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Acyclicity and Dynamic Stability: Generalizations and Applications

by

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Abstract

We study the asymptotic stability of infinite horizon concave programming problems. Turnpike theorems for this class of models generaly have to assume a low level of discounting. By generalizing our precedent work we provide a one-parameter family of verifiable conditions that guarantee convergence of the optimal paths to a stationary state. We call this property θ-acyclicity. In the one-dimensional case we show that supermodularity implies our property but not viceversa. In the multidimensional case supermodularity has no relevant implications for the asymptotic behavior of optimal paths. We apply θ-acyclicity to a pair of models which study firms' dynamic behavior as based on adjustment costs. The first is the familiar model of competitive equilibrium in an industry in the presence of adjustment costs. In the second case firms act strategically and we study the dynamic evolution implied by the closed-loop Nash equilibria. In both instances our criteria apply and allow us to obtain stability results that are much more general than those already existing in the literature.

1 Introduction

Turnpike theorems for infinite horizon concave programming problems typically depend on the level of discounting (see e.g. Cass and Shell (1976), McKenzie (1976), Rockafellar (1976), Scheinkman (1976), and McKenzie (1986) for an overview). They have the form: for given preferences and technology there exists a (low enough) degree of impatience such that all optimal trajectories are attracted to a unique stationary state. Indeed the level of discounting *must* be crucial for the general validity of the Turnpike property: we have shown in Boldrin and Montrucchio (1986) that any kind of dynamic behavior is optimal if the level of discounting can be appropriately chosen.

To obtain stability theorems which are independent of the discount factor additional properties (over and above concavity) must be added to the model. This is the case of Brock and Scheinkman (1976) (global positive definiteness of the "Q-matrix"), Araujo and Scheinkman (1977) (dominant diagonality of the infinite "Euler matrix"), Scheinkman (1978) (separability of the Hamiltonian between state and co-state variables) and Magill and Scheinkman (1979) (symmetry of the off-diagonal block of the Hessian matrix).

In Boldrin and Montrucchio (1988) we have provided our contribution to this line of research. We have shown that by writing the optimal growth model as a dynamic programming problem a natural binary relation is defined over the set of feasible states. A pair of capital stock vectors x and y satisfy this relation if, when starting at x, one obtains a higher value by moving to y than by staying at x. When such a binary relation is *acyclic* only "simple" (in a sense properly defined below) optimal trajectories are possible.

The purpose of the present work is two-fold. We generalize our previous result by introducing a one parameter family of conditions of which the old one was just a special case. In so doing we also provide new characterizations of the class of dynamic programming problems that are acyclic. Secondly, to show that our conditions are not vacuous, we apply them to the "costs of adjustment" model of the firm and obtain new stability results which are substantially more powerful than existing ones. In particular, by adopting the technique of Lucas and Prescott (1971), we provide stability conditions for the intertemporal competitive equilibrium of the whole industry when all prices are endogenously determined. Next we look at a game theoretical version of the same model. Using an equivalence result due to Dechert (1978, 1988) we study the artificial optimal control problem whose solution are closed-loop equilibria of the dynamic game. Once again our criteria can be applied to show that also these trajectories converge to a stationary point.

The property of acyclicity is partially related to that of supermodularity of the short run return function. Models with supermodular payoffs have recently attracted a wide attention for their applicability to a variety of economic and game theoretical issues (Topkis (1978, 1979) Vives (1990), Milgrom and Shannon (1991)). We discuss this relation in Section 3 and show that in the one dimensional case the set of supermodular payoff functions is strictly contained in the set of those that are acyclic. We also point out that while the supermodularity property deliveres powerful comparative statics results it is not very useful in a dynamic context. This is simply because only in the one dimensional case monotonicity of a map $x_{t+1} = f(x_t)$ implies convergence of the sequence $\{x_t\}_{t=0}^{\infty}$ to a fixed point of f. In other words: the fact that the optimal policy is monotone when the payoff function is supermodular will not guarantee that optimal paths are convergent.

The rest of the paper is organized in the following way. Section 2 contains a description of the abstract model and the statement and proof of the main theorems. In Section 3 the class of programming problems that satisfy our conditions is characterized. Section 4 studies the application to the costs of adjustment model. Section 5 concludes the paper. The most tedious proofs are relegated in the Appendix.

2 Intertemporal Optimization and Dynamic Stability

2.1 The Basic Theorem

The class of intertemporal optimization problems we want to study is described by problem (P) and assumptions (A1)-(A3).

$$W(x) = \max \sum_{t=0}^{\infty} V(x_t, x_{t+1})\delta^t$$
$$\text{s.to} : x_{t+1} \in \Gamma(x_t) \tag{P}$$
$$x_0 = x, \text{ given in } X.$$

(A1) $X \subset \mathbf{R}^n$ is convex and compact.

(A2) $\Gamma : X \to X$ is a continuous, compact-valued correspondence with convex graph, and $x \in \Gamma(x)$ for all $x \in X$.

(A3) $V : D \to \mathbf{R}$ is a continuous, concave function defined on $D = \{(x, y) \in \mathbf{R}^{2n}; x \in X$ and $y \in \Gamma(x)\}$. $V(x, \cdot)$ is strictly concave for every $x \in X$.

We gather here some well-known properties of (P) that are useful for our purposes (see Stokey, Lucas and Prescott (1989) and Boldrin-Montrucchio (1992) for details). The function $W : X \to \mathbf{R}$ is the value function associated to (P). It is (strictly) concave and continuous and it satisfies the relation:

$$W(x) = \max\{V(x, y) + \delta W(y) : y \in \Gamma(x)\} \tag{1}$$

Define as $\tau : X \to X$ the continuous map solving Eq. (1), i.e.:

$$W(x) = V(x, \tau(x)) + \delta W(\tau(x)). \tag{2}$$

We call τ the (optimal) policy function of (P). Using the Bellman Optimality Principle one can show that $\{x_t\}_{t=0}^{\infty}$ is a feasible sequence realizing the maximum for (P) if and only if it satisfies: $x_{t+1} = \tau(x_t)$, $x_0 = x$. The dynamic properties of τ will depend, ceteris paribus, on the magnitude of δ. More formally: for given X, Γ and V the map $\delta \mapsto \tau_\delta$ from the interval [0,1) into the space $C^0(X; X)$ is continuous (in the uniform topology). The dynamical system described on X by the iterates of τ may therefore have very different features at different values of δ. In particular, for suitable choices of V, periodic and even chaotic trajectories can be produced by τ at certain magnitudes of δ (compare Boldrin-Montrucchio (1986) and Boldrin-Montrucchio (1992) for details).

This is quite disturbing and, in many economic applications, also counterintuitive. One is therefore interested in finding conditions under which, independently of the discount factor, "complicated dynamics" cannot be solutions to (P). Boldrin-Montrucchio (1988) contains one such condition. It is based on the idea of obtaining "simple" dynamical systems as solutions to (P). To be precise we need a few definitions.

Definition 1. Let $f : X \to X$ be a continuous function inducing the dynamical system $x_{t+1} = f(x_t)$. The *non-wandering* set $\Omega(f)$ associated to f is defined as:
$\Omega(f) = \{x \in X: \forall$ neighborhood $\mathcal{B}$ of x and $T > 0,\ \exists\ t \geq T$ such that $f^t(\mathcal{B}) \cap \mathcal{B} \neq \emptyset\}$.
Here f^t denotes the t^{th} iterate of f, i.e., $f^t(x) = f(f^{t-1}(x)),\ f^0(x) = x$.

As the asymptotic behavior of the dynamical system $x_{t+1} = f(x_t)$ is described by $\Omega(f)$ it is clear that the latter can, in general, be very complex. It includes all the steady states, the periodic orbits, the strange attractors, etc. (see Boldrin and Montrucchio (1992), and references therein). Simple dynamics therefore requires a simple non-wandering set:

Definition 2. A dynamical system $f : X \to X$ is *simple* if $\Omega(f) = \text{Fix}(f)$, where $\text{Fix}(f) = \{x \in X : x = f(x)\}$.

Quite obviously simple dynamical systems have strong stability properties:

a) when $\text{Fix}(f)$ is a discrete set, every optimal trajectory converges to a fixed point;

b) when $\text{Fix}(f) = \{x^*\}$, x^* is a global attractor for the dynamical system f;

c) when $\text{Fix}(f) = \{x^*\}$ and x^* is locally stable then x^* is globally asymptotically stable.

Definition 3. To every binary relation $R \subset X \times X$ we associate its *transitive completion* $R^* \subset X \times X$ as follows: for any pair (x, y) in $X \times X$ we say yR^*x if there exists a finite sequence $(x_1, \ldots, x_N)$ of points belonging to X with $x_1 Rx$, $x_{i+1}Rx_i$ $(i = 1, \ldots, N-1)$ and yRx_N. Then R is *acyclic* if and only if R^* is antisymemtric, i.e., if yR^*x implies $x\not{R}^*y$.

In Boldrin-Montrucchio (1988) we proved the following theorems.

Theorem 1. Let $f : X \to X$ be a continuous map on the compact space X and R a binary relation over X. Assume that f and R satisfy the conditions:

(i) $f(x)Rx$ for all $x \in X$ such that $f(x) \neq x$;

(ii) R is open (as a subset of $X \times X$) and acyclic.

Then: $\Omega(f) = \text{Fix}(f)$.

Theorem 2. Let $\tau : X \to X$ solve (P). Then τ is simple for every $\delta \in [0,1)$ if V satisfies:

$$\sum_{t=1}^{N} V(x_t, x_t) \geq \sum_{t=1}^{N} V(x_t, x_{t+1}) \qquad (*)$$

for any finite sequence $\{x_1, \ldots, x_N\}$ of points in X ($x_{N+1} = x_1$ is understood in $(*)$).

2.2 Generalizations

We will now seek a generalization of Theorem 2. A crucial first step is to define a one-parameter family of binary relations $\mathcal{R}_\theta \subset X \times X$, with θ a number in the unit interval.

Definition 4. Let U: $X \times X \to \mathbf{R}$ be continuous, with $U(x, \cdot)$ strictly quasi-concave and X as in (A1). For a given number $\theta \in [0,1)$ define the binary relation $\mathcal{R}_\theta$ over X as:

$$y\mathcal{R}_\theta x \quad \text{iff} \quad U(x, (1-\theta)x + \theta y) < U(x, y)$$

for x, y in X.

The relation $\mathcal{R}_\theta$ is open as a set in $X \times X$ because U is continuous. Notice that $\tau(x)\mathcal{R}_\theta x$ whenever $\tau(x) \neq x$, where $\tau(x) = \text{Arg max}\,\{U(x,y) : y \in \Gamma(x)\}$. We should notice here that a binary relation such as the one we use is naturally induced by any well posed maximization problem and our methodology may therefore be applied to models other than (P). In the following we will convene to call θ-acyclic any U that induces a relation $\mathcal{R}_\theta$ which is acyclic.

Proposition 1. Under the hypotheses of Definition 4, $\theta \geq \theta'$ implies $\mathcal{R}_\theta \subset \mathcal{R}_{\theta'}$. Therefore if the relation $\mathcal{R}_\theta$ is acyclic for some $\theta = \bar{\theta}$, then it is acyclic for all $\theta \geq \bar{\theta}$.

Proof. Let $(x,y) \in \mathcal{R}_\theta$. Then: $U(x, (1-\theta)x + \theta y) < U(x,y)$. For x and y fixed consider the function $\lambda : [0,1) \to \mathbf{R}$ defined as $\lambda(\theta) = U(x, (1-\theta)x + \theta y)$. Then $(x,y) \in \mathcal{R}_\theta$ is equivalent to $\lambda(\theta) < \lambda(1)$. The (strict) quasi-concavity of $U(x, \cdot)$ guarantees that λ is quasi-concave and therefore: $\theta' \leq \theta$ implies: $\lambda(1) > \lambda(\theta) \geq \lambda(\theta')$, i.e., $U(x, (1-\theta')x + \theta' y) < U(x,y)$ and so $(x,y) \in \mathcal{R}_{\theta'}$. Q.E.D.

Proposition 2. If U is θ-acyclic for some $\theta \in [0,1)$ then, for any function $\psi : X \to \mathbf{R}$, the new function $V(x,y) = U(x,y) + \psi(x)$ is also θ-acyclic.

Proof. Obvious. Q.E.D.

Because of our concern with the asymptotic behavior of solutions to infinite horizon programing problem we are interested in the particular case in which the function $U(x,y)$ is defined as:

$$U(x,y) = V(x,y) + \delta W(y) \tag{3}$$

where V and W are as in problem (P) above. In other words we want to study the acyclicity of the relation:

$$yR_\theta x \quad \text{iff} \quad \{V(x,(1-\theta)x+\theta y) + \delta W((1-\theta)x+\theta y)\} < V(x,y) + \delta W(y) \tag{4}$$

for θ taking values in $[0,1)$.

This is quite difficult as one would need to know a lot about W, which is not the case most of the times. This provides the motivation for the study of a notion of acyclicity which is stronger than the one introduced in Definition 4. We have:

Definition 5. Let $V(x,y)$ be as in (A3). We say that it is *additively θ-acyclic* if the function $U(x,y) = V(x,y) + \delta W(y)$ is θ-acyclic for any concave $W(\cdot)$.

Once again if V is additively θ-acyclic for some $\theta \in [0,1)$, then the policy function τ will be simple (in the sense of Theorem 1) for all $\delta \in [0,1)$. A generalization of Theorem 2 can now be proved.

Theorem 3. Let V be as in (A3) and τ as defined in Eq. (2). If for any N and for any sequence of distinct points $\{x_1,\ldots,x_N\}$ in X one has:

$$\sum_{i=1}^{N} V(x_i,(1-\theta)x_i+\theta x_{i+1}) \geq \sum_{i=1}^{N} V(x_i,x_{i+1}) \tag{**}$$

for some $\theta \in [0,1)$, (with $x_{N+1} = x_1$), then V is additively θ-acyclic and τ is simple.

Proof. We will proceed by contradiction. Let (**) be satisfied for some $0 \leq \theta < 1$ and assume that $\mathcal{R}_\theta$ as defined in (4) is not acyclic for some concave W. This means that there exists a sequence of points $\{x_1,\ldots,x_N\}$ such that $x_{i+1}\mathcal{R}_\theta x_i$, $i = 1,\ldots,N$ (with $x_{N+1} = x_1$). In other words, for the given θ and chosen W one has:

$$V(x_i,(1-\theta)x_i+\theta x_{i+1}) + \delta W((1-\theta)x_i+\theta x_{i+1}) < V(x_i,x_{i+1}) + \delta W(x_{i+1})$$

for $i = 1,\ldots,N$. Summing over i:

$$\sum_{i=1}^{N} [V(x_i,(1-\theta)x_i+\theta x_{i+1}) + \delta W((1-\theta)x_i+\theta x_{i+1})] < \sum_{i=1}^{N} [V(x_i,x_{i+1}) + \delta W(x_{i+1})] \tag{5}$$

As W was assumed concave, we have:

$$\sum_{i=1}^{N} W((1-\theta)x_i + \theta x_{i+1}) \geq (1-\theta)\sum_{i=1}^{N} W(x_i) + \theta \sum_{i=1}^{N} W(x_{i+1}) = \sum_{i=1}^{N} W(x_i)$$

The latter implies that Eq. (5) may be rewritten as:

$$\sum_{i=1}^{N} V(x_i, (1-\theta)x_i + \theta x_{i+1}) < \sum_{i=1}^{N} V(x_i, x_{i+1})$$

which contradicts the hypothesis (**). Q.E.D.

Theorem 2 is obtained as a special case of Theorem 3 by setting $\theta = 0$. Properties analogous to those given in Proposition 1, Corollary and Proposition 2 also hold when V is additively θ-acyclic. They are illustrated next.

Proposition 3. If the function V, defined as in (A3), satisfies condition (**) for some $\theta = \bar{\theta}$, then it is additively θ-acyclic for all $1 > \theta \geq \bar{\theta}$.

Proof. The same argument used in the proof to Proposition 1 may be applied to the function $\lambda : [0,1) \to \mathbf{R}$ defined as: $\lambda(\theta) = \sum_{i=1}^{N} V(x_i, (1-\theta)x_i + \theta x_{i+1})$, for every given sequence $\{x_1, \ldots, x_N\}$. Q.E.D.

Proposition 4. If the function V, defined as in (A3), satisfies condition (**) for some $\theta \in [0,1)$ then for any function $\psi : X \to \mathbf{R}$ and any concave function $\phi : X \to \mathbf{R}$, the new function $\tilde{V}(x,y) = V(x,y) + \psi(x) + \phi(y)$ also satisfies condition (**)

Proof. One needs only to replicate the proof to Theorem 3. Q.E.D.

3 Characterization

3.1 The One-Dimensional Case

The case in which the state-space is one-dimensional is particularly simple to analyze and allows one to derive sharper results. Roughly speaking, the sharp characterization we are able to obtain may be seen as another implication of a famous theorem of Sarkovskij (1964) on the ordering of cycles of a map of the real line into itself. For our purposes Sarkovskij's theorem may be summarized in the following way. If $f : \mathbf{R} \to \mathbf{R}$ is continuous and has some cycles, then it must have cycles of order two. Alternatively: if f has no cycles of order two then it has no cycles whatsoever. The latter, together with Theorem 3, leads to:

Proposition 5. In problem (P) let $\dim(X) = 1$. If V satisfies:

$$V(x, (1-\theta)x + \theta y) + V(y, (1-\theta)y + \theta x) \geq V(x, y) + V(y, x)$$

for some $\theta \in [0, 1)$ and all x, y in $X, x \neq y$, then V is additively θ-acyclic.

Proof. As it is shown in Theorem 3 of Boldrin-Montrucchio (1988), Sarkovskij's theorem reduces the whole problem to ruling out cycles of period two, which is what the above condition does. Q.E.D.

It was pointed out in the introduction that certain similarities exist between those maximization problems that are acyclic and those that are supermodular. This relation is particularly straightforward in the one dimensional case.

Definition 6. Consider two pairs of points (x_1, x_2) and (y_1, y_2), with $x_2 \geq x_1$ and $y_2 \geq y_1$. A function $V(x, y)$ defined over $X \times X$ is called:

- *supermodular* if $V(x_1, y_1) + V(x_2, y_2) \geq V(x_1, y_2) + V(x_2, y_1)$, and
- *submodular* if $V(x_1, y_1) + V(x_2, y_2) \leq V(x_1, y_2) + V(x_2, y_1)$.

A function which is either super- or sub- modular will be called unimodular. We should stress that Definition 6 is not restricted to the case in which $\dim(X)$ is equal to one. The same is true for Proposition 6. The notion of super- (sub-) modularity may be derived from that of sub- (super-) additivity for a function defined on a lattice L, when $L = \mathbf{R}^n$, (see Marshall and Olkin (1979) for more details). The policy functions solving unimodular optimization problems have very neat monotonicity properties.

Proposition 6. Let $V : D \to \mathbf{R}$ be unimodular over $D = X \times X$. Then:

- if V is supermodular the poliy function $\tau(x)$ is non-decreasing; and
- if V is submodular the policy function $\tau(x)$ is non-increasing.

Proof. One should note, first of all, that $V(x,y)$ unimodular implies $U(x,y) = V(x,y) + \delta W(y)$ is also unimodular. Let us consider the supermodular case only, the other one being completely symmetric. Set $x_1 < x_2$ and assume, by absurd, that $\tau(x_1) > \tau(x_2)$. Set $y_2 = \tau(x_1) > y_1 = \tau(x_2)$. Supermodularity gives: $U(x_1, \tau(x_2)) + U(x_2, \tau(x_1)) \geq U(x_1, \tau(x_1)) + U(x_2, \tau(x_2))$. Strict concavity of V and the optimality principle together with the latter inequality gives:

$$U(x_1, \tau(x_1)) + U(x_2, \tau(x_2)) > U(x_1, \tau(x_2)) + U(x_2, \tau(x_1)) \geq$$

$$\geq U(x_1, \tau(x_1)) + U(x_2, \tau(x_2)),$$

a contradiction. Q.E.D.

The case $D \subset X \times X$ can be handled in the same way under mild additional restrictions, i.e. V unimodular over D and either $\tau(x)$ interior to $\Gamma(x)$ for all $x \in X$, or Γ an "increasing" correspondence with "free disposal". For a more extensive discussion of the implications for τ of the unimodularity of V see Boldrin and Montrucchio (1992, Ch. 7).

We can now characterize the relation between supermodularity and acyclicity in the one-dimensional case.

Proposition 7. Consider problem (P) with $\dim(X) = 1$. In this case if V is supermodular it is additively 0-acyclic.

Proof. We need only to show that the condition of Proposition 5 is satisfied for $\theta = 0$. Let x and y be two points in X with, say, $x < y$. In the definition of supermodularity set: $x_1 = y_1 = x$ and $x_2 = y_2 = y$. This gives: $V(x,x) + V(y,y) \geq V(x,y) + V(y,x)$ which is the desired inequality. Q.E.D.

The opposite is clearly not true as shown below in Example 1. This also proves that the set of acyclic functions is larger than the set of supermodular ones.

Example 1. Consider $V(x,y) = ax + by - 1/2Ax^2 - y^2 - xy - 1/2Bxy^2$ defined over $[0,1] \times [0,1]$. For the following set of parameters V satisfies assumption (A3): $A \geq 1/2, B \geq -2$ and $(2A-1) \geq B(B+2)$. This function also satisfies $V_2(x,y)(y-x) \leq V_2(y,x)(y-x)$

for all x and y in X, which is the one–dimensional version of the sufficient condition for acyclicity proved in Proposition 8 below. Furthermore, when $B > 0$ $V(x,y)$ is submodular whereas for $B < 0$ it is neither sub– nor supermodular.

3.2 The Multi–dimensional Case

Characterizing acyclic return functions is much more difficult in this case as nothing as powerful as the Sarkovskij theorem exists in dimension larger than one. Furthermore here the relation with supermodularity does not provide useful insights. Indeed this should be obvious. Supermodularity only guarantees that the policy function $x_{t+1} = \tau(x_t)$ is a non–decreasing one, i.e. that $x \leq y$ implies $\tau(x) \not\geq \tau(y)$. In dimension one this is enough to rule out cycles of period two (and therefore all other cycles) but it becomes an almost irrelevant property in higher dimensions.

Nevertheless some progresses can still be made under fairly general conditions.

Proposition 8. Assume V is C^1 and that hypotheses $(A1) - (A3)$ are satisfied. If for all sequences $(x_1, \ldots, x_N)$ and for all finite N:

$$\sum_{i=1}^{N} \Big(V_2(x_i, x_{i+1}) \cdot (x_{i+1} - x_i) \Big) \leq \epsilon < 0$$

then V is additively θ-acyclic for some $\theta \in [0,1)$. The former (with $\epsilon = 0$) is also a necessary condition for additive θ-acyclicity.

Proof. Set $\phi(\theta) = \sum_{i=1}^{N} V(x_i, (1-\theta)x_i + \theta x_{i+1})$. Then ϕ is strictly concave on $[0,1]$ because $V(x,\cdot)$ is concave by assumption. To prove necessity notice that $\phi(\theta) \geq \phi(1)$, because V is θ-acyclic. Hence $\phi'(1) \leq 0$, that is to say:

$$\phi'(1) = \sum_{i=1}^{N} \Big(V_2(x_i, x_{i+1}) \cdot (x_{i+1} - x_i) \Big) \leq 0$$

Conversely, suppose that the latter is satisfied as a strict inequality for any finite sequence $(x_1, \ldots, x_N)$. Given that $V(x,\cdot)$ is strictly concave and differentiable, we have

$$V(x_t, x_{t+1}) \leq V(x_t, \theta x_{t+1} + (1-\theta)x_t) + (1-\theta)V_2(x_t, \theta x_{t+1} + (1-\theta)x_t) \cdot (x_{t+1} - x_t)$$

Summing up over t:

$$\sum_{t=1}^{N} V(x_t, x_{t+1}) \leq \sum_{t=1}^{N} V(x_t, \theta x_{t+1} + (1-\theta)x_t) + M$$

where

$$M = (1-\theta)\Big(\sum_{t=1}^{N} V_2(x_t, \theta x_{t+1} + (1-\theta)x_t)\cdot(x_{t+1} - x_t)\Big) \leq 0,$$

for θ close enough to one. Hence:

$$\sum_{t=1}^{N} V(x_t, x_{t+1}) \leq \sum_{t=1}^{N} V(x_t, \theta x_{t+1} + (1-\theta)x_t)$$

as desired.   Q.E.D.

When the short-run return function satisfies a strong concavity assumption, a small variation on the above line of proof can also provide an estimate of the magnitude of the parameter θ. In our terminology a function $f : X \to \mathbf{R}$ is concave-β if there exists $\beta > 0$ such that $g(x) = f(x) + \beta/2 \mid\mid x \mid\mid^2$ is convex over X.

Proposition 9. Assume V is of class C^1 and $V(x, \cdot)$ is concave$-\beta$ for all $x \in X$. If there exists $0 < \eta < \beta$ such that:

$$\sum_{i=1}^{N} \langle V_2(x_i, x_{i+1}), (x_{i+1} - x_i)\rangle \ \leq -\eta \sum_{i=1}^{N} \mid\mid x_{i+1} - x_i \mid\mid^2$$

holds for all $(x_1, \ldots, x_N)$, then V is additively θ-acyclic for some θ near 1.

Proof. See Appendix.

4 Some Economic Applications

4.1 The Cost-of-Adjustments Model of the Firm

Stripped down to its bare essentials, the economic environment is the following.[1] A representative firm is considered which produces a single output by means of a vector of inputs and a time invariant technology. The critical feature of the model comes from the assumption that inputs are quasi-fixed: a variation in their utilized quantities will entail positive costs for the firm over and above the payment of the pure market prices. These "adjustment costs" depend on the magnitude of the change. Denote the vector of inputs at time $t = 0, 1, 2, \ldots$ with $x_t \in \mathbf{R}^n_+$.

All the $(n + 1)$ spot markets in which the output and inputs trade are perfectly competitive. The firm is infinitely lived and it perfectly foresees the sequences of prices $\{p_t\}_{t=0}^{\infty}$ and $\{q_t\}_{t=0}^{\infty}$, $p_t \in \mathbf{R}_+$, $q_t \in \mathbf{R}^n_+$, $t = 0, 1, \ldots$ that will clear the markets.

The technology is described by a concave production function $f(x_t)$ and a cost-of-adjustment function $g(x_t, x_{t+1})$ which are both time-invariant and have values expressed in units of output: $g(x_t, x_{t+1})$ is then the output foregone in period t in order to adjust the inputs level from x_t to x_{t+1} for the next period. Because quasi-fixed factors depreciate, we denote with Σ the $n \times n$ diagonal matrix with elements $0 \leq \sigma_i \leq 1$, $i = 1, \ldots, n$ representing the rates of depreciation for each coordinate of x. Finally, we denote with $I_t \in \mathbf{R}^n_+$ the gross purchases of factors during period t. The firm's total cash-flow in any period can then be written as:

$$G(x_t, x_{t+1}, p_t, q_t) = p_t[f(x_t) - g(x_t, x_{t+1})] - q_t \cdot I_t$$

The assumption of perfect capital-markets closes the model: for given sequences of prices $(\{p_t\}, \{q_t\})$ a profit-maximizing firm will follow the objective:

$$\max_{\{x_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} G(x_t, x_{t+1}, p_t, q_t)\delta^t$$

$$\text{s.to} : x_t \geq 0 \text{ all t}, \; x_0 \in \mathbf{R}^n_+$$

To get things going we assume that the price sequences $\{p_t\}$ and $\{q_t\}$ are constant over time and use the output as the numeraire: $p_t = 1$ and $q_t = q \in \mathbf{R}^n_+$ all $t = 0, 1, \ldots$. In subsection 4.2 we show how this hypothesis can be removed, and the central stability

[1] We refer the reader to, e.g. Brock and Scheinkman (1978), Gould (1968), Lucas (1967), Mortensen (1973), Treadway (1971) for details.

results retained, by using a technique introduced first in Lucas-Prescott (1971). Our second assumption constrains the input vectors within a compact and convex set $X \subset \mathbf{R}^n_+$. This, again, is of no harm to the generality of the argument, (see Brock-Scheinkman (1978, pp. 5–8) for details).

Finally we impose some additional structure on the costs of adjustment, by assuming that $g(x_t, x_{t+1})$ is convex and satisfies: $g(x_t, x_{t+1}) = g(x_{t+1} - x_t)$. This is the so-called case of *internal costs* (see Lucas (1967) and Mortensen (1973)).

The dynamic programming problem of the firm now is:

$$\max \sum_{t=0}^{\infty} [f(x_t) - g(x_{t+1} - x_t) - q \cdot (x_{t+1} - (I - \Sigma)x_t)]\delta^t$$

$$\text{s.to} : x_t \in X \text{ all } t, x_0 \text{ given.}$$

Proposition 10. The short-run return function:

$$V(x, y) = f(x) - g(y - x) - q \cdot (y - (I - \Sigma)x)$$

is additively θ-acyclic for all $1 > \theta \geq 0$.

Proof. Let $\{x_1, \ldots, x_N\}$ be any feasible sequence. We need to show that:

$$-Ng(0) \geq - \sum_{t=1}^{N} g(x_{t+1} - x_t)$$

holds, with $x_{N+1} = x_1$. This follows from concavity of $-g$, i.e.:

$$-g(0) = -g\left[\frac{1}{N}\sum_{t=1}^{N} g(x_{t+1} - x_t)\right] \geq -\frac{1}{N}\sum_{t=1}^{N}(x_{t+1} - x_t)$$

Q.E.D

We have already proved that in the case of internal adjustment-costs the optimal investment policy of the firm is a simple dynamical system at every level of the interest rate.

A second type of adjustment-cost function is studied in the literature, e.g. Brock-Scheinkman (1978) and Gould (1968). It assumes a convex cost function $g(x, y) = g(y - (I - \Sigma)x)$. In this case the equilibrium price for the quasi-fixed factors will be an increasing function of the amount demanded by the single firm: q is not taken parametrically any more, and the linear term $q \cdot (y - (I - \Sigma)x)$ will be incorporated in g. This seems hard to reconcile with the hypothesis of perfect competition: why should the choice of an individual firm affect market prices? The story behind these so called *external costs* should rely, we believe, on the existence of some market imperfection or on strategic behavior on the part of the firms, a case we examine later on in section 4.4 . Taking this caveat as understood we can consider the new objective:

$$\max \sum_{t=0}^{\infty}[f(x_t) - g(x_{t+1} - (I - \Sigma)x_t)]\delta^t$$

$$\text{s.to} : x_t \in X \text{ all } t,\ x_0 \text{ given}$$

Once again strict concavity guarantees the existence of a policy function describing the optimal program: $x_{t+1} = \tau(x_t)$. Unfortunately the simple result of the previous paragraph cannot be replicated here, as the presence of the element $(I - \Sigma)$ prevents the same argument from carrying through. Nevertheless there are still two significant cases in which we can establish a stability result for the model with external costs of adjustment.

Proposition 11. Assume that: $\Sigma = \sigma I$ holds for some scalar $\sigma \in [0,1]$, where I is the $n \times n$ identity matrix. Then the policy function τ is simple. The same is true even if convexity of g is replaced by quasi-convexity, as long as τ exists.

Proof. See Appendix.

Our second case is:

Proposition 12. Assume that for every vector $z = (z_1, \ldots, z_n)$ the cost function satisfies $g(z) = \Sigma_{i=1}^{n} g_i(z_i)$, with g_i convex for all $i = 1, \ldots, n$. Then the policy function is simple.

Proof. See Appendix.

Under the rationale for external costs we have given above, it should be clear that assuming this kind of separability in the cost function amounts to assuming that the cross-elasticities across the different input's markets are negligible.

4.2 The Industry Competitive Equilibrium

Consider now the competitive equilibrium over time of an industry composed by a fixed number M of different firms, producing the same kind of output. Retain the previous notation and denote with $f_j, g_j, \Sigma_j, y_{jt} = [f_j(x_{jt}) - g_j(x_{jt+1} - x_{jt})]$ the fundamentals of a generic firm $j = 1, \ldots, M$. Note that the input vector x_{jt} has coordinates $x^i_{jt}, i = 1, \ldots, n$.

We also need the additional notation:

$$I_{jt}^i = x_{jt+1}^i - (1-\sigma_j^i)x_{jt}^i, \qquad I_t^i = \sum_{j=1}^M I_{jt}^i\,, \qquad i = 1,\ldots,n\,.$$

$$y_t = \sum_{j=1}^M y_{jt}, \quad \text{and} \quad z_t = [x_{1t}^1,\ldots,x_{1t}^n,\ldots,x_{jt}^i,\ldots,x_{Mt}^n] \in Z\,.$$

where Z is a convex and compact subset of $\mathbf{R}^{nM}$. The equilibrium prices for output and inputs are determined as:

$$p_t = \phi(y_t),\ \phi' \le 0\,, \phi > 0; \quad \text{and} \quad q_t^i = H_i(I_t^i),\ H_i' \ge 0,\ H_i > 0,\ i = 1,\ldots,n$$

where ϕ is the inverse demand function for output and the H_i are the inverse supply functions of the n inputs.

Only the internal-cost case will be worked out in full detail: the case of external-costs does not present additional complication, other than those already discussed at the end of section 4.1. Each firm solves:

$$\max \sum_{t=0}^{\infty} \Big\{ p_t[f_j(x_{jt}) - g_j(x_{jt+1} - x_{jt})] - q_t \cdot (x_{jt+1} - (I - \Sigma_j)x_{jt}) \Big\} \delta^t \tag{6}$$

$$\text{s. to} : x_{jt} \in X_j,\ x_{j0} \in X_j \text{ given.}$$

The definition of competitive equilibrium is the usual one.

Definition 7. A set of sequences $\{x_{1t},\ldots,x_{Mt}\}_{t=0}^{\infty}$ and $\{p_t, q_t\}_{t=0}^{\infty}$ with $x_{jt} \in X_j$ all j and all t, $p_t \in \mathbf{R}_{++}, q_t \in \mathbf{R}_{++}^n$ all t is a *competitive equilibrium* for this industry if:

(i) $\{x_{jt}\}_{t=0}^{\infty}$ solves (6) for the given $\{p_t, q_t\}_{t=0}^{\infty}$ all $j = 1,\ldots,M$;

(ii) $p_t = \phi(\sum_{j=1}^M y_{jt})$ all t;

(iii) $q_t^i = H_i(\sum_{j=1}^M I_{jt}^i)$ all t and all $i = 1,\ldots,n$.

The programming problem (6) is not time invariant because of the exogenous terms p_t and q_t. However it is a routine exercise to derive an autonomous dynamic programming problem which has the competitive equilibrium sequence as its solution. To accomplish this we need a few more definitions. Set:

$$F(z_t, z_{t+1} - z_t) = \int_0^{y_t} \phi(L)dL, \text{ with } y_t = \sum_{j=1}^M [f_j(x_{jt}) - g_j(x_{jt+1} - x_{jt})]$$

$$H(z_{t+1}-(I-\tilde{\Sigma})z_t)=\sum_{i=1}^{n}\int_0^{I_t^i} H_i(L)d(L).$$

where $\tilde{\Sigma}$ is the $(n\times M)\times(n\times M)$ matrix having the M matrices Σ_j along its diagonal. Notice that $F: Z\times Z\rightarrow \mathbf{R}$ and $H: Z\times Z\rightarrow \mathbf{R}$, are well defined and continuous. It is also straightforward to verify that the function F defined above is concave over $Z\times Z$ while H is convex over $Z\times Z$.

We may now consider the following problem:

$$\max\sum_{t=0}^{\infty}[F(z_t,z_{t+1}-z_t)-H(z_{t+1}-(I-\tilde{\Sigma})z_t)]\delta^t \tag{7}$$

$$\text{s.to}: z_t\in Z \text{ all } t, z_0 \text{ given in } Z.$$

The reader should notice that, even if we have assumed only internal adjustment costs, endogenizing the input-prices created an external-costs problem for the whole industry. This is only natural, as the Pareto optimal solution cannot disregard the effect of the individual firms' demand on market prices. Moreover such "external costs" would arise in any case, even if no (internal or external) adjustment costs had been assumed at the firm level.

We would like to prove that the return function $V(z_t,z_{t+1})=F(z_t,z_{t+1}-z_t)-H(z_{t+1}-(I-\tilde{\Sigma})z_t)$ is additively θ-acyclic. The function H does not present any obstacle different from those we already addressed in the previous subsection. The case for F is somewhat more complicated. In fact F is of the type $h[f(x)-g(y-x)]$ where h is concave and increasing: this is not enough to make $F(x,y-x)$ θ-acyclic even if $f(x)-g(y-x)$ were such. Indeed the problem would not have arised if we had specified the return function of the firm j as:

$$p_t f_j(x_{jt})-g_j(x_{jt+1}-x_{jt})$$

This, indeed, appears to be the standard way in which the firm's problem has been formalized in most of the literature (e.g. Dechert (1988), Mortensen (1973), Scheinkman (1978)). In this case the function F has the simple form $F(z,z')=\sum_{j=1}^{M}f_j(x_j)+\sum_{j=1}^{M}g_j(y_j-x_j)$ which is obviously acyclic. Nevertheless, there are some disturbing problems with the economic interpretation of the cost function g in this case. In fact, the value $g(y-x)$ should now be expressed in units of the numeraire and there is nothing in the underlying economic problem that guarantees a convex and time invariant g. Finally, the very same notion of "internal costs" as "lost output" becomes much less intuitively appealing in this case.

A possible and somewhat more palatable alternative, used in Hayashi (1982) and Uzawa (1969), is to interpret the costs of adjustment as "lost inputs", define g_j as a function from

$X_j \times X_j$ into $\mathbf{R}^n_+$ and write the return function of a generic firm as:

$$p_t f_j(x_j) - q_t \cdot g_j(y_j - (I - \Sigma_j)x_j)$$

This leads to a "social maximum" problem of the type $F(z, z') = \sum_{j=1}^{M} f_j(x_j) + G(z' - (I - \tilde{\Sigma})z)$ which can be handled as we did in section 4.1 for the external costs case.

A third solution (which we follow in the rest of this paper) is to set the price of output equal to one in each period. This is the same as assuming that the demand function for such output is linear[2]. Notice, though, that the supply functions for inputs remain unrestricted. When we set $p_t = 1$ all t, the function $F(z_t, z_{t+1} - z_t)$ simplifies to:

$$F(z_t, z_{t+1} - z_t) = \int_0^{\sum_{j=1}^{M}[f_j(x_{jt}) - g_j(x_{jt+1} - x_{jt})]} dL = \sum_{j=1}^{M}[f_j(x_{jt}) - g_j(x_{jt+1} - x_{jt})] \tag{8}$$

In this case we can prove:

Proposition 13. The concave function $F : Z \times Z \to \mathbf{R}$ defined in Eq. (8), satisfies condition $(**)$ of Theorem 3 for $\theta \geq 0$.

Proof. See Appendix.

Proposition 14. The concave function $H : Z \times Z \to \mathbf{R}$ satisfies condition $(**)$ of Theorem 3 with $\theta \geq 0$, when one of the following occurs:

1) $\Sigma_j = \Sigma$ all $j = 1, \ldots, M$ (depreciation matrices are the same across firms);

2) for each $\tilde{H}_i = \int_0^{I_t^i} H_i(L)dL$ there exist M functions $h_j^i : X_j \to \mathbf{R}$ such that $\tilde{H}_i(I_t^i)$ can be written as: $\tilde{H}_i(I_t^i) = \sum_{j=1}^{M} h_j^i(I_{jt}^i)$ for all $i = 1, \ldots, n$.

Proof. See Appendix

We have therefore proved that under the maintained hypotheses, the Competitive Equilibrium sequence $\{(z_t, p_t, q_t)\}_{t=0}^{\infty}$ generated by the solution to (8) will exhibit a simple dynamic behavior.

4.3 Capital Accumulation Games

This class of dynamic games has been recently studied by Dockner and Takahashi (1988), Fershtman and Muller (1984, 1986) and Flaherty (1980) among others. We refer the reader to these papers for detailed discussions of the model's structure.

[2] We owe this observation to Hugo Hopenhayn

The physical environment is completely similar to the one we used in the industry equilibrium example. The distinct feature is the assumption that firms act strategically instead of competitively. One is therefore interested in the dynamic properties of the Nash equilibria associated to a set of initial conditions $(x_0^1, \ldots, x_0^j, \ldots, x_0^M)$.

Retain the notation and assumptions used in Section 4.2 and consider the case of external adjustment costs. Without loss of generality the jth firm's optimization problem can be written as:

$$\max \sum_{t=0}^{\infty} \delta^t \Big\{ \phi\Big(\sum_{i=1}^{M} f_i(x_{it})\Big) f_j(x_{jt}) - g_j(x_{jt+1} - (I - \Sigma_j)x_{jt}) \Big\} \tag{9}$$

$$\text{s.to} : x_{jt} \in X_j \ \forall t; \ x_{j0} \text{ given}, \quad \{x_{it}\}_{t=0}^{\infty} \text{ given for all } i \neq j\,.$$

The definition of equilibrium for this game is simple:

Definition 8. A *Nash Equilibrium* for the dynamic game described in (9) is a set of sequences $\{x_{jt}^*\}_{t=0}^{\infty}$, for $j = 1, \ldots, M$ such that

$$\sum_{t=0}^{\infty} \delta^t \Big\{ \phi\Big(\sum_{i=1}^{M} f_i(x_{it}^*)\Big) f_j(x_{jt}^*) - g_j(x_{jt+1}^* - (I - \Sigma_j)x_{jt}^*) \Big\} \geq$$

$$\geq \sum_{t=0}^{\infty} \delta^t \Big\{ \phi\Big(\sum_{i \neq j}^{M} f_i(x_{it}^*) + f_j(x_{jt})\Big) f_j(x_{jt}) - g_j(x_{jt+1} - (I - \Sigma_j)x_{jt}) \Big\}$$

for all feasible $\{x_{jt}\}_{t=0}^{\infty}$, and for all $i = 1, \ldots, M$.

We will now exploit a technique developed in Dechert (1988) to transform the game described in (9) into a control problem of the type (P). The latter will easily seen to be acyclic. In order to accomplish this we need one strong extra assumption, i.e. that the aggregated demand function is once again linear,

$$\phi\Big(\sum_{j=1}^{M} f_j(x_{jt})\Big) = P - \sum_{j=1}^{M} f_j(x_{jt}), \quad \text{with } P > 0$$

The return function for the artificial control problem is then:

$$V(z_t, z_{t+1}) = \sum_{j=1}^{M} \Big\{ [P - \sum_{i=1}^{j} f_i(x_{it})] f_j(x_{jt}) - g_j(x_{jt+1} - (I - \Sigma_j)x_{jt}) \Big\} \tag{10}$$

where $z_t \in \tilde{Z}$ is as defined in Section 4.2. It is a matter of fairly straightforward algebra to verify that if a sequence $\{z_t\}_{t=0}^{\infty} = \{x_{1t}, \ldots, x_{Mt}\}_{t=0}^{\infty}\}$ is a solution to the Dynamic Programming problem induced by (10), then it is also a (closed-loop) Nash Equilibrium for the game described in (9), (see Dechert (1988) for the details). Among the other things this implies the existence of M continuous maps $\tau_i : Z \to X_i$ for $i = 1, \ldots, M$ such that the individual firms' equilibrium strategies satisfy:

$$x_{it+1} = \tau_i(z_t), \quad i = 1, \ldots, M \tag{11}$$

As an immediate consequence of the results obtained in sections 4.1 and 4.2 we can conclude:

Proposition 15. Under the assumptions of either Proposition 11 or Proposition 12, the dynamical system (11), representing the closed-loop Nash equilibria of (9), is a simple one.

5 Conclusion

In this paper we have introduced and characterized a new condition for the dynamic stability of the solutions to infinite horizon optimal control problems. This condition substantially generalizes the one we had previously proposed in Boldrin and Montrucchio (1988), and it allows for a fairly elegant characterization of the class of dynamic programming problems satisfying it.

We have tested our approach by considering the dynamic theory of the firm, under both competitive and oligopolistic market conditions. In both cases we have been able to deliver strong stability results which followed as immediate application of our abstract condition. A brief comparison with the relevant literature should make it clear also in what sense our technique improves upon all the previous ones.

From the standpoint of economic applications the most affine article is Scheinkman (1978) where the continuous-time version of the internal-costs model is considered. He shows that every optimal path either converges to the boundary of the feasible set or to a unique, interior steady state. His basic argument is different from ours and cannot be applied to the external-costs model. He exploits the fact that in his model the Hamiltonian is separable in the state and co-state variables. Differentiability is also used. Further, he provides a solution to the industry-competitive equilibrium problem along the same lines we have used here, but he keeps input prices exogenous and fixed over time.

To the best of our knowledge Brock-Scheinkman (1978) is the only other global analysis of the discrete-time version of this problem. They take in explicit consideration only the external-costs, but the internal-costs is also covered by their technique. They add differentiability to our assumptions and obtain a global stability result for the interior solutions as a consequence of the negative quasi-definitiveness of a certain Hessian matrix. The later is satisfied, in general, for values of the discount factor close to one. Their theorem does not, therefore, exclude different behaviors for smaller values of δ.

Again for the discrete-time problem, an interesting local analysis has recently been carried on by Dasgupta (1985) (see also Dasgupta-McKenzie (1990)). While both articles look at the same class of models we also consider, their main concern is the relation between local stability and regularity of the optimal steady states. The ensuing results are therefore not comparable to those obtained here.

As for the game-theoretical version of the firm's problem the early study of Fershtman and Muller (1986) considered the continuous-time, one-dimensional version of the problem and is therefore not comparable. The only work we are aware of and which analyzes a general version of the discrete-time capital accumulation game is Dockner and Takahashi (1988). They study only open-loop Nash equilibria and their Turnpike theorem is obtained by applying the infinite-dimensional Dominant Diagonal condition of Araujo and Scheinkman (1977) to the set of first order conditions of the individual players. While they need not assume linearity of the aggregate demand function they must assume that the individual profit functions satisfy a dominat diagonal condition.

Appendix

Proof of Proposition 9. Concavity of $V(x,y)$ in y implies:

$$V(x_i, x_{i+1}) \leq V(x_i, (1-\theta)x_i + \theta x_{i+1}) + (1-\theta)V_2(x_i, (1-\theta)x_i + \theta x_{i+1}) \cdot (x_{i+1} - x_i)$$

and concavity-β in y implies (see Boldrin and Montrucchio (1992, App. C)):

$$(1-\theta)V_2(x_i, (1-\theta)x_i + \theta x_{i+1}) \cdot (x_{i+1} - x_i) \leq V_2(x_i, x_{i+1}) \cdot (x_{i+1} - x_i) + \beta(1-\theta)^2 \mid\mid x_{i+1} - x_i \mid\mid^2$$

Using these two inequalities and summing up along the index i, for $i = 1, \ldots, N$ we have:

$$\sum_{i=1}^{N} V(x_i, x_{i+1}) \leq \sum_{i=1}^{N} V(x_i, (1-\theta)x_i + \theta x_{i+1}) +$$

$$+ (1-\theta) \sum_{i=1}^{N} \Big(V_2(x_i, x_{i+1}) \cdot (x_{i+1} - x_i) \Big) + \beta(1-\theta)^2 \sum_{i=1}^{N} \mid\mid x_{i+1} - x_i \mid\mid^2$$

Our hypotheses imply that the sum of the last two terms on the right hand side is bounded above by:

$$-(1-\theta)[\eta - (1-\theta)\beta] \sum_{i=1}^{N} \mid\mid x_{i+1} - x_i \mid\mid^2$$

which is nonpositive for $\theta \geq 1 - \eta/\beta$. Q.E.D.

Proof of Proposition 11. We only need to show that $(**)$ is satisfied for some $\theta \in [0,1)$. In fact, we can prove it for $\theta = 0$, i.e.:

$$\sum_{i=1}^{N} g(\sigma x_i) \leq \sum_{i=1}^{N} g(x_{i+1} - (1-\sigma)x_i)$$

holds for all sequences $\{x_1, \ldots, x_N\}$ and all $\sigma \in [0,1]$. For given sequence and σ define the convex combination:

$$\sigma x_{i+1} = (1-\sigma)\sigma x_i + \sigma[x_{i+1} - (1-\sigma)x_i]$$

for $i = 1, \ldots, N$.

Convexity of g implies: $g(\sigma x_{i+1}) \leq (1-\sigma)g(\sigma x_i) + \sigma g[x_{i+1} - (1-\sigma)x_i]$. Summing up from $i = 1$ to $i = N$ and simplifying:

$$\sum_{i=1}^{N} g(\sigma x_i) \leq \sum_{i=1}^{N} g(x_{i+1} - (1 - \sigma(x_i)$$

As for the quasi-convex case, quasi-convexity implies: $g(\sigma x_{i+1}) \leq \text{Max}[g(\sigma x_i), g(x_{i+1} - (1-\sigma)x_i)]$ $i = 1, \ldots, N$. Hence there exists $\mu \in [0,1]$ such that

$$g(\sigma x_{i+1}) \leq (1-\mu)g(\sigma x_i) + \mu g(x_{i+1} - (1-\sigma)x_i)$$

$i = 1, \ldots, N$. Summing up and simplifying yields the desired inequality. Q.E.D.

Proof of Proposition 12. Depreciation factors can be different so $\Sigma = \text{diag}(\sigma_1, \ldots, \sigma_n)$. Let $x_1, \ldots, x_N$ be a sequence of feasible vectors in X, with coordinates $(x_i^1, \ldots, x_i^n)$ for $i = 1, \ldots, N$. Once again set:

$$\sigma_j x_{i+1}^j = (1-\sigma_j)\sigma_j x_i^1 + \sigma_j[x_{i+1}^j - (1-\sigma_j)x_i^j]$$

for every vector $i = 1, \ldots, N$ and each coordinate $j = 1, \ldots, n$.

Convexity of g_j implies:

$$g_j(\sigma_j x_{i+1}^j) \leq (1-\sigma_j)g_j(\sigma_j x_i^j) + \sigma_j g_j(x_{i+1}^j - (1-\sigma_j)x_i^j)\,,\ \forall_i, \forall_j$$

Summing along the index i we get:

$$\sum_{i=1}^{N} g_j(\sigma_j x_{i+1}^j) \leq (1-\sigma_j)\sum_{i=1}^{N} g_j(\sigma_j x_i^j) + \sigma_j \sum_{i=1}^{N} g_j(x_{i+1}^j - (1-\sigma_j)x_i^j)$$

for all $j = 1, \ldots, n$. After simplification we get:

$$\sum_{i=1}^{N} g_j(\sigma_j x_i^j) \leq \sum_{i=1}^{N} g_j(x_{i+1}^j - (1-\sigma_j)x_i^j)$$

all $j = 1, \ldots, n$. A second summation along the index j gives:

$$\sum_{j=1}^{n}\sum_{i=1}^{N} g_j(\sigma_j x_i^j) \leq \sum_{j=1}^{n}\sum_{i=1}^{N} g_j(x_{i+1}^j - (1-\sigma_j)x_i^j)\,.$$

Interchanging the order of summation and using the property of g given in the Proposition we get:

$$\sum_{i=1}^{N} g(\Sigma x_i) \leq \sum_{i=1}^{N} g(x_{i+1} - (I-\Sigma)x_i)$$

Once again the return function is additively θ-acyclic and the result follows. Q.E.D.

Proof of Proposition 13. Let the sequence $\{z_1, \ldots, z_N\}$ be given. We want to show that:

$$\sum_{t=1}^{N}\Big\{\sum_{j=1}^{M}[f_j(x_{jt}) - g_j(0)]\Big\} \geq \sum_{t=1}^{N}\Big\{\sum_{j=1}^{M}[f_j(x_{jt}) - g_j(x_{jt+1} - x_{jt}]\Big\}$$

holds true for any such sequence. The latter reduces to:

$$-\sum_{j=1}^{M}[Ng_j(0) - \sum_{t=1}^{N} g_j(x_{jt+1} - x_{jt})] \geq 0$$

which is satisfied because g_j is convex for all j. Q.E.D.

Proof of Proposition 14. When hypothesis 1) is realized H is the sum of n concave functions:

$$\int_0^{\sum_{j=1}^{M}(x^i_{jt+1} - (1-\sigma^i)x^i_{jt})} H_i(L)dL = G_i(z^i_{t+1} - (1 - \sigma^i I)z^i_t)$$

where z^i_t denotes the M-dimensional vector $[x^i_{1t}, \dots, x^i_{Mt}]$, for $i = 1, \dots, n$ and I is the $M \times M$ unitary diagonal matrix. Then G_i belongs to the class of functions already considered in Proposition 11, and condition $(**)$ is satisfied. When hypothesis 2) is realized H is totally separable and so each of the functions $\tilde{H}_i$ belongs to the class of functions considered in Proposition 12 and $(**)$ applies once again. Q.E.D.

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