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TRANSPORTATION TYPE PROBLEMS WITH QUANTITY DISCOUNTS

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ABSTRACT

It is known, to be real, that the per unit transportation cost from a specific supply source to a given demand sink is dependent on the quantity shipped, so that there exists finite intervals for quantities where price breaks are offered to customers. Thus, such a quantity discount results in non convex, piecewise-linear functional. In this paper algorithms are provided for solving the 'all unit' and the 'incremental' quantity discount, as well as, the 'fixed charge' problems. These algorithms are based upon a branch and bound solution procedure. The branches lead to ordinary transportation problems whose results are obtained utilizing the "cost operator" for one branch and "rim operator" for another branch so that these new problems are not resolved thus reducing computational time. Suitable illustrations and extensions are also provided.

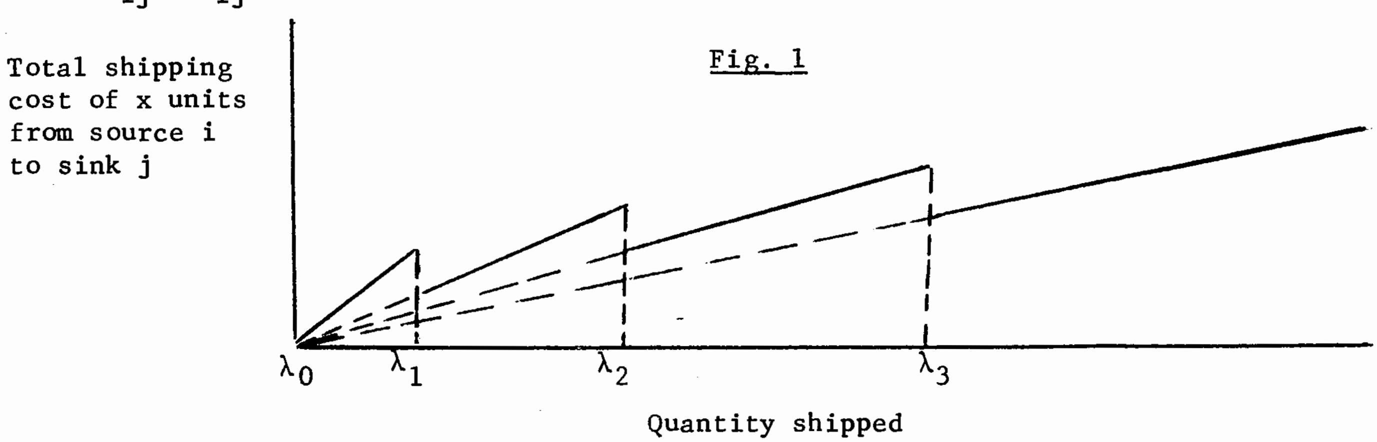
I. Introduction

In the real world, it is a common practice to offer discounts for the purchase of large quantities, and/or for shipment of large volumes of a given commodity. [6,2]

In this paper we analyze quantity discount problems, representing the "all unit" [5] and the "incremental quantity" discounted transportation problems. The details of the procedure are explained considering the 'all unit' quantity discount problem in Section 2, whereas the associated algorithm is presented in Section 3. Certain branch selection procedures and heuristics are provided in Section 4 with corresponding illustrations in Section 5. Based on this algorithm, we provide extensions in Section 6 where the "incremental quantity discounted problem" and the "fixed charge transportation problems" are addressed specifically.

The analysis will focus on the "all unit" quantity discount problem where the methodology concerning the general approach for solving the piecewise linear programming is developed.

Let $\lambda_{ij}^{0} = 0, \lambda_{ij}^{1}, \lambda_{ij}^{2},\ldots, \lambda_{ij}^{k}, \lambda_{ij}^{r} \leq \infty$, where $\lambda_{ij}^{k-1} < \lambda_{ij}^{k}$ (for k = 1,2,...,r) be such that if a quantity x_{ij} is shipped from source i to sink j (i = 1,2,...,m), (j = 1,2,...,n) and $\lambda_{ij}^{k-1} \leq x_{ij} < \lambda_{ij}^{k}$, then the per unit cost of the x_{ij} units is c_{ij}^{k}, and the total cost associated with shipping x_{ij} units is $c_{ij}^{k}x_{ij}$ where $c_{ij}^{k} > c_{ij}^{k+1}$. This result is illustrated in figure 1.



This problem may be solved with separable convex programming [11], but solving the transportation problem in this manner has the disadvantage of a large number of constraint equations being necessary.

Our approach is similar to the one suggested by Falk and Soland [4] for solving the general non-convex type math program, but, nevertheless, is more specialized and concerns the special type of non-convex math program, namely, the piecewise linear program.

II. The All Unit Quantity Discount Transportation Type Problem

The transportation type problem with all unit quantity discounts may be formulated as follows:

(1) Minimize $$Z = \sum_{j \in J} \sum_{i \in I} c^*_{ij}\, x_{ij}$$

(2) Subject to: $$\sum_{j \in J} x_{ij} = a_i \qquad \text{for } i \in I$$

(3) $$\sum_{i \in I} x_{ij} = b_j \qquad \text{for } j \in J$$

(4) $$0 \leq x_{ij} \leq \lambda^r_{ij} \qquad \text{for } i \in I \text{ and } j \in J$$

(5) $$c^*_{ij} = \begin{cases} c^1_{ij} & \text{if} \quad 0 = \lambda^0_{ij} \leq x_{ij} < \lambda^1_{ij} \\ c^2_{ij} & \text{if} \quad \lambda^1_{ij} \leq x_{ij} < \lambda^2_{ij} \\ c^k_{ij} & \text{if} \quad \lambda^{k-1}_{ij} \leq x_{ij} < \lambda^k_{ij} \\ c^r_{ij} & \text{if} \quad \lambda^{r-1}_{ij} \leq x_{ij} < \lambda^r_{ij} \leq \infty \end{cases}$$

where: $I = \{1,2,\ldots,i,\ldots,m\}$ set of sources

$J = \{1,2,\ldots,j,\ldots,n\}$ set of sinks

$R = \{1,2,\ldots,k,\ldots,r\}$ set of cost intervals

In order to facilitate the presentation, expressions (1) - (5) will be referred to as problem P^*.

The algorithm provided for solving P^* is basically a branch and bound type similar to the subtour elimination algorithm of the travelling salesman problem [7]. Here, instead of eliminating infeasible subtours, we eliminate all infeasibilities due to (5) until complete feasibility is restored.

Let us now define the following "initial Transportation Problem" P_0 which is given by (1) - (4) of P^* (note that constraint set (5) is not included), and all c^*_{ij} are replaced with c^r_{ij}. Since c^r_{ij} are the minimum values for every $(i,j) \in I \times J$, the solution to problem P_0 is a "better than optimal" solution. If, in addition, the solution matrix $X = \{x_{ij}\}$ satisfies $\lambda^{r-1}_{ij} \leq x_{ij} < \lambda^r_{ij}$ for every i,j, then the solution of problem P_0 is the optimum solution to problem P^*.

<u>Definition 1</u>: A solution $X = \{x_{ij}\}$ to the problem (1) - (4) is said to be "interval feasible" if all c^*_{ij} used in (1) are implied to be feasible due to the fact that x_{ij} lies in the feasible interval given by (5).

<u>Definition 2</u>: A solution $X = \{x_{ij}\}$ to the problem (1) - (4) provided $\lambda^{k-1}_{ij} \leq x_{ij} < \lambda^k_{ij}$ is said to be the "better than optimal solution" if that solution is an optimal solution to the problem where each $c^*_{ij} = c^\ell_{ij}$ in (1) where $k \leq \ell \leq r$. (Note that 'ℓ' may be different for different $(i,j) \in [(I \times J)]$.

It is true that the optimal solution to problem (1) - (5) should be both "better than optimal solution" and "interval feasible." Thus our algorithm presented here will always possess "better than optimality" criterion and proceed to restore "interval feasibility", similar to any dual algorithm which always has a "better than optimal solution" and approaches "primal feasibity."

III. A General Description of the Algorithm

In the first stage P_0 is solved.* Thus, if the "better than optimal" solution to P_0 is "interval feasible" it is also the optimal solution to P^* given by (1) - (5). If the "better than optimal" solution to P_0 is not "interval feasible" to P^* the algorithm proceeds as follows: Let x_{ij} be a value for which "interval feasibility" is violated. More specifically, suppose x_{ij} is "better than optimal" in the interval $\lambda_{ij}^{k-1} \leq x_{ij} < \lambda_{ij}^{k}$ and its associated cost parameter c_{ij}^{*} is not equal to c_{ij}^{k}. This condition leads to two branches (subproblem) as follows:

(i) In branch 1 the current c_{ij}^{*} is replaced by the "interval feasible" c_{ij}^{k} and an upper bound restriction in the form of $x_{ij} < \lambda_{ij}^{k}$ is implied. (Note that it is unnecessary to impose an explicit upper bound. If $x_{ij} \geq \lambda_{ij}^{k}$ at the optimum, then branch 1 is inferior to branch 2.)

(ii) In branch 2 a lower bound restriction of the form $x_{ij} \geq \lambda_{ij}^{k}$ is imposed and c_{ij}^{*} remains unchanged.

The two new transportation problems corresponding to (i) and (ii) are solved (see the solution procedure in Algorithm A1) to determine the lower bounds of Z for all interval-feasible better than optimal solutions in their respective subsets. If the optimal solution corresponding to any one subset is "interval feasible"

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*We have used the Srivinasan and Thompson [8] algorithm for solving P_0.

that basis from the two subsets but does not exclude any other interval feasible basis. The algorithm converges in a finite number of steps since the total number of bases for the constraint set given by (2) - (5) is finite and each iteration excludes at least one basis. Secondly the branching procedure results in a partition of the interval feasible basic feasible solutions in that subset, and thus the algorithm is expected to be efficient. Third and most importantly each subproblem is not completely resolved. Instead, we apply the "Operator Theory" [8] for the Transportation Problem which is utilized to generate the new solution for each subproblem with minor computational efforts.

Let x_{st} be one such x_{ij} where the interval feasibility is violated. (In Section IV we provide a heuristic for the choice of such x_{st}). First let us consider (i) where c^*_{st} is replaced by c^k_{st}. Let $c^k_{st} - c^*_{st} = \delta > 0$ (due to (6)) be the positive value to which the current cost c^*_{st} is to be increased. The optimal solution to this problem (where the entire data of the problem is unchanged except for the new cost $c^*_{st} + \delta$) is obtained by applying the "cell cost operator" [8,9] δC^+_{st} to the current problem P*.

This operation provides the new best optimal solution X and the optimal total cost Z for the revised problem.

Now let us consider the second subproblem (ii). Here the only change is the new lower bound imposed on x_{st}, i.e., $\lambda^k_{st} \leq x_{st}$. Let $x'_{st} = x_{st} - \lambda^k_{st}$ and $x'_{ij} = x_{ij}$ for all $(i,j) \in [I \times J] - \{(s,t)\}$ such that

(7) $$0 \leq x'_{ij} < \infty \qquad \text{for all } (i,j) \in [I \times J]$$

Substituting (7) in the current problem we have

(8) Minimize $$\sum_{j \in J} \sum_{i \in I} c^*_{ij} x'_{ij} + c^*_{st} \lambda^k_{st}$$

(9) Subject to $$\sum_{j \in J} x'_{ij} = a_i \qquad \text{for } i \in I \text{ and } i \neq s$$

(9a) $$\sum_{j \in J} x'_{sj} = a_s - \lambda^k_{st}$$

(10) $$\sum_{i \in I} x'_{ij} = b_j \qquad \text{for } j \in J \text{ and } j \neq t$$

(10a) $$\sum_{i \in I} x'_{it} = b_t - \lambda^k_{st}$$

(11) $$0 \leq x'_{ij} < \infty$$

The solution to this new problem (8) - (11) can be obtained by utilizing the "Rim Operator Theory" [8] where a cell rim operator δR^-_{st} is applied by equating $\delta = \lambda^k_{st}$ for one row s and a column t. Note, that in both subproblems we use the "operator theory of parametric programming for the transportation problem" [8,9] for computing the solutions to the branch problems efficiently.

IV. Branch Selection Procedure

In the process of searching for the best optimal solution to problem P^* we suggest two alternative selection rules: In the first alternative we select that branch which yields the lowest upper bound on the value of the objective function Z^*. This strategy is similar to the penalty proposed by Driebeek [3] and Tomlin [10] for solving integer programs. The above procedure tends to postpone the search in less promising branches by concentrating on branches which seem to yield a better solution value. Once our optimal "interval feasible" solution is found, branches yielding inferior upper bounds are eliminated and the number of steps required for solving P^* are reduced. The process of establishing upper bounds on the current subsets consists of two major stages. In the first stage a heuristic rule is used for selecting the variable to be branched upon in the next step. This stage is followed by a short search which constitutes the second stage. More specifically, let Ω represent the current set of cells $(i,j) \in [I \times J]$ where "interval feasibility" is violated. By introducing a cell (i,j) as the decision cell for branching, define Q_{ij} as the infeasibility index such that:

$$Q_{ij} = (c^k_{ij} - c^*_{ij})x_{ij} \tag{12}$$

and determine the most "interval infeasible" index:

$$Q_{max} = \max_{(i,j) \in \Omega} Q_{ij} = (c^k_{st} - c^*_{st})\, x_{st} \tag{13}$$

The variable x_{st} associated with Q_{max} is the variable used for branching in the next step. In branch (i) c^*_{st} is replaced with c^k_{st}, x_{st} may either stay in the basis or may become nonbasic. If x_{st} stays in the basis and the current solution is still optimal, the penalty P_{st} is equal to Q_{max} in (12), namely:

$$P_{st} = (c^k_{st} - c^*_{st})\, x_{st} \tag{14}$$

The heuristic given above takes negligible computational time. However, as a second alternative, if one is prepared for more computations, the following procedure provides a better heuristic. If x_{st} becomes non-basic, then a non-basic cell (s,ℓ) in the s^{th} row or a non-basic cell (k,t) in the t^{th} column replaces (s,t). Let T be the set of all such non-basic cells in the s^{th} row and t^{th} column and let α_s and β_t be the dual variables corresponding to row s and column t, respectively. Let $(u,v) \in T$. Then we could compute a penalty for each $(u,v) \in T$ given by $(c^*_{uv} - \alpha_u - \beta_v)\, x_{st}$ and choose the minimum of all such penalties corresponding to every $(u,v) \in T$. Comparing this with the earlier one given in (14) the cell to branch form can be ascertained. It is to be recognized that this procedure involves additional computations and one needs computational experiments for the choice of the best heuristic. Currently, we use the heuristic based on equation (14) and leave the rest for future testing.

Algorithm A1 given below summarizes the above discussion for this quantity discounted non-convex transportation problem.

<u>Algorithm A1</u>. Algorithm for finding the optimal solution to the "interval feasible" all unit quantity discounted non-convex transportation problem (1) - (6).

Initialization:

<u>Step 1</u>. Set up the problem P as presented in (1) - (4) with c^*_{ij} in (1) replaced by c^r_{ij} the smallest cost as given in the r^{th} interval $\lambda^{r-1}_{ij} \leq x_{ij} < \lambda^r_{ij} \leq \infty$. Let P_1 denote this problem above, and $\Omega_1 = \emptyset$ denote the set of cells (i,j) that are required to be <u>included</u> in the "final optimal interval feasible" solution. Let $\Psi_1 = \emptyset$ be the set of all cells $(i,j) \in [I \times J]$ that should be excluded from the "current optimal solution," where the interval in which the basic x_{ij} lies is the one corresponding to the

current lowest cost c_{ij}^*. Let $X_1 = \{x_{ij}\}$ be the optimum solution to P_1 with basis B_1 and the current optimal cost Z_1. (In this step we solve the transportation problem (1)-(4) and obtain the solution with $c_{ij}^* = c_{ij}^r$ for $\forall$ (i,j)). Let $S = \{1\}$ denote the set of problems under consideration and let $m = 1$ denote the total number of problems generated thus far.

Step 2. Choose problem P_k for which Z_k (Z_k is either the current value of the objective function or the current upper bound on that value depending on the branch selection rule used) is the smallest for $k \in S$. Let this problem have $c_{ij}^* = c_{ij}^u$. If B_k is interval feasible, i.e. satisfies the constraint set (5) for every $(i,j) \in B_k$, go to (8). Otherwise go to (3).

Step 3. Find the set of cells Λ where the cells (i,j) in basis B_k violate constraint set (5). Since P_k has costs $c_{ij}^* = c_{ij}^u$ (the lowest cost) the x_{ij} for $(i,j) \in B_k$ will satisfy (5) if $\lambda_{ij}^{u-1} \leq x_{ij} < \lambda_{ij}^u$ where u is the one determined by c_{ij}^* for each (i,j). (Note u may be different for different (i,j)'s). For each $(i,j) \in \Lambda$ find the infeasibility index Q_{ij} given by (12) and select the variable to be branched from as in (13). Let the cell corresponding to this variable be (s,t).

Step 4. Define P_{m+1} as the problem obtained from P_k by increasing the cost of c_{ij}^* to c_{st}^u, i.e. $\Omega_{m+1} = \Omega_k \cup \{(s,t)\}$ and let $\Psi_{m+1} = \Psi_k$. The problem P_{m+1} and its solution can be obtained from problem P_k by applying "cell cost operator" δC_{st}^+ (see Srinivasan and Thompson [8]) to P_k where $\delta = (c_{st}^u - c_{st}^*) > 0$. (Here c_{st}^u is the new cost due to the fact that $\lambda_{st}^{u-1} \leq x_{st} < \lambda_{st}^u$ and c_{st}^* is that value which was used as cost for cell (s,t) in problem P_k). Find the new basis B_{m+1} and Z_{m+1} also as per δC^+_{st} cost operator [8].

Step 5. Define P_{m+2} as the problem obtained from P_k by imposing a lower bounded constraint $x_{st} \geq \lambda^u_{st}$ to the current optimal basis B_k. Set $\Psi_{m+2} = \Psi_k \cup \{(s,t)\}$ and $\Omega_{m+2} = \Omega_k$. This solution is obtained by solving the same problem P_k except that the rim conditions (row and column totals) for s^{th} row and t^{th} column will be decreased by a value of λ^u_{st}. The solution for P_{m+2} is obtained by applying the cell rim operator δR^-_{st} [8]. If the new basis B_{m+2} still contains cell (s,t), change x_{st} value to $x'_{st} = x_{st} - \lambda^u_{st}$. The new optimal solution Z_{m+2} = the optimal solution to problem $P_{m+2} + c^*_{st}\lambda^k_{st}$ where c^*_{st} is the same current cost used in problem P_k for the cell (s,t).

Step 6. Denote the basic optimal solutions to P_{m+1} and P_{m+2} obtained from P_k as X_{m+1} and X_{m+2} with bases B_{m+1} and B_{m+2} respectively. Let Z_{m+1} and Z_{m+2} be the corresponding costs for problems P_{m+1} and P_{m+2}.

Step 7. Drop k from the set S and include (m+1) and (m+2) in S. Redefine m as (m+2) and go to (2).

Step 8. The optimal solution to the quantity discounted (interval feasible) final problem (1) - (6) is given by X_k and B_k with associated cost Z_k. Stop.

From a computational viewpoint it is unnecessary to store all the problems P_k for every $k \in S$. It is enough if the sets Ω_k and Ψ_k are stored for $k \in S$. With the original problem P and the cells in Ω_k and Ψ_k we know the current P_k. It is to be noted that in Step 5, while decreasing the row and column totals by λ^u_{st}, either one of the resultant row or column totals may become negative. This leads to an infeasible subproblem. In such an eventuality drop that branch from the list of active branches. Also in Step 7 drop k from the set S and include only (m+1) to the set S. Next, redefine m as (m+1) and go to step 2.

V. Illustration

In this section we will provide a simple illustration of the above algorithm. Consider the following three sources (m = 3), four destination (n = 4) transportation problem, which has quantity discounted transportation costs. The following Table 1 provides the data of the problem for different c^*_{ij} for the three levels of quantity discounts. We provide an example where cell upper bounds are also imposed.

Step 1. Table 2 provides optimal solution for the initial problem. In each cell (i,j) the value of x_{ij} is written in the northeast corner. If a cell is the basis then the corresponding c_{ij} is circled. Those celles which have x_{ij} as their upper bound have their corresponding c_{ij} underlined. Those at zero levels are left out without any entry posted in the northwest corner. The optimal dual variables u_i's for rows and v_j's for columns are given in southeast corner (N denotes large positive #). It is easy to check that the

TABLE 1

→ Destination ↓ Source	1	2	3	4	Warehouse Capacity
1	3 $[20 \leq x_{11} < \infty]$ 4 $[10 \leq x_{11} < 20]$ 5 $[0 \leq x_{11} < 10]$	6 $[10 \leq x_{12} \leq 15]$ 7 $[5 \leq x_{12} < 10]$ 8 $[0 \leq x_{12} < 5]$	3 $[27 \leq x_{13} < 60]$ 4 $[15 \leq x_{13} < 27]$ 5 $[5 \leq x_{13} < 15]$	One price bracket 4 Upperbound 30	80
2	Upperbound 20 (one price bracket) 6	5 $[65 \leq x_{22} < \infty]$ 6 $[20 \leq x_{22} < 65]$ 8 $[0 \leq x_{22} < 30]$	8 $[10 \leq x_{23} \leq 25]$ 9 $[5 \leq x_{23} < 10]$ 10 $[0 \leq x_{23} < 5]$	15 $[30 \leq x_{24} < \infty]$ 16 $[12 \leq x_{24} < 30]$ 17 $[0 \leq x_{24} < 15]$	90
	1 $[27 \leq x_{31} \leq \infty]$ 2 $[20 \leq x_{31} \leq 27]$ 3 $[0 \leq x_{31} \leq 30]$	3 $[60 \leq x_{32} < \infty]$ 4 $[30 \leq x_{32} < 60]$ 5 $[0 \leq x_{32} < 30]$	10 $[20 \leq x_{33} \leq 30]$ 11 $[10 \leq x_{33} < 20]$ 12 $[0 \leq x_{33} < 10]$	5 $[30 \leq x_{34} \leq 50]$ 6 $[20 \leq x_{34} < 30]$ 7 $[0 \leq x_{34} < 20]$	55
Market Demand	70	60	35	60	

TABLE 2

$v_j \rightarrow$ $u_i \downarrow$	3	0	3	7	Warehouse Capacity a_i
0	25 3 N	6 15	25 3 60	30 <u>4</u> 30	80
	20 <u>6</u> 20	60 5 75	10 8 25	15 N	90
	25 1 60	3 N	10 30	30 5 50	55
Market Demands $\rightarrow$	70	60	35	60	

X_1 is given above; $Z_1 = 945$; $S = \{1\}$ $m = 1$

solution is optimal for the costs given. Notice that cells (1,3), (2,2) and (3,1) are not <u>interval feasible</u>.

<u>Step 2</u>. Choose P_1; $Z_1 = 945$; B_1 is not interval feasible. Hence go to Step 3.

<u>Step 3</u>. The set of interval infeasible cells

$\Lambda = \{(1,3), (2,2), (3,1)\}$.

Infeasibility indices are

$$Q_{1,3} = (4-3) * 25 = 25$$
$$Q_{2,2} = (6-5) * 60 = 60 \quad \leftarrow$$
$$Q_{3,1} = (2-1) * 25 = 25$$

Thus the variable to branch from is x_{22}.

<u>Step 4</u>. P_2 is the problem where cell (2,2) is contained in the basis and the cost is changed so that it becomes interval feasible.

$\Omega_2 = \Phi \cup \{(2,2)\}$; $\Psi_2 = \emptyset$. Problem P_2's solution is obtained by applying cell cost operator [8], δC^+_{22} where $\delta = (6-5) = 1$.

Since the current problem is basis preserving, following [8], the new solution becomes $X^+ = X$ so that x_{ij} values are not altered, and

$$Z^+ = Z + \delta\ x_{22} = 945 + 1 * 60 = 1005$$

Form the set $\Omega = B - \{(2,2)\} = [(1,1), (1,3), (2,3), (3,1), (3,4)]$.

Following the notation of [8]

$$I_p = I_2 = \{1,2,3\};\ I_q = \emptyset$$
$$J_p = \{1,3,4\};\ J_q = \{2\}.$$

The maximum extent μ^+ to which c_{22} can be increased without changing the basis structure ("Basis preserving") will be as in equation (35) of [8].

$$\mu^+ = \text{Min} \begin{cases} (c_{ij} - u_i - v_j) \text{ for } (i,j) \in [(I_p \times J_q) \cap LB] \\ \\ (u_i + v_j - c_{ij}) \text{ for } (i,j) \in [(I_q \times J_p) \cap UB]. \end{cases}$$

Now the $(i,j) \in [(I_p \times J_q) \cap LB]$ are cells (1,2) and (3,2)

and $(i,j) \in [(I_q \times J_p) \cap UB] = \emptyset$

So that $\mu^+ = \text{Min}\,\{(6-0-0);\ (3+2-0)\}$

$= 5$ occurring at cell (3,2).

Thus the basis remains unchanged. The only change occurs in the optimal duals as given below. (Refer to equation (34) of [8])

$$u_i^+ = \begin{cases} u_i + \delta & \text{for } i \in I_p \\ \\ u_i & \text{for } i \in I_q \end{cases}$$

$$v_j^+ = \begin{cases} v_j - \delta & \text{for } j \in J_p \\ \\ v_j & \text{for } j \in J_q \end{cases}$$

Thus $u_1^+ = 1$; $u_2^+ = 6$; $u_3^+ = -1$

and $v_1^+ = 2$; $v_2^+ = 0$; $v_3^+ = 2$ and $v_4^+ = 6$.

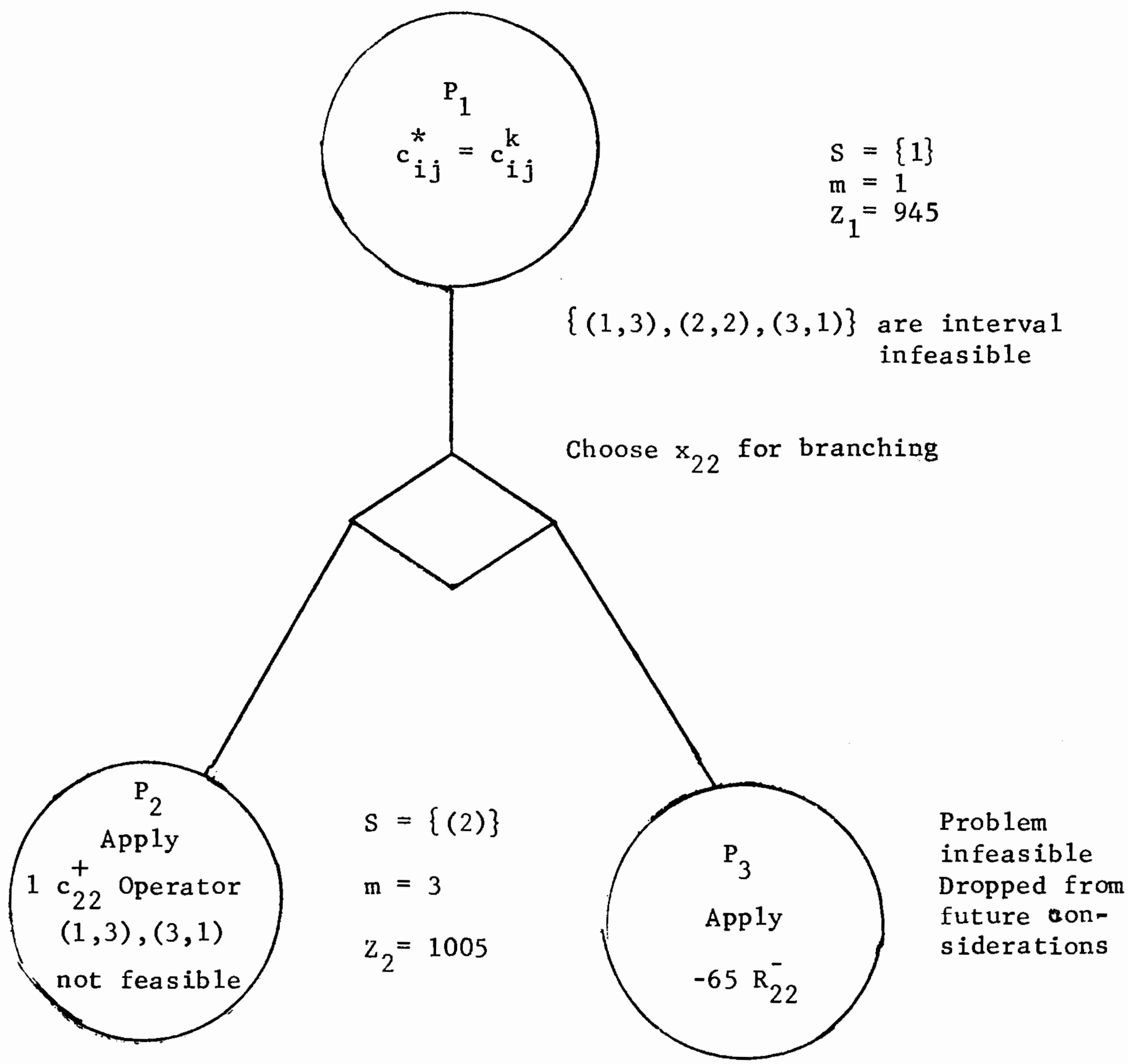
The new tableau for P_2 is presented in Table 3 with $Z_2 = 1005$.

TABLE 3

v_j → u_i ↓	2	0	2	6	Warehouse Capacity a_i ↓
1	25 3 N	6 15	25 3 60	30 4 30	80
6	20 6 20	60 5 75	10 8 25	15 N	90
-1	25 1 60	3 N	10 30	30 5 50	55
Market Demands →	70	60	35	60	

Step 5. P_3 is a new branch created by imposing the bound constraint $65 \leq x_{22} < \infty$ on P_1. This guarantees that (2,2) will be nonbasic. Now a_2 becomes 80 - 65 with $b_2 = 60 - 65 < 0$. Since demand must be nonnegative, this subproblem is infeasible. Thus, this branch and its associated subbranches are deleted from the list of active branches. Figure 2 summarizes the operations up to this point.

Figure 2



Step 6. P_2, P_3 are given. P_3 is discarded due to infeasibility. X_2 is given in Table 3. $Z_2 = 1005$;

Step 7. (1) is dropped from S and (2) is included in S. $m \Rightarrow m+1 = 2$. Go to Step 2.

Step 2. P_2 is not interval feasible. Go to Step (3).

Step 3. $\Lambda = \{(1,3)(3,1)\}$. $Q_{13} = Q_{35} = 25$. Arbitrarily choose x_{13} to branch from.

Step 4. P_3 is obtained from P_2 by increasing the cost of c_{13} to the interval feasible cost. Following similar cost operation, $1 \cdot c_{13}^{+}$ application, we see the operation is basis preserving. The new cost $Z_3 = 1005 + 1 * 25 = 1030$. The optimal duals change. The optimal primals do not change. The resultant optimal tableau is given in Table 4.

Step 5. P_4 is obtained from P_2 by imposing the lower bounded constraint $27 \leq x_{13} < \infty$. This creates a new $a_1 = 80 - 27 = 53$ and a new $b_3 = 35 - 27 = 8$. Now we apply cell rim operator δR_{13}^{-} with $\delta = 27$. From Thm. 2 of [8] the maximum extent μ^{-} that this operator can be applied to be basis preserving is 25. But since $\delta = 27$ we follow the method provided by [8]. The new Tableau is given in Table 5 with the new optimal primal and dual solutions

TABLE 4

$v_j \rightarrow$ / $u_i \downarrow$	1	0	2	5	Warehouse Capacity a_i
2	25 3 N	6 15	25 4 30	30 <u>4</u> 30	80
6	20 <u>6</u> 20	60 6 75	10 8 25	15 N	90
0	25 1 60	3 N	10 30	30 5 50	55
Market Demands $\rightarrow$	70	60	35	60	

TABLE 5

$u_i \downarrow$ \ $v_j \rightarrow$	5	0	2	9	Warehouse Capacity a_i
-2	23 3 $\boxed{N}$	6 $\boxed{15}$	3 $\boxed{30}$	30 $\underline{4}$ $\boxed{30}$	80
6	20 $\underline{6}$ $\boxed{20}$	60 6 $\boxed{75}$	8 8 $\boxed{25}$	2 15 $\boxed{N}$	90
-4	27 1 $\boxed{60}$	3 $\boxed{N}$	10 $\boxed{30}$	28 5 $\boxed{50}$	55
Market Demands $\rightarrow$	70	60	8	60	

Note that cell (1,3) has left the basis so that cell (2,4) is in the basis.

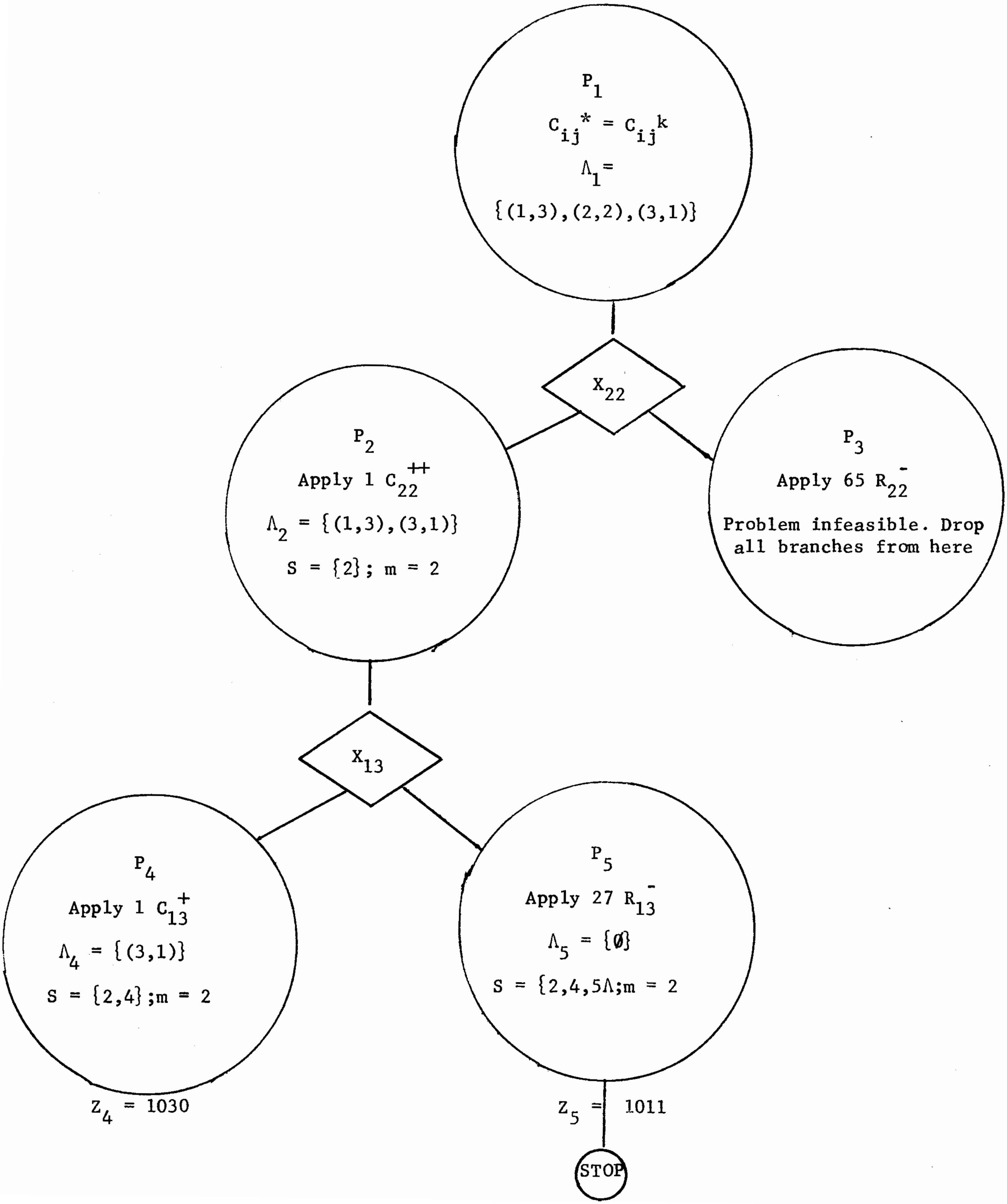
The optimal cost is Z = 930 + 81 = 1011.

Since this problem has the least total costs in all pendent branches and, nevertheless, it is interval feasible, it is the optimum solution.

Note that the cell (3,1) which was interval infeasible earlier became automatically feasible when the basis change occurred.

The results of branching and bounding are given in Figure 3.

Figure 3



P1
Cij* = Cij^k
Λ1 =
{(1,3),(2,2),(3,1)}
X22
P2
Apply 1 C22++
Λ2 = {(1,3),(3,1)}
S = {2}; m = 2
P3
Apply 65 R22-
Problem infeasible. Drop all branches from here
X13
P4
Apply 1 C13+
Λ4 = {(3,1)}
S = {2,4};m = 2
Z4 = 1030
P5
Apply 27 R13-
Λ5 = {∅}
S = {2,4,5Λ;m = 2
Z5 = 1011
STOP

VI. Extensions

(i) The Fixed Charge Problem [1]

In this section we outline an algorithm for solving the following fixed charge (transportation) problem:

$$\text{(15)} \qquad \min Z = \sum_i \sum_j c_{ij} x_{ij} + \sum_k \sum_l f_{kl} y_{kl}$$

$$\text{(16)} \qquad \text{s.t.} \quad \sum_j x_{ij} = a_i \qquad \text{for } i \in I$$

$$\text{(17)} \qquad \sum_i x_{ij} = b_i \qquad \text{for } j \in J$$

$$\text{(18)} \qquad y_{kl} = \begin{cases} 0 & \text{if } x_{kl} = 0 \\ 1 & \text{if } x_{kl} \geq 1 \end{cases}$$

where $k \in K$, $l \in L$, $i \in I$, $K \subseteq I$, $L \subseteq J$, and $x_{ij} \geq 0$.

The method used for solving the fixed charge problem is identical in principle to the one devised in Section IV for the all unit quantity discount type problem. Here, again, we start by solving a relaxed problem, the solution of which is interval infeasible, then, branch and bound procedure is applied to retain feasibility. In the following statements the algorithm is summarized:

Step 1: Let $y_{kl} = 0$ for all $k \in K$ and $l \in L$ and solve the relaxed transportation problem in (15) - (18).

Step 2: If the solution is interval infeasible, i.e., $x_{kl} > 0$, $y_{kl} = 0$, select the one variable among all interval infeasible variables for which f_{kl} is the largest.

Step 3: Branch from the variable selected in step 2 into two new problems: in branch 1 increase c_{kl} to an arbitrary large number making cell (k,l) an inadmissible cell and solve the new problem using cost operator. In branch 2 introduce a lower bound $x_{kl} \geq 1$ and increase the current value of the objective function by f_{kl}.

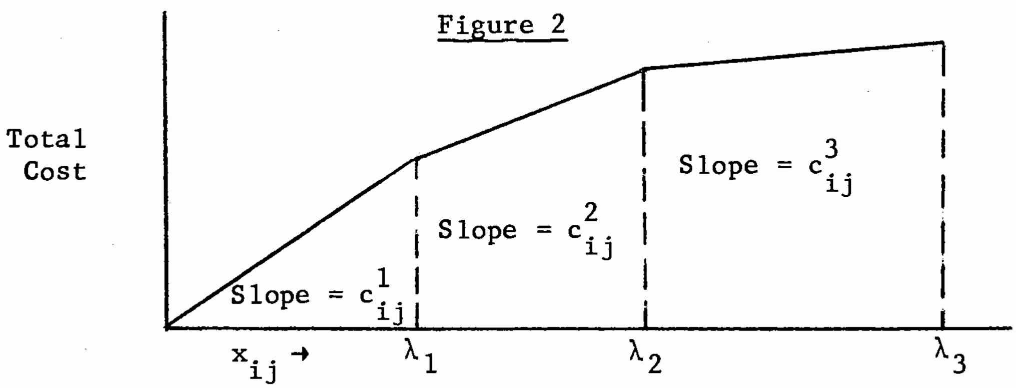
Step 4: Select the one branch with the smallest objective function value among all active branches.

Step 5: If the branch selected in step 4 is interval feasible, then stop. This is the optimum. Otherwise go to step 2.

(ii) The Incremental Quantity Discount Problem

The nonconvex cost structure of this problem is shown in Figure 2. To accomodate such a framework, x_{ij}^k must be $\geq \lambda_k$ before x_{ij}^{k+1} (the amount that can be shipped from i to j at a reduced cost c_{ij}^{k+1}) can be positive, etc.

Figure 2



The problem can be formulated as follows:

(19) $$\min \sum_i \sum_j c_{ij}^k x_{ij} + \sum_i \sum_j f_{ij}^k y_{ij}^k$$

(20) $$\text{s.t.} \quad \sum_j x_{ij} = a_i \qquad \text{for } i \in I$$

(21) $$\sum_i x_{ij} = b_j \qquad \text{for } j \in J$$

(22)

$$c_{ij}^{k} = \begin{cases} c_{ij}^{1} & \text{if } 0 = \lambda_{ij}^{0} \le x_{ij} < \lambda_{ij}^{1} \\ c_{ij}^{2} & \text{if } \lambda_{ij}^{1} \le x_{ij} < \lambda_{ij}^{2} \\ \cdot & \quad \cdot \quad \cdot \quad \cdot \\ \cdot & \quad \cdot \quad \cdot \quad \cdot \\ \cdot & \quad \cdot \quad \cdot \quad \cdot \\ c_{ij}^{r} & \text{if } \lambda_{ij}^{r-1} \le x_{ij} < \lambda_{ij}^{r} \le \infty \end{cases}$$

(23)

$$y_{ij}^{k} = \begin{cases} 1 & \text{if } \lambda_{ij}^{k-1} \le x_{ij}^{k} < \lambda_{ij}^{k} \\ 0 & \text{otherwise} \end{cases}$$

and $x_{ij} \ge 0$ for all $i \in I$ and $j \in J$

(24) $f_{ij}^{k} = [\sum_{v=1}^{k-1} c_{ij}^{v}(\lambda_{ij}^{v} - \lambda_{ij}^{v-1})] - c_{ij}^{k}\, \lambda_{ij}^{k-1}$ for all $k = 1,\ldots,r$.

A close examination of the above formulation reveals that the incremental quantity discount problem can be formulated and solved as a generalized fixed charge problem. In the following statements we outline the algorithm:

<u>Step 1</u>: Let $c_{ij}^{k} = c_{ij}$ and $y_{ij} = 0$ for all $i \in I$ and $j \in J$ and solve the transportation problem in (19) - (24).

<u>Step 2</u>: If the solution is interval infeasible, i.e., there is at least one variable x_{ij} such that $\lambda_{ij}^{k-1} \le x_{ij} < \lambda_{ij}^{k}$ and $y_{ij}^{k} = 0$ select the one variable among all interval infeasible variables for which f_{ij}^{k} is the largest.

<u>Step 3</u>: Branch from the variable selected in step 2 into two new problems: in branch 1 make $y_{ij}^{k} = 1$ (thereby increasing the current value of the

objective function by f_{ij}^{k} and introduce a lower bound $x_{ij} \geq \lambda_{ij}^{k}$. In branch 2 increase c_{ij}^{k} to c_{ij}^{k-1} and solve the problem in (19)-(24) using cost operator.

Step 4: Select the branch with the smallest objective function value from among all active branches.

Step 5: If the solution of the branch selected in step 4 is interval feasible--stop--this is the optimum. Otherwise go to step 2.

Discussion

In the above examples we showed how one can employ the branch and bound procedure for solving the piecewise linear programming problem. The algorithm outlined above yields an efficient procedure in cases where the linear program has a special structure (as in our examples) because of the important fact that the special structure is retained throughout the process. Our algorithm differs from the one suggested by Falk and Soland [4] whose approach may be characterized by the use of a convex combination of points to approximate the value of c_{ij}^{k} over a given range whereas our approach is characterized by the use of the marginal cost c_{ij}^{k} at a given range. Falk and Soland's [4] approach should, then, be considered as a more general framework for solving nonconvex programs, while our approach is specialized to the case where the nonconvex program is, nevertheless, piecewise linear.

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