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An Approach to Equilibrium Selection

by

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Abstract

We consider equilibrium selection in 2×2 bimatrix games with two strict Nash equilibria in a random matching framework. The players seek to maximize the discounted payoffs, but are restricted to make a short run commitment. Modelling the friction this way yields equilibrium dynamics of the behavior patterns in the society.

We define and characterize an absorbing and globally attractive state in this dynamics. It is shown that, as friction becomes arbitrarily small, a strict Nash equilibrium outcome becomes uniquely absorbing and globally attractive if and only if it satisfies the Harsanyi/Selten notion of risk-dominance criterion.

Keywords: Equilibrium Selection, Random Matching Games, Risk-dominance

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1. Introduction

We approach the problem of equilibrium selection in 2×2 bimatrix games with two strict Nash equilibria. This class of games, which contains pure coordination games and the battle of the sexes as special cases, is not only important in its own right, but also captures a variety of economic problems in their essentials. Examples include adoption of new technologies (Farrell and Saloner [1985]), choice among alternative media of exchange (Matsuyama, Kiyotaki, and Matsui [1991]), geographical distribution of cities (Krugman [1991a, 1991b]), Keynesian macroeconomics (Cooper and John [1988]), and economic development (Murphy, Shleifer and Vishny [1989] and Matsuyama [1991a, 1991c]). In spite of its central role in game theory and economics, the literature offers very few formal approaches to the problem of equilibrium selection. For instance, the most solution concepts proposed in the literature on refinements of Nash equilibria, such as the strategic stability of Kohlberg and Mertens (1986), have nothing to say about selection among strict Nash equilibria.

Our approach to this problem is to examine the stability of strict Nash equilibria in an explicitly dynamic context. To this end, we consider the society consisting of a continuum of anonymous agents and each agent plays the game repeatedly with an opponent randomly chosen from the population. All players maximize the expected discounted payoffs with one restriction; they need to make a short-run commitment to the action they chose. The opportunity to switch actions arrives stochastically; it follows a Poisson process, which is identical and independent across players. By modelling some friction this way, this dynamic game generates nontrivial equilibrium paths of the behavior patterns in the society, whose stationary states correspond to the Nash equilibria of the original game. We define and characterize the stationary

states of this dynamic game.

For example, suppose that the initial behavior patterns are in the neighborhood of a strict Nash equilibrium of the original game, say (L,L) in Figure 1. One can show that, in the presence of large friction, an equilibrium path is unique and the behavior patterns always converge to (L,L). In this sense, any strict Nash equilibrium is an absorbing state with sufficiently large friction. When friction is small, however, (L,L) may be fragile in that another equilibrium path exists, along which the behavior patterns move away from (L,L) and converge to (R,R). In other words, beliefs that the "band wagon" effects will induce all players to switch from L to R in the future may be consistent, thereby upsetting (L,L).

Our selection criterion is based on the following two observations. First, one can show for generic games that, as friction becomes arbitrarily small, the only one strict Nash equilibrium remains absorbing, while all other states become fragile. Second, the unique absorbing state has additional stability property, which we call globally attractive; that is, for any initial behavior patterns, there exists an equilibrium path along which the behavior patterns converge to the unique absorbing state. We view that these properties make the unique absorbing state a natural choice among strict Nash equilibria.

Interestingly enough, our selection criterion turns out to be equivalent to the Harsanyi and Selten (1988) notion of the risk dominance criterion; in any 2x2 bimatrix game with two strict Nash equilibria, a strict Nash equilibrium is uniquely absorbing (and globally attractive) if and only if it is risk dominant. Thus, our approach can be viewed as a dynamic justification of the Harsanyi/Selten risk-dominance criterion.

Some recent studies on learning and evolution have also addressed the question of how a particular equilibrium will emerge in a dynamic context. A very partial list of this literature includes Boylan (1990), Canning (1989), Friedman (1991), Fudenberg and Kreps (1988), Gilboa and Matsui (1991a), Matsui (1990), Matsuyama (1991b), Milgrom and Roberts (1990, 1991), Swinkels (1991), and Taylor and Jonker (1978). Although some convergence results are obtained, these studies do not offer an equilibrium selection criterion, since all strict Nash equilibria share the same dynamic properties in their models.

Foster and Young (1990) and Kandori, Mailath and Rob (1991) consider evolutionary models with constant flow of mutations, which generate Markov processes in the behavior patterns. It turns out that the stationary distribution of the Markov processes attaches probability one to the risk dominant outcome in the limit as the rate of mutation goes to zero. Another related work is Matsui (1991a), which considers a model with pre-play communication and shows that a Pareto inefficient strict Nash equilibrium is upset through the best response dynamics in a class of games called games of common interest. In these studies, the convergence to Nash equilibria are studied in the context of repeated play by myopic players. To be perfectly clear, we emphasize that our approach assumes that players form their beliefs in a highly rational manner; to use Binmore's (1990) terminology, we remain in the eductive context.¹

¹Some mention should be made of Kalai and Lehrer (1991). They consider an infinite repetition of a stage game between fixed players. Players have some priors over the opponents' repeated game strategies and try to maximize their expected discounted payoffs. They show that in spite of discrepancy in their initial beliefs, the actual sequence of actions converges to that of Nash equilibrium. Any Nash equilibrium appears as an outcome; their motivation is not to tell a story of equilibrium selection.

2. Symmetric Games

In this section, we restrict our attention to the symmetric game given in Figure 1. This game has two strict Nash equilibria, (L,L) and (R,R), as well as one mixed strategy equilibrium in which each player chooses L with probability $\mu = (d-c)/((a-b)+(d-c))$. Instead of analyzing this game in isolation, we envision that this game is played repeatedly in a society with a continuum of identical players. At every point in time, each player is matched to form a pair with another player, randomly drawn from the population, and they play the game anonymously. All players are highly rational and choose a strategy to maximize the expected discounted payoffs. Because of the anonymity, they are engaged in this maximization without taking into account strategic considerations such as reputation and retaliation.

The key assumption is that no player can switch actions at every point in time. Every player needs to make a commitment to a particular action in the short run. Following Matsuyama (1991a,b,c,d), we assume that the opportunity to switch actions arrives randomly; it follows the Poisson process with p being the mean arrival rate. Furthermore, it is assumed that the process is independent across the players and there is no aggregate uncertainty.² The strategy distribution in the society as of time t can be thus described as $x_t[L] + (1-x_t)[R]$, where x_t is the fraction of the players that are committed to action L as of time t . We simply call x_t the behavior pattern in the society. Because of the restriction imposed above, x_t changes continuously over time and the rate of change in x_t belongs to $[-px_t,$

²There are some technical problems concerning the law of large numbers with a continuum of i.i.d. random variables, as first pointed out by Feldman and Gilles (1985) and Judd (1985). Boylan (1991a,b) and Gilboa and Matsui (1991b) discuss these issues in the context of random matching games and offer some possible solutions.

$p(1-x_t)]$. Furthermore, any feasible path necessarily satisfies $x_0 e^{-pt} \leq x_t \leq 1 - (1-x_0)e^{-pt}$, where the initial condition, x_0 , is given exogenously, or "by history."

When the opportunity to switch arrives, players choose the action which results in the higher expected discounted payoffs, knowing the future path of x as well as their own inability of switching actions continuously. Since the strategy distribution as of time t is $x_t[L] + (1-x_t)[R]$, the value of playing action L instead of R as of time t is equal to

$$\{ax_t + c(1-x_t)\} - \{bx_t + d(1-x_t)\} = \{(a-b) + (d-c)\}(x_t - \mu) ,$$

and thus players, given the opportunity, commit to play L if $V_t > 0$ and to play R if $V_t < 0$ and are indifferent if $V_t = 0$, where

$$V_t \equiv (p+\theta) \int_0^\infty (x_{t+s} - \mu) e^{-(p+\theta)s} ds , \quad (1)$$

with $\theta > 0$ being the discount rate. Therefore, $\{x_t\}_{t=0}^\infty$ is an equilibrium path from x_0 if its right-hand derivative exists and satisfies

$$\frac{d^+ x_t}{dt} \in \begin{cases} \{p(1-x_t)\} & \text{if } V_t > 0, \\ \{-px_t, p(1-x_t)\} & \text{if } V_t = 0, \\ \{-px_t\} & \text{if } V_t < 0, \end{cases} \quad (2)$$

for all $t \in [0, \infty)$. Equation (2) states that all players currently playing action R (resp. L), if given the opportunity, switch to L (resp. R), when $V_t >$ (resp. $<$) 0.

It is straightforward to show that $x = 0$, μ , and 1 are the only stationary states of the dynamics (1) and (2); that is, $x \in [0, 1]$ is a stationary state if and only if it is a Nash equilibrium of the original game.

We use (1) and (2) to study the stability of the Nash equilibria.

Since there are generally multiple equilibrium paths from a given initial condition, one needs to be specific about what the stability means. It is thus necessary to introduce some terminologies.³

Definitions:

- i) $x \in [0,1]$ is accessible from $x' \in [0,1]$, if there exists an equilibrium path from x' that reaches or converges to x .
- ii) $x \in [0,1]$ is absorbing if there is a neighborhood of x , U , such that any equilibrium path from U converges to x .
- iii) $x \in [0,1]$ is fragile if it is not absorbing, that is, if there is a neighborhood of x , U , such that there exists an equilibrium path from x that leaves U after a finite time.
- iv) $x \in [0,1]$ is globally attractive if it is accessible from any $x' \in [0,1]$.

By definition, if an absorbing state, x , is globally attractive, then it is a unique absorbing state in $[0,1]$ and any state in $[0,1] \setminus \{x\}$ is fragile. (The definitions do not rule out the possibility that a state may be both fragile and globally attractive, or that a state may be uniquely absorbing but not globally attractive. As will be shown below, however, these situations never exist and a state is uniquely absorbing if and only if it is globally attractive in the dynamics considered in this paper.) Finally, define the degree of friction by $\delta \equiv \theta/p$, the expected duration of the commitment (with

³Alternatively, we could have borrowed a variety of stability concepts in the set-valued differential equations, such as "Absorbent Stable Sets (ASS)" of Gilboa and Samet (1991). We have chosen to avoid introducing such a formality, however, given the simple structure of our dynamics. One can show that any absorbing point, taken as a singleton set, is an ASS.

the unit of time is normalized so that the discount rate is equal to one).

Lemma 1.

- a) $x = 0$ is globally attractive if and only if $(1+\delta)/(2+\delta) \leq \mu < 1$,
- b) $x = 1$ is globally attractive if and only if $0 < \mu \leq 1/(2+\delta)$,
- c) $x = 0$ is absorbing if and only if $1/(2+\delta) < \mu < 1$,
- d) $x = 1$ is absorbing if and only if $0 < \mu < (1+\delta)/(2+\delta)$.

Proof. See the appendix.

Lemma 1 implies that there exists at least one and at most two absorbing states. Furthermore, a strict Nash equilibrium is globally attractive if and only if it is uniquely absorbing.⁴ In other words, if $x = 1$ is accessible from $x = 0$, then $x = 0$ is not accessible from $x = 1$, and vice versa. Thus, Lemma 1 can be rephrased as:

Proposition 1.

- a) (R,R) is uniquely absorbing and globally attractive if $(1+\delta)/(2+\delta) \leq \mu < 1$; (L,L) is uniquely absorbing and globally attractive if $0 < \mu \leq 1/(2+\delta)$; both (L,L) and (R,R) are absorbing if $1/(2+\delta) < \mu < (1+\delta)/(2+\delta)$.
- b) For any $\mu \in (0,1)$, both (L,L) and (R,R) are absorbing for a sufficiently large $\delta > 0$.
- c) If $\mu \in (1/2, 1)$, (R,R) is uniquely absorbing and globally attractive for a sufficiently small $\delta > 0$; If $\mu \in (0,1/2)$, (L,L) is uniquely absorbing and globally attractive for a sufficiently small $\delta > 0$.

⁴The assumption of a strictly positive discount rate is crucial for these results. As long as $-1 < \delta \leq 0$, (1) and (2) give a well defined dynamics. In this case, every state in $[0,1]$ becomes both fragile and globally attractive, if $(1+\delta)/(2+\delta) \leq \mu \leq 1/(2+\delta)$. (In the terminology of Gilboa and Samet, the entire space, $[0,1]$, becomes an Absorbent Stable Set.)

Figure 2 illustrates Proposition 1. What b) states is that any strict Nash equilibrium is absorbing in the presence of large friction.⁵ More interestingly, unless $\mu = 1/2$, one strict Nash equilibrium becomes fragile, while the other becomes globally attractive, as friction goes to zero. If $\mu > 1/2$, there is an equilibrium path that traverses from (L,L) to (R,R); that is, even if (L,L) is the initial behavior patterns in this society, there exist consistent beliefs, with which the behavior patterns converge to (R,R) and thereby upsetting (L,L). On the other hand, if the initial behavior patterns are given by (R,R), no consistent beliefs can upset this behavior patterns. In this sense, (R,R) dominates (L,L) if $\mu > 1/2$. Likewise, (L,L) dominates (R,R) if $\mu < 1/2$.

It should be noted that the condition, $\mu > 1/2$, is equivalent to $d - c > a - b$; the deviation loss associated with (R,R) is larger than the deviation loss at (L,L). That is, in the terminology of Harsanyi/Selten, (R,R) risk dominates (L,L). Similarly, (L,L) risk-dominates (R,R) if $\mu < 1/2$. In sum, a Nash equilibrium of the symmetric game given in Figure 1 is a unique absorbing (and globally attractive) state in the presence of sufficiently small friction, if and only if it satisfies the risk-dominant notion of Harsanyi/Selten.

To grasp the intuition behind these results, it is useful to consider a slightly more general game in which the payoff difference of playing L instead of R is given by $\pi(x_t)$, where π is a strictly increasing function and satisfies $\pi(0) < 0$ and $\pi(1) > 0$. (The pairwise random matching game is a

⁵In the limit as δ goes to infinity, the dynamics (1) and (2) are equivalent to the best response dynamics proposed in Gilboa and Matsui (1991); see also Matsui (1990) and Matsuyama (1991c). Every strict Nash equilibrium is absorbing in the best response dynamics.

special case in which $\pi(x) = x - \mu$.) The outcome (L,L) can be upset when the players have an incentive to deviate for a feasible path from $x = 1$. Because of the monotonicity of π , the incentive to deviate is the strongest if all players are anticipated to switch from L to R in the future, or $x_t = e^{-pt}$. Thus, the condition for $x = 1$ being fragile is

$$V_0 = (p+\theta) \int_0^{\infty} \pi(e^{-pt}) e^{-(p+\theta)t} dt \leq 0 . \quad (3)$$

As seen from this expression, an increase in the expected duration of the commitment (a small p) has the two opposite effects. On one hand, it reduces the effective discount rate; the players are more concerned about the future when making decisions. On the other hand, it reduces the rate of change in the behavior patterns so that the current strategy distribution becomes more important in calculating the expected discounted payoffs. The strictly positive discount rate, $\theta > 0$, implies that the second effect always dominates the first, since (3) can be rewritten to:

$$V_0 = (1+\delta) \int_0^1 \pi(x) x^\delta dx \leq 0 , \quad (4)$$

by letting $x = e^{-pt}$. Condition (4) means that the expected discounted payoff of choosing action L when all other players are anticipated to switch from L to R is given by the weighted average of π . Note that, as x moves from 1 to 0, the players attach more weight to a higher value of x with a large degree of friction, $\delta = \theta/p$. In the limit as δ goes to infinity, $V_0 = \pi(1) > 0$ so that (4) is violated; or (L,L) becomes absorbing with a sufficiently large friction.

Similarly, starting from $x = 0$, the incentive to deviate is the strongest when all players are anticipated to switch from R to L in the

future, or $x_t = 1 - e^{-pt}$, so that the condition for $x = 0$ being fragile is given by

$$V_0 = (p+\theta) \int_0^{\infty} \pi(1-e^{-pt}) e^{-(p+\theta)t} dt \geq 0 ,$$

or

$$V_0 = (1+\delta) \int_0^1 \pi(x) (1-x)^\delta dx \geq 0 . \quad (5)$$

Thus, as x moves from 0 to 1, the players attach more weight to a lower value of x with a large degree of friction, $\delta = \theta/p$. In the limit as δ goes to infinity, $V_0 = \pi(0) < 0$ so that (5) is violated; or (R,R) becomes absorbing with a sufficiently large friction.

The two conditions, (4) and (5), are mutually exclusive for any $\delta > 0$ so that at least one of the two strict Nash outcomes is absorbing. Furthermore, in the limit as δ goes to zero, (4) and (5) become

$$\int_0^1 \pi(x) dx < 0 , \quad (6)$$

and

$$\int_0^1 \pi(x) dx > 0 , \quad (7)$$

respectively. For the pairwise random matching game, $\pi(x) = x - \mu$ and (6) and (7) are equal to $\mu > 1/2$ and $\mu < 1/2$, respectively. This shows why only one strict Nash outcome remains absorbing as the friction goes to zero for generic games. When the expected duration of the commitments becomes extremely small and the behavior patterns can move between 0 and 1 arbitrarily fast (but are not able to jump between them), all that matters is the average payoff differences. If action L performs better than R on average, then (L,L) is absorbing, while (R,R) is fragile. Note that the uniqueness of the absorbing

state in the limit does not depend on the linearity of the payoff differences.⁶

The above discussion also points out the significant difference between the logic behind our result and that of Kandori, Mailath and Rob (1991). Recall that their model is based on the repeated play by myopic players and the constant flow of mutations, so that the stationary distribution of the behavior patterns depends on the size of the basins of attraction. Their selection criterion coincides with the Harsanyi/Selten risk-dominance criterion, because the risk dominant outcome has a larger basin of attraction. On the other hand, we rely on the rational calculations by players. Our selection criterion coincides with the Harsanyi/Selten criterion because deviating from the risk dominant outcome always implies a payoff loss, whereas there exist consistent conjectures with which deviating from the risk dominated outcome leads to a gain in the expected payoffs.

3. Asymmetric Games

In this section, we extend our analysis to the class of asymmetric games given in Figure 3. Again, there are two strict Nash equilibria, (L_1, L_2) and (R_1, R_2) , and one mixed strategy Nash equilibrium in which player i plays L_i with probability $\mu_j \equiv (d_j - c_j) / \{(a_j - b_j) + (d_j - c_j)\}$, where $i \neq j$. As in the previous section, we consider the random matching framework, but the players are now divided into two groups of equal size, 1 and 2. Each player from group 1 (player 1) is randomly matched with a player from group 2 (player 2) to play the game under the same restriction with the previous case. Let x_t^i

⁶The dynamics of the game with nonlinear payoff differences are considered in Matsuyama (1991d).

($i = 1, 2$) denote the fractions of players i who play L_i as of time t . Then, the equilibrium dynamics of the behavior patterns $\{(x_t^1, x_t^2)\}_{t=0}^{\infty}$ are described by

$$\frac{d^+ x_t^i}{dt} \in \begin{cases} \{p(1-x_t^i)\} & \text{if } V_t^i > 0, \\ [-px_t^i, p(1-x_t^i)] & \text{if } V_t^i = 0, \\ \{-px_t^i\} & \text{if } V_t^i < 0, \end{cases} \quad (8)$$

where

$$V_t^i = (p+\theta) \int_0^{\infty} (x_{t+s}^j - \mu_i) e^{-(p+\theta)s} ds, \quad (i, j, = 1, 2, i \neq j) \quad (9)$$

as well as the initial condition, (x_0^1, x_0^2) .

As before, the set of stationary states of (8) and (9) is $\{(0,0), (\mu_2, \mu_1), (1,1)\}$, which is identical to the set of Nash equilibria of the original game. The definition of accessible, absorbing, fragile, and globally attractive can be directly extended into the dynamics on $[0,1]^2$.

To state the properties of $(0,0)$ and $(1,1)$, or equivalently (R_1, R_2) and (L_1, L_2) , let us define the following partition of $(0,1)^2 = A(\delta) + B(\delta) + C(\delta)$ (see Figure 4):

$$\begin{aligned} A(\delta) &\equiv \{(\mu_1, \mu_2) \in (0,1)^2: \mu_2 \geq F_\delta(\mu_1)\}, \\ B(\delta) &\equiv \{(\mu_1, \mu_2) \in (0,1)^2: 1 - \mu_2 \geq F_\delta(1 - \mu_1)\}, \\ C(\delta) &\equiv \{(\mu_1, \mu_2) \in (0,1)^2: 1 - F_\delta(1 - \mu_1) < \mu_2 < F_\delta(\mu_1)\}, \end{aligned}$$

where

$$F_\delta(X) = \begin{cases} f_\delta(X), & \text{if } 0 < X \leq \frac{1+\delta}{2+\delta}, \\ f_\delta^{-1}(X), & \text{if } \frac{1+\delta}{2+\delta} \leq X < 1, \end{cases}$$

and

$$f_\delta(X) = 1 - (2+\delta)\delta\left(\frac{X}{1+\delta}\right)^{1+\delta}.$$

Simple algebra shows that $F_\delta(X)$ is strictly decreasing, strictly concave and

$$\lim_{X \rightarrow 0} F_\delta(X) = 1, \quad \lim_{X \rightarrow 1} F_\delta(X) = 0, \quad F_\delta\left(\frac{1+\delta}{2+\delta}\right) = \frac{1+\delta}{2+\delta}, \quad F'_\delta\left(\frac{1+\delta}{2+\delta}\right) = -1.$$

Lemma 2.

- a) $(0,0)$ is globally attractive if and only if $(\mu_1, \mu_2) \in A(\delta)$.
- b) $(1,1)$ is globally attractive if and only if $(\mu_1, \mu_2) \in B(\delta)$.
- c) $(0,0)$ is absorbing if and only if $(\mu_1, \mu_2) \in (0,1)^2 \setminus B(\delta) = A(\delta) + C(\delta)$.
- d) $(1,1)$ is absorbing if and only if $(\mu_1, \mu_2) \in (0,1)^2 \setminus A(\delta) = B(\delta) + C(\delta)$.

Proof. See the appendix.

Again, Lemma 2 implies that there is at least one and at most two absorbing state and that a state is uniquely absorbing if and only if it is globally attractive. Thus, one can rephrase it as:

Proposition 2.

- a) (R_1, R_2) is uniquely absorbing and globally attractive if $(\mu_1, \mu_2) \in A(\delta)$;
 (L_1, L_2) is uniquely absorbing and globally attractive if $(\mu_1, \mu_2) \in B(\delta)$;
Both (R_1, R_2) and (L_1, L_2) are absorbing if $(\mu_1, \mu_2) \in C(\delta)$.
- b) For any $(\mu_1, \mu_2) \in (0,1)^2$, both (R_1, R_2) and (L_1, L_2) are absorbing for a sufficiently large $\delta > 0$.
- c) If $\mu_1 + \mu_2 < 1$, (L_1, L_2) is uniquely absorbing and globally attractive for a sufficiently small $\delta > 0$. If $\mu_1 + \mu_2 > 1$, (R_1, R_2) is uniquely absorbing and globally attractive for a sufficiently small $\delta > 0$.

Proof:

- a) This follows directly from Lemma 2.

b) Note that $\lim_{t \rightarrow \infty} F_\delta(X) = 1$ for $0 < X < 1$ monotonically. Thus, $C(\infty) = (0,1)^2$, from which b) follows from a).

c) For any $\delta > 0$, $F_\delta(X) \geq 1 - X$ and $\lim_{t \rightarrow 0} F_\delta(X) = 1 - X$ for $0 < X < 1$.

Therefore, $(\mu_1, \mu_2) \in A(\delta)$ for a sufficiently small $\delta > 0$ if $\mu_1 + \mu_2 > 1$, and

$(\mu_1, \mu_2) \in B(\delta)$ for a sufficiently small $\delta > 0$ if $\mu_1 + \mu_2 < 1$. Q.E.D.

Figure 4 illustrates Proposition 2a). It shows that, for a given δ , if the unique mixed strategy equilibrium is close to (L_1, L_2) , then (R_1, R_2) is absorbing. That is, for any initial behavior patterns, there is an equilibrium path that converges to (R_1, R_2) , and, if any initial behavior patterns are in the neighborhood of (R_1, R_2) , any equilibrium path converges to (R_1, R_2) . Similarly, for a given δ , if the unique mixed strategy equilibrium is sufficiently close to (R_1, R_2) , then (L_1, L_2) is absorbing. If the unique mixed strategy equilibrium belongs to $C(\delta)$, on the other hand, both strict Nash equilibria are absorbing. These regions, $A(\delta)$ and $B(\delta)$ shrink as δ becomes large, and, in the limit as friction goes to infinity, disappear. Thus, as Proposition 2b) states, in the presence of large friction, both strict Nash equilibria become absorbing. Proposition 2c), on the other hand, states that, as friction goes to zero, one strict Nash equilibrium becomes fragile and the other becomes globally attractive.

Proposition 2c) also states that (L_1, L_2) becomes absorbing if $\mu_1 + \mu_2 < 1$, which is equivalent to $(1-\mu_1)(1-\mu_2) > \mu_1\mu_2$, or $(a_1-b_1)(a_2-b_2) > (d_1-c_1)(d_2-c_2)$; that is, the product of deviation losses associated with (L_1, L_2) is larger than the product of deviation losses at (R_1, R_2) . That is, (L_1, L_2) is absorbing in the presence of small friction if and only if it is risk-dominant in the sense of Harsanyi-Selten. The intuition behind these

results is analogous to the case of the symmetric case and therefore omitted here.

Appendix

Proof of Lemma 1. To prove the "if" part of a) and the "only if" part of d), it suffices to demonstrate that, if $(1+\delta)/(2+\delta) \leq \mu < 1$, a feasible path from $x = 1$ to $x = 0$, $x_t = e^{-pt}$ satisfies the equilibrium condition, that is $V_t \leq 0$ for all t along this path. This can be checked as follows:

$$V_t = (p+\theta) \int_0^{\infty} (e^{-p(t+s)} - \mu) e^{-(p+\theta)s} ds = e^{-pt} \left(\frac{1+\delta}{2+\delta} \right) - \mu \leq 0.$$

To prove the "if" part of d) and the "only if" part of a), it suffices to prove that, if $0 < \mu < (1+\delta)/(2+\delta)$, the equilibrium path is unique and converges to $x = 1$ for x_0 sufficiently close to 1. Note that any feasible path from x_0 satisfies $x_t \geq x_0 e^{-pt}$. Therefore, if $\mu(2+\delta)/(1+\delta) < x_0 < 1$,

$$V_0 \geq (p+\theta) \int_0^{\infty} (x_0 e^{-ps} - \mu) e^{-(p+\theta)s} ds = x_0 \left(\frac{1+\delta}{2+\delta} \right) - \mu > 0.$$

This implies $x_0 \leq x_t < 1$, and $V_t > 0$ for all t . Thus, $x_t = 1 - (1-x_0)e^{-pt}$, and $\lim_{t \rightarrow \infty} x_t = 1$. This proves a) and d). The proof of b) and c) follows similarly, due to the symmetry. Q.E.D.

Proof of Lemma 2. The proof is divided into three parts.

Part 1. Proof that $(0,0)$ is globally attractive if $(\mu_1, \mu_2) \in A(\delta)$:

Without loss of generality, we assume $\mu_1 \leq \mu_2$, which can be further divided into the two cases: 1-A) $(1+\delta)/(2+\delta) \leq \mu_1 \leq \mu_2$, and 1-B) $f_\delta(\mu_1) \leq \mu_2$, and $(1+\delta)/(2+\delta) > \mu_1$.

1-A) $(1+\delta)/(2+\delta) \leq \mu_1 \leq \mu_2$: it suffices to show $(x_t^1, x_t^2) = (x_0^1 e^{-pt}, x_0^2 e^{-pt})$ is an equilibrium path for any $(x_0^1, x_0^2) \in [0,1]^2$, which can be checked as follows:

for $i, j = 1, 2, i \neq j$,

$$V_t^i = (p+\theta) \int_0^\infty (x_0^i e^{-p(t+s)} - \mu_i) e^{-(p+\theta)s} ds = x_0^i e^{-pt} \left(\frac{1+\delta}{2+\delta} \right) - \mu_i \leq 0.$$

1-B) $f_\delta(\mu_1) \leq \mu_2$, and $(1+\delta)/(2+\delta) > \mu_1$: If $x_0^2 \leq \mu_1(2+\delta)/(1+\delta)$, one can show

$(x_t^1, x_t^2) = (x_0^1 e^{-pt}, x_0^2 e^{-pt})$ is an equilibrium path converging to $(0,0)$, as in 1-

A). Suppose $x_0^2 > \mu_1(2+\delta)/(1+\delta)$. We show that a feasible path from (x_0^1, x_0^2)

to $(0,0)$, defined by

$$x_t^1 = \begin{cases} 1 - (1-x_0^1) e^{-pt} & \text{if } t < T, \\ [1 - (1-x_0^1) e^{-pT}] e^{-p(t-T)} & \text{if } t \geq T, \end{cases} \quad \text{and} \quad x_t^2 = x_0^2 e^{-pt}$$

where T satisfies $x_0^2 e^{-pT} = \mu_1 \left(\frac{2+\delta}{1+\delta} \right) < 1$, is an equilibrium path. First,

$$V_t^1 = (p+\theta) \int_0^\infty (x_0^2 e^{-p(t+s)} - \mu_1) e^{-(p+\theta)s} ds = x_0^2 e^{-pt} \left(\frac{1+\delta}{2+\delta} \right) - \mu_1,$$

so that $V_t^1 > 0$ if $t < T$; $= 0$ if $t = T$; < 0 if $t > T$. Second, let y_t be

defined by

$$y_t = \begin{cases} 1 & \text{if } t < T, \\ e^{-p(t-T)} & \text{if } t \geq T, \end{cases}$$

Note that y_t is nonincreasing and $y_t \geq x_t^1$ for all t . Therefore,

$$\begin{aligned} V_t^2 &= (p+\theta) \int_0^\infty (x_{t+s}^1 - \mu_2) e^{-(p+\theta)s} ds \leq (p+\theta) \int_0^\infty (y_{t+s} - \mu_2) e^{-(p+\theta)s} ds \\ &\leq (p+\theta) \int_0^\infty (y_s - \mu_2) e^{-(p+\theta)s} ds = 1 - \mu_2 - \frac{1}{2+\delta} e^{-(p+\theta)T} \\ &= 1 - \mu_2 - \frac{1}{2+\delta} \left[\left(\frac{\mu_1}{x_0^2} \right) \left(\frac{2+\delta}{1+\delta} \right) \right]^{1+\delta} \leq f_\delta(\mu_1) - \mu_2 \leq 0. \end{aligned}$$

Part 2. Proof that $(1,1)$ is absorbing if $(\mu_1, \mu_2) \in B(\delta) + C(\delta)$.

Without loss of generality, we assume $\mu_1 \leq \mu_2 < f_\delta(\mu_1)$, which implies $\mu_1 < (1+\delta)/(2+\delta)$. First, note that, for any feasible path, if $x_t^2 > \mu_1(2+\delta)/(1+\delta)$, then $x_{t+s}^2 \geq x_t^2 e^{-ps}$, and

$$V_t^1 \geq (p+\theta) \int_0^\infty (x_t^2 e^{-ps} - \mu_1) e^{-(p+\theta)s} ds = x_t^2 \left(\frac{1+\delta}{2+\delta} \right) - \mu_1 > 0.$$

This implies that, for $x_0^2 > \mu_1(2+\delta)/(1+\delta)$, $V_t^1 > 0$ for all $t < T$, where T satisfies $x_0^2 e^{-pT} = \mu_1 \left(\frac{2+\delta}{1+\delta} \right) < 1$. Thus,

$$x_t^1 \geq \begin{cases} 1 - (1-x_0^1) e^{-pt} & \text{if } t < T, \\ [1 - (1-x_0^1) e^{-pT}] e^{-p(t-T)} & \text{if } t \geq T, \end{cases}$$

for all $t > 0$. Since the right hand side is continuous in x_0^1 , one can choose x_0^1 sufficiently close to 1 so that, for any $\epsilon_1 > 0$,

$$\begin{aligned} V_0^2 &\geq (p+\theta) \int_0^\infty (y_s - \mu_2) e^{-(p+\theta)s} ds - \epsilon_1 = 1 - \mu_2 - \frac{1}{2+\delta} e^{-(p+\theta)T} - \epsilon_1 \\ &= 1 - \mu_2 - \frac{1}{2+\delta} \left[\left(\frac{\mu_1}{x_0^2} \right) \left(\frac{2+\delta}{1+\delta} \right) \right]^{1+\delta} - \epsilon_1. \end{aligned}$$

Therefore, for any $\epsilon_2 > 0$, by choosing x_0^2 sufficiently close to 1,

$$V_0^2 \geq f_\delta(\mu_1) - \mu_2 - \epsilon_1 - \epsilon_2 > 0.$$

This shows that there exists a neighborhood of (1,1) such that $V_0^1, V_0^2 > 0$, thus (1,1) is absorbing.

Part 3.

From Part 1 and Part 2, a) and d) follow immediately. b) and c) can be proved similarly, due to the symmetry. Q.E.D.

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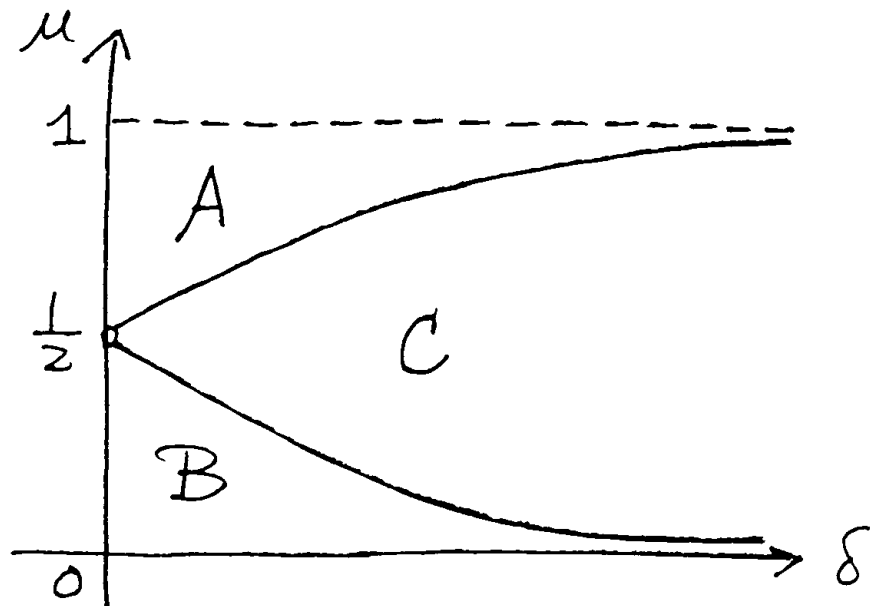
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Figure 1:

$a > b, c < d$

	L	R
L	a, a	c, b
R	b, c	d, d

Figure 2



A: (R, R) is uniquely absorbing and globally attractive.

B: (L, L) is uniquely absorbing and globally attractive.

C: (R, R) and (L, L) are both absorbing.

Figure 3:

$$a_i > b_i$$

$$c_i < d_i$$

	L_2	R_2
L_1	a_1, a_2	c_1, b_2
R_1	b_1, c_2	d_1, d_2

Figure 4

