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MULTIPRODUCT FIRMS:
A NESTED LOGIT APPROACH

by

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Abstract

The paper proves the existence of a symmetric equilibrium with multiproduct firms using a nested logit model of demand. The demand model is parameterized by two variables which characterize different dimensions of preference for variety. These reflect intragroup heterogeneity and intergroup heterogeneity, a group (or nest) being the set of products produced by a firm. There are then two dimensions to market performance; the total number of firms and the range of products produced per firm. It is shown that the market equilibrium involves an excessive number of firms, but each firm provides too few products.

Introduction

Many firms sell more than one product, from supermarkets to Heinz to

General Motors. There is remarkably little economic analysis of multiproduct

firms. One reason is the analytical complexity of any oligopolistic model of

such firms. Showing existence of equilibrium for example is very difficult

since firms must choose several decision variables (such as the prices of their

products). Another problem is the appropriate choice of a demand model.

Exceptions in the literature include the contestability models of Baumol,

Panzar and Willig (1982), which have led to great advances in thinking about

the cost side (e.g. economies of scope) but for which strategic interaction is

rather minimal. Brander and Eaton (1984) have looked at duopoly product choice

from a possible constellation of four products arranged in two groups of two

products each, with intragroup cross elasticities of demand being more elastic

than intergroup ones. They show two types of (Cournot) equilibrium exist with

firms producing two products each, with products either interlaced or

segmented.

Spatial (or characteristics) models have also been used for the demand

side. Champsaur and Rochet (1989) have proved the existence of a duopoly

equilibrium in a model of vertical product differentiation, with each firm

producing a non-overlapping continuum of the product spectrum. Martinez-Giralt

and Neven (1988) have used a version of the circle model to show duopolists

will prefer to sell single products because price competition becomes too

intense when selling more than one product each.[1]

A recent paper by Raubitschek (1987) uses a CES approach to model the

demand side, and allows for an arbitrary number of firms. However, she assumes

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[1] Champsaur and Rochet (1989) find the same result under a similar assumption on
preferences.

that all products owned by a firm are operated individually by independent profit-maximizing managers. While the model moves in the right direction, it ignores coordination of pricing decisions across products owned by a single firm. In what follows we assume instead that all pricing decisions are coordinated by the firm to maximize overall profit.

The model we use for the demand side is a generalization of the multinomial logit model.[2] This is the nested logit model which was first introduced by Ben-Akiva (1973). The model allows for different degrees of substitutability between products. Products which are close substitutes constitute a "nest", and products in different nests are less close substitutes. The nested logit is parameterized by two constants which capture the degree of substitutability at these two different levels. Aggregate preference for produce diversity therefore involves two dimensions. We assume here that the products produced by each firm constitute a separate nest.

One important question in the realm of product differentiation is the bias in the diversity of products provided as compared to the social optimum variety (Spence (1976), Dixit and Stiglitz (1977)). When firms are not constrained by assumption to produce one produce each, there are two dimensions to this problem. First, there is the total number of firms. Second, there is the product range per firm. These magnitudes depend on the two preference for diversity parameters.

Another important question concerns the existence of equilibrium. In a remarkable paper, Caplin and Nalebuff (1991) have delineated a broad class of

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[2] The multinomial logit model has been previously analyzed in an oligopoly context by Anderson and de Palma (1992), and by Besanko, Perry and Spady (1990). It is an example of the discrete choice approach proposed in Perloff and Salop (1985). For some recent applications of the logit in marketing, see Choi, DeSarbo and Harker (1990), Katahira (1990) and McFadden (1986).

models for which a price equilibrium exists for single product firms. For the specific example here we provide an existence proof for multiproduct firms. We also extend the proof to show existence of equilibrium in a two stage game in which firms choose product ranges anticipating subsequent price competition in the second stage of the game.

In Section II gives the backdrop to the nested logit demand system and describes its parameters. In Section III the main results are presented for the multiproduct firm model, and in Section IV market performance is evaluated. Section V provides concluding comments.

II. The Nested Logit Model

Suppose there are n product groups, product group i consisting of m_i products, i = 1...n. The expected demand for product k belonging to product group i is

$$X_{ik} = NP_{ik} \tag{1}$$

where N is the number of consumers and P_{ik} is the probability that a consumer selects product k from group i.

The nested logit formulation provides a closed form expression for P_{ik}. An intuitive derivation of the model is given below, and follows Ben-Akiva (1973). Consumer choice can be seen as a sequential process in which first a group i (or "nest") is chosen with probability P_i, and then, conditional on the choice of this nest, a particular product k is selected with probability $P_{k|i}$. Hence $P_{ik} = P_i P_{k|i}$. The two choice levels are described by logit models. The choice within a nest is given by the multinomial logit demand model as

$$P_{k|i} = \frac{\exp[(a_{ik} - p_{ik})/\mu_2]}{\sum_{\ell=1}^{m_i} \exp[(a_{i\ell} - p_{i\ell})/\mu_2]} , \quad k = 1 \ldots m_i \qquad (2)$$

where $a_{i\ell}$ is the "quality" of product ℓ in nest i, and $p_{i\ell}$ is its price. The parameter $\mu_2 \geq 0$ represents the degree of heterogeneity of products within the nest. When $\mu_2 \rightarrow 0$, products become perfect substitutes in that all demand addressed to nest i is satisfied by the product for which the net quality, $a_{i\ell} - p_{i\ell}$, is greatest. The larger μ_2 is, the greater is the degree of heterogeneity of products within the nest.

The multinomial logit model (2) can be derived by assuming that consumer choice, conditional upon having selected nest i, can be modeled by utility maximization. The (indirect) utility of a consumer buying product k in nest i is given as

$$u_{ik} = a_{ik} + y - p_{ik} + \mu_2 \epsilon_{ik} , \quad k = 1 \ldots m_i \qquad (3)$$

where y is consumer income[3] and the ϵ_{ik}'s are random variables which are independently and identically double exponentially distributed, i.e., $Prob(\epsilon_{ik} < x) = \exp(-\exp - x)$.[4] The probability that product k is selected (from nest i) is $Prob(U_{ik} = \max_\ell U_{i\ell}, \ell=1\ldots m_i)$, which yields the logit form (2). The attractiveness of nest i for the consumer is measured by the expected benefit,

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[3] It is assumed that prices are low enough for consumers to be able to afford all variants. When this constraint is satisfied, the demand for each product in the differentiated goods sector under study is independent of income.

[4] The mean of the corresponding density function is Euler's constant, γ, and the variance is $\pi^2/6$.

$A_i = E(\max_\ell(a_{i\ell} - P_{i\ell} + \mu_2 \epsilon_{i\ell}))$, which, for the double exponential distribution

is (up to an additive constant)

$$A_i = \mu_2 \ln \sum_{\ell=1}^{m_i} \exp[(a_{i\ell} - P_{i\ell})/\mu_2], \quad i=1\ldots n. \tag{4}$$

The choice of a specific nest i is also given by a logit form, with heterogeneity parameter μ_1:

$$P_i = \frac{\exp(A_i/\mu_1)}{\sum_{j=1}^{n} \exp(A_j/\mu_1)} \quad i=1\ldots n. \tag{5}$$

This form can be generated by assuming the nest chosen is the one for which

$$u_j = A_j + \mu_1 \epsilon_j \quad i=1\ldots n \tag{6}$$

is greatest, with ϵ_j again independently and identically double exponentially distributed.

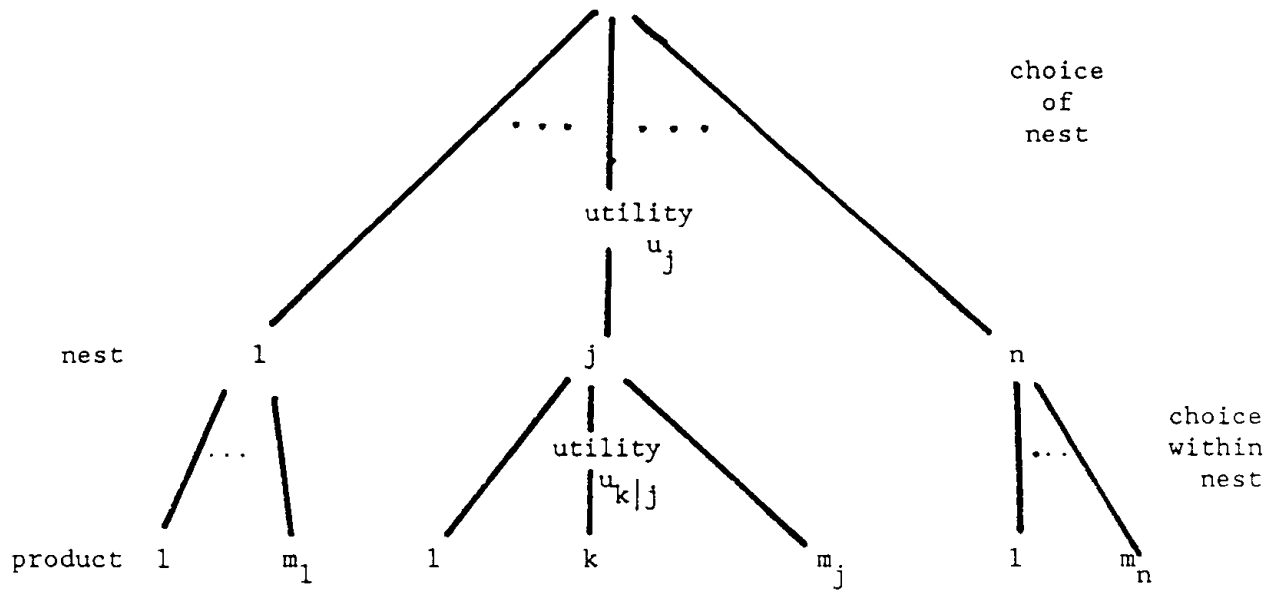The sequential choice process is illustrated in Figure 1.

Figure 1. Sequential choice process for nested logit model, with
$u_j = A_j + \mu_1 \epsilon_j$ and $u_{k|j} = a_{jk} + y - p_{jk} + \mu_2 \epsilon_{jk}$.

In the sequential choice derivation of the model, consumers do not know

the realization of the match values, $\mu_2 \epsilon_{ik}$, associated with nest i.   Instead,

they know the expected benefit, A_i; that is, they know they will pick the best

alternative once the nest is chosen, but they do not know beforehand which

alternative that will be, nor do they know its actual value. For example, if

the nests are restaurants, and the products are particular meals, consumers

form estimates of expected benefits, and pick the best of the available choices

on arrival. The two stage process can also be seen as a simplified choice

procedure which reduces the alternatives the individual must simultaneously

consider.

An alternative derivation of the model assumes that all match values are

known in advance. Indeed, McFadden (1978) has shown that the nested logit

model is consistent with the (non-sequential) maximization of a random utility

function of the form $u_{ik} = a_{ik} + y - p_{ik} + \epsilon_{ik}$ (with i=1...n and k=1...m_i), and

arises from a specific correlation between the random elements ϵ_{ik}. The distribution function depends on two parameters, μ_1 and μ_2, with $\mu_1 \geq \mu_2$, and is given by $F(x_1 \ldots x_J) = \exp[-H(x_1 \ldots x_J)]$ with $H(x_1 \ldots x_J) =$

$$\sum_{j=1}^{n} \left[\sum_{\ell=1}^{m_j} \exp(-x_\ell/\mu_2)\right]^{\mu_2/\mu_1} \text{ where } J = \sum_{j=1}^{n} m_j .$$

The parameter μ_1 can be viewed as a measure of *heterogeneity of product groups*; whereas μ_2 represents *intragroup heterogeneity*. The condition $\mu_1 \geq \mu_2$, has an intuitive interpretation: the products within a group are more similar than products belonging to different groups. Note that the cross price derivative for products in the same nest is $\partial P_{ik}/\partial p_{i\ell} = P_i P_{k|i} P_{\ell|i} [\frac{1}{\mu_2} - \frac{(1 - P_i)}{\mu_1}]$ with $\ell \neq k$; therefore the cross price derivatives are always positive provided that $\mu_1 \geq \mu_2$. When $\mu_1 = \mu_2$, all products are equally "close" and groups therefore cannot be distinguished: it can easily be verified that (1) reduces to the simple multinomial logit model with J products. As noted above, the case $\mu_2 = 0$ means the products within each nest are perfect substitutes.

To see the role of μ_1 and μ_2, suppose all products are priced at p and have the same quality a; moreover each nest comprises m products so that there are nm products in total. The cross price elasticity of demand for products in different nests is $\frac{p}{nm\mu_1}$, while for products within the same nest it is $\frac{p}{nm}(\frac{n}{\mu_2} - \frac{n - 1}{\mu_1})$. Clearly the latter exceeds the former for $\mu_1 > \mu_2$, and they are equal for $\mu_1 = \mu_2$.

The consumer surplus associated with the nested logit model is (cf (4))

$$CS = \mu_1 \ln \sum_{j=1}^{n} \exp[A_j/\mu_1] + y ;$$

it can be verified using Roy's identity that the choice probabilities P_{ik} can be recovered. When prices are all equal to p, qualities are all equal to a, and the number of products in each nest is m, (7) reduces to

$$CS = \mu_1 \ln(n) + \mu_2 \ln(m) + a - p + y \qquad (8)$$

so that consumer surplus increases with the number of clusters, n, and with the number of products per cluster, m.

The nested logit model has been used to describe demand when options are clustered. For example, in transportation science commuter decisions have been modelled with departure time choice in the first stage and route choice at the second stage (Ben-Akiva and Lerman (1985)). In urban economics, residential location has been described by choice of neighborhood at the first stage and a specific dwelling at the second stage (McFadden (1978)). Telephone demand can be divided into choice of company and then choice of service option (Train, McFadden and Ben-Akiva (1987)). The next section investigates multiproduct firm equilibrium with the nested logit model.

III. The Nested Logit Model with Multiproduct Firms

The nested logit model is now used to describe competition among firms selling multiple products. It is assumed that the products sold by any firm are closer substitutes for each other than they are for products sold by

different firms.[5] That is, μ_2 is the degree of heterogeneity across products of any given firm, whereas μ_1 is the heterogeneity across firms, with $\mu_1 \geq \mu_2$.[6] The demand system is the one described in Section II, where n is the number of firms and m_j is the number of products produced by firm j, j - 1...n. It is assumed that all products have the same quality, a.

The equilibrium concept can be viewed as a three-stage process. In the first stage firms decide whether to enter the market. In the second stage those which have entered decide how many products to produce. The last stage is the price equilibrium. We want to find a symmetric equilibrium in the second stage, for a given number of entrants n. To do so, suppose n-1 of the firms choose m_2 products each, and consider the decision problem of the first firm, which is to choose m_1 products. Let c be marginal cost, F be the fixed cost per product, and K be the set-up cost per firm. The following lemma is used to find the symmetric equilibrium.

Lemma. Consider the nested logit model and suppose Firm 1 sells m_1 products and the remaining n-1 firms sell m_2 products each. There exists a unique price equilibrium at which Firm 1 sells all its products at the same price, p_1, and all other firms sell their products at the common price p_2.

The argument behind the proof is as follows. In Appendix 1 it is shown that the first order conditions imply that Firm 1 will sell all its products at p_1 and all other firms charge p_2 per product. Hence Firm 1's profit is

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[5] An alternative assumption, which may fit some markets better, is to suppose product groups (nests) consist of products produced by different firms.

[6] In the context of the two stage decision process envisaged by Ben-Akiva (1973) and described in Section III, consumers can be viewed as first selecting a firm, and then selecting one of its products.

$$\Pi_1 = N(p_1 - c)\mathbb{P}_1 - m_1 F - K \qquad (9)$$

with

$$\mathbb{P}_1 = \frac{\exp[\frac{\mu_2}{\mu_1}\ln(m_1)]}{(n-1)\exp[\frac{\mu_2}{\mu_1}\ln(m_2) - \chi] + \exp[\frac{\mu_2}{\mu_1}\ln(m_1)]} \qquad (10)$$

where $\chi = \frac{p_2 - p_1}{\mu_1}$. Solving the first order conditions to find the price

equilibrium gives

$$p_1 - c = \frac{\mu_1}{1-\mathbb{P}_1} - \frac{\mu_1}{(n-1)\mathbb{P}_2} \qquad (11)$$

and

$$p_2 - c = \frac{\mu_1}{1-\mathbb{P}_2} \qquad (12)$$

for all firms other than Firm 1, where $\mathbb{P}_1 + (n-1)\mathbb{P}_2 = 1$. Note that the

symmetric solution to (11) and (12) when $m_1 = m_2$ yields a mark-up equal to

$\mu n(n-1)$, which is independent of the common size of product ranges. There are

two effects at work when m rises. On the one hand, more products on the market

tends to depress the market price. On the other hand each firm internalizes

the effect of price changes on its own products, which in itself is a force which leads prices to rise. Here these effects exactly cancel.[7]

It has still to be shown that the prices p_1 and p_2 do constitute a price equilibrium for given m_1 and m_2. It seems intractable to address this problem by showing the Hessian of the profit function is negative definite at the first order conditions. Likewise, the results of Caplin and Nalebuff (1991) apply only to single product firms. In Appendix 1 we use an alternative method of proof. This first shows that there is a unique solution to the first order conditions for each individual firm as well as a unique solution to the whole system of first order conditions. We then construct a hypercube $[c,\bar{p}]^{m_i}$, in firm i's price strategy space and show that i's profit is maximized in the interior of the hypercube. Firm i's profit therefore has a unique global maximizer which is in the interior of $[c,\bar{p}]^{m_i}$. Since this global maximizer satisfies i's first order conditions, there is a unique Nash equilibrium for the price subgame.

Using the price equilibrium expression in (9) gives

$$\Pi_1 = \frac{N\mu_1}{n-1}\left(\frac{m_1}{m_2}\right)^{\frac{\mu_2}{\mu_1}} \exp\gamma - m_1 F - K . \tag{13}$$

Hence,

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[7] We expect that introducing outside alternatives (for example as in Anderson and de Palma (1992) or Besanko, Perry and Spady (1990) will cause equilibrium mark-up to rise with m. In this case more consumers will be drawn from the outside alternative since there is greater product variety as m rises.

$$\frac{d\Pi_1}{dm_1} = \frac{N\mu_1}{n-1}\left(\frac{m_1}{m_2}\right)^{\frac{\mu_2}{\mu_1}}\left[\frac{\mu_2}{\mu_1}\frac{1}{m_1} + \frac{d\chi}{dm_1}\right]\exp\chi - F \qquad (14)$$

and, at a symmetric equilibrium with $m_1 = m_2 = m^*$, $\dfrac{d\chi}{dm_1}$ equals

$$\frac{-\mu_2}{\mu_1}\frac{1}{m_1}\frac{n}{n^2 - n + 1} \quad \text{so that (14) is solved by}$$

$$m^* = N\frac{\mu_2}{F}\frac{n-1}{n^2 - n + 1} \quad . \qquad (15)$$

This implies that the total product set-up costs (m^*F) are independent of set-up cost per product, F. Clearly, for any n, there exist values of N, μ_2 and F such that each firm only chooses a single product. The greater is μ_2, the greater is the variety provided by each firm. As expected (15) is decreasing in n for $n \geq 2$. The corresponding equilibrium prices are

$$p^* = c + \frac{\mu_1 n}{n-1} \quad . \qquad (16)$$

Indeed, this is the price equilibrium whenever product ranges are the same size; this observation explains why p^* is independent of F. Note that, μ_2 is the key heterogeneity parameter in determining m^* in (15) since it measures the relative attractiveness of the product ranges per se (see (8)). Similarly, μ_1 is the parameter which determines p^* because μ_1 measures the degree of heterogeneity across firms, and $A_i = A_j$ for all i, j=1...n at a symmetric equilibrium. The proof of existence of the symmetric equilibrium for the product range game is given in Appendix 2, so that we can state the following proposition.

Proposition 1. *Consider the nested logit model. There exists a symmetric equilibrium to the two stage product range then price game with n firms each choosing m^* products as given by (15) and then choosing prices p^* as given by (16).*

Substituting (15) into (13) and setting equal to zero yields the free entry equilibrium number of firms as the solution to

$$\frac{N[(\mu_1 - \mu_2)(n - 1)^2 + \mu_1 n]}{(n - 1)(n^2 - n + 1)} - K , \tag{17}$$

where the left-hand side is decreasing in n (since $\mu_1 \geq \mu_2$) so that there is a unique solution (the condition $n \geq 2$ is also used in the comparative static results that follow).

Higher product fixed costs, F, have no effect on equilibrium firm numbers. Doubling F halves product ranges, m^*, for a given number of firms, n, and keeps total product fixed costs constant. Since prices are independent of F, and total demand is constant, net revenues are unchanged as m changes. This means that neither costs nor revenues change (at the short run equilibrium) with F so profits are unaltered.

The comparative statics with respect to the preference for diversity parameters are summarized in the following proposition.

Proposition 2. *The long-run equilibrium number of firms for the nested logit model is increasing with the degree of heterogeneity across product groups (μ_1) and decreasing with the degree of heterogeneity within product groups (μ_2). The range of products offered per firm falls with μ_1 and rises with μ_2.*

The equilibrium number of firms is decreasing with μ_2, since a higher value of μ_2 causes firms to offer wider product ranges (for n fixed) - in a symmetric equilibrium, price doesn't change so that profits are eaten away via this extra competition for customers. As μ_1 rises, on the other hand, holding n fixed, consumer loyalty to firms rises so prices rise, and product ranges are unchanged. The resulting higher profitability leads to more entry: as further firms enter, firms now start to offer smaller product ranges. That is, *an increase in μ_1 ultimately causes more firms, each offering fewer products; whereas a rise in μ_2 leads to the opposite result.*

IV. Market Equilibrium and Social Optimum

We can now derive the first best social optimum configuration of n and m. Note the first best is equivalent to the second best subject to a zero profit constraint since adding a constant to all prices serves only to convert consumer surplus to profits.

Maximization of the surplus function for the nested logit given in (3) gives (for interior solutions):

$$m^0 = N \frac{\mu_2}{nF} \tag{18}$$

for given n, and

$$m^0 = \frac{\mu_2 K}{F(\mu_1 - \mu_2)} \tag{19}$$

when $n = n^0$, given by

$$n^0 = \frac{N(\mu_1 - \mu_2)}{K} . \tag{20}$$

Note that $n^0 = 1$ when $\mu_1 = \mu_2$ (the simple multinomial logit case) since extra firms bring no variety benefit per se. In this case, from (8), m^0 becomes $N\mu_2/F$ so the optimum involves a single firm (incurring a set-up cost, K) producing all the desired products.

The comparative statics are qualitatively similar to those of the free entry equilibrium. The comparison between optimum and equilibrium is summarized by:

Proposition 3. For the nested logit model, the market solution provides too many firms ($n^* > n^0$) but too few products per firm ($m^* < m^0$). The total number of products produced is too few ($n^* m^* < n^0 m^0$).

This result is also interesting because the previous literature has measured diversity only as the number of products, which there is the same as the number of firms. In the nested logit formulation, there are two measures of taste for variety, μ_1 and μ_2 (see (8)); and there are two dimensions to the diversity provided by the market, the number of firms and the product range per firm.

To understand the intuition behind the proposition, consider first the simple multinomial logit model with multiproduct firms (or $\mu_1 = \mu_2$ in the nested logit). When a firm introduces a new product, this product will take some of its demand from that firm's existing products. At the margin the incentive to introduce products is reduced because the firm internalizes this cannibalization. Since the number of products is about right for the single

product multinomial model (this result holds more generally under monopolistic competition when products are symmetric substitutes - see Deneckere and Rothschild (1992)), the number of products will tend to be insufficient for the multiproduct firm model with fixed firm numbers. When $\mu_2 < \mu_1$, this underentry will be exacerbated since when a firm introduces a new product this product is a closer substitute for the firm's other products and will capture much of its demand from the firm's existing products and relatively little from other firms. Here the business stealing effect is smaller and the tendency for underentry is increased.

As another experiment, we can suppose the number of products per firm is exogenously fixed (and equal for all firms), but the free entry number of firms is determined by the zero profit condition. In this case the result is excessive entry by a single firm, just as for single product firms. The reason is that the equilibrium price is independent of the symmetric range of products per firm so the multiproduct model is formally equivalent to the single product one.

To reinforce this idea that equilibrium will involve too many firms, consider a case where all products are symmetric substitutes. The *optimal* number of firms is one (assuming the firm set-up cost, K, exceeds zero); the equilibrium number is typically greater, and indeed in the multinomial logit may be much greater. Clustering firms' products in nests increases the optimal number of firms, but the over-entry force remains strong.

V. Conclusions

The demand model used in this paper, the nested logit, would seem to be a useful tool for modelling multiproduct firms. On the theoretical side, we have shown equilibrium existence for the model (albeit in a pure symmetric case),

which is a preliminary step to extending the Caplin and Nalebuff (1991) existence results for single product firms. On the empirical side, the nested logit has already been used as a demand model for differentiated options, and the current work might hopefully provide a basis for studies incorporating *endogenous* pricing decisions by oligopolistic competitors. It is relatively easy to introduce different marginal costs and perceived qualities into the model (see Section II) to allow for asymmetric situations. It is also possible to relax the assumption that total demand is fixed, by introducing "outside" alternatives, along the lines suggested in Anderson and de Palma (1992).

As far as we know, the comparison between equilibrium and optimum has not been addressed in the literature for anything other than single-product firms. Raubitschek (1987) goes part of the way by allowing for multiple products, but she assumes the number of firms is fixed. She concludes that since there are too few products in the Spence (1976) model on which her analysis is based, there will a fortiori be too few products in her set-up since, when deciding whether to introduce a new product, a firm will internalize part of the "business stealing" effect, the business stealing effect being a force toward having too many products. In this light, it is useful to compare the results for the multiproduct firm model with those of the single product model described in Anderson and de Palma (1992). In the single product model there are too many products (although only one too many). In the multiproduct model, for fixed n, there are too few products. This indicates that the internalization of business stealing in the multiple product context suffices to bring about the opposite conclusion.

For the problem we consider, with endogenous firm numbers as well as product ranges, an entrant firm is associated with three types of externality. There is the standard business stealing externality (an entrant does not

account for the detrimental effect on existing firms' profits) which is a tendency for overentry. Then there is the consumer surplus externality whereby an entrant cannot extract the whole surplus associated with producing its product range, and this is a tendency toward underentry. In our context there is an *additional negative externality* which has to do with total variety. An entrant also causes existing firms to contract their product ranges (at least in the present model), reducing the variety offered by them. For firms, at the margin the net value of an additional product in the range is zero (via the first order condition for profit maximizing choice of product range). However, the net social value of these lost products is positive due to the consumer surplus associated with them. This suggests there is an additional force tending toward insufficient product variety and overentry of firms. Indeed, our final result, the net effect, is overentry of firms, and this is combined with product ranges which are not broad enough.

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Appendix 1. Existence of a Price Equilibrium with Multiproduct Firms for Given Product Ranges[1]

Here we show there is a unique price equilibrium for the subgame in which all firms but firm 1 sell m_2 products, whereas firm 1 sells m_1 products. For the nested logit formulation, firm i's profit is given by

$$\Pi_i = N \sum_{k=1}^{m_i} (p_{ik} - c) \mathbb{P}_{ik} - m_i F - K, \quad i=1\ldots n,$$

where $\mathbb{P}_{ik} = \mathbb{P}_i \mathbb{P}_{k|i}$, as given by (2) and (5). The first order condition with respect to p_{ik} yields

$$p_{ik} - c = \mu_2 + [1 - \frac{\mu_2}{\mu_1}(1 - \mathbb{P}_i)] \sum_{h=1}^{m_i} (p_{ih} - c) \, \mathbb{P}_{h|i}, \quad k=1\ldots m_i, \; i=1\ldots n,$$

so that the price is the same for all firm i's products (and is necessarily positive since $\mu_1 \geq \mu_2$). The first order conditions therefore reduce to

$$p_1 - c = \frac{\mu_1}{1-\mathbb{P}_1} \quad \text{and} \quad p_i - c = \frac{\mu_1}{1-\mathbb{P}_i}, \quad i=2\ldots n, \; \text{where}$$

$$\mathbb{P}_j = \frac{m_j^{\mu_2/\mu_1} \exp[-p_j/\mu_1]}{D}, \quad j=1\ldots n, \; \text{and}$$

$$D = m_2^{\mu_2/\mu_1} \sum_{j=2}^{n} \exp[-p_j/\mu_1] + m_1^{\mu_2/\mu_1} \exp[-p_1/\mu_1].$$

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[1] We would like to thank Jacques Thisse for encouraging and helping us with this proof.

Hence $p_j = p_2$ for $j=2...n$, and the first order conditions become

$$\frac{p_1 - c}{\mu_1} = 1 + \frac{M \exp\chi}{n - 1}$$

and

$$\frac{p_2 - c}{\mu_1} = 1 + \frac{1}{n - 2 + M \exp\chi}$$

where we have defined $M = \left(\dfrac{m_1}{m_2}\right)^{\mu_2/\mu_1}$ and $\chi = \dfrac{p_2 - p_1}{\mu_1}$.

Clearly, the first expression solves for a unique p_1 given any p_2; the second yields a unique p_2 for any given p_1. Subtracting the latter expression from the former yields

$$\chi = \frac{1}{n - 2 + M \exp \chi} - \frac{M \exp \chi}{n - 1} \qquad (A.1)$$

Since the left hand side of this expression is linearly increasing in χ and the right hand side is decreasing, (A.1) has a unique solution in χ. Hence there is a unique solution (p_1^*, p_2^*) to the first order conditions.

Let $\underset{\sim}{p}_1^*$ be the m_1-dimensional vector with all components equal to p_1^* and $\underset{\sim}{p}_2^*$ the m_2-dimensional vector with all components equal to p_2^*. Now, we wish to show that $\underset{\sim}{p}_1^*$ and $\underset{\sim}{p}_2^* = ... \underset{\sim}{p}_n^*$ is an equilibrium of the price subgame defined from (m_1, m_2).

Consider any firm i. The price derivative of Π_i with respect to p_{ik} is given by

$$\frac{\partial \Pi_i}{\partial p_{ik}} = N\mathbb{P}_{ik}\left[1 + \left(\frac{1}{\mu_2} - \frac{1-\mathbb{P}_i}{\mu_1}\right)\sum_{h=1}^{m_i}(p_{ih} - c)\mathbb{P}_{h|i} - \frac{p_{ik} - c}{\mu_2}\right] \ .$$

Since $\mu_1 \geq \mu_2$, it follows immediately that

(i) $\left.\dfrac{\partial \Pi_i}{\partial p_{ik}}\right|_{p_{ik}=c} > 0$, $k=1\ldots m_i$.

Moreover, it can be shown that

(ii)   there exists $\bar{p}$ such that $\left.\dfrac{\partial \Pi_i}{\partial p_{ik}}\right|_{p_{ik}=\bar{p}} < 0$ for all $p_{ih} \leq \bar{p}$, whatever $h \neq$

k.

The argument is as follows. When $p_{ik} = \bar{p}$, we have

$$\left.\frac{\partial \Pi_i}{\partial p_{ik}}\right|_{p_{ik}=\bar{p}} = N\mathbb{P}_{ik}\left[1 + \left(\frac{1}{\mu_2} - \frac{1-\mathbb{P}_i}{\mu_1}\right)\sum_{h=1}^{m_i}(p_{ih} - c)\mathbb{P}_{h|i} - \frac{\bar{p} - c}{\mu_2}\right]$$

$$\leq N\mathbb{P}_{ik}\left[1 + \left(\frac{1}{\mu_2} - \frac{1-\mathbb{P}_i}{\mu_1}\right)(\bar{p} - c) - \frac{\bar{p} - c}{\mu_2}\right] \text{ since } \sum_{h=1}^{m_i}\mathbb{P}_{h|i} = 1$$

$$\leq N\mathbb{P}_{ik}\left[1 - \left(\frac{1-\mathbb{P}_i}{\mu_1}\right)(\bar{p} - c)\right]$$

$$\leq N\mathbb{P}_{ik}\left[1 - \frac{\kappa}{\mu_1}(\bar{p} - c)\right]$$

where $1 - \mathbb{P}_i \geq 1 - \sup_{\mathbb{P}_i} \mathbb{P}_i - \left. 1 - \mathbb{P}_i \right|_{\mathbb{P}_i \to 0} - \kappa > 0$. Consequently, for $\bar{p}$ large

enough the last expression is negative for all $i=1 \ldots n$, which completes the

proof.

Clearly, $\pi_i(p_i, p_{-i}^*; \underline{m})$ has a global maximizer p_i^M in the compact set

$[c, \bar{p}]^{m_i}$. If p_i^M belongs to the boundary of $[c, \bar{p}]^{m_i}$, at least one component of

p_i^M is equal either to c or to $\bar{p}$. But then, property (i) or (ii) would imply

that $\pi_i(p_i, p_{-i}^*; \underline{m})$ is higher at some point in $]c, \bar{p}[^{m_i}$, a contradiction. Hence

p_i^M must belong to $]c, \bar{p}[^{m_i}$ and is, therefore, a solution to the first order

conditions applied to π_i. Since p_i^* is the only solution of these conditions,

$p_i^* = p_i^M$ so that p_i^* maximizes $\pi_i(p_i, p_{-i}^*; \underline{m})$.

$\square$

Appendix 2. Existence of a Symmetric Product Range Equilibrium with Multiproduct Firms

Here we prove the assertion that $m_1 = m^*$, where m^* is given by (15), is

the number of products that maximizes Firm 1's profit, (13), given all other

(n-1) firms choose m^* products. By construction the derivative $d\Pi_1/dm_1$ as

given by (14) is zero when evaluated at $m_1 = m^*$, (and furthermore m^* is

uniquely determined).

We wish to show that the profit function $\Pi_1(m_1, m^*)$ is strictly

quasi-concave in m_1. We know from (13) that

$$\Pi_1 = \frac{N\mu_1}{n-1}\Delta - m_1 F - K \tag{A.2}$$

with $\Delta = \left[\dfrac{m_1}{m^*}\right]^{\frac{\mu_1}{\mu_2}} \exp\ [(p^* - p_1^*)/\mu_1]$.

Clearly,

$$\frac{d\Delta}{dm_1} = \Delta\left[\frac{\mu_2}{\mu_1}\frac{1}{m_1} + \frac{d}{dm_1}\left(\frac{p^* - p_1^*}{\mu_1}\right)\right] . \tag{A.3}$$

Furthermore, (A.1) implies that

$$\frac{p^* - p_1^*}{\mu_1} = \frac{-\Delta^2 - (n - 2)\Delta + n - 1}{(\Delta + n - 2)(n - 1)}$$

from which it follows that

$$\frac{d}{dm_1}\left(\frac{p^* - p_1^*}{\mu_1}\right) = -\left[\frac{1}{(\Delta + n - 2)^2} + \frac{1}{n - 1}\right]\frac{d\Delta}{dm_1} . \tag{A.4}$$

Combining (A.3) and (A.4) then gives

$$\frac{d\Delta}{dm_1} = \frac{\mu_2}{\mu_1}\frac{\Delta}{m_1}\varphi^{-1} \tag{A.5}$$

where

$$\varphi = \frac{\Delta}{(\Delta + n - 2)^2} + \frac{\Delta}{n - 1} + 1 > 1. \tag{A.6}$$

Differentiating (A.4) with respect to m_1 and using (A.5) yields

$$\frac{\partial \Pi_1}{\partial m_1} = \frac{N\mu_1}{n-1} \frac{d\Delta}{dm_1} - F = \frac{N}{n-1} \frac{\mu_1}{m_1} \Delta \varphi^{-1} - F$$

while the second derivative is given by

$$\frac{\partial^2 \Pi_1}{\partial m_1^2} = -\left(\frac{1}{m_1} \frac{\partial \Pi_1}{\partial m_1} + F\right) + \frac{N}{n-1} \frac{\mu_2}{m_1} (\varphi^{-1} - \Delta \varphi^{-2} \frac{d\varphi}{d\Delta}) \frac{d\Delta}{dm_1} . \qquad (A.7)$$

We can now evaluate (A.7) at any solution of $\dfrac{\partial \Pi_1}{\partial m_1} = 0$ (so that we have $\dfrac{d\Delta}{dm_1} = \dfrac{F(n-1)}{N\mu_1}$):

$$\left. \frac{\partial^2 \Pi_1}{\partial m_i^2} \right|_{\frac{\partial \Pi_i}{\partial m_1} = 0} = \frac{F}{m_1}\{-1 + \frac{\mu_2}{\mu_1} (\varphi^{-1} - \Delta \varphi^{-2} \frac{d\varphi}{d\Delta})\} . \qquad (A.8)$$

Since $\mu_1 \geq \mu_2$, this expression is negative when

$$\varphi - \Delta \frac{d\varphi}{d\Delta} < \varphi^2 \qquad (A.9)$$

where (from (A.6))

$$\frac{d\varphi}{d\Delta} = \frac{-\Delta + n - 2}{(\Delta + n - 2)^3} + \frac{1}{n-1} . \qquad (A.10)$$

Substituting (A.10) in (A.9) and using (A.6), we obtain

$$1 + \frac{2\Delta^2}{(\Delta + n - 2)^3} < \left[\frac{\Delta}{(\Delta + n - 2)^2} + \frac{\Delta}{n - 1} + 1 \right]^2$$

which can be shown to hold (recalling $n \geq 2$) by expanding the right hand side of the inequality. Hence (A.8) is negative.

□