \quad &= (\alpha_{k+1}, (u_1(\alpha_1) - u_1(\alpha_k) + b + (M - M_E(F'))/n\,) \\ &= (\alpha_{k+1}, (1/n)[M + \delta + (n-2)(u_1(\alpha_1) - u_1(\alpha_k) + b)]\,)\end{aligned}$$

Which corresponds to the following utility level:

$$U_1(\Phi^*_1(F', M)) \quad = u_1(\alpha_{k+1}) + (1/n)\,[M + \delta + (n-2)(u_1(\alpha_1) - u_1(\alpha_k) + b)]$$

$$= u_1(\alpha_k) - a + (1/n)\,[M + \delta + (n-2)(u_1(\alpha_1) - u_1(\alpha_k) + b)]$$

$$= (1/n)\,[\,M + \delta + (n-2)\,u_1(\alpha_1) + 2\,u_1(\alpha_k) + (n-2)\,b - n\,a].$$

Then, if case 2.2.1 applies $U_1(\Phi^*_1(F, M)) - U_1(\Phi^*_1(F', M)) = (1/n)\,(n-1)\,(a - b) > 0$ since $a > b$ and if case 2.2.1 applies $U_1(\Phi^*_1(F, M)) - U_1(\Phi^*_1(F', M)) = (1/n)\,(u_1(\alpha_1) - u_1(\alpha_k) + b - \delta + (n-1)\,(a - b)) > 0$ since $a > b$ and $\delta < u_1(\alpha_1) - u_1(\alpha_k) + b$.So agent 1 may be worse off by reporting $u_1'(\alpha_k) - u_1'(\alpha_{k+1}) = b < a = u_1(\alpha_k) - u_1(\alpha_{k+1})$. This completes the proof of the theorem.•

Finally, the next proposition provides an algorithm to compute an allocation selected by our solution and shows that it is of polynomial time complexity. (For definitions of "algorithm" and "time complexity" the reader is referred to, say, Lawler (1976).)

PROPOSITION 11

There exists an algorithm, whose time complexity is polynomial, that computes an element of Φ^* and the utility profile corresponding to Φ^* for any given economy. (With rational data.)

Proof:

Given $e = (F, M)$, consider the following procedure to construct an element of the set $\Phi^*(F, M)$.

First step: Find the α-efficient distributions of objects. Given $F = (Q, A, u_Q)$, let $G_F = (Q \cup A, QxA)$ be a directed bipartite graph, where every agent and every object are represented by a node ($Q \cup A$ is the set of nodes) and each agent is connected to each object by an arc ($QxA = \{ (i, \alpha_j) : i \in Q$ and $\alpha_j \in A\}$ is the set of arcs). We define the weight of an arc (i, α_j) to be $u_i(\alpha_j)$. Then the problem of finding an α-efficient distribution of objects can be thought of as a weighted matching problem:

"Given an arc-weighted bipartite graph, find a matching for which the sum of the arcs is maximum." [1]

This can be done in $O(n^3)$ steps. (See, for instance, Lawler (1976), pp.201-207.)

Second step: Find $M_E(F)$. Following the proof of theorem 1 $M_E(F) = \sum_{i=1}^{n} m_i^*$, therefore we have to find, for each $i \in Q$:

$$m_i^* = \max \sum_{t=1}^{T-1} k^{\sigma}_{r(t)r(t+1)}$$

$$\text{s.t.} \quad r(1) = i$$

$$(r, T) \text{ is any path in } G_\sigma$$

for any α-efficient σ. This problem can be written as (n-1) different "shortest paths problems" [2]:

$$m_i^* = \max_{j \in Q} \; \{ -\min \sum_{t=1}^{T-1} -k^{\sigma}_{r(t)r(t+1)} \quad \text{s.t. } r(1) = i \;\; (r, T) \in G_\sigma \text{ and } r(T) = j \}$$

By lemma 1 we know that if σ is α-efficient the graph G_σ has no cycles with positive weight. Given this condition a solution can be found in $O(n^3)$ operations. (See Lawler (1976), pp.82-89.) Once we have all the shortest paths between all pairs of nodes, we have to select, for each $i \in Q$, the shortest path from i, which requires $O(n^2)$ steps.

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1 Lawler (1976), p. 183.

2 Lawler (1976), p. 59.

Third step: Compute an allocation in $\Phi^*(F, M)$. Given σ, compute $m_i^{\sigma} = m_i^* + (M-M_E(F))/n$ to obtain $z \equiv (\sigma, m_i^{\sigma}) \in \Phi^*(F, M)$. This can obviously be done in linear time.

Finally note that the time complexity of the algorithm is $O(n^3)$: step 1 requires $O(n^3)$ operations, step 2 requires $O(n^3)$ as well, while step 3 is of linear time complexity•

6.COMMENTS ON CONSISTENCY

Tadenuma and Thomson (1989.a) define a set of properties that may be required of any solution. Among the set of properties they use, consistency plays an important role. The properties based on consistency define the solution's behavior when changes in the number of agents and objects occur.

Because the availability of money (infinitely divisible good) is necessary for the existence of envy-free allocations in economies with indivisible goods, the individual amounts of money that agents get in an envy-free allocation will depend on the values that individuals assign to each and every indivisible object in the economy as we have seen in the proof of existence. From our results we see that when the economy gets smaller (an agent $i \in Q$ and a bundle $(\sigma(i), m_i)$ leave the economy) the bundles that the original envy-free allocations assigned to the remaining agents are envy-free allocations for the new sub-economy; yet we may have more envy-free allocations since now the restrictions on this set have been relaxed. In particular, the minimal amount of money to have envy-free allocations may decrease, because now the paths with maximal total weight may be at most as long as before and the weights have not changed.

Hence it seems too restrictive to require that an envy-based normative solution would be invariant with respect to these changes: when an agent leaves the economy there are less sources of envy, and therefore one may wish to reallocate the money in order to get a better allocation. The following example illustrates this point.

Example 3:

Consider the following economy $e = (F, M) \in \varepsilon$, where $Q = \{1, 2, 3\}$, $A = \{ \alpha_1, \alpha_2, \alpha_3 \}$, $u_Q = \{ [u_1(\alpha_1) = 10, u_1(\alpha_2) = 5, u_1(\alpha_3) = 0], [u_2(\alpha_1) = 2, u_2(\alpha_2) = 7, u_2(\alpha_3) = 5], [u_3(\alpha_1) = 6, u_3(\alpha_2) = 1, u_3(\alpha_3) = 4] \}$ and $M = 11$.

There is a unique α-efficient distribution of objects for this economy, and it defines the following allocation of objects: $[(1, \alpha_1), (2, \alpha_2), (3, \alpha_3)]$; and the minimal necessary amount of money for envy-free allocations with nonnegative transfers to exist, given the α-efficient distribution of objects, is $M^{\sigma}_{E}(F) = \sum_{i=1}^{n} m^{\sigma}_{i} = 2$, where $m^{\sigma} = (0, 0, 2)$. If we allocate the remaining amount of money equally, $M - M^{\sigma}_{E}(F) = 9$, among the agents we obtain the unique allocation chosen by our solution:

$$\Phi^*(e) = \{ (\alpha_1, 3), (\alpha_2, 3), (\alpha_3, 5) \}.$$

Consider the sub-economy of e with respect to $Q' = \{2, 3\}$ and $z = \Phi^*(e)$, i.e.: $Q' = \{2, 3\}$, $A' = \{ \alpha_2, \alpha_3 \}$, $u_Q = \{ [u_2(\alpha_2) = 7, u_2(\alpha_3) = 5], [u_3(\alpha_2) = 1, u_3(\alpha_3) = 4] \}$ and $M' = 8$.

For this sub-economy, the minimal amount of money for envy-free allocations with nonnegative transfers to exist is $M^{\sigma}_{E}(t^{z}_{Q'}(F)) = 0$. The unique allocation chosen by our solution is $[(\alpha_2, 4), (\alpha_3, 4)] \in \Phi^*(t^{z}_{Q'}(F))$.

One may wonder, why does the solution change the amounts of money assigned to agents 2 and 3, when agent 1 leaves the economy? Clearly, because in the sub-economy there is no envy generated by the distribution of objects, and therefore we can just distribute the money equally, whereas when agent 1 and object α_1 are in the economy, we have to use money to compensate the envy generated by the distribution of objects. •

The analysis of the set of envy-free allocations for economies with indivisibilities show that the size and shape of this set depends on the individual valuations of the indivisible objects. If we want to select among envy-free allocations, it is logical that the sources of envy play an important role. In our model envy is originated by the values that individuals assign to the objects in the economy. Therefore, changes in the sets of objects or agents of the economy imply changes in the sources of envy. Thus the idea of consistency is not too appealing from the viewpoint of envy-minimization. Although we have only analyzed a particular case, we can extend the conclusions to the general case, since it is easy to see that we will always have more sources of envy to compensate for, the larger the number of agents we have in the economy.

7. GENERALIZATIONS OF THE MODEL

We can apply our solution to economies with a larger number of agents than objects by defining as many "empty" objects as needed to equate the number of objects and agents in the economy. Any agent that receives one of these "empty" objects in an allocation will derive the utility of having none of the existing objects plus the amount of money given at that allocation. The utility of having no object may differ among the agents. Once we have an economy with the same number of objects and agents we can apply our solution as shown before.

When the number of objects is larger than the number of agents some of our results do not hold anymore. It has been already shown (Svensson (1983)) that in this case there may exist inefficient envy-free allocations. Since we are interested in a selection of the set of envy-free allocation we will only consider envy-free allocations that are also efficient, i.e., envy-free allocations generated from α-efficient distributions of objects. However, even when restricting attention to such allocations theorem 1 does no longer hold, since for different α-efficient distributions of objects the minimal amount of money to derive an envy-free allocation with nonnegative transfers may differ.

<u>Example 4</u>:

Consider the following economy $e = (F, M) \in \mathcal{E}$, where $Q = \{1, 2, 3\}$, $A = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, $u_Q = \{[u_1(\alpha_1) = 3, u_1(\alpha_2) = 1, u_1(\alpha_3) = 0, u_1(\alpha_4) = 1], [u_2(\alpha_1) = 2,$

$u_2(\alpha_2) = 1$, $u_2(\alpha_3) = 0$, $u_2(\alpha_4) = 0]$, $[u_3(\alpha_1) = 0$, $u_3(\alpha_2) = 2$, $u_3(\alpha_3) = 1$, $u_3(\alpha_4) = 0]\}$ and $M \in \Re$.

There are two α-efficient distributions of objects for this economy, σ and σ', and they define the following allocations of objects: $[\sigma(1) = \alpha_1, \sigma(2) = \alpha_2, \sigma(3) = \alpha_3)]$ and $[\sigma'(1) = \alpha_4, \sigma'(2) = \alpha_1, \sigma'(3) = \alpha_2)]$. The minimal amount of money for envy-free allocations with nonnegative transfers to exist given σ is $M_E^{\sigma}(F) = \sum_{i=1}^{n} m_i^{\sigma} = 3$, where $m^{\sigma} = (0, 1, 2)$ and given σ' is $M_E^{\sigma'}(F) = \sum_{i=1}^{n} m_i^{\sigma'} = 2$, where $m^{\sigma'} = (2, 0, 0)$.•

For instance, in the example given above Φ^* will choose σ', because $M_E^{\sigma'}(F) = 2 < 3 = M_E^{\sigma}(F)$, which gives the following allocation:

$$\Phi^*(F, M) = \{ (\alpha_4, (M+4)/3), (\alpha_1, (M-2)/3), (\alpha_2, (M-2)/3) \}.$$

In the literature we find generalizations of this model by introducing fictitious agents when the number of objects in the economy exceeds the number of agents (Alkan, Demange and Gale (1991)). When this is the case, we have to compensate real agents for the envy they feel with respect to fictitious ones (agents who do not exist). Furthermore, the envy of these fictitious agents is considered as important as the envy of the real ones in the computation of the solutions. Our generalization does not include any kind of fictitious agents and, by considering only α-efficient distributions of objects, agents will never want to exchange the objects assigned to them by the solution for the ones left over. One may define a variation of our solution for this case that still satisfies most of the properties of the original Φ^*, such as: invariance with respect to shifts in the utility scale of any agent, monotonicity with respect to money and an appropriately defined version of "not obviously manipulable".

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