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PROPERTIES OF DYNAMIC TRAFFIC EQUILIBRIUM INVOLVING
BOTTLENECKS, INCLUDING A PARADOX AND METERING

by

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Abstract

Braess and others have shown that creating a new link in a congested network, or adding capacity to an existing link, can raise total travel costs if drivers switch routes. We show that a paradox can also result when routes are fixed but users choose when to travel. As is true of the Braess paradox, the paradox here arises when the inefficiency due to underpricing of congestion increases by more than the direct benefit of the new capacity.

For a corridor with two groups of drivers, we show that expanding capacity of an upstream bottleneck raises travel costs when the reduction in congestion upstream is more than offset by increased congestion downstream. Metering can thus improve efficiency. Optimal upstream capacity is equal to or smaller than downstream capacity for the user equilibrium. Total construction costs equal total variable travel costs when capacities are optimal and construction costs are independent of scale.

Notational Glossary

(in alphabetic order)

Greek characters

α	Unit cost of in-vehicle travel time
β	Unit cost of arriving at work early
θ	β/α
ν	n_1/n_2
σ_1	s_1/s_d
σ_2	s_2/s_d
τ_g	Set of times during which group g departs

English characters

$C_g(t)$	Travel cost of a group g commuter departing at time t
\bar{C}_g	Equilibrium travel cost for group g
d	Subscript denoting downstream bottleneck
d_g	Departure rate of group g from residential area
g	Index of groups
i	Index of bottlenecks
k_i	Marginal cost of expanding bottleneck i
n_g	Number of commuters in group g
$Q_i(t)$	Number of commuters queued behind bottleneck i at time t
r_d	Arrival rate of commuters at downstream bottleneck
s_d	Capacity of downstream bottleneck
s_1	Capacity of upstream bottleneck serving group 1
s_2	Capacity of upstream bottleneck serving group 2
s_i^*	Optimal long-run capacity of bottleneck i in user equilibrium
t	Time
$t_A(t)$	Arrival time for commuter departing at time t
t^*	Desired arrival time at work
t_0^g	Earliest departure time for group g
t_e^g	Last departure time for group g
$T_g(t)$	Trip duration for commuter of group g departing at time t
1	Subscript denoting upstream bottleneck serving group 1
2	Subscript denoting upstream bottleneck serving group 2
TC	Aggregate travel costs

INTRODUCTION

Since Braess [8] it has been well known that adding capacity to a congested network can raise total travel costs. Braess demonstrated this by supposing a link is created that connects two routes in parallel running between a common origin and destination. Sternberg and Zangwill [27] extended the analysis to an arbitrary network and showed that adding a link increases costs if a determinantal condition is met that "... is about as likely to occur as not occur". (p.32)

It is also possible for an increase in capacity to reduce welfare when only one of the travel alternatives is congested. Suppose there is a roadway along a route served also by rail. If travel costs by road rise with the number of drivers, while rail costs are independent of ridership, then in equilibrium just that number of travelers will drive to raise travel costs by road to equal costs by rail. If road capacity is now expanded, users will shift to the road until it is as congested as before.

But if the railway has to balance its budget, the loss of revenue will force it to increase fares and cut service, inducing more passengers to switch so that travel costs on both modes end up higher than before the road expansion. This phenomenon, first noted by Downs [10] and Thomson [29], has been called the "Downs-Thomson paradox" by Mogridge, Holden, Bird and Terzis [20]. For later discussions see [13], [17] and [19].

The situations just described involve networks with alternative routes or modes on at least one of which unpriced congestion occurs. The user equilibrium is inefficient because travelers make decisions without taking into account their impact on system costs. The paradox arises when an investment causes users to alter their behavior in such a way that the

increased efficiency loss from underpricing of congestion exceeds the direct benefit of the improvement.

Several other examples of paradoxical behavior have been identified in the literature. Arnott and Mackinnon [5] consider a monocentric city in which the number of commuters and their routes are given, but the length of their trips is endogenous. An increase in the proportion of city land devoted to transportation creates an increase in lot size, causing the city to expand and more commuting traffic to occur at any given distance from the centre.

Another paradox, discussed by Sheffi [24, Ch. 10], can occur under stochastic network loading when a relatively costly alternative is marginally improved: travelers may switch to it in sufficient numbers that total costs increase. This phenomenon is driven by randomness of perceived travel times, rather than unpriced congestion.

Finally, Harker [12] and Catoni and Pallottino [9] have recently shown that, on a network with fixed link capacities and a fixed O-D demand matrix, total travel costs can be higher than in user equilibrium when some users generate an appreciable fraction of the demand, and select routes so as to internalize the congestion costs they impose on themselves.

In this paper we show that a paradox can result when the number of drivers, as well as their origins, destinations, travel modes and routes are all fixed, but they may choose *when* to travel. To demonstrate this we consider a Y-shaped travel corridor with one bottleneck on each arm and a third bottleneck downstream. Two groups of commuters use the corridor, one entering each arm and passing through the corresponding upstream bottleneck as well as the bottleneck downstream on their way to work. We show that expanding one of the upstream bottlenecks may induce drivers to alter their departure times in such a way as to increase the sum of total queueing and

schedule delay costs.

As is true of most other paradoxes, the paradox here arises because of unpriced congestion; however the behavioral adjustment that generates the perverse increase in costs is different. As a corollary of the result that increasing capacity can raise costs we show that metering access to an upstream bottleneck can improve welfare. Traffic engineers have long known that ramp metering can increase efficiency, either by preventing flow from exceeding capacity, or by encouraging drivers to use alternative surface routes. (See, for example, [1], [16, Ch. 25] and [24, p.77].) Here, neither of these factors is at work; the benefits of metering come from adjustments in the times at which individuals travel.

The paradox we identify is reminiscent of, but distinct from, that studied by Smeed [26], who claimed that drivers on a given route can arrive earlier by leaving later. Ben-Akiva and de Palma [7] showed that such a result cannot happen under realistic traffic behavior assumptions. Our demonstration that delaying vehicles can decrease total travel costs revitalizes Smeed's conjecture, but within a modified framework in which travel cost includes schedule delay as well as travel time.

The two-bottleneck corridor model we consider is similar to that employed by Kuwahara [18] in a recent paper. (We wrote the first version of this paper before seeing Kuwahara's, and have modified our notation somewhat to conform to his.) Kuwahara provides a comprehensive derivation of the time pattern of departures and queueing at the two bottlenecks under various assumptions about the distribution of work start times of drivers and their travel time and schedule delay cost functions. He also suggests ramp metering as a way to alleviate a certain equity problem (described later) but does not solve for equilibrium travel costs, consider optimal capacity or identify the paradox.

In Section 1 we specify the model and in Section 2 demonstrate the paradox. Ramp metering is considered in Sections 3 and 4 and optimal capacities in Section 5. Concluding remarks are made in Section 6.

1. THE MODEL

The model is similar to Kuwahara's (1990); thus the description will be brief. We consider a network, shown in Figure 1, with two upstream bottlenecks with maximum service rates or capacities s_1 and s_2 , and one downstream bottleneck with capacity s_d . If the arrival rate of vehicles at bottleneck 1 exceeds s_1 a queue, Q_1 , develops behind it. There is assumed to be no congestion elsewhere on the network, or on access routes to and from residential areas and workplaces. Without loss of generality free-flow travel times are set to zero, so that travel time consists only of queueing time. It is assumed that queue discipline is first-in, first-out (FIFO) and that the service rate of a bottleneck, when saturated, is independent of the number of drivers queued up. The physical length of the queue behind a bottleneck is immaterial as long as vehicles are not backed up so far as to interfere with bottlenecks upstream. Consistent with our assumptions of constant free-flow travel time and a maximum service rate independent of queue length, empirical studies have found that travel speed on freeways declines only slightly with flow until capacity is approached, and that the discharge rate of vehicles from a queue is equal to or only slightly below free-flow capacity (see, e.g., [6], [11], [14] and [15]).

Individuals live at different locations relative to the corridor. Group g enters upstream of bottleneck g , $g = 1, 2$. Choice of entry point for each driver is assumed fixed. Drivers travel one per car. The departure rate from

home of group g (the rate at which it enters the corridor) at time t is $d_g(t)$.

Drivers are assumed to incur costs from travel time and from schedule delay: arriving at work earlier or later than desired. All have the same travel cost function and the same desired arrival time, t^* . For simplicity it is also assumed that late arrival is not permitted. Let α denote cost per unit of travel time and β cost per unit time of early arrival. (For there to exist a deterministic equilibrium it is necessary to assume $\alpha > \beta$; the case $\alpha \leq \beta$ is considered for a single bottleneck in [2]). The travel cost of a group g commuter departing at time t can then be written

$$C_g(t) = \alpha T_g(t) + \beta [t^* - (t + T_g(t))], \quad (1)$$

where $T_g(t)$ is trip duration. If $Q_i(t)$ is the number of drivers in the queue behind bottleneck i at time t then the trip duration of a driver in group g departing at time t is

$$T_g(t) = \frac{Q_g(t)}{s_g} + \frac{Q_d \left(t + \frac{Q_g(t)}{s_g} \right)}{s_d}, \quad (2)$$

where $t + \frac{Q_g(t)}{s_g}$ is the time at which the driver arrives at bottleneck d .

2. THE CAPACITY EXPANSION PARADOX

In the model the route taken by each driver is predetermined. However, drivers can choose *when* to leave for work. Equilibrium on a day-to-day basis obtains when no driver can reduce his travel costs by departing at a different time. In this subsection we derive the equilibrium departure rate and travel costs of each group on the assumption that s_1 is sufficiently large that it is not binding. The network reduces then to a corridor with one upstream bottleneck 2 and one downstream bottleneck d ; queueing time for group 1 is

$$T_1(t) = \frac{Q_d(t)}{s_d}. \quad (3)$$

We focus on this simplified case in Section 2 because the paradox can be illustrated most clearly this way.

Let τ_g be the set of times during which group g departs, and let \bar{C}_g be its equilibrium travel cost. By definition, equilibrium obtains if and only if

$$C_g(t) \begin{cases} = \bar{C}_g & \text{for } t \in \tau_g, \\ \geq \bar{C}_g & \text{for } t \notin \tau_g, \end{cases} \quad g = 1, 2. \quad (4)$$

Clearly, $\bar{C}_2 \geq \bar{C}_1$, since someone in group 1 can always arrange to reach bottleneck d at the same time as someone in group 2, thereby incurring the same queueing time at bottleneck d , and the same schedule delay, but no queueing time at bottleneck 2. Thus, either $\bar{C}_2 = \bar{C}_1$ or $\bar{C}_2 > \bar{C}_1$. (These cases are referred to as Case 2 and Case 3 respectively by Kuwahara [18, Section 2.1].) We first derive equilibrium on the assumption that $\bar{C}_2 > \bar{C}_1$ and identify the parameter conditions under which it obtains. Then we consider $\bar{C}_2 = \bar{C}_1$.

Case A: $\bar{C}_2 > \bar{C}_1$

In equilibrium a queue must exist at bottleneck 2 throughout τ_1 , since otherwise someone in group 2 could traverse bottleneck 2 when it has no queue and incur the same cost as someone in group 1 departing at the same time. By (4), $C_1(t)$ must be constant during τ_1 . Differentiating (1) with respect to t and using (3), the equilibrium condition (4) implies for $t \in \tau_1$

$$\dot{Q}_d(t) = \frac{\beta}{\alpha - \beta} s_d \quad (5)$$

where the dot denotes a time derivative. Given a fixed and deterministic bottleneck service rate the number of vehicles in the queue follows the law of motion

$$\dot{Q}_d(t) = r_d(t) - s_d, \quad (6)$$

where r_d is the aggregate arrival rate of both groups at the downstream bottleneck. Since $Q_2(t) > 0$ for $t \in \tau_1$,

$$r_d(t) = s_2 + d_1(t) \quad \text{for } t \in \tau_1, \quad (7)$$

and hence from (5), (6) and (7)

$$d_1(t) = \frac{\alpha}{\alpha-\beta} s_d - s_2 \quad \text{for } t \in \tau_1. \quad (8)$$

Since $d_1(t) > 0$, a necessary condition for $\bar{C}_2 > \bar{C}_1$ is

$$\frac{s_2}{s_d} < \frac{\alpha}{\alpha-\beta}. \quad (9)$$

We now show that τ_1 and τ_2 are as indicated in Figure 2, with $\tau_g = [t_0^g, t_e^g]$, where subscript 0 (zero) indicates the beginning of departures and e the end. The last driver in each group arrives at work on time; that is $t_A(t_e^1) = t_A(t_e^2) = t^*$, where $t_A(\cdot)$ denotes arrival time. (Figure 2 is drawn with $t_e^2 > t_0^1$, but $t_e^2 \leq t_0^1$ is also possible.) A proof is given by the following four lemmas.

$$(L1) \quad t_0^2 < t_0^1.$$

Proof: If not, the first group 2 driver would depart home at the same time as someone in group 1 and arrive at work at the same time, thus incurring the same travel cost, a contradiction.

$$(L2) \quad t_A(t_e^2) = t^*.$$

Proof: $t_A(t_e^2) \leq t^*$ since otherwise the last drivers in group 2 would arrive late and incur an infinite travel cost. If $t_A(t_e^2) < t^*$, anyone in group 2 could delay departure until bottleneck 2 cleared, thereby incurring the same queueing time and schedule delay as the driver departing at t_e^2 but less delay at bottleneck 2, a contradiction.

$$(L3) \ t_A(t_e^1) = t^*.$$

Proof: $t_A(t_e^1) \leq t^*$ for the same reason as $t_A(t_e^2) \leq t^*$. If $t_A(t_e^1) < t^*$ then for an interval of time after t_e^1 , $r_d(t) = s_2$ and $\dot{Q}_d(t) = s_2 - s_d < \frac{\beta}{\alpha - \beta} s_d$ by (9). Given (5) this would mean that group 1 drivers would be better off departing after t_e^1 , a contradiction.

$$(L4) \ t_e^2 < t_e^1.$$

Proof: This follows from (L2), (L3) and the fact that there is a queue at bottleneck 2 during τ_1 .

t_0^1 , t_e^1 , t_0^2 and t_e^2 can be solved with equations (10)-(13) that follow.

From (L3)

$$t^* - t_e^1 = \frac{Q_d(t_e^1)}{s_d}. \quad (10)$$

By (8)

$$t_e^1 - t_0^1 = \frac{n_1}{\frac{\alpha}{\alpha - \beta} s_d - s_2}. \quad (11)$$

Since bottleneck 2 is fully utilized from t_0^2 to t_e^1 ,

$$t_e^1 - t_0^2 = \frac{n_2}{s_2}. \quad (12)$$

Finally,

$$t_e^1 = t_e^2 + \frac{Q_2(t_e^2)}{s_2}. \quad (13)$$

It turns out that the welfare properties of equilibrium depend on whether $s_2 > s_d$ or $s_2 \leq s_d$. We begin with the case $s_2 > s_d$, for which the promised capacity expansion paradox occurs.

Case A1: $\bar{C}_2 > \bar{C}_1, s_2 > s_d$

The equilibrium departure and arrival distributions for Case A1 are shown in Figure 3 for a numerical example. The unit cost of travel time is set at $\alpha = 6.0$. β follows from Small's [25, Table 2, model 1] estimate of $\beta/\alpha = 0.61$. t^* is set at 8:30 (times are written in decimal form). Other parameters are chosen in order to obtain Case A1.

By (L1), only group 2 departs initially. Differentiating (1) and using (2), the equilibrium condition (4) implies

$$-\beta + (\alpha - \beta) \left(\frac{\dot{Q}_2(t)}{s_2} \right) + \frac{\dot{Q}_d[t + Q_2(t)/s_2]}{s_d} \left(1 + \frac{\dot{Q}_2(t)}{s_2} \right) = 0 \quad \text{for } t < t_0^1.$$

As shown below, $d_2(t) > s_2$. Thus, $\dot{Q}_2(t) = d_2(t) - s_2$ and $\dot{Q}_d(t) = s_2 - s_d$.

Making the substitutions one obtains

$$d_2(t) = \frac{\alpha}{\alpha - \beta} s_d \quad \text{for } t < t_0^1.$$

That $d_2(t) > s_2$ is confirmed by condition (9). Given $\dot{Q}_d(t) = s_2 - s_d$, $Q_d(t_0^1) = (s_2 - s_d)(t_0^1 - t_0^2)$, or by (11) and (12)

$$Q_d(t_0^1) = (s_2 - s_d) \left(\frac{n_2}{s_2} - \frac{n_1}{\frac{\alpha}{\alpha - \beta} s_d - s_2} \right). \quad (14)$$

When group 1 starts to travel, it departs at rate $d_1(t)$ given by (8). $Q_d(t)$ continues to rise at the same rate as before t_0^1 . The travel cost of group 2 remains constant provided the queue length at bottleneck 2 remains constant; thus $d_2(t) = s_2$. Now

$$Q_d(t_e^1) = Q_d(t_0^1) + \int_{t_0^1}^{t_e^1} \dot{Q}_d(t) dt,$$

or by (5), (11) and (14)

$$Q_d(t_e^1) = n_1 + n_2 - \frac{n_2 s_d}{s_2}. \quad (15)$$

Given that all group 2 incurs the same cost as the first driver to depart

$$\bar{C}_2 = C_2(t_0^2) = \beta \left(t^* - t_0^2 \right) = \beta \left[\left(t^* - t_e^1 \right) + \left(t_e^1 - t_0^2 \right) \right],$$

which by (10), (12) and (15) reduces to

$$\bar{C}_2 = \beta \left(\frac{n_1 + n_2}{s_d} \right). \quad (16)$$

This reflects the fact that $n_1 + n_2$ users flow through the downstream bottleneck d between t_0^2 and t^* . Similarly,

$$\bar{C}_1 = C_1(t_e^1) = \alpha \frac{Q_d(t_e^1)}{s_d},$$

or using (15)

$$\bar{C}_1 = \alpha \left(\frac{n_1 + n_2}{s_d} - \frac{n_2}{s_2} \right) = \beta \frac{n_1 + n_2}{s_d} + (\alpha - \beta) \frac{n_1 + n_2}{s_d} - \alpha \frac{n_2}{s_2}. \quad (17)$$

Thus, by (16) and (17)

$$\bar{C}_2 - \bar{C}_1 = \alpha \frac{n_2}{s_2} - (\alpha - \beta) \left(\frac{n_1 + n_2}{s_d} \right).$$

Since by assumption $\bar{C}_2 > \bar{C}_1$, consistency requires

$$\frac{s_2}{s_d} < \frac{\alpha}{\alpha - \beta} \frac{n_2}{n_1 + n_2}. \quad (18a)$$

Introducing the dimensionless variables $\theta \equiv \beta/\alpha$, $\nu \equiv n_1/n_2$ and $\sigma_2 \equiv s_2/s_d$, (18a) can be written

$$\sigma_2 < \tilde{\sigma}_2 \equiv \frac{1}{1 - \theta} \frac{1}{1 + \nu}. \quad (18b)$$

This condition is more stringent than (9) above, assuring that (9) is satisfied if (18) is. Total travel costs are

$$TC = n_1 \bar{C}_1 + n_2 \bar{C}_2 = (\alpha n_1 + \beta n_2) \left(\frac{n_1 + n_2}{s_d} \right) - \frac{\alpha n_1 n_2}{s_2}. \quad (19)$$

Clearly, $\frac{\partial TC}{\partial s_d} < 0$, but $\frac{\partial TC}{\partial s_2} > 0$: expanding capacity of the downstream bottleneck

d reduces costs, but paradoxically expanding the upstream bottleneck 2 is harmful.

To see intuitively why the paradox occurs, note first from (16) that the travel cost of group 2 does not depend on s_2 ; expanding s_2 merely transfers queueing time spent by group 2 downstream. Expanding s_2 however increases travel costs for group 1. The last group 1 individual to depart faces a queue equal to the number of individuals who have arrived at the bottleneck, n_1+n_2 , minus the number who have got through, $\left(\frac{n_2}{s_2}\right)s_d$. Increasing s_2 lengthens this queue.

Looked at another way: expansion of s_2 allows group 2 to arrive at a greater rate downstream, thereby partially crowding out group 1. This lengthens the period over which group 1 departs and hence increases its costs. In another paper ([3]) we have derived the dynamic user equilibrium for a model with the same cost specification as here, but on a network with two routes in parallel that do not merge. No paradox results in this case, suggesting that it is dependent on having a series configuration.

Case A2: $\bar{C}_2 > \bar{C}_1$, $s_2 \leq s_d$

To save space, derivations for the remaining two cases are not provided here (see Appendix A in [4]). The equilibrium departures and arrivals are shown in Figure 4 for the same parameters as in Figure 3, except for $s_2 = 0.80$ and $s_d = 1.00$. Figures 3 and 4 converge in the limit as s_2 approaches s_d , in which case the downstream bottleneck operates at capacity without a queue building up during the initial period when only group 2 is departing.

Individual and total costs for all 3 cases are collected in Table 1. In Case A2, trip costs are

$$\bar{C}_1 = \frac{\alpha\beta n_1}{\alpha(s_d - s_2) + \beta s_2}, \quad (20)$$

$$\bar{C}_2 = \beta \left[\frac{n_2}{s_2} + \frac{n_1\beta}{\alpha(s_d - s_2) + \beta s_2} \right], \quad (21)$$

$$TC = \beta \left(\frac{n_2^2}{s_2} + \frac{n_1(\alpha n_1 + \beta n_2)}{\alpha(s_d - s_2) + \beta s_2} \right). \quad (22)$$

As in Case A1, $\frac{\partial TC}{\partial s_d} < 0$. The consistency condition (18) for Case 1 also applies. But now the costs of group 2 depend on s_2 as well as s_d . Parameter values for which cases A1 and A2 obtain are shown in Figure 5. As indicated by the shaded region, the paradox ($\frac{\partial TC}{\partial s_2} > 0$) occurs throughout Case A1, and in Case A2 within the parameter region bounded by the curves $\tilde{\sigma}_2$ (defined in (18b)) and

$$\hat{\sigma}_2 \equiv \left[1 - \theta + \left(\nu(\nu + \theta)(1 - \theta) \right)^{1/2} \right]^{-1}. \quad (23)$$

From (20) it is clear that increasing s_2 makes group 1 worse off, as was true of Case A1. It is not obvious whether group 2 is better off. The first term in (21) is the cost that group 2 would incur if only the upstream bottleneck were present. The second term is the additional cost imposed by the downstream bottleneck. Increasing s_2 (and σ_2) reduces the first term but increases the second. However, it is easy to show that for $\sigma_2 < \text{Min}[1, \tilde{\sigma}_2]$, $\partial \bar{C}_2 / \partial s_2 < 0$, so that expanding the upstream bottleneck does make group 2 better off. But within the region $(\hat{\sigma}_2, \tilde{\sigma}_2)$, the loss to group 2 outweighs the gain to group 1: a capacity expansion reduces welfare overall.

Case B: $\bar{C}_2 = \bar{C}_1$

This case, in which the two groups incur equal travel costs, occurs when condition (18) is not satisfied. There is no queue at bottleneck 2; s_2 is not binding and total travel costs are the same as if everyone belonged to group 1:

$$TC = \frac{\beta(n_1+n_2)^2}{s_d}. \quad (24)$$

Evidently, $\frac{\partial TC}{\partial s_d} < 0$ and $\frac{\partial TC}{\partial s_2} = 0$, so that no paradox results.

Cases A1, A2 and B are mutually exclusive and collectively exhaustive (see Figure 5). Therefore the necessary conditions identified in each case are also sufficient. Total and individual travel costs are unique by construction.

The effects of capacity expansion are summarized in:

PROPOSITION 1

Total travel cost decreases when the downstream bottleneck is expanded, but may increase or decrease, depending on parameter values, when the upstream bottleneck is expanded:

$$\frac{\partial TC}{\partial s_d} < 0,$$

$$\frac{\partial TC}{\partial s_2} \begin{cases} < 0 \\ > 0 \\ = 0 \end{cases} \text{ iff } \begin{cases} \sigma_2 < \min [1, \hat{\sigma}_2], \\ \min[1, \hat{\sigma}_2] < \sigma_2 < \tilde{\sigma}_2, \\ \sigma_2 > \tilde{\sigma}_2, \end{cases}$$

where $\tilde{\sigma}_2$ and $\hat{\sigma}_2$ are defined by (18b) and (23) respectively.

3. RAMP METERING

According to Proposition 1, increasing the capacity of the upstream bottleneck 2 raises total travel costs under some circumstances. Turning this around, it follows that in these instances *reducing* s_2 would be beneficial, which suggests metering as a policy control. Proposition 1 shows that travel costs are everywhere a decreasing function of downstream capacity, indicating that the *downstream* bottleneck should never be metered. However, it is conceivable that metering the access rate of group 1 to the downstream bottleneck s_1 would be helpful.

In this section we investigate the effects of metering, first of group 2 on the assumption that s_1 is not binding, then of group 1 on the assumption that s_2 is not binding, and finally of both groups. We ignore the infrastructure and operation costs of metering. As envisaged here, metering does not conform to the principle specified in [16, p.786] "to limit the number of vehicles entering the freeway so that the demand on the freeway will not exceed its capacity". The intent here is to reduce queueing rather than prevent it. Indeed, we show in Section 4 that it is never optimal to expand a bottleneck to the point where queueing at it is eliminated. Problems of ensuring adequate storage space for queued vehicles, and congestion on surface routes, are also ignored.

In Case A, where s_2 is binding, group 2 incurs a higher travel cost than group 1 because it is restricted in the rate it can access the downstream bottleneck, whereas group 1 is not. Kuwahara [18, p.227] calls this an "equity problem" and suggests as a possible countermeasure ramp metering of group 1. In fact, there is no equity problem as such if drivers living upstream are compensated for their greater travel costs by lower housing costs, as should happen in the long run if residents are mobile. Accordingly, we treat metering here as a means of increasing overall efficiency, rather than improving the welfare of a particular group.

(a) Metering of group 2 (s_1 not binding)

The optimal metering policy for this case follows immediately from Figure 5 and Proposition 1. If $\sigma_2 > \min[1, \hat{\sigma}_2]$ the upstream bottleneck 2 should be metered until $\sigma_2 = \min[1, \hat{\sigma}_2]$. (Note that if $\sigma_2 > \tilde{\sigma}_2$, metering has no effect until $\sigma_2 = \tilde{\sigma}_2$ and then becomes beneficial.) If $\sigma_2 \leq \min[1, \hat{\sigma}_2]$, metering is unwarranted.

These conditions apply under the assumption that s_1 is not binding. The question of which bottleneck to meter (if there is a choice) and by how much is considered later.

Metering of group 1

Metering of group 1 when s_2 is not binding is formally identical to metering of group 2 when s_1 is not binding, only with subscripts 1 and 2 interchanged. Corresponding to Case A1 there is a new Case $\overline{A1}$ with $\bar{C}_2 < \bar{C}_1$ and $s_1 > s_d$, and corresponding to Case A2 there is a new Case $\overline{A2}$ with $\bar{C}_2 < \bar{C}_1$ and $s_1 \leq s_d$. Case B is the same as before.

(b) Both upstream bottlenecks s_1 and s_2 binding

In the corridor configuration considered above in which the upstream bottleneck serving group 1 was not binding, group 1 departed at rate $\frac{\alpha}{\alpha-\beta} s_d - s_2$. We now assume

$$s_1 < \frac{\alpha}{\alpha-\beta} s_d - s_2 \quad (25)$$

so that s_1 is binding as well as s_2 . We also assume

$$s_1 + s_2 > s_d, \quad (26)$$

since otherwise *downstream* capacity would not be binding, and the two groups would not interact. Four new cases emerge under these assumptions, which we call AA1, AA2, $\overline{AA1}$ and $\overline{AA2}$. Since $\overline{AA1}$ and $\overline{AA2}$ are mirror images of AA1 and AA2, we need consider only the latter two.

Case AA1: $\bar{C}_2 > \bar{C}_1, s_2 > s_d$

Equilibrium travel costs for this case can be shown (see Appendix B of ADL (1991)) to be:

$$\bar{C}_1 = \alpha \left(\frac{n_1 + n_2}{s_d} - \frac{n_2}{s_2} \right) + \left(\alpha - (\alpha-\beta) \frac{s_1 + s_2}{s_d} \right) \frac{n_1}{s_1}, \quad (27a)$$

$$= \beta \left(\frac{n_1 + n_2}{s_d} \right) - \left(\alpha - (\alpha - \beta) \frac{s_2}{s_d} \right) \left(\frac{n_2}{s_2} - \frac{n_1}{s_1} \right), \quad (27b)$$

$$\bar{C}_2 = \beta \left(\frac{n_1 + n_2}{s_d} \right), \quad (28)$$

$$TC = \beta \frac{(n_1 + n_2)^2}{s_d} - \left(\alpha - (\alpha - \beta) \frac{s_2}{s_d} \right) \left(\frac{n_2}{s_2} - \frac{n_1}{s_1} \right) n_1. \quad (29)$$

For this to be consistent with the premise $\bar{C}_2 > \bar{C}_1$ we require by (27b) and

(28) that $\frac{n_2}{s_2} > \frac{n_1}{s_1}$: upstream capacity per user must be higher for group 1 than

group 2. Equation (28) reveals that \bar{C}_2 is the same as without metering (equation (16)), but \bar{C}_1 is larger (compare (27a) with (17)) since the term in the second pair of brackets in (27a) is positive by condition (25). Thus,

in Case AA1 metering of group 1 is harmful. Furthermore, $\frac{\partial \bar{C}_2}{\partial s_2} = 0$, $\frac{\partial \bar{C}_1}{\partial s_1} > 0$

and $\frac{\partial TC}{\partial s_2} > 0$: in Case AA1 metering of group 2 is desirable.

Case AA2: $\bar{C}_2 > \bar{C}_1$, $s_2 \leq s_d$

User costs in this case are (see Appendix B in [4]):

$$\bar{C}_1 = \beta \left(\frac{n_1 + n_2}{s_d} \right) - \beta \frac{s_2}{s_d} \left(\frac{n_2}{s_2} - \frac{n_1}{s_1} \right), \quad (30)$$

$$\bar{C}_2 = \beta \left(\frac{n_1 + n_2}{s_d} \right) + \beta \frac{s_d - s_2}{s_d} \left(\frac{n_2}{s_2} - \frac{n_1}{s_1} \right), \quad (31)$$

$$TC = \beta \frac{(n_1 + n_2)^2}{s_d} + \frac{\beta}{s_d} \left[n_2 s_d - (n_1 + n_2) s_2 \right] \left(\frac{n_2}{s_2} - \frac{n_1}{s_1} \right). \quad (32)$$

Evidently, $\frac{\partial \bar{C}_1}{\partial s_1} < 0$, $\frac{\partial \bar{C}_2}{\partial s_1} > 0$: in Case AA2, metering group 1 benefits group 1

but raises costs for group 2. Furthermore,

$$\frac{\partial TC}{\partial s_1} \cong \frac{s_d}{n_1 + n_2} - \frac{s_2}{n_2} \cong (1 + \nu)^{-1} - \sigma_2. \quad (33)$$

In Case AA2, metering group 1 reduces total travel costs if and only if downstream capacity per person for the two groups combined is greater than upstream capacity per person in group 2. Furthermore, metering group 2 is beneficial if and only if

$$\sigma_2 > \sqrt{\frac{1}{\nu(1+\nu)}} \sqrt{\sigma_1}. \quad (34)$$

As in Case AA1, $\bar{C}_2 > \bar{C}_1$ iff $\frac{n_2}{s_2} > \frac{n_1}{s_1}$.

Case $\overline{AA1}$ with $\bar{C}_2 < \bar{C}_1$ and $s_1 > s_d$ is obtained from case AA1 by

interchanging the role of the bottlenecks. Case $\overline{AA2}$ with $\bar{C}_2 < \bar{C}_1$ and $s_1 \leq s_d$ is obtained similarly from case AA2. These cases require no discussion. It is straightforward to establish that metering *both* groups is not optimal for any of the four cases in which both upstream bottlenecks are binding. It is also readily checked that in all cases \bar{C}_1 and \bar{C}_2 are nondecreasing functions of n_1 and n_2 . Newell [22, pp. 167-8] gives an example of a network in static equilibrium, with 3 groups, 2 origins and 2 destinations, for which an increase in the size of one of the groups *reduces* travel costs for another group. It remains to be seen whether a 'paradox' of this sort can arise for dynamic equilibrium in more complex networks.

This completes examination of the various equilibrium configurations individually. We now turn to consideration of the optimal metering strategy.

4. OPTIMAL METERING

In this section we characterize the globally optimal metering policy; that is the optimal effective capacities $(s_1^*, s_2^*) \leq (s_1, s_2)$ for given initial capacities s_1, s_2 and s_d . A comprehensive analysis would be tedious because there are 9 equilibrium cases and because total costs are not a concave

function of s_1 and s_2 . A reasonably complete picture can nevertheless be derived from Figure 6, in which the various equilibrium cases are depicted as a function of σ_1 and σ_2 for particular values of α , β and ν . Figure 6 is constructed on the assumption that

$$\tilde{\sigma}_2 = \frac{1}{1-\theta} \frac{1}{1+\nu} > 1 \quad \text{and} \quad \tilde{\sigma}_1 = \frac{1}{1-\theta} \frac{\nu}{1+\nu} > 1.$$

The other three possibilities yield similar figures and conclusions, but some of the 9 equilibrium configurations are missing from them.

The parameter values for which metering one upstream bottleneck or the other reduces total costs are indicated by the shaded areas. Arrows point in the direction of falling costs. As noted above, metering both groups is never optimal: the shaded areas do not intersect. However, if equilibrium begins in region B, where neither upstream bottleneck is binding, it is always desirable to meter one of the groups. This is clear in Figure 5, which shows (in the case where only s_2 is binding) that metering group 2 is desirable if equilibrium begins in region B, whatever the value of ν , and whether equilibrium passes through both Case A1 and Case A2 as s_2 is reduced, or only one of them.

The optimal metering policy can be deduced by inspection of Figure 6.

Upstream capacity per person is the same for the two groups (*i.e.* $\frac{s_2}{n_2} = \frac{s_1}{n_1}$) along the ray $\sigma_2 = \frac{1}{\nu} \sigma_1$, extending from the origin through points P_0 and P_3 . This ray is bounded by four cases: AA1, AA2, $\overline{AA1}$ and $\overline{AA2}$. It is readily checked from the respective equations (29) and (32) that total costs are constant along the ray, and hence equal to costs in region B. Expanding s_1 and s_2 proportionally along the ray reduces queueing upstream at the expense of queueing downstream, leaving total travel costs unchanged.

Total costs are also constant along the lines P_0P_1 and P_0P_5 . Using the arrows as a guide it follows by a process of elimination that total costs reach a global minimum either at point P_2 (or on the line extending north of it) or at point P_4 (or on the line extending east of it). As drawn, the respective equilibria are of type $\overline{A2}$ and $A2$. Were $\nu < \bar{\nu}$, P_4 would lie on the boundary between $A1$ and $A2$, and were $1/\nu < \bar{\nu}$, P_2 would lie on the boundary between $\overline{A1}$ and $\overline{A2}$.

Using equations (19) and (22) it is straightforward, though tedious, to show that total costs are lower at P_2 than P_4 iff $\nu > 1$. Thus, it is optimal to meter the larger group. The optimal metering policy is asymmetric in that, at either P_2 or P_4 , upstream capacities per person are unequal for the two groups

If either or both of P_2 and P_4 are infeasible for some practical reason, the optimal metering policy has to be deduced by searching over the feasible subregion of the region: $0 \leq s_1^* \leq s_1$, $0 \leq s_2^* \leq s_2$. At most one of the two groups should be metered. Nevertheless, metering can be beneficial even if both upstream capacities are initially below their globally optimal values. For example, if the initial equilibrium is at P_7 , costs can be reduced by metering group 1 to move the equilibrium to P_6 .

For the example in Figure 6, $\nu < 1$, so that metering group 2 is optimal. Moving from region B to P_4 reduces total costs by 12.2%. Moving to point P_2 by metering group 1 reduces costs by 8.7%. And moving from P_8 with $(\sigma_1, \sigma_2) = (2.46, 0.1)$ to $P_6 = (0.9, 0.1)$ by metering group 1 reduces costs by 4.1%. Thus, for this example at least, the gains from metering are small but appreciable.

The general results on optimal metering derived here are summarized in

PROPOSITION 2

(a) If the capacities of the upstream bottlenecks serving groups 1 and 2 are not binding and sufficiently large, then it is optimal to meter the larger group; specifically:

$$\text{if } \nu < 1 \text{ then } \sigma_2^* = \min[1, \hat{\sigma}_2], \quad \sigma_1^* = \sigma_1,$$

$$\text{if } \nu > 1 \text{ then } \sigma_1^* = \min[1, \hat{\sigma}_1], \quad \sigma_2^* = \sigma_2,$$

where $\hat{\sigma}_2$ and $\hat{\sigma}_1$ are given respectively by (23) and its symmetric counterpart, and σ_1 and σ_2 are the original capacity ratios.

(b) It is never optimal to meter both groups.

5. OPTIMAL CAPACITIES FOR THE USER EQUILIBRIUM

Section 2 demonstrated that expanding capacity can increase costs. In this section we take the analysis further by deriving optimal capacities s_2^* and s_d^* on the assumption that s_1 is not binding. Amongst other results we show that capacity should never be increased to the point where queueing is eliminated at either bottleneck.

Optimal capacities s_2^* and s_d^* are defined by the solution to

$$\text{Min}_{s_2, s_d} \text{TC}(s_2, s_d) + k_2 s_2 + k_d s_d, \quad (34)$$

where marginal construction costs k_2 and k_d are assumed constant, but allowed to differ between bottlenecks because, for example, land acquisition costs may vary with location.

The solution to (34) (derived in Appendix C of [4]) is described by

PROPOSITION 3

In user equilibrium there exists $\nu^c \equiv \frac{\theta}{2} \left[\left(1 + \frac{4}{1 - \theta + k_2/k_d} \right)^{1/2} - 1 \right]$ such that:

(a) For $\nu \leq \nu^c$, optimal capacities of the upstream and downstream bottlenecks are equal: $s_d^* = s_2^*$.

For $\nu > \nu^c$, optimal downstream capacity is larger than optimal upstream capacity: $s_d^* > s_2^*$.

- (b) It is never optimal to expand either bottleneck to the point where queueing there is eliminated.
- (c) With optimal capacities, total travel costs equal total capacity construction costs.

A discussion of each part of Proposition 3 follows.

(a) If $\nu = 0$, so that all drivers live upstream of bottleneck 2, then $s_d^* = s_2^*$, since in a corridor with a single entry point and queueing congestion, travel costs are determined solely by the bottleneck with smallest capacity. Conversely, if ν is very large it is not worth investing much in the upstream bottleneck just to service a few drivers, and $s_d^* > s_2^*$.

Proposition 3 reveals in addition to this that optimal capacities are equal even when a nontrivial fraction of commuters lives downstream of s_2 . The reason is that in Case A2, where $s_2 < s_d$, no queue accumulates downstream until group 1 departs. Increasing s_2 thus benefits group 2, by allowing it to reach work faster, without harming group 1 until after t_0^1 . For ν sufficiently small, the overall benefit net of construction costs remains positive until $s_2 = s_d$ at the boundary between Cases A1 and A2: see Figure 5.

(b) Since it is not economical to eliminate congestion at either bottleneck, commuters living upstream incur higher travel costs than those living downstream. This result is sensitive to the assumption that all drivers have the same desired arrival time at work. If there is a distribution of desired times, as Newell [23] and Kuwahara [18] have assumed, travel costs will not vary hyperbolically with capacity, and it may be cost-effective to eliminate congestion, as Vickrey [30] showed for a single bottleneck via a numerical

example.

(c) The equality of travel costs and construction costs is an instance of a well known result in the literature (see, for example, [21] and [28]). It follows from the *assumption* that construction costs are homogeneous of degree 1 in capacity, and the *result* that total travel costs are homogeneous of degree -1. The equality of travel costs and construction costs is reminiscent of Lord Kelvin's rule that, at the optimum, the cost of building an electrical wire equals the energy lost over the life of the wire due to heat. (This brings to mind the early days on network analysis in transportation and operations research where analogies were made with electrical circuits.)

We offer the conjecture that equality of total travel and construction costs, when all capacities are optimal, holds for any network with queueing congestion when there are constant costs of capacity expansion. The argument is as follows: Solve for equilibrium on a network. Then halve the capacities of all the bottlenecks. Take each commuter in the pre-change equilibrium and double his schedule delay and queueing costs. It is conjectured that the resulting travel pattern is the post-change equilibrium. This would establish that $TC(\cdot)$ is homogeneous of degree -1 in all capacities, and hence that the congruence of travel and construction costs applies to any network.

6. CONCLUDING REMARKS

Braess [8] and others have shown that introducing a new link in a congested network, or adding capacity to an existing link, can raise total travel time. This paradox is due to the fact that congestion is unpriced, and route usage socially nonoptimal. In this paper we show that a similar paradox may result when modes and routes are fixed, but users are free to choose when to travel. Specifically, a capacity expansion can increase travel costs,

defined as the sum of travel time and schedule delay costs. Costs rise when drivers alter their travel times in such a way that the efficiency loss due to mispriced travel increases by more than the direct benefit of the capacity expansion.

To demonstrate this we consider a Y-shaped highway corridor with one bottleneck on each arm and a third bottleneck downstream. Two groups of commuters use the corridor, each passing through one of the upstream bottlenecks and then the bottleneck downstream. We show that expanding capacity of one of the upstream bottlenecks can raise total travel costs because reduced congestion upstream is more than offset by increased congestion downstream. Metering access in order to reduce effective upstream capacity can thus improve efficiency. If neither group is initially constrained by upstream capacity then it is efficient to meter the larger group.

In the static setting considered by Braess, a capacity expansion may increase travel time cost if at least two travelled routes between the same O-D pair share a link. In our dynamic framework, drivers have no choice of route in geographic space. However, each departure time can be seen as a temporal 'route', and similar to Braess we find that travel cost inclusive of schedule delay costs may increase as capacity is increased.

Regarding optimal capacity we show that for user equilibrium the upstream bottleneck should be no larger than the one downstream. In [4] we have shown that for the system optimum the discrepancy is sharper: optimal upstream capacity per user living upstream is strictly smaller than downstream capacity per person in both groups.

We also proved that, assuming bottlenecks of optimal capacity and constant capacity expansion costs, total construction costs equal total

variable travel costs. This follows from the result that variable travel costs are homogeneous of degree -1 in capacity, which derives from the assumption that congestion entails queueing behind bottlenecks.

The paradox identified here is robust to relaxation of some limiting assumptions. In particular, one can show that expanding upstream capacity can still raise travel costs when (a) there is a distribution of desired arrival times (the same for the two groups), and (b) late arrivals are permitted. This is to be expected since costs are a continuous function of the distribution of desired arrival times (assumed degenerate in our analysis) and of the penalty for late arrival (which we assumed to be infinite).

While the corridor considered here is very simple, the results and insights derived from it should aid in the study of more complex networks. For example, the congruence of construction and travel costs should apply to any network under the aforementioned assumptions on technology. It may be possible, by comparing total travel costs on a network to its total capacity costs, to determine whether there is over- or under-investment in capacity for the network as a whole. It may also be possible to identify general rules under which capacity expansion raises costs, as did Sternberg and Zangwill [27] for static equilibria in which drivers choose routes. The analogy between dynamic models and static models within a space-time framework would be one line of attack. There is a clear need, when congestion is underpriced, to model the entire traffic network and all behavioral margins of adjustment when considering a capacity expansion, or indeed any other policy.

Case	A1	A2	B
C_1	$\alpha \left(\frac{n_1 + n_2}{s_d} - \frac{n_2}{s_2} \right)$	$\beta \frac{\alpha n_1}{\alpha(s_d - s_2) + \beta s_2}$	$\beta \left(\frac{n_1 + n_2}{s_d} \right)$
C_2	$\beta \left(\frac{n_1 + n_2}{s_d} \right)$	$\beta \left(\frac{n_2}{s_2} + \frac{\beta n_1}{\alpha(s_d - s_2) + \beta s_2} \right)$	$\beta \left(\frac{n_1 + n_2}{s_d} \right)$
TC	$(\alpha n_1 + \beta n_2) \left(\frac{n_1 + n_2}{s_d} \right) - \frac{\alpha n_1 n_2}{s_2}$	$\beta \left(\frac{n_2^2}{s_2} + \frac{n_1 (\alpha n_1 + \beta n_2)}{\alpha(s_d - s_2) + \beta s_2} \right)$	$\beta \left(\frac{(n_1 + n_2)^2}{s_d} \right)$

Table 1: Total travel costs in user equilibrium

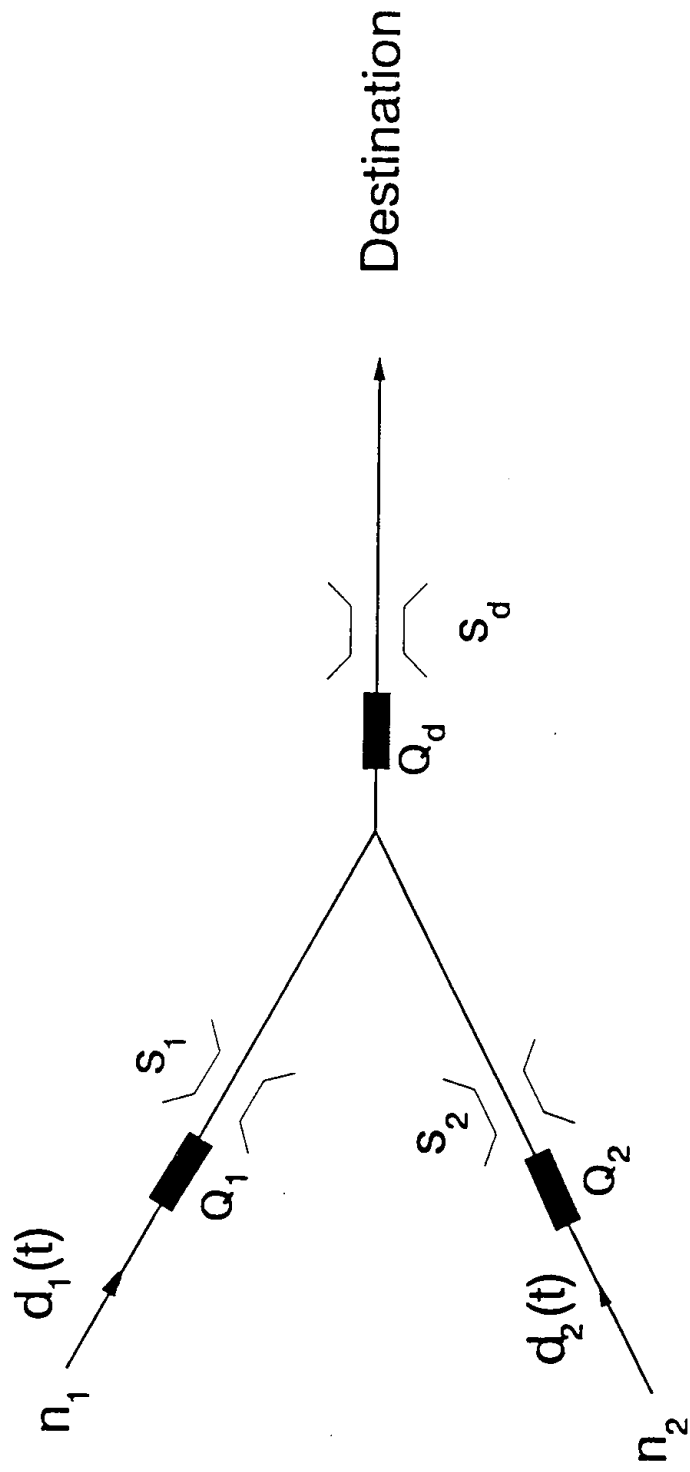


Figure 1: The bottleneck configuration

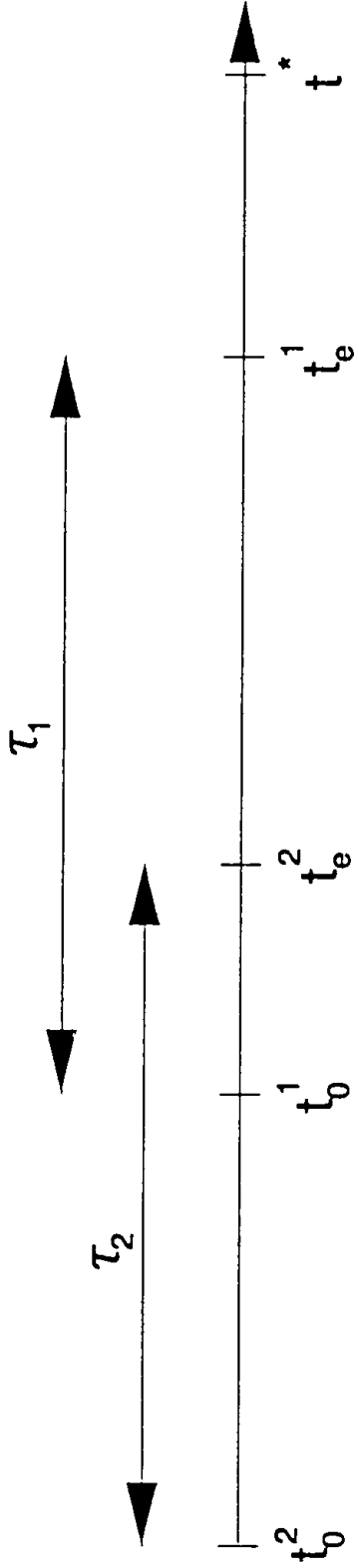
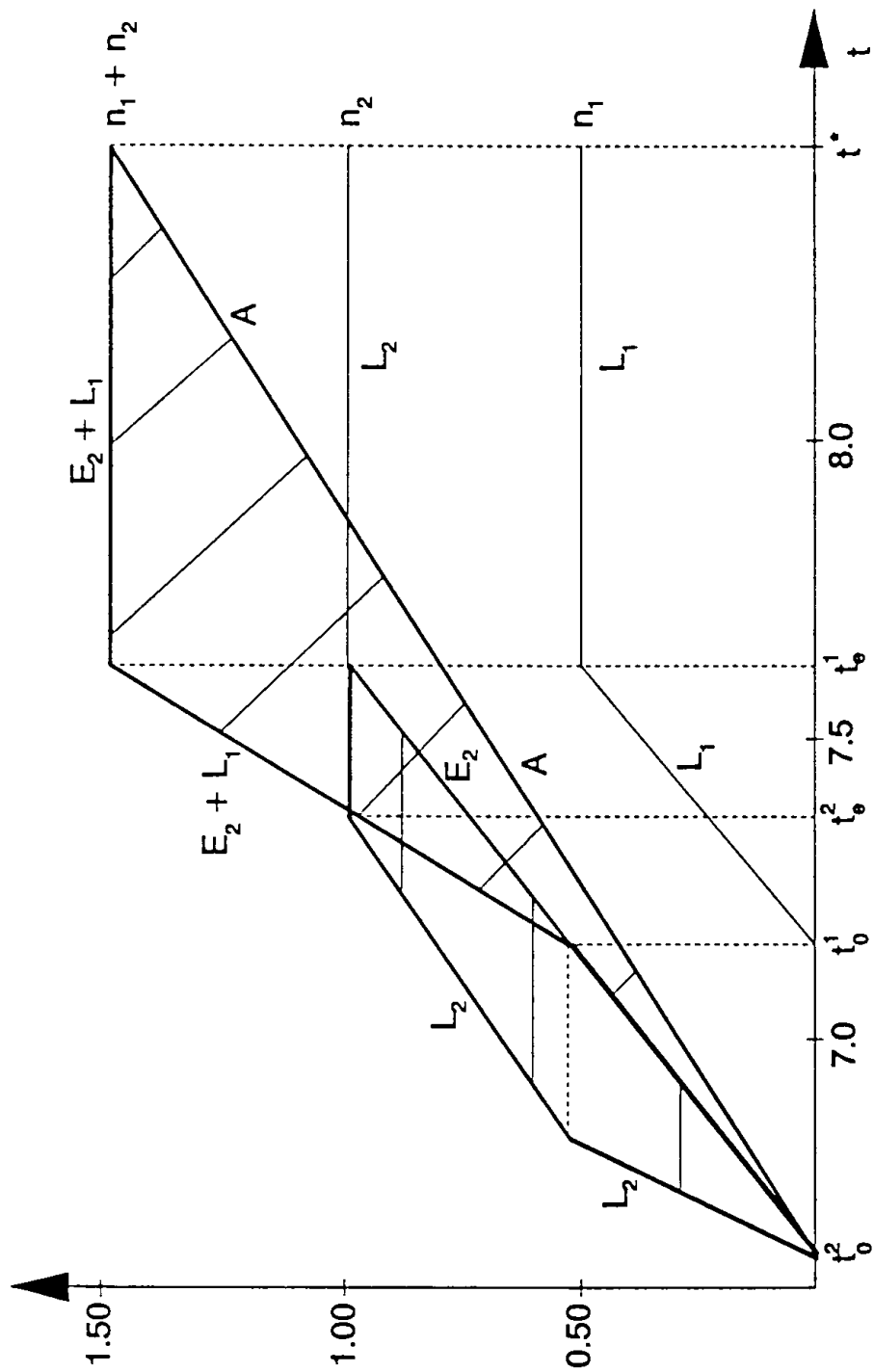
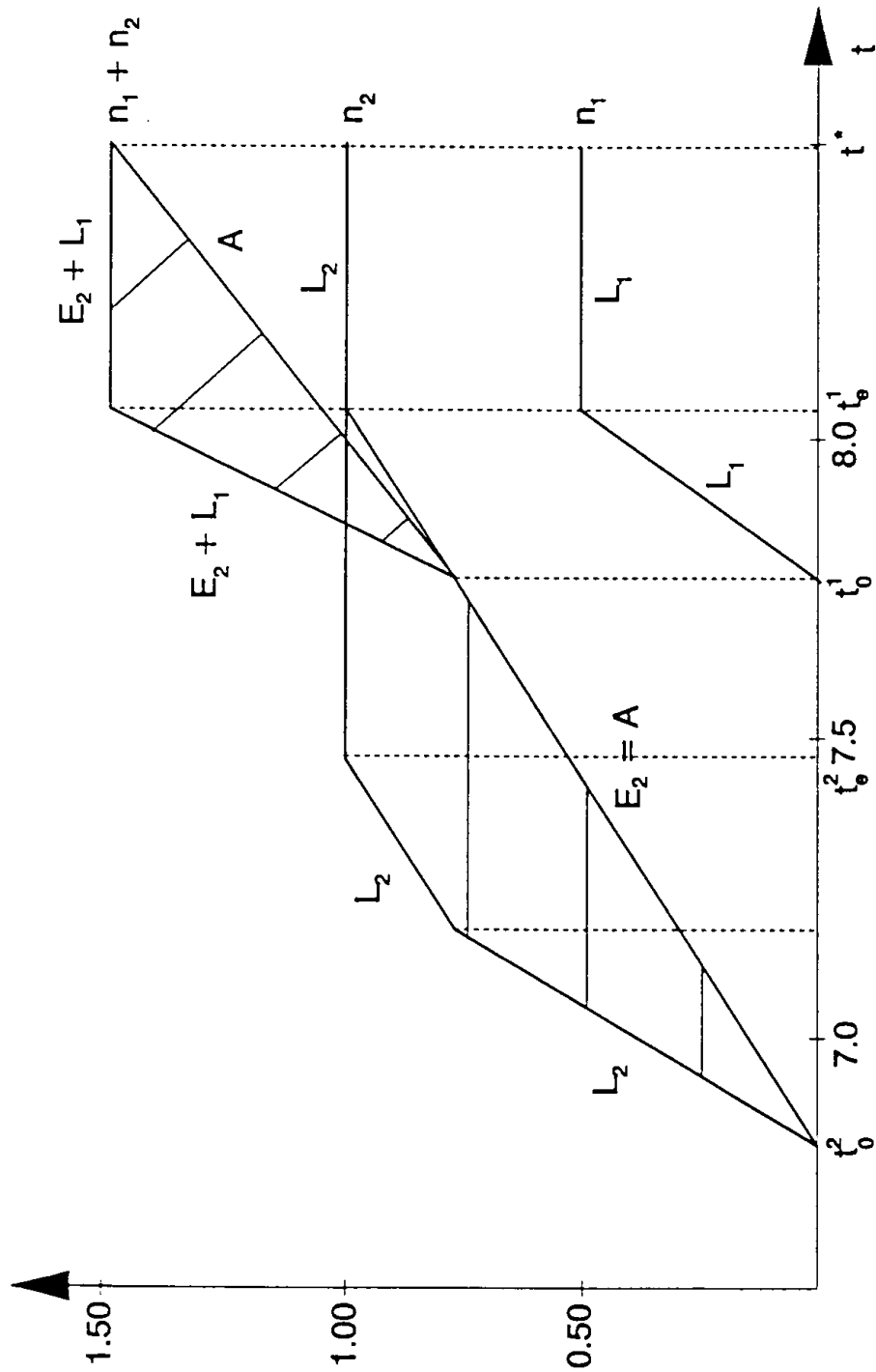


Figure 2. The time pattern of departures in Case A



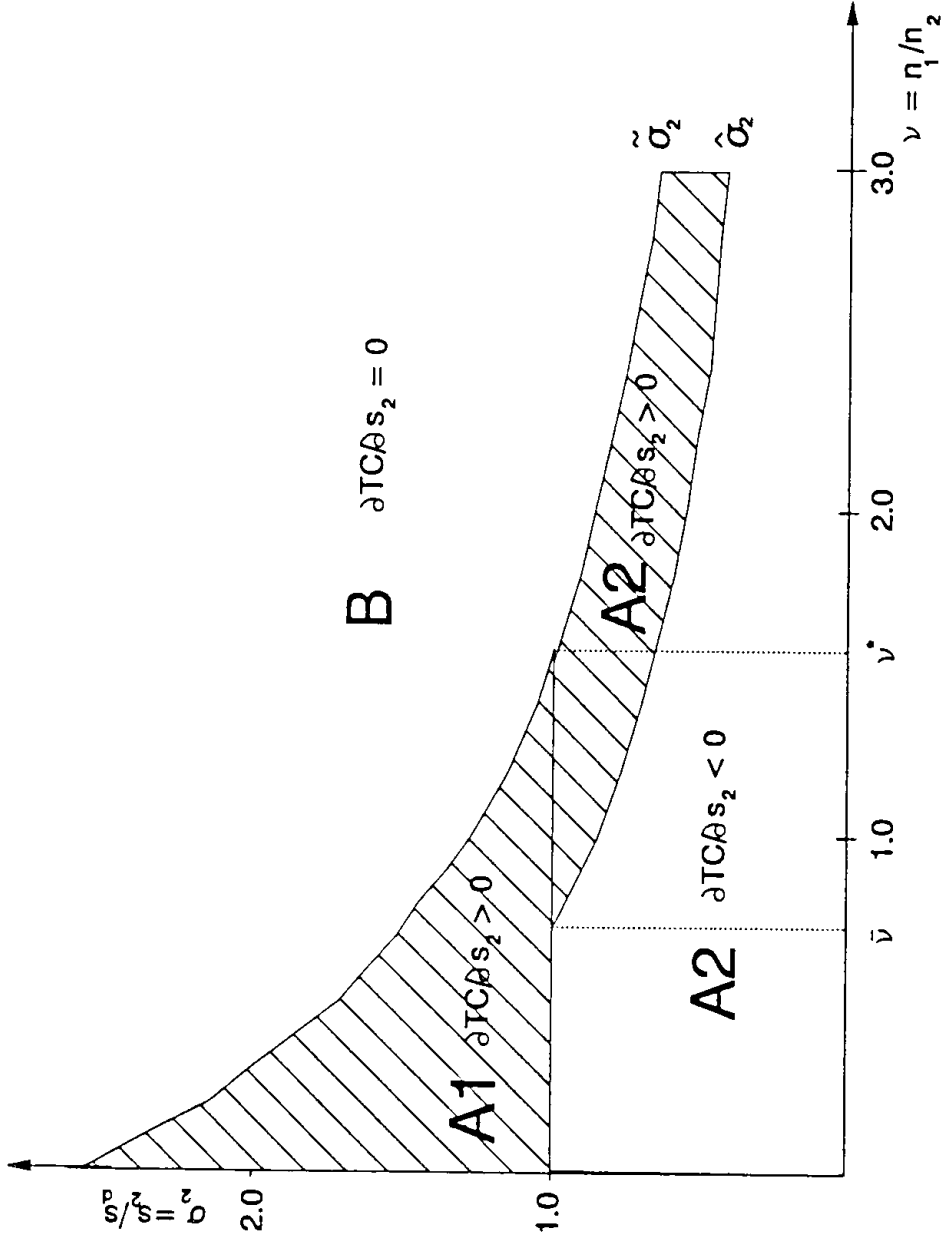
$\alpha=6.00, \beta=3.66, s_2=1.00, s_d=0.80$
 $n_1=0.5, n_2=1.0, t^*=8.5$
 L_1 = Cumulative departures of group 1
 L_2 = Cumulative departures of group 2
 E_1 = Cumulative exits from bottleneck 2
 E_2 = Cumulative arrivals at destination

Figure 3. Cumulative departures and arrivals. Case A1: $\bar{C}_2 > \bar{C}_1, s_2 > s_d$



$\alpha=6.00, \beta=3.66, s_2=0.80, s_d=1.00$
 $n_1=0.5, n_2=1.0, t^*=8.5$
 L₁ = Cumulative departures of group 1
 L₂ = Cumulative departures of group 2
 E₂ = Cumulative exits from bottleneck 2
 A = Cumulative arrivals at destination
 [Horizontal lines] Queue at upstream bottleneck 2
 [Diagonal lines] Queue at downstream bottleneck d

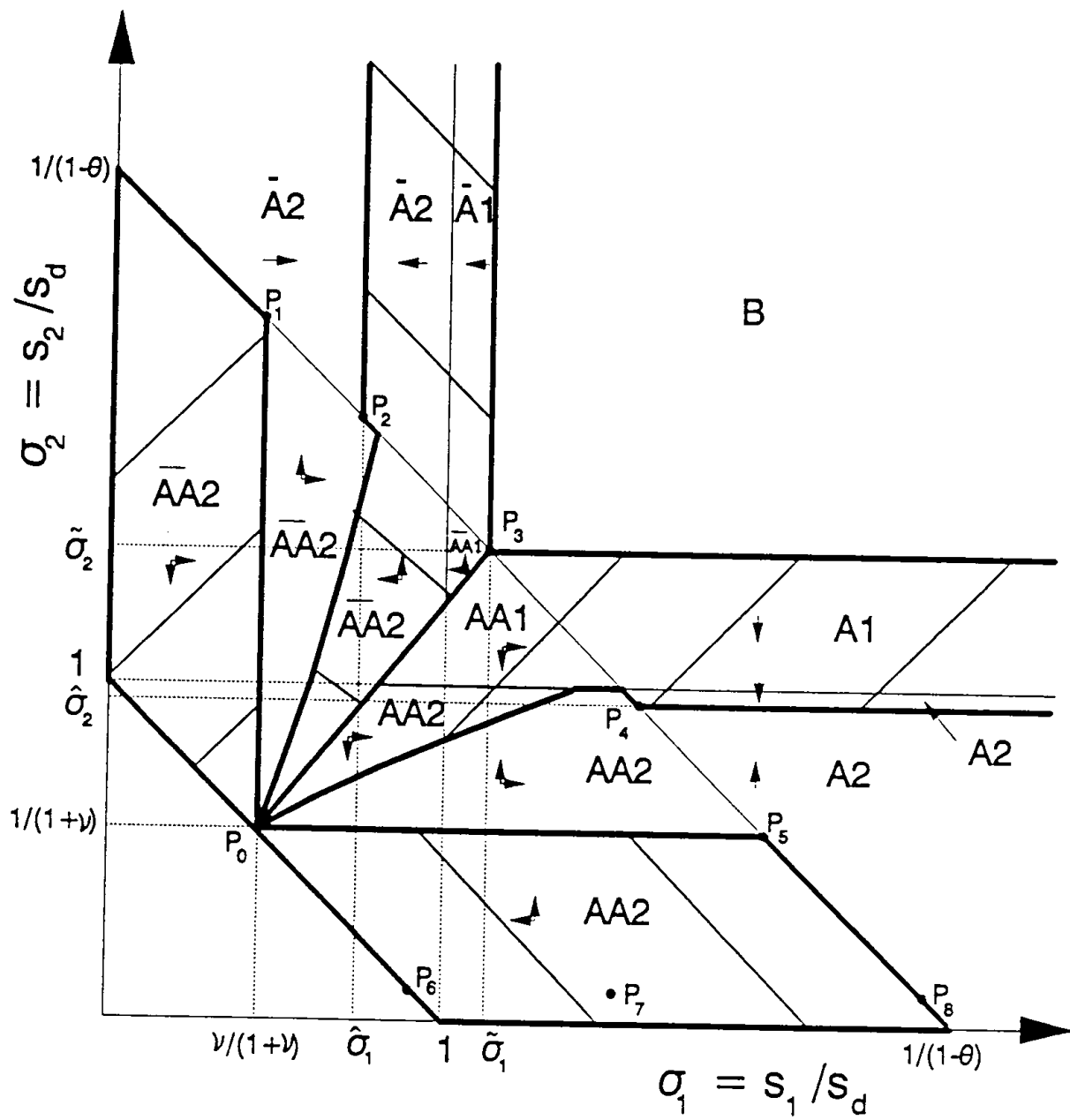
Figure 4. Cumulative departures and arrivals. Case A2: $\bar{C}_2 > \bar{C}_1, s_2 \leq s_d$



$$\bar{\nu} = (\theta/2)[(1+4/(1-\theta))^2 - 1], \quad \nu^* = \theta/(1-\theta)$$

$$\hat{\sigma}_2 = [1-\theta + \nu(\nu+\theta)(1-\theta)^{1/2}]^{-1}, \quad \tilde{\sigma}_2 = [(1-\theta)(1+\nu)]^{-1}$$

Figure 5. Parameter values for which cases A1, A2 and B occur



$\alpha=6.00, \beta=3.66, \theta=\beta/\alpha=0.61, \gamma=n_1/n_2=0.8$

Metering s_1 beneficial
 Metering s_2 beneficial

Figure 6. Conditions under which metering reduces travel costs

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