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SYMMETRY EXTENSIONS OF "NEUTRALITY"

II: PARTIAL ORDERING OF DICTIONARIES

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SYMMETRY EXTENSIONS OF "NEUTRALITY" II: PARTIAL ORDERING OF DICTIONARIES

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ABSTRACT. For $n \geq 3$ candidates, a system voting vector $\mathbf{W}^n$ specifies the positional voting method assigned to each of the $2^n - (n+1)$ subsets of two or more candidates. While most system voting vectors need not admit any relationships among the election rankings; the ones that do are characterized here. The characterization is based on a particular geometric structure (an algebraic variety) that is described in detail and then used to define a partial ordering "$\blacktriangleleft$" among system voting vectors. The impact of the partial ordering is that if $\mathbf{W}_1^n \blacktriangleleft \mathbf{W}_2^n$, then $\mathbf{W}_2^n$ admits more kinds of (single profile) voting paradoxes than $\mathbf{W}_1^n$. Therefore the partial ordering provides a powerful, computationally feasible way to compare system voting vectors. The basic ideas are illustrated with examples that completely describe the partial ordering for $n = 3$ and $n = 4$ candidates.

1. INTRODUCTION

This is the second of a three part study devoted toward completely characterizing all possible election outcomes over all subsets of candidates that can occur for all possible choices of positional election procedures with any profile. In the first part [4] it is shown that when the effects of neutrality are examined over families of subsets of candidates, new kinds of "super-symmetries" emerge. The importance of these symmetries is that they create the election relationships among the different subsets of candidates. So, by using these symmetries to assign certain positional voting methods to specified subsets of candidates, we can ensure the existence of election relationships. (We assume, of course, that the same voters sincerely rank the candidates of the various subsets.)

To underscore this assertion, in [4] I show that the well known relationships between the Borda Count election for n candidates and the majority vote elections for the $\binom{n}{2}$ pairs of candidates extend to large numbers of other classes of positional voting methods. In fact, for any s satisfying $2 \leq s < n$ and for *any* positional voting method for s-candidate subsets, there is an unique positional n-candidate voting method which permits relationships among the election rankings of n candidates and the $\binom{n}{s}$ subsets of s candidates. In the third part of this study [5], the technical material needed to compute all possible election outcomes for any choice of a system voting vector and to prove the major conclusions of this study are developed.

In this current article, the emphasis is

1. to describe new types of relationship that emerge from positional voting methods,
2. to characterize all possible system voting vectors that ensure election relationships among the different subsets of candidates,

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3. to describe the (stratified) geometry formed by the system voting vectors, and
4. to use the geometric stratification to define a partial ordering for positional voting methods – a partial ordering that determines which systems admit more kinds and numbers of voting paradoxes.

Most of the definitions used in this article are given in [4].

One goal of this article is to answer and/or extend comments in [S].[1] As an example, a natural extension of a classical problem from choice theory is to determine whether there exists a positional procedure so that a specified family of subsets of candidates must admit election relationships. For instance, no relationship can exist among the election rankings of the two subsets defining the family $\mathcal{F} = \{\{c_1, c_2, c_3\}, \{c_4, c_5, c_6\}\}$ because neither set has anything to do with the other. On the other hand, the significant overlap of candidates among the subsets of $\mathcal{F}^* = \{\{c_1, c_2, c_3\}, \{c_2, c_3, c_4\}, \{c_3, c_4, c_1\}\}$ suggests that some sort of relationship must hold. To illustrate, if a given profile results in the election rankings of $c_1 \succ c_2 \succ c_3$ and $c_2 \succ c_3 \succ c_4$, then it would seem to be impossible for the ranking of the third set to be the reversed $c_4 \succ c_3 \succ c_1$; but this can happen. Indeed, no matter what positional voting systems are used, the family $\mathcal{F}^*$ need not admit any relationship whatsoever among the three election rankings. This means that we can arbitrarily choose a ranking for each of the three subsets and then, for any choices of positional voting methods, there exists a profile so that each chosen ranking is the election ranking.

As demonstrated, it is not obvious whether a given family $\mathcal{F}$ must admit election relationships. However, we now know the minimal conditions for a family to admit relationships; the subsets of $\mathcal{F}$ must satisfy the set theoretic conditions developed in [3]. But, as emphasized in [3], even if $\mathcal{F}$ satisfies these minimal conditions, it can be that *election relationships among the subsets of $\mathcal{F}$ emerge iff each subset of candidates is tallied with the Borda Count (BC).* So, what can be said about a family $\mathcal{F}$ if the BC is not used? What stronger conditions ensure that $\mathcal{F}$ is relationship admitting for non-BC positional voting vectors? This problem is completely solved here.

By characterizing the relationship admitting families $\mathcal{F}$ (Sect. 2), we complete the goal of (1). (A surprise is that the derived conditions are computed with elementary algebra.) Then, armed with this theory and the results from [4], the mission of part (2) is fulfilled; all system voting vectors that must admit election relationships – either good or bad – are characterized.

A main result from [S] asserts that the set of relationship admitting system voting vectors form a lower dimensional (algebraic) subset, α^n, in the space of all system voting vectors. It is further reported in [S] that α^n has a stratified geometric structure. The appropriate mathematical tools to describe the algebraic variety α^n is not developed in [S], so the assertion is not proved. But these required tools are based on ideas needed to solve parts (1) and (2), so the description of the stratified structure of α^n along with a discussion of implications of the geometry is in Section 3. As suggested by the constructions in [4] and in this paper, the analysis involves the actions of various symmetry groups.[2] To keep the discussion from

[1] Following the lead of [4], reference [S] represents the three references [1,2,3].

[2] The underlying mathematical structure depends on symmetry considerations, including groups and subgroups of wreath products, that are not commonly used by researchers in social choice. To make this paper accessible to a wider audience, my proofs and discussions are based on more standard arguments. However, I strongly recommend that readers familiar with wreath products arguments reinterpret the assertions in this series in the more abstract setting. In this way, insight is gained about how and why these results extend to more general choice settings.

becoming overly abstract, when a branch of α^n is described, the corresponding branch of the simpler α^4 is described in detail. By doing so, we discover that even though α^4 has the simplest non-trivial structure of all α^n's, it is surprisingly complicated.

The importance of the stratified structure for positional voting methods is that the geometry captures the intricate connections among system voting vectors. Indeed, this geometry is used to define a partial ordering among the system voting vectors that characterizes which methods provide more relief from election paradoxes. Of particular importance is that this partial ordering is not just a theoretical existence assertion; with the geometric construction developed in Section 3 any set of system voting vectors can be compared! Examples showing how to do this are found at the end of Section 4.

2. $\mathcal{F}$ – COMPOSITE VECTORS

The theorems in Section 2 of [4] come close to characterizing all system vectors that admit election relationships. The remaining situations are motivated by examining the subtle BC relationships among, say, the BC rankings of the $\binom{4}{3} = 4$ three-candidate subsets with $n = 4$ candidates. (See [2].) As emphasized in [4], the origin of these desirable BC relationships is the BC symmetry acquired by virtue of the BC being the aggregation of the pairwise contests. To underscore this fact, the notation adopted in [4] for the BC vector $\overrightarrow{B}^n = (n-1, n-2, \ldots, 0)$ is $\overrightarrow{w}^n((1,0))$. (See [4] for a precise definition.)

To exploit the fact that the BC election tally for a set of candidates is uniquely determined by the sums of the pairwise majority vote outcomes, consider what happens if c_1 beats c_2 in a pairwise election with the vote $m_1 > m_2$. The values m_1 and m_2 become key terms to compute the BC tally for every subset of candidates that includes the pair $\{c_1, c_2\}$! (See [4].) Because the same values reappear in the computations of different election outcomes, the tallies of these different subsets of candidates must be related. In particular, with enough overlap (of pairs) of candidates among the $\mathcal{F}$ subsets, we must expect the pairwise vote tallies to force relationships among the outcomes of the different subsets; consequently, election relationships must emerge.

To understand the ideas, observe that if $\mathcal{F}_3$ represents the four sets of three candidates constructed with $n = 4$ candidates, then each pair of candidates is in half of the $\mathcal{F}_3$ subsets. Are the BC rankings of these subsets related? They are ([S]), but the analysis is more delicate than suggested by the example. After all, $\mathcal{F}^*$ from the introductory section does not admit relationships even though some pairs of candidates belong to $\frac{2}{3}$ of the $\mathcal{F}^*$ subsets.

As a tool to develop intuition about when relationships occur, observe that the voting vector $\overrightarrow{w}^k \doteq (w_1, w_2, \ldots, w_k)$ requires each voter to divide $\sum_{j=1}^{k} w_j$ points among the k-candidates in a specified fashion. As ν voters spread $\nu \sum_{j=1}^{k} w_j$ points among the candidates, if we know the vote totals for $k-1$ of the candidates, we know the last candidate's vote total – she gets the rest of the votes. So, in a pairwise vote with the voting vector $(1, 0)$; if one candidate gets m_1 votes, then the other gets $m_2 = \nu - m_1$ votes. Thus the election outcomes of the $\binom{n}{2}$ pairs of candidates are completely described with the $\binom{n}{2}$ variables where only the vote of a designated candidate in each pairwise contest is specified.

Applying the same reduction to the family $\mathcal{F}_3$ consisting of the $\binom{4}{3}$ subsets of three candidates, observe that the BC election outcome for two candidates of a subset determines the BC outcome for the last candidate. Thus the BC outcome for the 4 three-candidate subsets involves eight equations. But the BC outcome for each equation is uniquely determined by the values of the $\binom{4}{2} = 6$ variables coming from the $\binom{4}{2} = 6$ pairwise elections. This creates an overdetermined system of *eight equations with six variables, so the values of at least two equations are uniquely determined by the values of the other six.* Consequently, there must exist BC election rankings among these four three-candidate subsets; these new relationships are in addition to the known ones whereby the BC ranking if each set of three (or more) candidates must be related to the rankings of the pairs!

Example. Two examples are offered to further illustrate the equation counting approach. The first one explains the relationship of the BC election with that of the pairs. With the BC election of $n \geq 3$ candidates, the $\binom{n}{2}$ pairwise elections define the $\binom{n}{2}$ basic variables and $\binom{n}{2}$ equations. The BC election for n candidates defines $n-1$ equations. Thus, the family of $\binom{n}{2}$ pairs and the set of n candidates defines the overdetermined system of $n-1+\binom{n}{2}$ equations in $\binom{n}{2}$ variables. The overdetermined system forces BC election relationships which are, of course, the assertions that a Condorcet winner cannot be BC bottom-ranked, that she is BC ranked above a Condorcet loser, etc.

For the family $\mathcal{F}^* = \{\{c_1, c_2, c_3\}, \{c_2, c_3, c_4\}, \{c_3, c_4, c_1\}\}$ from the introductory section, the BC tallies for the three $\mathcal{F}^*$ subsets requires six equations. The number of independent variables – the number of pairs of candidates in the three subsets – also is six. There is no reason to expect the outcomes of six equations in six unknowns to be related, and, in general, they are not. This is why $\mathcal{F}^*$ does not admit election relationships.

Suppose $\mathcal{F}^*$ is augmented by adding $\{c_1, c_2, c_3, c_4\}$ to $\mathcal{F}^*$. No new pairs of candidates are involved, so the augmented family defines nine equations in six variables. The overdetermined system forces election relationships to occur. □

These ideas generalize. If the BC admits election relationships because it is the aggregation of the pairwise votes, then election relationships should emerge whenever a positional voting method aggregates the votes of simpler methods. Because $\vec{w}^k(\vec{w}^s)$ is the aggregate of $\vec{w}^s$ elections [4], election relationships should emerge when $\vec{w}^k(\vec{w}^s)$ is used with "enough" of the k-candidate subsets in $\mathcal{F}$ that contain the same s-tuples of candidates. This is because the $\vec{w}^s$ tallies should dictate the tallies of the different $\vec{w}^k(\vec{w}^s)$ elections. The answer to "What is 'enough'?" follows from the equation and variable counting exercise.

To illustrate what to expect, for $n = 5$ candidates let $\mathcal{F}$ be the $\binom{5}{4}$ four-candidate subsets. Each four-candidate election is defined by three equations, so the election outcomes of the $\binom{5}{4}$ sets requires $3\binom{5}{4} = 15$ equations. If each election is based on the voting vector $\vec{w}^4(\vec{w}^3)$ where $\vec{w}^3 \neq \vec{w}^3((1, 0))$, then the four-candidate tallies are uniquely determined by the $\vec{w}^3$ tallies of the $\binom{5}{3}$ three-candidate elections. This means that the three-candidate election outcomes are the independent variables for the four-candidate elections. To count the number of independent variables, each three-candidate set is determined by two equations where the values of these equations become the relevant independent variables. As there are $\binom{5}{3} = 10$ three-candidate subsets, there are 20 variables. Because there is no reason to expect relationships for a system of 15 equations in 20 variables, there is no reason to expect election relationships.[3] To force election relationships, it appears we need more subsets of

[3]The caution is due to the concern deferred to later in this section that 15 equations in 20 variables *can* admit election relationships if the equations are linearly dependent. A related but more general issue is whether election relationships can be introduced though mechanisms other than this aggregation process. The surprising fact, as proved in this three part study, is that the equation counting approach is the only method!

candidates (more equations) that *do not involve new triplets of candidates* (i.e., that do not introduce more independent variables).

Another way to obtain more equations (subsets of candidates) is to introduce additional candidates. So, consider the family of $\binom{6}{4} = 15$ four-candidate subsets constructed from $n = 6$ candidates. If the voting vector $\vec{w}^4(\vec{w}^3)$ is used with each set, then the election outcomes for these 15 sets are determined by 45 equations with $2\binom{6}{3} = 40$ variables. The resulting overdetermined system requires the values of at least five equations to be expressed in terms of the 40 others, so election relationships must exist. Of course, in order for the five relationships to occur, all of the equations must use the same three-candidate independent variables. Namely, the $\binom{6}{4}$ subsets must be assigned the same voting vector $\vec{w}^4(\vec{w}^3)$.

Relationships for the above family still emerge even should one four-candidate subset deviate from the pattern by using a different voting vector. Here the system remains overdetermined with 42 equations in 40 variables, so there exist at least two possible election relationships. On the other hand, if two subsets are assigned different voting vectors, the resulting system is not overdetermined (39 equations in 40 variables), so it is not clear whether any election relationships remain.

It is important to note that the existence of election relationships among the 15 four-candidate subsets does *not* require the three-candidate elections to be tallied with $\vec{w}^3$! This is because the election relationships among the four-candidate subsets are based on what *would be* the $\vec{w}^3$ tallies if it had been used. If $\vec{w}^3$ is used with the three-candidate elections, then extra equations based on the same variables are introduced, so even more election relationships emerge!

Finally, return to the family of $\binom{5}{4}$ four-candidate subsets. Rather than increasing the number of equations, election relationships can be forced by decreasing the number of independent variables. To do so, choose the voting vector $\vec{w}^3 = \vec{w}^3((1,0))$ so that $\vec{w}^4 = \vec{w}^4((1,0))$ is the BC voting method. Now the relevant independent variables are the outcomes of the pairwise elections, rather than the triplets. In this situation, the number of independent variables is decreased to $\binom{5}{2} = 10$ independent variables, so we have an overdetermined system of 15 equations with 10 variables. Consequently the family of $\binom{5}{4}$ four-candidate subsets must admit election relationships if the BC is used. This is a reason the BC admits more relationships than any other method.

2.2. The start of a theory.

The first goal is to convert the "equation and variable counting" intuition into a theory to determine when a family must admit election relationships and for what kinds of voting vectors. The idea is clear; the subsets of a family $\mathcal{F} = \{S_1, \dots, S_k\}$ determine the number of equations. In [3] this is called (for different but related reasons) *the dimension of* $\mathcal{F}$. More precisely,

$$dim(\mathcal{F}) = \sum_{S_j \in \mathcal{F}} (|S_j| - 1). \tag{2.1}$$

Example. For $n = 5$, if $\mathcal{F}$ is the family of $\binom{5}{4}$ four- candidate subsets, then $dim(\mathcal{F}) = 3\binom{5}{4} = 15$.

For $n = 5$, let $\mathcal{F}_5$ be the family of all subsets of four or more candidates. $\mathcal{F}_5$ differs from the above family in that it includes the set of all five candidates, so $dim(\mathcal{F}_5) = 19$. □

To count the number of independent variables, we need to count the numbers of pairs, triplets, etc. involved in the aggregation process.

Theorem 2.1. *For the family $\mathcal{F} = \{S_1, \dots, S_k\}$, suppose that S_j is assigned the Λ reduced voting vector $\overrightarrow{w}^{|S_j|}(\overrightarrow{w}^{t_1}, \dots, \overrightarrow{w}^{t_z}, \Lambda)$, $j = 1 \dots, k$. For each S_j, list all possible t_i-candidate subsets that can be constructed, $i = 1, \dots, z$. Next, count the number of distinct subsets created in this fashion over all choices of $S_j \in \mathcal{F}$. If this number is less than $dim(\mathcal{F})$, then $\mathcal{F}$ must admit election relationships.*

Example. Suppose $n = 8$ and let $\mathcal{F}$ consist of all six and seven candidate subsets. For each $S \in \mathcal{F}$, assign the voting vector $\overrightarrow{w}^{|S|}((3,1,0),(1,0);\{1,-\frac{1}{4}\})$. (Because of the negative value, a Condorcet loser has the advantage over a Condorcet winner if she also does well in each $(3,1,0)$ election ([4]).) Here $dim(\mathcal{F}) = 188 > 140 = \{$the number of triplets and pairs that can be created from the subsets of $\mathcal{F}\}$. There must exists at least 48 election relationships among the subsets of $\mathcal{F}$.

As another example, suppose all sets in $\mathcal{F}$ are BC ranked and pairs in $\mathcal{F}$ are majority ranked. As the number of pairs is $\binom{n}{2} = \frac{n(n-1)}{2}$, if $dim(\mathcal{F}) > \binom{n}{2}$, then BC election relationships must exist. This is a result from [3]. □

While Theorem 2.1 is a sufficient condition, it is not a necessary condition for the existence of election relationships. The problem is that the defining equations for the tallies of the sets in $\mathcal{F}$ can admit reductions that reduce the effective number of variables.

Example. a. The system of three equations in five unknowns,

$$x + y + z + u + v = s_1$$
$$x + z + u + v = s_2$$
$$x + 4y + z + u + v = s_3$$

admits the relationship $s_3 = s_2 + 4(s_1 - s_2)$.

b. For $n = 5$, the above $\mathcal{F}_5$ (all subsets of four or more candidates) has dimension 19. If $\overrightarrow{w}^{|S_k|}(\overrightarrow{w}^3)$ is assigned to each subset, we have 19 equations and 20 variables so it is not clear whether election relationships must exist with this three-fold symmetry. On the other hand, with the four-fold symmetry introduced by assigning $\overrightarrow{w}^4$ to the four-candidate subsets and $\overrightarrow{w}^5(\overrightarrow{w}^4)$ to the five-candidate set creates a system of 19 equations with 15 independent variables, so there must be election relationships. But, because a three-fold symmetry admits at least as many election relationships as a four-fold symmetry, it appears that the dimension counting argument is incomplete. In other words, we must believe that the three-fold symmetry also admits election relationships for $\mathcal{F}_5$. The proof of this assertion follows from the relationship ([4]) $\overrightarrow{w}^5(\overrightarrow{w}^4(\overrightarrow{w}^3)) \approx \overrightarrow{w}^5(\overrightarrow{w}^3)$ that requires any election relationship admitted by the four-symmetry to be automatically inherited by the three-fold symmetry. Consequently, the tally equations based on the three- fold symmetry must admit reductions described by the four-fold symmetry.

c. The family $\{\{c_1, c_2, c_3\}, \{c_3, c_4, c_5\}, \{c_1, c_2, c_4\}, \{c_1, c_4\}, \{c_1, c_5\}, \{c_2, c_4\}, \{c_2, c_5\}\}$ defines ten equations in ten unknowns with the BC, so there is no reason to expect relationships. Yet, as shown in [3], this family does admit an election relationship. To prove this, one could use a laborious algebraic analysis of the ten equations in ten unknowns, or the simpler *cyclic dimension* argument of [3]. □

The goal is to discover the appropriate necessary and sufficient conditions on the subsets of a family $\mathcal{F} = \{S_1, \dots, S_k\}$ so that if $\overrightarrow{w}^{|S_j|}(\overrightarrow{w}^{t_1}, \dots, \overrightarrow{w}^{t_z}, \Lambda)$ is used with S_j, $j = 1, \dots, k$, then there must exist election relationships.[4] The conditions developed below extend the cyclic dimension of a family (developed in [3]), but they are expressed in a significantly different form. The difference is needed because the BC admits only a few potential paradoxes (i.e., unexpected lists of election rankings) relative to other methods, so it is simpler to emphasize the BC election rankings that can occur – this leads to the derivation of the cyclic dimension.[5] On the other hand, if a non-BC positional method admits election rankings, the numbers of potential paradoxes remains sufficiently large to make it more economical to emphasize what election relationships are not admitted, rather than those that are. This dual philosophy[6] is the source of the following derivation.

Definition 2.1. Assume given a family $\mathcal{F} = \{S_1, \dots, S_k\}$ and solitary voting vectors $\{\overrightarrow{w}^{t_1}, \dots, \overrightarrow{w}^{t_z}\}$ where $t_1 > t_2 > \dots t_z \geq 2$. Assume that the assigned voting vectors are $\{\overrightarrow{w}^{|S_j|}(\overrightarrow{w}^{t_1}, \dots, \overrightarrow{w}^{t_z}, \Lambda)\}$ for all $|S_j| > t_1$, and that $\overrightarrow{w}^{t_i}$ is used whenever there is an t_i and a $S_j \in \mathcal{F}$ so that $|S_j| = t_i$. Let $\mathcal{E}(\mathcal{F}) = \cup_{S_j \in \mathcal{F}} S_j$ be the set of all candidates that are in any of the subsets in $\mathcal{F}$. The $\{\overrightarrow{w}^{|S_j|}(\overrightarrow{w}^{t_1}, \dots, \overrightarrow{w}^{t_z}, \Lambda)\}$ *extended family*, $\mathcal{F}^E$, is the collection of all subsets that can be constructed from $\mathcal{E}(\mathcal{F})$ of cardinality $t_z, \dots,$ of cardinality t_1, and of cardinality greater than t_1. □

Example. Let $\mathcal{F} = \{\{c_1, c_2, c_3, c_4\}, \{c_2, c_3, c_4, c_5\}\}$. If both sets use the voting vector $\overrightarrow{w}^4(\overrightarrow{w}^3)$, then $\mathcal{E}(\mathcal{F}) = \{c_1, c_2, c_3, c_4, c_5\}$, and $\mathcal{F}^E$ consists of all $\binom{5}{3}$ subsets of three candidates, all $\binom{5}{4}$ subsets of four candidates, and the set of five candidates. □

Definition 2.2. Let S be a subset of the given n candidates. The c_i *indicator vector* for S is the vector

$$\mathbf{v}_{i,S} = \frac{|S| - 1}{|S|}\mathbf{e}_i - \sum_{c_k \in S, c_k \neq c_i} \frac{\mathbf{e}_k}{|S|}$$

where $\mathbf{e}_i \in R^n$ is the unit vector with unity for the ith component. □

Example. For $n = 5$ candidate, the c_2 indicator vector for $S_1 = \{c_1, c_2, c_3, c_4\}$ is $\mathbf{v}_{2,S_1} = (-\frac{1}{4}, \frac{3}{4}, -\frac{1}{4}, -\frac{1}{4}, 0)$, the c_4 indicator vector for $S_2 = \{c_2, c_3, c_4, c_5\}$ is $\mathbf{v}_{4,S_2} = (0, -\frac{1}{4}, -\frac{1}{4}, \frac{3}{4}, -\frac{1}{4})$, and the c_4 vector for $S_3 = \{c_1, c_4, c_5\}$ is $\mathbf{v}_{4,S_3} = (-\frac{1}{3}, 0, 0, \frac{2}{3}, -\frac{1}{3})$. □

Definition 2.3. 1. For a given family $\mathcal{F}$ and solitary vectors $\{\overrightarrow{w}^{t_1}, \dots, \overrightarrow{w}^{t_z}\}$ where $t_1 > \dots > t_z$, assume that the voting vectors assigned to sets in $\mathcal{F}$ are the reduced Λ composite vectors $\{\overrightarrow{w}^{|S_j|}(\overrightarrow{w}^{t_1}, \dots, \overrightarrow{w}^{t_z}, \Lambda)\}$ for all $S_j \in \mathcal{F}$ so that $|S_j| > t_1$, and $\overrightarrow{w}^{t_i}$ is the assigned voting vector for any $S_j \in \mathcal{F}$ for which there is an t_i so that $|S_j| = t_i$. The family $\mathcal{F}$ is $\{\overrightarrow{w}^k(\overrightarrow{w}^{t_1}, \dots, \overrightarrow{w}^{t_z}, \Lambda)\}$ *relaxed* if for each subset $S \in \mathcal{F}^E$, $|S| > t_1$, scalars $d_{i,S}$ can be found so that for all $S_j \in \mathcal{F}^E \backslash \mathcal{F}$ the equality

$$\sum_{i \in S_j \subset S} d_{i,S}\mathbf{v}_{i,S_j} = 0, \tag{2.2}$$

[4] What needs to be done is clear from the example – we need to find whether there exist algebraic reductions in the system. Therefore, for a first reading, this material can be skipped.

[5] The cyclic dimension is the dimension of a vector space designed to capture the binary relationships of pairs of candidates within the different subsets of candidates.

[6] The important vector space is the space of vectors normal to the space of binary, triplet, etc. relationships among the candidates within the different subsets of candidates.

but for at least one $S_j \in \mathcal{F}$, we have that

$$\sum_{i \in S_j \subset S} d_{i,S}\mathbf{v}_{i,S_j} \neq 0. \tag{2.3}$$

2. If for $\mathcal{F}$ and $\{\overrightarrow{w}^{|S_j|}(\overrightarrow{w}^{t_1}, \dots, \overrightarrow{w}^{t_z}, \Lambda)\}$, we have either that

$$dim(\mathcal{F}) > \sum_{i=1}^{z} \binom{k}{t_i}(t_i - 1) \tag{2.4}$$

where $k = |\mathcal{E}(\mathcal{F})|$; or

$$\mathcal{F} \text{ is } \{\overrightarrow{w}^{k}(\overrightarrow{w}^{t_1}, \dots, \overrightarrow{w}^{t_z}, \Lambda)\} \text{ relaxed} \tag{2.5}$$

then these vectors are called $\mathcal{F}$ composite voting vectors. □

Example. a. Let $n = 5$ and let $\mathcal{F}$ be the collection of all subsets of three or more candidates except $\{c_1, c_3, c_5\}, \{c_1, c_2, c_3\}$, and $\{c_1, c_2, c_4\}$. Choose a solitary voting vector $\overrightarrow{w}^3$ and suppose $\overrightarrow{w}^3$ is assigned to all subsets of three candidates of $\mathcal{F}$, while $\overrightarrow{w}^{|S|}(\overrightarrow{w}^3)$ is the assigned voting vector for all other $S \in \mathcal{F}$. To show that this is a relaxed family, let $d_{1,\{1,2,3,4,5\}} = -d_{1,\{1,3,4,5\}} = -d_{1,\{1,2,3,4\}} = 1$, and all of other $d_{i,S_j} = 0$. We need to show that with these choices of $\{d_{i,S_j}\}$, Eqs. 2.2 and 2.3 are satisfied. To see the computation for $\{c_1, c_2, c_3\}$, note that this set involves only the coefficients $d_{1,\{1,2,3,4,5\}}, d_{1,\{1,2,3,4\}}$, so we have

$$[d_{1,\{1,2,3,4,5\}} + d_{1,\{1,2,3,4\}}]\mathbf{v}_{1,\{1,2,3\}} = 0.$$

A similar computation holds for the other sets in $\mathcal{F}^E$ that are not in $\mathcal{F}$. On the other hand, the set $\{c_1, c_2, c_3, c_4, c_5\} \in \mathcal{F}$, and the above computation leads to

$$d_{1,\{1,2,3,4,5\}}\mathbf{v}_{1,\{1,2,3,4,5\}} \neq 0.$$

Therefore, for any $\overrightarrow{w}^3$, this family is $\overrightarrow{w}^k(\overrightarrow{w}^3)$- $\mathcal{F}$ composite.

The dimension definition also can be used to prove that these voting vectors are $\mathcal{F}$ composite. This is because

$$dim(\mathcal{F}) = 4\binom{5}{5} + 3\binom{5}{4} + 2[\binom{5}{3} - 3] > 2\binom{5}{3}.$$

It is fairly easy to show that if a system of vectors is composite because the family satisfies the dimension requirement (Eq. 2.4), then the family is relaxed. However, there are many examples of relaxed families that do not satisfy the dimension condition Eq. 2.5.

b. For any $n > s \geq 2$ let $\mathcal{F}$ be the family consisting of the $\binom{n}{s}$ subsets of s candidates and at least one other set with more than s candidates. In computing the value of $dim(\mathcal{F})$, the $\binom{n}{s}$ subsets of s candidates contribute the value $(s-1)\binom{n}{s}$. As the extra set adds a positive amount to $dim(\mathcal{F})$, it follows immediately that

$$dim(\mathcal{F}) > (s-1)\binom{n}{s}.$$

Therefore, the assignment of $\overrightarrow{w}^s$ to the subsets of s candidates, and $\overrightarrow{w}^{|S|}(\overrightarrow{w}^s)$ to the extra set ensures that these voting vectors are $\mathcal{F}$-composite voting vectors.

c. Let $n = 4$ and $\mathcal{F}$ be the four subsets of three candidates. This family satisfies Eq. 2.4 for the binary symmetries, so it follows that it is $\overrightarrow{w}^3((1,0))$ relaxed. However, to illustrate Eqs. 2.2, 2.3, and to create an example for the reader interested in comparing the cyclic symmetry dimension with the above definition, the assertion that this family is $\overrightarrow{w}^3((1,0))$ relaxed is shown directly.

The family $\mathcal{F}^E$ consists of all subsets of two or more candidates. The scalars $d_{1,\{1,2,3\}} = d_{3,\{1,2,3\}} = d_{2,\{1,2,4\}} = d_{4,\{1,2,4\}} = d_{1,\{1,3,4\}} = d_{3,\{1,3,4\}} = d_{2,\{2,3,4\}} = d_{4,\{2,3,4\}} = 1$ satisfy the conditions of the definition. For instance, consider some pair, say $S = \{c_2, c_4\}$. We have that

$$d_{2,\{1,2,4\}}(\mathbf{e}_2 - \mathbf{e}_4) + d_{2,\{2,3,4\}}(\mathbf{e}_2 - \mathbf{e}_4) + d_{4,\{1,2,4\}}(\mathbf{e}_4 - \mathbf{e}_2) + d_{4,\{2,3,4\}}(\mathbf{e}_4 - \mathbf{e}_2) = 0.$$

However, when we consider a set of three variables, say $\{c_1, c_2, c_3\}$ we obtain the outcome

$$d_{1,\{1,2,3\}}(2\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3) + d_{3,\{1,2,3\}}(2\mathbf{e}_3 - \mathbf{e}_1 - \mathbf{e}_2) = (-2\mathbf{e}_2 + \mathbf{e}_1 + \mathbf{e}_3) \neq 0.$$

Therefore, this family is BC- composite.

d. We now come to a surprise. One of the first families discussed in this article is

$$\mathcal{F} = \{\{c_1, c_2, c_3, c_4\}, \{c_1, c_2, c_3, c_4, c_5\}, \{c_1, c_2, c_4, c_5\}, \{c_1, c_3, c_4, c_5\}, \{c_2, c_3, c_4, c_5\}\},$$

the family of all 5 subsets of four candidates for $n = 5$, where the equation counting approach gave little hope for election relationships to exist. However, with the above argument, it can be shown that $\mathcal{F}$ is $\overrightarrow{w}^k(\overrightarrow{w}^3)$ relaxed so there are election relationships. □

The following is one of the major conclusions of this article.

Theorem 2.2. *A system voting vector $\mathbf{W}^n \in \alpha^n$ iff it contains an $\mathcal{F}$-composite voting vector for some family $\mathcal{F}$.*

Because this theorem completely characterizes all possible ways to define system voting vectors so that relationships more exist among the election outcomes, it plays a critical role in characterizing the positional voting vectors.

Example. To illustrate this theorem, suppose for $n = 5$ that

i. $\overrightarrow{w}^5((1,0,0))$ is assigned to the five- candidate subset,

ii. $\overrightarrow{w}^4((1,0,0))$ is assigned to the sets

$$\{c_1, c_2, c_3, c_4\},\ \{c_1, c_3, c_4, c_5\},\ \{c_2, c_3, c_4, c_5\},\ \{c_1, c_2, c_4, c_5\}$$

while $\overrightarrow{w}^4((1,1,0))$ is assigned to the remaining four-candidate subset, and

iii. the plurality vote is used with the sets $\{c_1, c_2, c_3\}, \{c_2, c_4, c_5\}, \{c_1, c_4, c_5\}$ while

iv. the antiplurality vote is used with the rest of the three-candidate subsets.

This assignment of voting vectors admits election relations because the family

$$\mathcal{F} = \{\{c_1, c_2, c_3, c_4, c_5\},\ \{c_1, c_2, c_3, c_4\},\ \{c_1, c_3, c_4, c_5\},\ \{c_1, c_2, c_4, c_5\},$$
$$\{c_2, c_3, c_4, c_5\},\ \{c_1, c_2, c_3\},\ \{c_2, c_4, c_5\},\ \{c_1, c_4, c_5\}\}$$

satisfies the condition $dim(\mathcal{F}) = 21 > 2\binom{5}{3} = 20$. This is the only election relationship among all possible families.

As a second example for $n = 6$, suppose

i. $\overrightarrow{w}^6((1, 1, 0))$ is assigned to the six-candidate subset,
ii. $\overrightarrow{w}^5((1, 1, 0, 0))$ is assigned to each five- candidate subset,
iii. the BC is assigned to each four-candidate subset, and
iv. the anti-plurality vector is assigned to each three-candidate subset.

For this system voting vector, several different kinds of relationships occur. For instance,

i. there are relationships between the ranking of the six- candidate subset and $\binom{6}{3}$ three-candidate subsets,
ii. there are election relationships among the $\binom{6}{5}$ five-candidate subsets and among the $\binom{6}{4}$ four-candidate subsets, and
iii. there are election relationships between each set of four candidates and the $\binom{4}{2}$ pairs of candidates that can be constructed from this set. □

Corollary 2.3. *Let S be a subset of $k > 3$ candidates, $C \subset S$, and $\overrightarrow{w}^k(\overrightarrow{w}^{t_1}, \dots, \overrightarrow{w}^{t_z}, \Lambda)$ be the voting vector assigned to S where $t_z \geq |C|$. Let $\mathcal{F}_{S,C}$ be the family consisting of S and all subsets $V \subset S$ where $C \subset V$ and $|V| = t_i$, $i = 1, \dots, z$. The family $\mathcal{F}_{S,C}$ admits election relationships.*

Example. Suppose that $C = \{c_1\}$ and that S is the set of all n candidates. Then, for any Λ composite voting vector, there is a relationship among the election rankings of the family $\mathcal{F}_{S,\{c_1\}}$. This relationship is described in [4] in terms of the different kinds of Condorcet winners and losers that c_1 can be. □

3. The Stratified Structure of α^n

To describe the geometric, stratified structure of α^n, assume all voting vectors are expressed in a Borda normalized form. (See [4] for a definition.) As this normalization $\overrightarrow{w}^k = (k-1, w_2, w_3, \dots, w_{k-1}, 0)$ allows $k-2$ degrees of freedom (the choices of $w_2, \dots, w_{k-1}$) for the selection of a particular $\overrightarrow{w}^k$, there are

$$\gamma(n) = \sum_{j=3}^{n} (j-2)\binom{n}{j}. \tag{3.1}$$

degrees of freedom in the selection of the $2^n - (n+1)$ voting vectors defining a system voting vector. In the obvious way, a system voting vector can be identified with a point in $R^{\gamma(n)}$.

Example. For $n = 3$ the voting vector for each of the $\binom{3}{2}$ pairs of candidates is $(1, 0)$ so there are zero degrees of freedom in the choice of these voting vector. What remains is the three-candidate subset where the normalized voting vector $\overrightarrow{w}^3 = (2, w_2, 0)$ has $\gamma(3) = 1$ degrees of freedom in the choice of the value of w_2.

For $n = 4$ candidates, the $\gamma(4) = 6$ degrees of freedom arise in the following manner. There are zero degrees of freedom to choose the $\binom{4}{2}$ voting vectors for the pairs of candidates and a single degree of freedom for the voting vector for each of the $\binom{4}{3}$ three-candidate subsets. (This is the choice of the value of w_2 for each voting vector $(2, w_2, 0)$; so it accounts for four of the six degrees of freedom.) The remaining two degrees of freedom involve choosing two variables w_2, w_3 from the normalized form $(3, w_2, w_3, 0)$ for the four-candidate set. This completely describes the six degrees of freedom for the choice of the system voting vector $\mathbf{W}^4$.

For $n = 5$, the $\gamma(5) = 23$ degrees of freedom reflect the $\binom{5}{3}$ choices of w_2 for each $(2, w_2, 0)$ three-candidate voting vector, the $\binom{5}{4}$ choices of w_2, w_3 for each $(3, w_2, w_3, 0)$ four-candidate voting vector, and the choices of w_2, w_3, w_4 for the five-candidate normalized voting vector $(4, w_2, w_3, w_4, 0)$. $\square$

The space of system voting vectors is the subset of $R^{\gamma(n)}$ determined by the constraints imposed on the variables so that the associated vector is a voting vector. For instance, to make $(2, w_2, 0)$ a voting vector, it must be that $0 \leq w_2 \leq 2$. Similarly, to make $(3, w_2, w_3, 0)$ a voting vector, we need $3 \geq w_2 \geq w_3 \geq 0$.

Definition 3.1. For a subset of candidates S_j where $|S_j| = k$, let

$$R(S_j) = \{(s_1, s_2, \cdots s_{k-2}) \in R^{k-2} \mid k - 1 \geq s_1 \geq s_2 \geq \cdots \geq s_{k-2} \geq 0\}.$$

The *positional voting pyramid for n candidates* is the set

$$PVP^{\gamma(n)} = \prod_{S_j, |S_j|>2} R(S_j). \quad \square \tag{3.2}$$

According to the definition, $PVP^{\gamma(n)} \subset R^{\gamma(n)}$ is a closed subset of $R^{\gamma(n)}$ with non-empty interior. To identify a point in $PVP^{\gamma(n)}$ with a system voting vector, use the obvious one-to-one correspondence based on the fact that the coordinates of $q \in PVP^{\gamma(n)}$ correspond to the unspecified values of the voting vectors that define $\mathbf{W}^n$. Thus, $PVP^{\gamma(n)}$ becomes *the space of system voting vectors* where α^n is a subset of $PVP^{\gamma(n)}$.

Example(α^3, α^4). a. $PVP^{\gamma(3)} = [0, 2]$ where the boundary point $q = 0$ corresponds to the plurality voting vector $(2, 0, 0)$ and the other boundary point $q = 2$ corresponds to the anti-plurality vector $(2, 2, 0)$. The system voting vector $\mathbf{B}^3$, where the BC voting vector for the set of three candidates is $\overrightarrow{w}^3((1, 0)) = (2, 1, 0)$, corresponds to the point $q = 1 \in PVP^{\gamma(3)}$. Thus, according to [S], $\alpha^3 = \{1\} \subset PVP^1$. This midpoint of the line segment $[0, 2]$ constitutes the complete description of α^3.

b. For $n = 4$, the normalized voting vector for each three-candidate subset is $(2, w_2, 0)$ while for four candidates the normalized voting vector is $(3, w_2, w_3, 0)$. The free variables are the four choices of w_2 (one for each three-candidate subset) and the w_2, w_3 components for the four- candidate set. Let

$$(x_1, x_2, x_3, x_4; y_1, y_2) \in PVP^{\gamma(4)} = PVP^6 \tag{3.3}$$

be where the x_j component is the value of w_2 for the voting vector assigned to the three-candidate subset that does *not* include c_{5-j}. The y components are the values to be selected in $\overrightarrow{w}^4$; $3 \geq y_1 = w_2 \geq y_2 = w_3 \geq 0$. So, the positional voting pyramid is the six-dimensional space

$$PVP^{\gamma(4)} = [0, 1]^4 \times \{(y_1, y_2) \mid 3 \geq y_1 \geq y_2 \geq 0\}.$$

c. The entry $(1, 0, 1, 2; 2, 0) \in PVP^6$ represents where the BC $((2, 1, 0))$ is used with the subsets $\{c_1, c_2, c_3\}$, $\{c_1, c_3, c_4\}$ $(x_1 = x_3 = 1)$, the plurality vote $((2, 0, 0))$ is used with $\{c_1, c_2, c_4\}$ $(x_2 = 0)$, the anti-plurality vote $((2, 2, 0))$ is used with $\{c_2, c_3, c_4\}$ $(x_4 = 2)$, and the voting vector for the set of four candidates is $(3, 2, 0, 0)$.

d. The 23-dimensional space $PVP^{\gamma(5)}$ is given by the product

$$[0, 2]^{\binom{5}{3}} \times \{(y_1, y_2) | 3 \geq y_1 \geq y_2 \geq 0\}^{\binom{5}{4}} \times \\ \times \{(z_1, z_2, z_3) | 4 \geq z_1 \geq z_2 \geq z_3 \geq 0\}. \quad \square$$

3.2. The stratified structure of α^n.

The following assertion plays an important role in our construction of $\alpha^n \subset PVP^{\gamma(n)}$. The technical terms of this description are illustrated in the examples.

Theorem 3.1. *For $n \geq 3$, α^n is a closed subset (an algebraic variety) of $PVP^{\gamma(n)}$ consisting of the finite union of smooth (algebraic) lower dimensional surfaces. For each non-negative integer $k < \gamma(n)$, there is at least one surface of α^n with dimension k. Each k dimensional component has a parametric representation in k variables expressed in algebraic equations with degree bounded by k. Each k-dimensional component of α^n, $1 \leq k < \gamma(n) - 1$ is in the boundary for two or more connected components of α^n with dimension larger than k. The point $\beta_2 \in PVP^{\gamma(n)}$ representing the BC system voting vector, $\mathbf{B}^n$, is a boundary point of all components of α^n.*

If β is a surface of α^n with positive dimension and if p is a boundary point of β (i.e., if $p \in \partial(\beta)$), then either p is in a lower dimensional surface of α^n, or $p \in \partial(PVP^{\gamma(n)})$. If $dim(\beta) > 0$, then β has a boundary point that is not on $\partial(PVP^{\gamma(n)})$.

As described by this theorem, emanating from each lower dimensional surfaces of α^n is not just a single higher dimensional surface, but several higher dimensional surfaces. A useful image is to view each component as an "edge" connecting several higher dimensional components; a situation not that dissimilar from the binder of a book connecting the pages. Then, each of these pages are the "edges" for the even higher dimensional surfaces.

Theorems 2.2, 3.1 and theorems from [4] and [S] are used to construct α^n. The idea is to use an inductive approach by starting with a lower dimensional surface β of α^n and then determine those components of α^n with dimension $dim(\beta) + 1$ with β as a boundary. A detailed description of α^4 accompanies the introduction of each new branch of the algebraic variety.

As an aside, remember that the importance of α^n is that its structure captures the intricate connections between different kinds of system voting vectors in α^n. The geometry and connection properties of α^n explain the relationships among different system voting vectors and lead (in Section 4) to the definition of the partial ordering for system voting vectors.

3.3. The zero and one-dimensional components of α^n.

Our construction of α^n starts with the only zero dimensional component; the point β_2 which represents the system voting vector $\mathbf{B}^n$. (The subscript emphasizes that β_2 is based on a binary symmetry property.) Starting with this zero-dimensional component of α^n, the next step is to find all α^n components with dimension $1 + dim(\beta_2) = 1$. According to Theorem 3.1, all one-dimensional components have β_2 as a boundary point.

A one-dimensional component of α^n is a curve defined by the single degree of freedom coming from the definition of the relation admitting system vector. According to Theorem 2.2 and Theorem 2.1 of [4], such a one-dimensional curve in α^n must be based on the choice of a single $\overrightarrow{w}^3$. Of the two ways this can occur, one is to assign $\overrightarrow{w}^3$ to a specified three-candidate subset and the BC to all subsets of candidates. The second approach is to find all families that are $\overrightarrow{w}^3$ composite; all other subsets of candidates are assigned the BC.

According to Theorem 2.1 of [4], $\overrightarrow{w}^3$ composite family is defined by $\{\mathbf{W}^n(\overrightarrow{w}^3)\} \in \alpha^n$; such a system voting vector is based on a three-fold symmetry. This class admits the interpretation of being, at least locally, a perturbation of $\mathbf{B}^n$ because $\mathbf{B}^n = \mathbf{W}^n((1,0))$ is a limiting, special case corresponding to $\overrightarrow{w}^3 = (2,1,0)$. (This perturbation assertion manifests the assertion in Theorem 3.1 that β_2 must be in the boundary of all surfaces constituting α^n.)

To represent this system as a one-dimensional set $\beta_3 \subset \alpha^n \subset PVP^{\gamma(n)}$, use the fact that each voting vector is uniquely and algebraically defined by the sole degree of freedom, w_2, in the choice of $\overrightarrow{w}^3$. (The notation β_3 is selected to denote the three-fold symmetry of the voting vectors.) Indeed, each of the two connected portions of β_3 is a segment of a *straight line* passing through β_2 and terminating on the boundary of $PVP^{\gamma(n)}$. That the parametric representation is first order follows from the definition of $\{\mathbf{W}^n(\overrightarrow{w}^3)\}$. There are two line segments parameterized by the choice of w_2 in $\overrightarrow{w}^3 = (2, w_2, 0)$ where one segment has the restriction $0 \le w_2 < 1$, while the other has $1 < w_2 \le 2$. So, $\beta_3 \subset PVP^{\gamma(n)}$ is identified with the one-dimensional set of system voting vectors

$$\{\mathbf{W}^n(\overrightarrow{w}^3) | \overrightarrow{w}^3 \ne \overrightarrow{B}^3 \text{ is a voting vector}\}.$$

A simple way to construct β_3 is first to define the straight line segment corresponding to $\mathbf{W}^n((2, w_2, 0))$ in $PVP^{\gamma(n)}$ and then remove the center point β_2 which represents $\mathbf{B}^n$. In other words, the BC defines a point, and passing through this point is a straight line of positional voting methods that admit election relationships. This construction indicates the typical situation for the surfaces of α^n. By use of Theorem 3.1, a surface corresponding to a designated class of system voting vectors from α^n is defined. Then, lower dimensional components of α^n divide the surface into several components. In other words, a fixed parametric, algebraic representation is given for the surface. Restrictions on the parameter divide the surface into components. To distinguish between the two situation, we call the union of these components the *component type*. Thus, the class $\{\mathbf{W}^n(\overrightarrow{w}^3)\}$ defines one component type of α^n with two components.

Example(α^4). For $n = 4$, $\mathbf{B}^4 = \mathbf{W}^4((1,0))$ is the system vector where $(1,0)$ is assigned to all pairwise elections, where $(2,1,0)$ is assigned to each of the four subsets of three candidates, and where $(3,2,1,0)$ is used with the set of four candidates. Using the above notation, $\mathbf{B}^4$ corresponds to the point $\beta_2 = (1,1,1,1;2,1) \in PVP^{\gamma(4)}$.

The component type β_3 corresponds to all system voting vectors $\mathbf{W}^4((2,s,0)) \in \alpha^4$, $0 \le s \le 2; s \ne 2$ where $(1,0)$ is assigned to all pairwise elections, $(2,s,0)$ is assigned to all elections with three candidates, and $(3, 1+s, s, 0)$ is assigned to the set of all four candidates. Thus, these system voting vectors $\mathbf{W}^4((2, s, 0)$, $0 \leq s \leq 2$, are identified with the two line segments

$$\beta_3 = \{(s, s, s, s; 1+s, s)\} \subset PVP^6,\ s \in [0,1) \cup (1,2].\} \tag{3.4}$$

There are two interesting facts to notice.

1. The end points of the line segment are on the boundary of $PVP^{\gamma(4)}$, but the natural projection of this segment into the component $R(\{c_1, c_2, c_3, c_4\})$ is in the interior of $R(\{c_1, c_2, c_3, c_4\})$. In particular, those voting vectors represented by points on the boundary of $R(\{c_1, c_2, c_3, c_4\})$, such $(3,0,0,0), (3,3,0,0)$ and $(3,3,3,0)$ are not in the projection of this set. This geometric fact is closely related to the assertions from [4] asserting that certain positional voting vectors cannot be expressed as a Λ composite voting vector $\overrightarrow{w}^k(\overrightarrow{w}^{t_1}, \ldots, \overrightarrow{w}^{t_z}, \Lambda)$, thus they cannot admit election relationships. (In fact, it is an exercise to show that if a vector of $R(\{c_1, c_2, c_3, c_4\})$ is not in this projection, then it can not be expressed as $\overrightarrow{w}^4(\overrightarrow{w}^3)$.)
2. The midpoint of this line, where $s = 1$ so the point is $(1,1,1,1;2,1)$, is identified with $\beta_2 = \mathbf{B}^4$. Therefore, β_2 is the boundary points of $\beta_3 = \{(s, s, s, s; 1+s, s) | s \in [0,1) \cup (1,2]$ that divides the component type into two components. □

Returning to the description of α^n, we seek other choices of system voting vectors which admit relationships and which correspond to one dimensional sets emanating from β_2. Following the above approach, because our set is one-dimensional, our search is restricted to system voting vectors where only one choice of $\overrightarrow{w}^3$ is free to be selected; all subsets of candidates not based on this choice of $\overrightarrow{w}^3$ must be assigned the BC. Moreover, the stratified structure requires one of the boundary points of this one-dimensional line to be β_2, so (at least locally) such a class of system voting vectors can be viewed as being a perturbation of $\mathbf{B}^n$. According to Theorem 2.1 of [4], [S], and Theorem 3.1 of this paper, one way to achieve a single degree of freedom is to modify the $\mathbf{B}^n$ system vector by replacing the BC voting vector assigned to a specified three-candidate subset with a $\overrightarrow{w}^3 \neq \overrightarrow{w}^3((1,0))$ voting vector. Such a vector is in α^n because election relationships are admitted among the election rankings of all subsets of candidates except the specified three-candidate subset. There are $\binom{n}{3}$ choices for this set of three candidates, so there are $\binom{n}{3}$ different component types. Each component type is divided by the β_2, so it leads to two line segments. Namely, as the parameterization of these lines is based on the choice of w_2 for the $\overrightarrow{w}^3$ assigned to the designated three-candidate subset, one straight line component has $w_2 < 1$, while the other has $w_2 > 1$. This defines $2\binom{n}{3}$ components of α^n with boundaries β_2 and on $\partial(PVP^{\gamma(n)})$.

These $2\binom{n}{3}$ one dimensional curves have the properties that

1. the curves are pairwise disjoint from one another and from β_3,
2. each component type has two components,
3. for each line segment, one boundary point is β_2 and the other is a point in $\partial PVP^{\gamma(n)}$, and
4. there is a symmetry relating the $\binom{n}{3}$ component types; a permutation σ interchanging some two coordinates for the triplets maps one component type onto another. (This

last assertion holds because the permutation changes which triplet has the BC voting vector replaced by $\overrightarrow{w}^3$.)

Example. (α^4). One component type, given by

$$\{(s, 1, 1, 1; 2, 1) | s \in [0, 1) \cup (1, 2]\},$$

is where the BC assigned to all sets except $\{c_1, c_2, c_3\}$ and $(2, s, 0)$, $s \neq 1$ is assigned to $\{c_1, c_2, c_3\}$. The other three lines of this type in α^4 are obtained by permuting which of the first four coordinates has the parameter s. Notice that β_2 is a boundary point for each line; it is the lower dimensional component of α^4 that divides each component type into two components. By the geometry, these line segments come out of β_2 in a pin-wheel fashion. This is an aspect of the stratified structure of α^4 and α^n combined with the symmetry. □

There are many other one-dimensional straight lines in α^n that emerge from β_2. They are found, of course, with Theorem 2.2. According to this theorem, if a system voting vector contains an $\overrightarrow{w}^3$ composite family of voting vectors, then this system voting vector is in α^n. So, find all such families of subsets. Then, each family $\mathcal{F}$ defines a one-dimensional component type consisting of two components that are separated by β_2; as above, each component depends upon whether the chosen $\overrightarrow{w}^3$ has $w_2 < 1$, or $w_2 > 1$. These straight line segments correspond to where $\overrightarrow{w}^3$ is assigned to each $S \in \mathcal{F}$ where $|S| = 3$, and $\overrightarrow{w}^{|S|}(\overrightarrow{w}^3)$ is assigned to each $S \in \mathcal{F}$ where $|S| > 3$. All $S \notin \mathcal{F}$ must be assigned the BC voting vector.

As an illustration of a $\overrightarrow{w}^3$-$\mathcal{F}$ composite family of voting vectors, recall that a special case of such families is where a particular subset C of candidates are contained in all sets. (See Corollary 2.3.) So, choose a particular set of candidates, say, $C = \{c_1, c_2\}$, and assign $\overrightarrow{w}^{|S|}(\overrightarrow{w}^3)$ to each subset S of three or more candidates that contains C. Any set not containing C is assigned the BC voting vector. This assignment defines a component type of α^n with two components. As another choice, assign $\overrightarrow{w}^n(\overrightarrow{w}^3)$ to the subset of all n candidates, $\overrightarrow{w}^3$ to each subset of three candidates that includes C, and the BC for all remaining sets. This assignment defines another pair of straight line segments in α^n.

Example (α^4). For $n = 4$, the above construction introduces only $\binom{4}{1}$ new component types with a total of $2\binom{4}{1}$ components (all straight line segments) in α^4. This is because if $|C| > 1$, then we obtain the set of system voting vectors $\{\mathbf{W}^4(\overrightarrow{w}^3)\}$ analyzed above. Also, because of the limited number of subsets of candidates available with only $n = 4$ candidates, the families defined in terms of such a C are the only possibilities for a $\overrightarrow{w}^3$-$\mathcal{F}$ composite family of voting vectors.

Each component type is obtained by choosing a particular $C = \{c_i\}$. Assign the BC to the unique set of three candidates that does not include the designated candidate; the remaining three subsets of three candidates are assigned $\overrightarrow{w}^3$ and the set of four candidates is assigned $\overrightarrow{w}^4(\overrightarrow{w}^3)$. For instance, the component type

$$\{(1, s, s, s; 1 + s, s) | s \in [0, 1) \cup (1, 2]\}$$

arises when $C = \{c_4\}$. The other three component types (where each produces two line segments) correspond to the three remaining permutations of the first four components of the vector.

The construction of α^5 admits added possibilities. This is because with the extra candidate, there are more admissible families. As an example, the family of all sets of four candidates is an admissible family. Therefore, among the component types, given by one dimensional curves in α^5 coming out of β_2, we now have situations where $\vec{w}^4(\vec{w}^3)$ is assigned to each of the five sets of four candidates, and the BC is assigned to all other sets of candidates. □

3.4. The two-dimensional components.

By use of Theorem 2.2, we know we have completely characterized all one-dimensional α^n components. Now consider the two-dimensional components of α^n – surfaces based on the availability of two degrees of freedom for a system vector in α^n. From Theorem 2.2 and Theorem 3.1 of [4], these degrees of freedom can arise in three ways.

1. After a solitary voting vector $\vec{w}^4$ is selected, the voting vectors for certain subsets of candidates is based on this choice; all other subsets of candidates are assigned the BC voting vector. This requires finding all families that are $\vec{w}^4$ relaxed, or by assigning $\vec{w}^4$ to a particular four-candidate subset.
2. Two solitary voting vectors $\vec{w}^3$ and $\vec{w'}^3$ are selected. The voting vector for certain subsets of candidates are based either on $\vec{w}^3$, while other subsets are based on $\vec{w'}^3$; all other subsets of candidates are assigned the BC voting vector. This is done by finding combinations of families that are $\vec{w}^3$ and/or $\vec{w'}^3$ relaxed, and assigning these vectors to specified three-candidate subsets.
3. The two degrees of freedom come from a choice of $\vec{w}^3$ and the choice of a scalar λ_1. Then certain subsets of candidates are assigned the voting vector $\vec{w}^{|S|}(\vec{w}^3, (1,0), \Lambda = \{\lambda_1, 1-\lambda_1\})$; all other subsets are assigned the BC voting vector.

We know that each of these assignments leads to a two-dimensional component type α^n where its boundary must contain β_2 and some one-dimensional branches of α^n. Secondly, a pattern has been established by the construction of the one-dimensional curves of α^n: either the rankings of the set assigned the non-BC voting vector is not related to the rankings of the other sets, or this voting vector defines a $\mathcal{F}$-composite voting vector. To illustrate, start with the first possibility where the two degrees of freedom arise from a choice of $\vec{w}^4$. For the first possibility, choose a set of four candidates. For this set, define the class of system voting vectors where the voting vector assigned to the specified set is a $\vec{w}^4 \neq (3,2,1,0)$; the BC voting vector is assigned to all remaining subsets of candidates. This defines $\binom{n}{4}$ two-dimensional surfaces where the only boundary point that is not on $\partial(PVP^{\gamma(n)})$ is β_2. The remaining possibilities involve choosing all possible families that admit a $\vec{w}^4$-$\mathcal{F}$ composite voting vector. Assign $\vec{w}^4$ to each set S in this family where $|S| = 4$, and $\vec{w}^{|S|}(\vec{w}^4)$ to each set S in this family for which $|S| > 4$. All other subsets of candidates are assigned the BC voting vector.

Example (α^4). Of the two choices, only the first is possible for $n = 4$. This one possible set is given by

$$\{(1,1,1,1;s,t)|\,3 \geq s \geq t \geq 0,\ (s,t) \neq (2,1)\} \subset \alpha^4.$$

For $n = 5$, there are several families that admit a $\overrightarrow{w}^4 - \mathcal{F}$ composite voting vector. One is the family of all subsets of four or five candidates. Five others are given by a choice of c_j; this is the family of $1 + \binom{4}{3}$ subsets of four or more candidates that include the c_j. $\square$

Another two-dimensional component type corresponds to the class of voting vectors $\{\mathbf{W}^n(\overrightarrow{w}^3, (1,0), \Lambda)\} \subset \alpha^n$. One of the two parameters defining this surface is the value of w_2 in the definition of $\overrightarrow{w}^3$, the other is the ratio $t = \lambda_1, \lambda_2 = 1 - t$. It follows immediately that

$$\text{as } t \to 0,\ \mathbf{W}^n(\overrightarrow{w}^3, (1,0), \Lambda) \to \mathbf{W}^n(\overrightarrow{w}^3). \tag{3.5}$$

At the other extreme,

$$\text{as } t \to 1,\ \mathbf{W}^n(\overrightarrow{w}^3, (1,0), \Lambda) \to \mathbf{B}^n = \mathbf{W}^n((1,0)). \tag{3.6}$$

This last assertion illustrates the first part of Theorem 3.1 In any case, the set of system voting vectors $\{\mathbf{W}^n(\overrightarrow{w}^3, (1,0), \Lambda)\}$ defines three different two-dimensional components of $PVP^{\gamma(n)}$ depending on whether $t < 1$, $0 < t < 1$, $1 < t$. It follows from the discussion in [4] that a voting vector in each component type leads to significantly different kinds of Condorcet properties that are admitted by the election rankings. Notice how the edges of these components either belong to $PVP^{\gamma(n)}$ or they are one-dimensional components of α^n. For example, the two edges $s = 0, 2; 0 < t < 1$, of one component are in $\partial PVP^{\gamma(n)}$. Another edge (where $t = 0, s \in [0,1) \cup (1,2]$) defines β_3. Again, this illustrates the stratified form of α^n. As another illustration of part 1 of Proposition 3.1, the remaining two edges ($t = 1; s \in [0,1) \cup (1,2]$ and $0 \leq t \leq 1, s = 1$ of this component is the point β_2. Thus, this two dimensional surface starts on the closure of β_3, arcs through $PVP^{\gamma(n)}$ to return to the point $\mathbf{B}^n$. A similar description holds for the other two component types corresponding to $t < 0, t > 1$.

This particular family $\{\mathbf{W}^n(\overrightarrow{w}^3, (1,0), \Lambda)\}$ can be modified to create a large number of related classes of voting vectors. To do this, instead of assigning $\overrightarrow{w}^3$ to all set of three candidates and $\overrightarrow{w}^{|S|}(\overrightarrow{w}^3, (1,0), \Lambda)$ to sets S where $|S| > 3$, these assignments are made only to certain specified subsets. Namely, for any family $\mathcal{F}$ that admits a $\{\overrightarrow{w}^3, (1,0)\}$ composite voting vector, the above assignment is made. All other sets are assigned the BC voting vector.

Example(α^4). Let $n = 4$ and let t represent the value of λ_1, so $\lambda_2 = 1 - t$. The two parameters s, t define the class $\mathbf{W}^4(\overrightarrow{w}^3, (1,0), \Lambda) \in \alpha^4$ and each of the three corresponding component types are given by a surface with the quadratic representation

$$\{(s,s,s,s;(1-t)(s+1) + 2t, (1-t)s + t)|\ s \in [0,1) \cup (1,2]\} \subset PVP^{\gamma(4)}\} \tag{3.7.}$$

where the three component types are determined by whether $t < 0$, $0 < t < 1$, $t > 1$. The boundary of each component surface includes β_3 and β_2. Each component type is divided into two components depending on whether $0 \leq s < 1$, or $1 < s \leq 2$.

Next, choose a particular candidate c_j. Modify the above procedure by assigning the BC to the one three-candidate subset that does not include $\{c_j\}$. For each choice of c_j we have three new two-dimensional component types of α^4 with two components. For example, the component type corresponding to the choice of c_2 is

$$\{(s, s, 1, s; (1-t)(s+1) + 2t, (1-t)s + t) | s \in [0, 1) \cup (1, 2]\}$$

where, again, the three types are determined by whether $t < 0$, $0 < t < 1$, $t > 1$.

The above introduces fifteen new component types for α^4 with a total of 30 components.

For $n = 5$, the additional subsets of candidates admit more imaginative choices of families that admit a $\overrightarrow{w}^3, (1, 0) - \mathcal{F}$ - composite voting vector, thus, different sorts of two dimensional component types emerge. □

A third possibility, where two voting vectors $\overrightarrow{w}^3$ and $\overrightarrow{w'}^3$ are selected, is handled in the same fashion. First consider where each class of voting vectors is assigned to a different subset of three candidates and the BC is assigned to all other subsets. Here, the election rankings of the designated two subsets need not have any relationship whatsoever with the rankings of the remaining subsets. This leads to $4[(\binom{n}{3})(\binom{n}{3} - 1)]/2$ choices of component types and, as each component type has four components, to $4[(\binom{n}{3})(\binom{n}{3} - 1)]/2$ components of α^n. Notice that the boundaries of such a family consist of the two one dimensional lines given by where a non-BC voting vector is assigned to only one subset of three candidates.

A fourth possibility is where one $\overrightarrow{w}^3$ voting vector is assigned to a particular set of three candidates, where $\overrightarrow{w'}^3$ or $w'^{|S|}(\overrightarrow{w'}^3)$ is assigned to the subsets of a family $\mathcal{F}$ that admits a $\overrightarrow{w'}^3$–$\mathcal{F}$ composite voting vector, and where the BC voting vector is assigned to all other subsets. For each such family, the two one-dimensional boundaries consist of where the non-BC vector is assigned to only one subset of three candidates, and where the subsets of a family $\mathcal{F}$ admit a $\overrightarrow{w'}^3$–$\mathcal{F}$ composite voting vector.

The remaining possibility is where there are two disjoint families $\mathcal{F}_1$ and $\mathcal{F}_2$ where one admits a $\overrightarrow{w}^3$–$\mathcal{F}_1$ composite voting vector and the other admits a $\overrightarrow{w'}^3$–$\mathcal{F}_2$ composite voting vector; the BC is assigned to all remaining subsets.

Example(α^4). An example of the third possibility is where the two sets of three candidates are $\{c_1, c_2, c_3\}$ and $\{c_1, c_2, c_4\}$ is given by the parametric representation

$$\{(s, t, 1, 1; 2, 1) \in PVP^{\gamma(4)} | s, t \in [0, 1) \cup (1, 2]\}.$$

This two- dimensional component type defines four two-dimensional components of α^4. As there are six choices $([\binom{4}{3})(\binom{4}{3} - 1)]/2)$ of the pairs, this defines six component types with 24 two-dimensional components for α^4.

For the fourth stated possibility in α^4, one choice of $\overrightarrow{w}^3$ defines a family of subsets where there are election relationships, and the other does not. For instance, if the family is a $\{c_1\}$ composite family, then the four components of the two dimensional surface are given by the set

$$\{(s, s, s, t; 1 - s, s) | s, t \in [0, 1) \cup (1, 2]\}.$$

As there are four ways to choose the subset represented by t, there are four new component types for α^4. Each component type has four components, so this adds 16 connected components to α^4.

For $n = 4$ candidates, there cannot be two disjoint families where each admits a $\mathcal{F}$-composite voting vector. Therefore, the last possibility cannot occur in α^4. On the other hand, such a situation does occur in α^5. For instance, $\mathcal{F}_1$ could consist of all $\binom{5}{3}$ subsets of three candidates and the set of all five candidates, while $\mathcal{F}_2$ consists of all $\binom{5}{4}$ subsets of four candidates. □

3.5. The higher dimensional components.

Now that all possible one and two dimensional components of α^n are found, the same procedure is used to find the three and higher dimensional components of α^n. By use of Theorem 2.2, the specified dimension of the component of α^n determines the kinds of voting vectors that can be used to replace BC voting vectors for specified subsets of candidates. There are two ways to determine how to use choose the subsets of candidates. One is where the rankings of the subset will be independent of the rankings of the other sets, and the other is based on families of candidates that admit composite voting vectors with the specified degrees of freedom. In choosing the non-BC voting vectors, we use solitary voting vectors or reduced Λ composite voting vectors. A particular choice of a family and signs of the Λ components defines a *component type*. Each component type is further subdivided into components according to the choice of the solitary vectors $\overrightarrow{w}^{t_i}$. To see this division, recall from [4] that if $\overrightarrow{w}^{t_j}$ is a solitary vector, then it must belong to one of the two components $BVV^{t_j}\backslash\mathcal{L}W(t_j, t_j - 1)$; corresponding to the choice of component for each solitary voting vector is a component of α^n.

To illustrate, consider $\overrightarrow{w}^k(\overrightarrow{w}^5, \overrightarrow{w}^4, \overrightarrow{w}^3, (1,0), \Lambda)$. There are $2^4 - 1$ sign combinations where not all λ_j's are negative, so there is a potential of $2^4 - 1$ component types for each choice of a relaxed family. Furthermore, each component type is further subdivided into 2^3 components reflecting the two different regions admitted by the first three solitary voting vectors.

This approach is the obvious generalization of the procedure established by the above discussion for the two-dimensional components, so it is carried out below only for α^4.

Example(α^4). If a three-dimensional components of α^4 is based on the choice of a single $\overrightarrow{w}^k$, then this voting vector must admit three degrees of freedom; i.e., we must use $\overrightarrow{w}^5$. This, of course, is impossible as $\overrightarrow{w}^5$ requires five candidates while only four are available. Thus, there are only three ways to choose the basic voting vectors in order to obtain the required three degrees of freedom for $n = 4$.

1. One degree of freedom is given by each of three choices of a voting vector $\overrightarrow{w}^3$.
2. Two degrees of freedom come from the choice of a $\overrightarrow{w}^4$, and the last degree of freedom occurs by choosing a $\overrightarrow{w}^3$.
3. The choice of a $\overrightarrow{w}^3$ to define a $\overrightarrow{w}^4(\overrightarrow{w}^3, (1,0), \{t, 1-t\})$ voting vector uses two degrees of freedom. The final degree of freedom comes from choosing another $\overrightarrow{w'}^3$. Once the choice of voting vectors have been isolated, the basic theorems are used to determine how to assign voting vectors to subsets of candidates. Again, the fact that only a limited

number of subsets can be defined by $n = 4$ candidates plays an important role in the analysis. To illustrate where three choices of voting vectors $\vec{w}^3$ are made, notice that the limited number of subsets of candidates makes it impossible to use any of these voting vectors $\vec{w}^3$ to define a composite voting vector. These vectors must be assigned, therefore, to individual subsets of three candidates; all other sets are assigned the BC. This assignment has the effect of eliminating the rankings of this set from being related to the rankings of the other subsets of candidates. Each assignment defines a three-dimensional component type which has 2^3 three- dimensional components. The dividing (two-dimensional) boundaries defining these components correspond to where one of the three designated sets now are assigned the BC. As there are $\binom{4}{3}$ such assignments, we add four three-dimensional component types and 32 three-dimensional components to α^4. Notice that the two dimensional boundaries of these sets correspond to where one of these sets is reassigned the BC and two of them have a non-BC voting vector. As an illustration, the following α^n component corresponds to where $\{c_1, c_2, c_3\}$ is the only three- candidate subset assigned the BC.

$$\{(1, s, t, z; 2, 1) | s, t, z \in [0, 1) \cup (1, 2]\}.$$

Again, with voting vectors based on choices of $\vec{w}^4$, and $\vec{w}^3$, a restriction is imposed by the relative scarcity of subsets of candidates. Because of the limited number of subsets of candidates for $n = 4$, these choices of $\vec{w}^4$ and $\vec{w}^3$ cannot be used to define composite voting vectors. Therefore, the only choice is to assign $\vec{w}^4$ to the set of all four candidates, and $\vec{w}^3$ to a specified subset of three candidates. As there are four choices for the subset of three candidates, this defines four new three-dimensional component types with a total of eight three-dimensional components have been added to α^4. One such component type where the chosen subset is $\{c_1, c_2, c_3\}$ is

$$\{(s, 1, 1, 1; t, z) | s \in [0, 1) \cup (1, 2], 3 \geq t \geq z \geq 0, (t, z) \neq (2, 1)\}.$$

The final way to obtain three degrees of freedom is to use $\vec{w}^4(\vec{w}^3, (1, 0), \{t, 1 - t\})$ and $\vec{w'}^3$. The only way the first composite vector can define relationships for $n = 4$ is if there is a family based on a set $C = \{c_j\}$ as described in Corollary 2.1. The one three-candidate subset not containing c_j is assigned $\vec{w'}^3$. Twelve α^4 component types are defined in this manner where each component type has four three-dimensional components. This construction, then, adds 48 three-dimensional components to α^4. For instance, the following component type is based on a $C = \{c_1\}$.

$$\{(s, s, s, z; t(1 + s) + 2(1 - t), ts + 1 - t) | s, z \in [0, 1) \cup (1, 2], t \in (0, 1)\}.$$

The boundaries of this set include the two dimensional surface given by the c_j composite voting vector (where $z = 1$). Two related component types are where $t < 0$ and $t > 1$.

Because of the limited number of subsets when $n = 4$, the four dimensional surfaces of α^4 are even easier to analyze. To obtain four degrees of freedom, we must either choose four vectors $\overrightarrow{w}^3$ or choose two vectors $\overrightarrow{w}^3$ and one $\overrightarrow{w}^4$. For the first setting, each three candidate subset is assigned a $\overrightarrow{w}^3$ voting vector, and the BC is assigned to the set of four candidates. This defines one four-dimensional component type with $2^4 = 16$ four-dimensional components of α^4. This component type has the representation

$$\{(s,t,u,v;2,1) \mid s,t,u,v \in [0,1) \cup (1,2]\}.$$

When a four dimensional component is defined in terms of two $\overrightarrow{w}^3$ vectors and one $\overrightarrow{w}^4$, we need to select the two three-candidate subsets that remain BC ranked. Each choice defines a component type that has 2^2 connected four-dimensional components. Together, this adds six component types and 24 four-dimensional components to α^4. The set

$$\{(1,s,t,1;z,u) \mid s,t \in [0,1) \cup (1,2], 3 \geq z \geq u \geq 0\}$$

is the component type where $\{c_1,c_2,c_3\}$ and $\{c_2,c_3,c_4\}$ are BC ranked.

The final possibilities are the five-dimensional components of α^4. These can occur only when only one set of three candidates is assigned the BC voting vector, and the assignment of the remaining voting vectors does not define a composite voting vector. (The situation where a composite voting vector is admitted defines a portion of the boundary for this set. If fact, this boundary divides this set into two components.) Each choice defines a component type, and each component type has 2^3 components. Thus, there are four five-dimensional component types and 64 five-dimensional components of α^4. The following component type is where $\{c_1,c_2,c_3\}$ is assigned the BC voting vector.

$$\{(1,s,t,u;v,z) \mid s,t,u \in [0,1) \cup (1,2], 3 \geq v \geq z \geq 0\} - \\ \{(1,s,s,s;t(1+s)+2(1-t), ts+1-t) \mid s \in [0,1) \cup (1,2], t \in [0,1]\}. \square$$

3.6. The structure of α^3 and α^4.

To summarize, α^3 has the trivial structure of a single (zero- dimensional) component consisting of the point β_2.

The structure of α^4 is significantly more complicated than α^3 as indicated by the following table listing the number of components and component types of α^4. One measure of the increased complexity is that α^3 has only one component while α^4 has 282 of them. The numbers rapidly escalate with the value of n.

(3.8)

Dimension	Components	Component types
0	1	1
1	18	9
2	71	26
3	88	20
4	40	7
5	64	4
Total	282	67

. $\square$

4. The Partial Ordering of System Voting Vectors

The stratified structure of α^n leads to one of the main goals of this article – the introduction of a partial ordering for the system voting vectors.

Definition 4.1. For $\mathbf{x}_1, \mathbf{x}_2 \in PVP^{\gamma(n)}$, $\mathbf{x}_2$ *is coarser than* $\mathbf{x}_1$, denoted by $\mathbf{x}_1 \blacktriangleleft \mathbf{x}_2$, if one of the following two conditions are satisfied.

1. $\mathbf{x}_1 \in \alpha^n$ but $\mathbf{x}_2 \notin \alpha^n$.
2. Both $\mathbf{x}_1, \mathbf{x}_2 \in \alpha^n$. Let β be the component of α^n that contains $\mathbf{x}_2$. Then $\mathbf{x}_1 \in \partial(\beta)$, but $\mathbf{x}_1 \notin \beta$.

Let $\mathbf{W}^n$ and $\mathbf{W}^{*n}$ be system voting vectors that are identified in the natural fashion, respectively, with $\mathbf{x}_1, \mathbf{x}_2 \in PVP^{\gamma(n)}$. If $\mathbf{x}_1 \blacktriangleleft \mathbf{x}_2$, then we say that $\mathbf{W}^n \blacktriangleleft \mathbf{W}^{*n}$.

If both $\mathbf{x}_1, \mathbf{x}_2$ are interior points of the same component type of α^n, then we say that both vectors are equally coarse. This is denoted by $\mathbf{x}_1 \blacksquare \mathbf{x}_2$. If $\mathbf{W}^n$ and $\mathbf{W}^{*n}$ are system voting vectors identified, respectively, with $\mathbf{x}_1, \mathbf{x}_2 \in PVP^{\gamma(n)}$, and if $\mathbf{x}_1 \blacksquare \mathbf{x}_2$, then we say that $\mathbf{W}^n \blacksquare \mathbf{W}^{*n}$. □

Proposition 4.1. *The binary relationship* $\blacktriangleleft$ *defines a partial ordering on* $PVP^{\gamma(n)}$.

The proof of the proposition follows directly from the stratified structure of α^n.

To explain what this partial ordering means, we need some basic definitions from [S] and [4]. With n candidates, there are $2^n - (n+1)$ subsets of two or more candidates, and a system voting vector, $\mathbf{W}^n$, indicates the positional voting method assigned to each set. A *word*, $f(\mathbf{p}, \mathbf{W}^n)$, defined by profile $\mathbf{p}$ is a listing of the $2^n - (n+1)$ election outcomes; there is a unique ranking for each subset of candidates. A *dictionary* is a listing of all possible words. Namely,

$$\mathcal{D}(\mathbf{W}^n) = \{f(\mathbf{p}, \mathbf{W}^n) \mid \text{all possible profiles } \mathbf{p}\}.$$

It is easy to show that all dictionaries contain those words where the ranking of each subset of candidates is inherited from the ranking of the set of all n-candidates. Consequently, any other word in a dictionary corresponds to where a profile defines election rankings that could be viewed as being counter-intuitive or paradoxical. It follows, therefore, that the fewer the entries in a dictionary, the fewer the paradoxical words that are admitted; that is, the election results are more predictable. The ultimate extreme is where a dictionary equals $\mathcal{U}^n$. ($\mathcal{U}^n$ is the set of all possible listings of rankings. Namely, for each subset of candidates, arbitrarily choose a ranking – thus the rankings among subsets need not have anything to do with one another. The collection of all possible listings of rankings constructed in this way is $\mathcal{U}^n$.) For more discussion, examples, and applications, see [S].

The following shows that if $\mathbf{W}^n \blacktriangleleft \mathbf{W}^{*n}$, then more numbers and kinds of paradoxes are admitted by $\mathbf{W}^{*n}$ than by $\mathbf{W}^n$. Conversely, $\mathbf{W}^n$ enjoys all of the election relationships permitted by $\mathbf{W}^{*n}$ and then some additional ones.

Theorem 4.2. *1. If* $\mathbf{W}^n \blacksquare \mathbf{W}^{*n}$, *then*

$$\mathcal{D}(\mathbf{W}^n) = \mathcal{D}(\mathbf{W}^{*n}). \tag{4.1}$$

2. If $\mathbf{W}^n \blacktriangleleft \mathbf{W}^{*n}$, *then*

$$\mathcal{D}(\mathbf{W}^n) \subset \mathcal{D}(\mathbf{W}^{*n}). \tag{4.2}$$

3. If $\mathbf{W}^n \not\blacktriangleleft \mathbf{W}^{*n}$ *and* $\mathbf{W}^{*n} \not\blacktriangleleft \mathbf{W}^n$, *then* $\mathcal{D}(\mathbf{W}^n) \not\subset \mathcal{D}(\mathbf{W}^{*n})$ *and* $\mathcal{D}(\mathbf{W}^n) \not\subset \mathcal{D}(\mathbf{W}^{*n})$. *In other words, there are words in each dictionary that are not in the other.*

4. If $\mathbf{W}^n \notin \alpha^n$, *then* $\mathcal{D}(\mathbf{W}^n) = \mathcal{U}^n$.

This theorem, then, gives a complete characterization of which system voting vectors admit any kind of relationship among the election rankings. As part 1 asserts, an election ranking is admitted iff the system voting vector can be expressed as one of the vectors in the constructed set α^n. This theorem is surprising in that the words (election rankings and election relationships) depend more on the component type than on the entries of a system voting vector. For instance, let $\mathbf{W}^4_1$ be where each three- candidate subset is plurality ranked and $\overrightarrow{w}^4((1,0,0)) = (3,1,0,0)$ is assigned to the four-candidate set. Next, let $\mathbf{W}^4_2$ be the system voting vector where the three-candidate subsets are anti-plurality ranked while $\overrightarrow{w}^4((1,1,0)) = (3,3,2,0)$ is assigned to the four-candidate subset. Both system voting vectors belong to the same one-dimensional component type of α^4 (but different components), so, according to the theorem, both have the same dictionary! This means that if the profile $\mathbf{p}$ defines a word for $\mathbf{W}^4_1$, then there is a profile $\mathbf{p}'$ which yields for each set of candidates the same election ranking when $\mathbf{W}^4_2$ is used.

An immediate corollary of Theorem 4.2 is that unity plus the number of component types in α^n is the number of different dictionaries that can be created. (The added value of unity is required for the dictionary for all system voting vectors not in α^n; this dictionary is the universal set $\mathcal{U}^n$.) The following is a special case based on the derivation in Section 3.

Corollary 4.3. *For $n = 3$ candidates, there are only two possible types of dictionaries for system voting vectors. The first is $\mathcal{D}(\mathbf{B}^3) \subset \mathcal{U}^3$, and the other is $\mathcal{D}(\mathbf{W}^3) = \mathcal{U}^3$. Namely, all dictionaries agree with $\mathcal{U}^3$ if $\mathbf{W}^3 \not\approx \mathbf{B}^3$.*

For $n = 4$ candidates, there are 68 different dictionaries for system voting vectors. All of these dictionaries contain the Borda Dictionary $\mathcal{D}(\mathbf{B}^4)$, all are contained in the dictionary for plurality voting which is $\mathcal{U}^4$.

This corollary means, for example, that the seemingly impossible problem of characterizing everything that can happen for all choices of system voting vectors has been reduced from an infinite dimensional task to a finite dimensional one. After all, if we know the words in each of the 68 dictionaries for four candidates, we know everything that can occur for four candidates with all possible choices of voting vectors.[7] Of course, 68 different dictionaries still constitutes a large number of possibilities. So, a further reduction is given (in Corollary 4.4) after I indicate how to use Theorem 4.2.

Of theoretical interest, Theorem 4.2 answers our earlier questions about the system voting vectors. It reaffirms ([S]) that $\mathbf{B}^n$ is the "best choice" because $\mathbf{B}^n$ minimizes the number and the kinds of single profile paradoxes that ever could occur. This is a direct

[7] The description of what words are in each dictionary is a basic theme of [5].

consequence of the theorem because for any $\mathbf{W}^n \neq \mathbf{B}^n$, we have $\mathbf{B}^n \blacktriangleleft \mathbf{W}^n$. This ordering holds because β_2 is a boundary point of all α^n components.

It also follows from Theorem 4.2, the partial (rather than a total) ordering, and the structure of α^n that there does not exist a natural "second best system voting vector." Support for this assertion can be seen from the structure of the one-dimensional components of α^n. For instance, if there were a second best system voting vector, then, by Theorem 4.1, it must belong to a one-dimensional component type of α^n. But, there are several different disjoint one-dimensional component types of α^n, $n \geq 3$, so there are several different choices of "second best" system voting vector, and each choice leads to a different dictionary. Of course, added conditions could be imposed to force the selection of a particular one-dimensional branch of α^n. For instance, if the BC is not to be assigned to any subset of candidates, then the only one one-dimensional branch of α^n that remains admissible is the one corresponding to $\mathbf{W}^3(\overrightarrow{w}^3)$. With this restriction, any system vector from the class $\{\mathbf{W}^3(\overrightarrow{w}^3)\}$ is "second best." Indeed, with this restriction, β_3 is in the boundary of all admissible system vectors.

The "third best", "fourth best," etc., choices become even more complicated. This is because β_3 is in the boundary of several different two-dimensional component types. In other words, at each level, new branches emerge from each component type. Thus selections must be based on imposing appropriate restrictions to isolate particular branches of the partial orderings.

4.2. The computations for an example.

While Theorem 4.2 is of obvious theoretical interest, what improves its value is that it is accompanied by a simple computation scheme to allow specified choices of system voting vectors to be compared. The computational scheme is to determine which α^n component type contains a given system voting vector. Then, system vectors can be compared by comparing the component types of α^n. At each step, the computations involve nothing more difficult than the use of elementary algebra.

Example(α^4). To illustrate how to use this partial ordering, twelve different system voting vectors are given and then ordered. The importance of this example is to illustrate the required computational steps in the restricted setting afforded by $n = 4$ candidates. The same approach, of course, extends to order any set of system voting vectors for any $n \geq 3$.

For $n = 4$, the first five system voting vectors have the $(3, 1, 0)$ assigned to all three-candidate subsets, so they differ in the assignment of a procedure for the four-candidate subset. The assigned vectors are

System Vector	Choice of $\overrightarrow{w}^4$
$\mathbf{W}^4_1$	$(12, 7, 3, 0)$
$\mathbf{W}^4_2$	$(15, 9, 4, 0)$
$\mathbf{W}^4_3$	$(9, 5, 2, 0)$
$\mathbf{W}^4_4$	$(3, 2, 1, 0)$
$\mathbf{W}^4_5$	$(12, 8, 3, 0)$

The next set of five system voting vectors have $(3,1,0)$ assigned to all three-candidate subsets except $\{c_1, c_2, c_3\}$. Therefore, these system vectors differ in the assignment of a voting vector for $\{c_1, c_2, c_3\}$ and for the four-candidate subset.

System Vector	Choice of $\overrightarrow{w}^3$	Choice of $\overrightarrow{w}^4$
$\mathbf{W}_6^4$	$(2,1,0)$	$(12,7,3,0)$
$\mathbf{W}_7^4$	$(2,1,0)$	$(3,2,1,0)$
$\mathbf{W}_8^4$	$(2,1,0)$	$(9,5,2,0)$
$\mathbf{W}_9^4$	$(1,0,0)$	$(9,5,2,0)$
$\mathbf{W}_{10}^4$	$(1,0,0)$	$(15,9,4,0)$

The final two system vectors have the $(3,1,0)$ assigned to all three- candidate subsets except $\{c_2, c_3, c_4\}$; here the BC is assigned. Then for the four-candidate set, $\mathbf{W}_{11}^4$ assigns $(12,7,3,0)$ while $\mathbf{W}_{12}^4$ assigns $(9,5,2,0)$.

To start the comparisons, note that $\overrightarrow{w}^4((3,1,0)) = (9,5,2,0)$, so $\mathbf{W}_3^4 = \mathbf{W}^4((3,1,0))$. The vector $\overrightarrow{w}^3 = (3,1,0)$, which has the normalized form $(2,\frac{2}{3},0)$, is a solitary vector so $\mathbf{W}_3^4$ is in a one-dimensional component of α^4 corresponding to the set of system voting vectors $\mathbf{W}^4(\overrightarrow{w}^3)$. Thus $\mathbf{W}_3^4$ belongs to the component of α^4 given by

$$\beta^3 = \{(s,s,s,s;1+s,s)|s \in [0,1).$$

Next note that $(12,7,3,0) = \overrightarrow{w}^4((3,1,0) + \overrightarrow{w}^4((1,0)) = (9,5,2,0) + (3,2,1,0)$ and $(15,9,4,0) = \overrightarrow{w}^4((3,1,0) + 2\overrightarrow{w}^4((1,0)) = (9,5,2,0) + (6,4,2,0)$. Therefore, both system voting vectors belong to the two-dimensional component of α^4 corresponding to the class $\mathbf{W}^4(\overrightarrow{w}^3, (1,0), \Lambda)$. Both voting vectors are special cases of the two-dimensional α^4 component

$$\beta^1 = \beta^2 = \{(s,s,s,s;(1-t)(s+1)+2t,(1-t)s+t)|\, s \in [0,1),\ o < t < 1\}.$$

Moreover, $\beta^3 \subset \partial(\beta^2)$ (which results when $t=0$), so we have that $\mathbf{W}_3^4$ corresponds to a boundary point of the component containing $\mathbf{W}_1^4$ and $\mathbf{W}_2^4$. Therefore,

$$\mathbf{W}_3^4 \blacktriangleleft \mathbf{W}_1^4 \blacksquare \mathbf{W}_2^4.$$

With respect to the election rankings, this means that

$$\mathcal{D}(\mathbf{W}_3^4) \subset \mathcal{D}(\mathbf{W}_2^4) = \mathcal{D}(\mathbf{W}_1^4) \subset \mathcal{U}^4.$$

Now turn to $\mathbf{W}_4^4$. Because the four sets of three candidates does not admit a $\overrightarrow{w}^3$ composite voting vector, and because the BC, rather than $\overrightarrow{w}^4((3,1,0)$ is used with the set of four candidates, there is nothing special about the assignment of $(3,1,0)$ to these sets; any $\overrightarrow{w}^3$ would have the same effect. Therefore, $\mathbf{W}_4^4$ belongs to the four-dimensional component of α^4

$$\beta^4 = \{(s,u,v,z;2,1)|\, s,u,v,z \in [0,1)\}.$$

Notice that the only boundaries of this set correspond to where one of the four variables assumes the value 1; this is where one of the sets is assigned the BC voting vector. So,

$$\beta^i \not\subset \partial(\beta^4), \quad i = 1, 2, 3.$$

This means that although $\mathbf{W}^4_4 \in \alpha^4$, it is not comparable to any of the system voting vectors $\{\mathbf{W}^4_j\}^3_{j=1}$. Therefore, there are words in the dictionary $\mathcal{D}(\mathbf{W}^4_4)$ that are not in the three dictionaries already discussed, and each of these dictionaries has words that are not in $\mathcal{D}(\mathbf{W}^4_4)$.

Next consider $\mathbf{W}^4_5$. The key term is the voting vector $(12, 8, 3, 0)$ assigned to the set of all four candidates. If $\mathbf{W}^4_5 \in \alpha^4$, then it must be that $(12, 8, 3, 0) = s\overrightarrow{w}^4((3, 1, 0) + u(3, 2, 1, 0)$ for some choice of non-negative scalars s, u. Simple algebra shows that this is not the case. Therefore, $\mathbf{W}^4_5 \notin \alpha^4$, so $\mathbf{W}^4_j \blacktriangleleft \mathbf{W}^4_5$ for $j = 1, 2, 3, 4$. Moreover, $\mathcal{D}(\mathbf{W}^4_5) = \mathcal{U}^4$. This means that for any rankings assigned to the $2^4 - 5 = 11$ subsets of candidates, there is a profile so that the election rankings are the chosen ones. Also, $\beta^5 = PVP^4\backslash\alpha^4$.

The voting vector $\mathbf{W}^4_6$ introduces a new feature: it differs from $\mathbf{W}^4_1$ only in that the BC is assigned to $\{c_1, c_2, c_3\}$ rather than $(3, 1, 0)$. As such, one might expect the dictionary of one of these voting vectors to contain the other. However, this is not the case – $\mathbf{W}^4_1$ and $\mathbf{W}^4_6$ are not comparable. As such, there are words in each dictionary that are not in the other one. To see why this is so, notice that with $\mathbf{W}^4_6$ the choice of the voting vector for three of the sets of three candidates and the set of all four candidates creates a special case of a c_4 - composite voting vector based on the parameters $\overrightarrow{w}^3$ and Λ. Thus, this voting vector belongs to the two-dimensional component of α^4 given by

$$\beta^6 = \{(1, s, s, s; (1-t)(s+1) + 2t, (1-t)s + t) | s \in [0, 1), 0 < t < 1\}.$$

Since β^6 is two-dimensional, it cannot be in the boundary of β^2. Moreover, it is not a same component type as β^2. Similarly, $\beta^3 \not\subset \partial(\beta^6)$. The component β^4 is of higher dimension, but $\partial(\beta^4) = \{\beta_2\}$, so $\beta^6 \not\subset \partial(\beta^4)$. Therefore, we have that any system vector from $\{\mathbf{W}^4_j\}^4_{j=1}$ and $\mathbf{W}^4_6$ are not comparable with this partial ordering. Thus, there exist words in $\mathcal{D}(\mathbf{W}^4_6)$ that are not in the other four dictionaries, and there are words in the other four dictionaries that are not in $\mathcal{D}(\mathbf{W}^4_6)$.[8] However, $\mathbf{W}^4_6 \blacktriangleleft \mathbf{W}^4_5$.

Still another feature arises when $\mathbf{W}^4_7$ is considered; this changes the voting vector assigned to the set of four candidates from $(12, 7, 3, 0)$ used with $\mathbf{W}^4_6$ to the BC. However, this

[8]For example, there is a word in $\mathcal{D}(\mathbf{W}^4_1)$ with the ranking $c_1 \succ c_2 \succ c_3$ for the set $\{c_1, c_2, c_3\}$. However, the majority vote rankings have $c_3 \succ c_1, c_3 \succ c_2$. Thus, c_3 is the Condorcet winner for the subset $\{c_1, c_2, c_3\}$. Clearly, such a ranking is not admissible for $\mathbf{W}^4_6$ as it assigns the BC to $\{c_1, c_2, c_3\}$. In the other direction, there exist words in $\mathcal{D}(\mathbf{W}^4_6)$ that have c_1 top-ranked in all sets of two and three candidates, yet it is bottom ranked in the set of four candidates. Such a word cannot be in $\mathcal{D}(\mathbf{W}^4_j)$ for $j = 1, 2, 3$. There is a word in $\mathcal{D}(\mathbf{W}^4_4)$ that has the Condorcet winner for $\{c_1, c_2, c_3\}$ bottom ranked for this subset of three candidates (so this word in not in $\mathcal{D}(\mathbf{W}^4_6)$). Likewise, there is a word in $\mathcal{D}(\mathbf{W}^4_6)$ where c_1 is the Condorcet winner, yet c_1 is bottom ranked in the set of all four candidates; this cannot occur for $\mathbf{W}^4_4$. The existence of these words requires the material that is developed in [5]. Nevertheless, these examples are consistent with the assertions about $\overrightarrow{w}^k$–Condorcet winners described in [4].

simple change destroys the relationship enjoyed by the voting vectors assigned to the family of subsets that include c_4. As such, this forces $\mathbf{W}^4_7$ to belong to the *three-dimensional* component

$$\beta^7 = \{(1, s, u, v; 2, 1) | s, u, v \in [0, 1)\}.$$

It follows immediately that $\beta^7 \subset \partial(\beta^4)$, so $\mathbf{W}^4_7 \blacktriangleleft \mathbf{W}^4_4$.

Using this same type of analysis, the rest of the system voting vectors are seen to belong to the following components of α^4.

Voting Vector	Component of α^4	Dimension
$\mathbf{W}^4_7$	$\beta^7 = \{(1, s, u, v; 2, 1)\}$	3
$\mathbf{W}^4_8$	$\beta^8 = \{(1, s, s, s; s+1, s)\}$	1
$\mathbf{W}^4_9$	$\beta^9 = \{(u, s, s, s; s+1, s)\}$	2
$\mathbf{W}^4_{10}$	$\beta^{10} = \{(1, s, s, s; (1-t)(s+1)+2t, (1-t)s+t)\}$	2
$\mathbf{W}^4_{11}$	$\beta^{11} = \{(s, s, s, 1; (1-t)(s+1)+2t, (1-t)s+t)\}$	2
$\mathbf{W}^4_{12}$	$\beta^{12} = \{(s, s, s, 1; s+1, s)\}$	1

The following set theoretic relationships occur for these α^4 components.

$$\beta^3 \subset \partial(\beta^2), \quad \beta^7 \subset \partial(\beta^4),$$
$$\beta^8 \subset \partial(\beta^6), \quad \beta^8 \subset \partial(\beta^9),$$
$$\beta^6 \subset \partial(\beta^{10}), \quad \beta^{12} \subset \partial(\beta^{11}).$$

These set theoretic containments lead to the following branches of the partial ordering:

$$\mathbf{B}^4 \blacktriangleleft \mathbf{W}^4_3 \blacktriangleleft \mathbf{W}^4_1 \blacksquare \mathbf{W}^4_2 \blacktriangleleft \mathbf{W}^4_5$$
$$\mathbf{B}^4 \blacktriangleleft \mathbf{W}^4_7 \blacktriangleleft \mathbf{W}^4_4 \blacktriangleleft \mathbf{W}^4_5$$
$$\mathbf{B}^4 \blacktriangleleft \mathbf{W}^4_8 \blacktriangleleft \mathbf{W}^4_6 \blacktriangleleft \mathbf{W}^4_{10} \blacktriangleleft \mathbf{W}^4_5$$
$$\mathbf{B}^4 \blacktriangleleft \mathbf{W}^4_{12} \blacktriangleleft \mathbf{W}^4_{11} \blacktriangleleft \mathbf{W}^4_5$$

The partial ordering defines a corresponding relationship among the dictionaries.

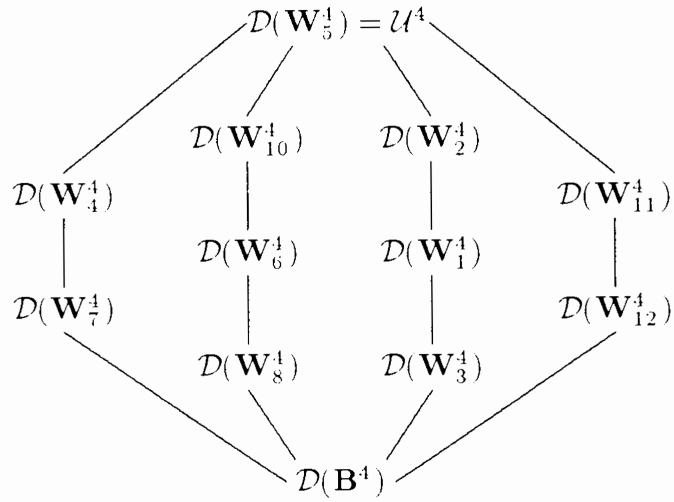

Figure 1. The partial ordering

4.3. Symmetry equivalence classes.

From the above example, one might suspect that another symmetry reduction is possible. To illustrate, using neutrality, a profile defining the word

$$\{c_1 \succ c_2, c_1 \succ c_3, c_2 \succ c_3; c_3 \succ c_2 \succ c_1\}$$

can be modified to construct a profile defining the word

$$\{c_3 \succ c_2, c_3 \succ c_1, c_2 \succ c_1; c_1 \succ c_2 \succ c_3\}.$$

System voting vectors for $n \geq 4$ candidates admit a related, but more subtle situation. For instance, notice that the only difference between the above system voting vectors $\mathbf{W}^4_8$ and $\mathbf{W}^4_{12}$ is that the first assigns the BC to the set $\{c_1, c_2, c_3\}$ and $(3, 1, 0)$ to the other three-candidate sets, while the second assigns the BC to the set $\{c_2, c_3, c_4\}$ and $(3, 1, 0)$ to the other three-candidate sets. The symmetry to convert one situation into the other is to interchange the subscripts 1 and 4. This symmetry is further manifested in the similarities of the component types β^8, β^{12} which contain, respectively, $\mathbf{W}^4_8, \mathbf{W}^4_{12}$. Moreover, it is easy to prove that a similar similarity holds between $\mathcal{D}(\mathbf{W}^4_8)$ and $\mathcal{D}(\mathbf{W}^4_{12})$; symmetries that require applying the permutation to the sets of candidates.

Definition 4.2. Let σ be a permutation of $\{c_1, \dots, c_n\}$. Define

$$\sigma(\mathbf{W}^n) = \sigma((\overrightarrow{w}^{|S_1|}, \dots, \overrightarrow{w}^{|S_{2^n-(n+1)}|})) = ((\overrightarrow{w}^{|\sigma(S_1)|}, \dots, \overrightarrow{w}^{|\sigma(S_{2^n-(n+1)})|})). \tag{4.3}$$

For a word $\mathcal{W} \in \mathcal{D}(\mathbf{W}^n)$, let $\sigma(\mathcal{W})$ be the element of U^n obtained by permuting the indices of each symbol in $\mathcal{W}$ according to the permutation σ.

Define

$$\sigma(\mathcal{D}(\mathbf{W}^n)) = \{\sigma(\mathcal{W}) | \mathcal{W} \in \mathcal{D}(\mathbf{W}^n)\}. \quad \square \tag{4.4}$$

Example. The permutation $\sigma = (1, 3, 2)$ changes the index 1 to 3, 3 to 2, and 2 to 1. Now suppose a system voting vector $\mathbf{W}^4$ assigns the BC to the four-candidate set with the following assignments of $\overrightarrow{w}^3$ for the three-candidate subsets.

Subset	$\overrightarrow{w}^3$
$\{c_1, c_2, c_3\}$	$(1, 0, 0)$
$\{c_1, c_2, c_4\}$	$(1, 1, 0)$
$\{c_1, c_3, c_4\}$	$(2, 1, 0)$
$\{c_2, c_3, c_4\}$	$(3, 1, 0)$.

Here $\sigma(\mathbf{W}^4)$ is the system voting vector where the BC is assigned to the four-candidate set with the following assignment for the three-candidate subsets.

Subset	$\overrightarrow{w}^3$
$\{c_1, c_2, c_3\}$	$(1, 0, 0)$
$\{c_1, c_2, c_4\}$	$(3, 1, 0)$
$\{c_1, c_3, c_4\}$	$(1, 1, 0)$
$\{c_2, c_3, c_4\}$	$(2, 1, 0)$.

This assignment follows because the four-candidate set and $\{c_1, c_2, c_3\}$ are invariant under σ, so they retain the same voting vector. (For example,

$$\sigma(\{c_1, c_2, c_3\}) = \{c_{\sigma(1)}, c_{\sigma(2)}, c_{\sigma(3)}\} = \{c_3, c_1, c_2\} = \{c_1, c_2, c_3\}.)$$

Also note that $\sigma(\{c_1, c_3, c_4\}) = \{c_{\sigma(1)}, c_{\sigma(3)}, c_{\sigma(4)}\} = \{c_2, c_3, c_4\}$, so the voting vector $(2, 1, 0)$ is reassigned by σ from the set $\{c_1, c_3, c_4\}$ to the set $\{c_2, c_3, c_4\}$.

The effects of σ for $n = 3$ on the word $\mathcal{W} = \{c_1 \succ c_2, c_1 \succ c_3, c_2 \succ c_3; c_3 \succ c_2 \succ c_1\}$ are $\sigma(\mathcal{W}) = \{c_{\sigma(1)} \succ c_{\sigma(2)}, c_{\sigma(1)} \succ c_{\sigma(3)}, c_{\sigma(2)} \succ c_{\sigma(3)}; c_{\sigma(3)} \succ c_{\sigma(2)} \succ c_{\sigma(1)}\}$ or $\{c_3 \succ c_2, c_3 \succ c_2, c_1 \succ c_2; c_2 \succ c_1 \succ c_3\}$.

Definition 4.3. Two component types β^1, β^2 from α^n are *symmetry related* if there exists a permutation σ of the n indices such that for each system voting vector $\mathbf{W}^n \in \beta^1$, we have that $\sigma(\mathbf{W}^n) \in \beta^2$.

Two dictionaries $\mathcal{D}(\mathbf{W}_1^n), \mathcal{D}(\mathbf{W}_2^n)$ are *symmetry related* if there exists a permutation σ of the n indices such that

$$\sigma(\mathcal{D}(\mathbf{W}_1^n)) = \mathcal{D}(\mathbf{W}_2^n). \quad \square$$

It is trivial to show that both of these symmetry relationships are equivalence relationships and that

$$\sigma(\mathcal{D}(\mathbf{W}^n)) = \mathcal{D}(\sigma(\mathbf{W}^n)).$$

Therefore we can speak about symmetry equivalence classes of component types and symmetry equivalence classes of dictionaries. From the definition, truly different kinds of election outcomes occur only when words from dictionaries from different symmetry equivalence classes are compared.

Corollary 4.3. *1. The number of symmetry equivalence classes of dictionaries exceeds the number of symmetry equivalence classes of component types of α^n by one.*

2. For $n = 3$, there are only two symmetry equivalence classes of dictionaries: $\mathcal{D}(\mathbf{B}^3)$ and U^3. For $n = 4$, there are 21 symmetry equivalence classes of dictionaries.

By use of symmetry, the number of different types of dictionaries for $n = 4$ candidates is significantly reduced from 68 to 21. This symmetry can be used to provide other kinds of reductions; for instance, it leads to a related partial ordering, etc. Details are left to the interested reader.

What remains is to find a way to characterize the words in these dictionaries. This is the purpose of the third part [5] of this study.

References

1. Saari DG (1989), A dictionary for voting paradoxes. J Econ Theory 48:443-475

2. Saari DG (1990), The Borda Dictionary. Soc Choice Welfare 7: 279-317

3. Saari DG (1991), Relationship admitting families of candidates. Soc Choice Welfare 8: 21-50

4. Saari DG (1991), Symmetry extensions of 'neutrality'I: Advantage to the Condorcet Loser. NU Discussion Paper.

5. Saari DG (1992), Symmetry extensions of 'neutrality'III: The vector spaces.

6. Saari, D. G. (1991), Millions of election rankings from a single profile, NU Discussion Paper.

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