DISCUSSION PAPER NO. 97

AN ECONOMIC SYSTEM COMBINING MARKET AND PLANNING CONTROL SUBSYSTEMS

by

Robert L. Welch

July 30, 1974

AN ECONOMIC SYSTEM COMBINING MARKET AND PLANNING CONTROL SUBSYSTEMS

Robert L. Welch*

An economic system is a pattern of relationships among individuals through which decisions are made concerning economic variables (goods and services). A casual examination of existing economic systems reveals that any economic system is a combination of a number of control subsystems, each characterized by different types of relationships among the participants. Three such subsystems which economists have investigated are the market, the planning and the voting subsystems. The purpose of this paper is to investigate the properties of an economic system combining market and planning subsystems.

Economic systems differ according to the importance of the decision making role of each of these subsystems. In democratic market economies, such as the United States, the voting and market subsystems have primary importance. Nevertheless, a planning system also exists both at the corporate level, where large integrated firms (such as General Motors) must coordinate the decisions of subsidiaries so as to achieve the overall corporate objective, at the government level, where decisions need to be

---

* The author is a Visiting Lecturer at the University of California at Santa Barbara. This paper is an extension of work completed in his Ph.D. dissertation, "Multilevel Systems and Markets: Two Alternative Solutions to the Computation Problem in Decentralized Economic Planning," Northwestern University, June, 1974. This paper benefits from comments and suggestions provided by John Ledyard, as well as Ted Groves, John Roberts, Larry Westphal and D. L. Brito.

coordinated between different branches of the executive, and at the corporate-government level, where the government must procure goods and services from corporations. In a "centrally planned" economy, such as the Soviet Union, the planning system is most important. However, both voting and market systems do have a role.1/ In the economic systems of Eastern Europe, the market subsystem has played an increasingly important role in decision making as reforms have been implemented. In the democratic socialist economies, all three subsystems have important roles.

However, a survey of theoretical economic literature does not reveal an impression of economic systems consisting of a mixture of different control subsystems. Rather, the impression is that economic systems are either pure market or pure planning systems.2/ Each subsystem is studied in a ceteris paribus situation where external variables that influence decisions within the subsystem in question are considered constant. The effect of variables determined by the model on these exogenous variables through other spheres of the economic system are ignored.

In each of these models of a pure subsystem certain desirable properties of the subsystem in question are established. For example, the existence of a competitive market equilibrium, the Pareto optimality of that

---

1/ Kornai and Martos [20] also have argued that both capitalist and socialist economies combine market and planning subsystems. See [20], page 511.

2/ Kornai and Martos [20] is an exception as are Richter [27], Foley [10] and Fouregard [11], although the latter papers are not concerned with the operation of the adjustment mechanisms of the component subsystems.

equilibrium, the convergence of a market adjustment process to that equilibrium, informational decentralization and informational efficiency have been shown to be properties of the market subsystem.[3/] Convergence of the planning adjustment process to a social optimum and the feasibility of the plan at each stage in that process are properties possessed by many of the proposed models of the planning subsystem.[4/] These properties are shown to hold if certain conditions on the economic environment (technology and preferences) are present.

When these conditions are absent, then, it is argued, there are theoretical reasons for a mixed system combining planning and market subsystems. The view of Western economists has been that whenever conditions of market failure exist--i.e., externalities and increasing returns--then government intervention is justified. Eastern economists have taken the standpoint that in general a planning system is preferred to the capitalist market system. However, in view of the informational decentralization and the informational efficiency of the market system, it may be desirable to combine markets and planning in order to improve the overall economic system. Both of these arguments for a mixed system are based on the belief that the already established properties of markets and planning systems will continue to hold under the new order, when the two subsystems are combined in a more general system.

3/ See, for example, Debreu [5], Hurwicz [15], Arrow [2], Diamond [6], and Fisher [9].

4/ Examples of planning mechanisms are found in Aoki [1], Arrow and Hurwicz [3], Heal [13, 14], Malinvaud [22, 23], Weitzman [31], and Younes [33].

However, without a model of a mixed system, such a belief may not be appropriate. For example, consider an economy of n commodities. Suppose inactivity is possible for each firm in the economy (that is, each firm can produce a vector of net outputs all of whose components are zero). If all n commodities belong to the market subsystem, then this condition together with profit distribution among consumers will establish Walras law through the summation of individual budget constraints. But in the situation where n_I of the commodities are allocated within the planning subsystem and the remaining $n-n_I$ goods are allocated by markets (for example, the n_I goods may be public goods) inactivity for marketed goods may not be possible if the production of these n_I goods is fixed in the planning subsystem at some non-zero level. Walras law in the market subsystem is then a condition that holds at a market equilibrium, but not the familiar identity holding out of equilibrium. Since Walras law is an important part of most existence proofs of a "general" market equilibrium, the possibility of a mixed system once again poses the question of existence of an equilibrium for the market subsystem.[5/]

In this paper we investigate whether a mixed system can be designed that combines the properties of market and planning systems. We will specify conditions under which existence of market subsystem equilibria, convergence of the market subsystem and of the planning subsystem can be demonstrated within the context of a mixed economic system. Furthermore,

---

5/ In their proofs of existence of an equilibrium in an economy with public goods both Richter [27] and Roberts [28] noted this difficulty with Walras law. Dreze and de la Vallee Poussin [7] pointed out the difficulty of integrating their mechanism for allocating public goods with a market system that allocates more than one private good.

we shall show that the mixed system provides for some informational efficiency over a pure planning system provided certain key commodities are included among those that are allocated on markets. Thus, our mixed system lends some support for the Eastern view towards combining market and planning systems.

In designing a mixed system it is necessary that one be able to distinguish between the market and planning subsystems. One suggestion is to base this distinction on the type of information flowing to individual firms and consumers. If this information is prices of various commodities, then the subsystem is a market subsystem. If this information consists of quantity allocations of goods and services, then the subsystem is a planning subsystem.[6/] However, this distinction is not satisfactory. Suppose in both systems these messages are adjusted in a centralized fashion. The interpretation is that the agent who adjusts prices is the Walrasian auctioneer, or helmsman, and the agent who adjusts quantities is the central planner. However, an alternate interpretation could be that the agent who adjusts prices is the central planner as in the Arrow and Hurwicz [3] and the Malinvaud [22] planning models, and the other agent is the market auctioneer who follows a Marshallian market adjustment rule. Hence,an inspection of the kind of messages flowing to the periphal agents does not provide a sufficient basis for distinguishing between the planning and market subsystems. Kornai (in Chapter 23 of [19]) has suggested a distinction based

---

6/ Heal presents such an interpretation of his allocation mechanism in [14].

on the direction of the flow of messages within the subsystem. The market subsystem consists primarily of horizontal information flows involving two peripheral agents (such as a firm and a consumer, or two different firms) in "immediate and informative contact." The planning system consists primarily of vertical information flows between subordinates and superordinates within a hierarchical organization of economic agents. Rather than firms in contact with other firms or consumers, firms are in contact with an industrial organization and these industrial organizations are in contact with the central planning bureau.

In the mixed system we present here, the Kornai distinction between market and planning subsystems is adopted. One consequence of adopting such a basis for distinction between market and planning systems is that the market adjustment process cannot be of the tatonnement variety because tatonnement is based on vertical rather than horizontal information flows. In the mixed system we consider, the planning system consists of vertical information flows--prices computed by the central agency on a class of commodites (called "planned goods") and net production proposals computed by firms for these planned goods. Allocations of the remaining commodites (referred to as "marketed goods") are determined by communication among firms within the market subsystem.

In Section I we present a model of a "pure production" economy. There is no consumption of commodities by private individuals. Rather, the central planner has a preference ordering of final consumption of the planned goods. Marketed goods are primary and intermediate products used in the production of planned goods.

In Section II the adjustment mechanism for the planning subsystem is presented. This process is similar to the decentralized planning procedure utilizing memory which Malinvaud introduced in Section 5 of [22]. At each iteration the center computes accounting prices for the planned goods only. Taking these prices as given, firms, communicating in markets, determine proposals for the planned goods and allocations of the marketed goods. For purposes of Section II, these markets are assumed to adjust to an equilibrium (existence is shown in Theorem 1) consisting of prices and allocations for marketed goods and proposals of planned goods that clear markets and maximize firm profits at the combined prices provided by the center and the market. The equilibrium proposals are the firms' messages to the center. These proposals are then used by the central planner to compute new prices at the next iteration.

In Section III we consider the mixed system as a planning procedure in its own right. It is demonstrated that the mixed procedure satisfies the criteria established by Malinvaud for evaluating planning procedures. The mixed procedure is well defined and generates a feasible plan at each iteration whose utility value converges monotonically (Theorem 2) to the maximum value attainable. In the next section we consider the mixed system as a reform of the Malinvaud "pure" planning system. The utilization of markets provides for an extension of the class of environments considered by Malinvaud and for some informational efficiency. Finally, in the last section, we consider the procedure presented in this paper as the mixed system described by Kornai. Here we provide an adjustment mechanism for the market subsystem which relies on the horizontal information flows of a system of bilateral trades (convergence of this process is provided by Theorem 3) similar to the model developed by Feldman in [8].

The appendices contain a dictionary of mathematical symbolization and proofs to some useful lemmata.

I A Model of the Economy

The n commodities of the economy are partitioned into two classes: n_1 goods allocated by the planning subsystem and $n_2 = n-n_1$ goods allocated by markets. Preferences of the central planning agency over final consumption of the n_1 planned goods are represented by a utility function $U(x):X \to \mathcal{R}$ where x is an n_1 vector of final consumption of planned goods and X is the set of admissible consumption bundles. There is no final consumption of marketed goods. Rather, these commodities are intermediate products and primary inputs used in the production of planned goods. Initial endowments of planned and marketed goods are denoted by ω_1 and ω_2 respectively.

There is a finite set $\mathcal{K}$ of K firms. An action of firm k is an n vector of net outputs, y_k (if an element of y_k is negative, then it is an input). The set of vectors y_k technologically feasible for firm k is represented by Y_k. Initial endowments of marketed goods, ω_2, have been allocated among the firms so that $Y_k \ni (0,\omega_{2k})$ where $\sum_{k\in\mathcal{K}} \omega_{2k} = \omega_2$.[7/]

The objective of the economic system is to find a consumption programme $\bar{x}^*$ and a production programme $\Pi\, \bar{y}_k^*$ that solve:

$$\text{Max } U(x) \text{ subject to}$$

$$\begin{aligned} x - \sum_{k\in\mathcal{K}} y_{1k} &\leq \omega_1 \\ - \sum_{k\in\mathcal{K}} y_{2k} &\leq 0 \end{aligned} \tag{1}$$

$$x \in X,\ y_k = (y_{1k},y_{2k}) \in Y_k,\ \forall\, k \in \mathcal{K}.$$

7/ The above definitions are summarized in Appendix A.

The class of economic environments we consider is described by the following assumptions.

A.1 Each production set, Y_k, is closed and convex.

A.2 The aggregate production set $\mathcal{Y} \equiv \sum_{k \in \mathcal{K}} Y_k$ is smooth [8/] and has a non-empty interior.

A.3 (Irreversibility) $(0^+ \text{cl}\,\mathcal{Y}) \cap (0^+ \text{cl} - \mathcal{Y}) = \{0\}$.

A.4 (Necessary input) [9/]

i $\mathcal{Y}_2 \cap \Omega$ is bounded

ii $\mathcal{Y}_1(y_2)$ is bounded, whenever $y_2 \geqq 0$.

A.5 (Positive marginal product) Suppose $y_2', y_2'' \in \mathcal{Y}_2$ and $y_2' < y_2''$. Then for each $y_1'' \in \mathcal{Y}_1(y_2'')$ there is a $y_1' \in \mathcal{Y}_1(y_2')$ such that $y_1' > y_1''$.

A.6 i X is closed, convex and bounded from below

ii $U: X \to \mathcal{R}$ is continuous, quasi-concave and increasing.

A.7 (Initial Knowledge) The central agency knows a plan $P^o = \{x^o, \Pi\, y^o_{1k}\}$ such that $x^o \in \text{int}\, X$, $x^o - \sum_{k \in \mathcal{K}} y^o_{1k} \leqq \omega_1$ and $\prod_{k \in \mathcal{K}} y^o_{1k} \in \mathcal{A}$, where

$$\mathcal{A} \equiv \{ \prod_{k \in \mathcal{K}} y_{1k} \mid \exists\, y_{2k} \ni (y_{1k}, y_{2k}) \in Y_k \text{ and } \sum_{k \in \mathcal{K}} y_{2k} \geqq 0 \} \tag{2}$$

8/ The definition of a smooth set is given in Appendix A, as are definitions of other mathematical concepts and notation.

9/ $\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_1(y_2)$ are projections of the aggregate production set $\mathcal{Y}$. See Appendix A for definitions.

A.8 (Dispersion of Information) Except as mentioned in A.7, each firm only has information concerning its own production set, and the central planning agent only has information concerning its preference ordering U, the consumption set X and the initial endowments of planned goods, ω_1.

A.1 and A.2 are standard assumptions implying non-increasing returns and differentiability of the aggregate production function. Since each Y_k contains $(0,\omega_{2k})$, irreversibility as defined in Debreu [5] implies that $\mathcal{Y} \cap -\mathcal{Y} = \{(0,\omega_2)\}$ from which condition A.3 follows. A.4 is satisfied if, among the marketed goods, there is a necessary input, ℓ, such that once ℓ has been allocated among the firms $(\Sigma\, y_{2k\ell} = y_{2\ell} \geqq 0)$, production (and disposal within plants) of marketed goods (A.4i) and planned goods (A.4ii) is bounded. An example of such a commodity is labor. By A.5, a positive "marginal product" relationship between marketed goods and planned goods is postulated. If the level of production of marketed goods is reduced (level of input is increased) then the output of planned goods can be increased (input of planned goods can be reduced). Standard restrictions are placed on the consumption set, X, and the utility function, U, in A.6. To guarantee feasibility of the plans generated by the planning procedure, the central bureaucracy is assumed in A.7 to know a feasible plan, P^o. $\mathcal{T}$ is the set of attainable planned goods production programmes.

For the problem (1) to be meaningful, it must have a solution. Since U is continuous (by A.6ii), (1) will have a solution if the set

$$X \cap (\{\Sigma y_{1k} \mid \Pi\, y_{1k} \in \mathcal{T}\} + \{\omega_1\} - \Omega) \qquad (3)$$

is compact. Since, by A.6i, X is closed and bounded from below, this intersection will be compact if $\mathcal{Y}$ is compact. Later we shall want $\mathcal{Y}$ to be convex as well.

Lemma 1 Under A.1, 3 and 4, the set of attainable planned goods production programmes, $\mathcal{Y}$, defined in (2), is compact and convex.

Proof:

Let

$$\mathcal{Y}^* \equiv \Pi Y_k \cap \mathcal{M} \tag{4}$$

where

$$\mathcal{M} \equiv \{\Pi(y_{1k}, y_{2k}) \in \mathcal{R}^{nK} \mid \Sigma\, y_{2K} \geqq 0\}. \tag{5}$$

By A.1, the cartesian product, ΠY_k is closed and convex. Since the half space $\mathcal{M}$ is closed and convex, $\mathcal{Y}^*$ is also. Notice that $\mathcal{Y}$ is the image of $\mathcal{Y}^*$ under the projection mapping $\mathcal{P}_1^*$ (see appendix A for definition). Since $\mathcal{P}_1^*$ is a linear transformation, $\mathcal{Y}$ is convex. Furthermore, if $\mathcal{Y}^*$ is bounded, as well as closed then $\mathcal{Y}$ is compact by continuity of $\mathcal{P}_1^*$.

To see that $\mathcal{Y}^*$ is bounded, we first demonstrate that the set

$$\mathcal{B} \equiv \mathcal{Y} \cap (\mathcal{R}^{n_1} \times \mathcal{R}_+^{n_2}) \tag{6}$$

is bounded. Suppose $y \in 0^+\text{cl}\,\mathcal{B}$. Then there is an $x \in \mathcal{B}$ such that $(x + \lambda y) \in \text{cl}\,\mathcal{B}$ for every $\lambda \geqq 0$. Since the projection map $\mathcal{P}_2$ is continuous, $\mathcal{P}_2\,\text{cl}\,\mathcal{B} \subset \text{cl}\,\mathcal{P}_2\mathcal{B} = \text{cl}\,(\mathcal{Y}_2 \cap \Omega)$. Thus $x_2 + \lambda y_2 \equiv \mathcal{P}_2(x+\lambda y)$ belongs to the closure of $\mathcal{Y}_2 \cap \Omega$ for every $\lambda \geqq 0$. By the necessary input assumption A.4i, $\mathcal{Y}_2 \cap \Omega$ is bounded. Hence the direction of recession of $\mathcal{B}$ in the marketed good space, y_2, is zero. Then $((x_1 + \lambda y_1), x_2) \in \text{cl}\, \mathcal{B}$ for every $\lambda \geqq 0$, where $x_1 + \lambda y_1$ is the projection $\check{\theta}_1(x + \lambda y)$. Clearly $((x_1 + \lambda y_1), x_2) \in (\text{cl}\, \mathcal{B} \cap \mathcal{R}^{n_1} \times \{x_2\})$. But $(\text{cl}\, \mathcal{B} \cap \mathcal{R}^{n_1} \times \{x_2\}) = \text{cl}\, (\mathcal{B} \cap \mathcal{R}^{n_1} \times \{x_2\})$ since $\mathcal{B} \cap \mathcal{R}^{n_1} \times \{x_2\} \neq \emptyset$. Again, by continuity of θ_1, $\theta_1 \text{cl}(\mathcal{B} \cap \mathcal{R}^{n_1} \times \{x_2\}) \subset \text{cl}\, \theta_1(\mathcal{B} \cap \mathcal{R}^{n_1} \times \{x_2\}) = \text{cl}\, \mathcal{Y}_1(x_2)$. Thus $x_1 + \lambda y_1 \in \text{cl}\, \mathcal{Y}_1(x_2)$ for every $\lambda \geq 0$. But, by A.4ii, the direction of recession, y_1, in the planned goods space is also zero. Hence, the recession cone $0^+ \text{cl}\, \mathcal{B} = \{0\}$ and $\mathcal{B}$ is bounded by Theorem 8.4 in Rockafellar [29].

Now, suppose $\{\Pi y_k^q\}$ is an arbitrary sequence in $\mathcal{A}^*$. Since, for each q, $\Pi y_k^q \in \mathcal{A}^*$, $y_k^q \in Y_k$ for each k and $\Sigma_k^q \in \mathcal{B}$. By boundedness of $\mathcal{B}$, Assumption A.3 and Lemma B.1, the sequence $\{\Pi\, y_k^q\}$ is bounded. It follows that $\mathcal{A}^*$ is bounded. qed

II The Mixed Market Planning Procedure

The algorithm for the planning subsystem is a variant of the Malinvaud planning procedure utilizing central memory found in [22]. At stage s in the planning process, the center has accumulated information on the set of attainable planned goods production programmes, $\mathcal{A}$. This information comes from the initially known programme Πy^{o}_{1k} in A.7 and from the firms' proposals of planned goods production from previous iterations, y^{τ}_{1k}, $\tau = 1,\ldots,s-1$. This information is used by the central planner in the computation of a "paper" plan $P^{s} = (\bar{x}^{s}, \Pi \bar{y}^{s}_{1k})$. Shadow prices for planned goods are then computed and sent to the firms. The firms communicate among themselves via markets for allocations of the marketed goods and production programmes of planned goods. The latter constitute the firms' responses to the center. More formally:

Center

At stage s the center solves the following problem:

$$\text{Max } U(x) \quad \text{subject to} \tag{7}$$

$$x - \sum_{k \in \mathcal{N}} y_{1k} \leqq \omega_1$$

$$x \in X \quad \text{and} \quad \Pi y_{1k} \in \mathcal{A}^{s}$$

where

$$\mathcal{A}^{s} = \{ z \in R^{Kn_1} \mid z = \sum_{\tau=0}^{s-1} \lambda^{\tau} (\Pi y^{\tau}_{1k}) \ ; \ \sum_{\tau=0}^{s-1} \lambda^{\tau} \in \mathcal{L} \}. \tag{8}$$

The set $\mathcal{A}^s$ represents the center's knowledge, at stage s, of the attainable planned goods production set, $\mathcal{A}$.

Let $P^s \equiv \{ \bar{x}^s, \Pi\ \bar{y}^s_{1k} \}$ solve (7). To see that P^s exists, define the sets

$$\varphi^s \equiv \{ \Sigma\ y_{1k} \mid \Pi\ y_{1k} \in \mathcal{A}^s \} \tag{9}$$

$$\Gamma^s \equiv \varphi^s + \{ \omega_1 \} - \Omega \quad . \tag{10}$$

Then (7) can be written as

$$\text{Max } U(x) \text{ s.t. } x \in X \cap \Gamma^s \ . \tag{11}$$

Since $\mathcal{A}^s$ is compact by construction, φ^s is compact and Γ^s is closed and bounded above. By A.6i, $X \cap \Gamma^s$ is compact. Since U is continuous by A.6ii, the solution P^s exists. This proves

Lemma 2 Under A.6, the central problem (7) has a solution.

Given the solution P^s, the center then finds the normalized price vector, $p^s_1 \in \mathcal{S}$, that satisfies:

$$\begin{aligned} &(\alpha) \quad p^s_1 x > p^s_1 \bar{x}^s \quad \text{for all } x \in X \text{ such that } U(x) > U(\bar{x}^s), \\ &(\beta) \quad p^s_1\ \Sigma\ y_{1k} \leqq p^s_1\ \Sigma\ \bar{y}^s_{1k} \quad \text{for all } \Pi y_{1k} \in \mathcal{A}^s, \\ &(\gamma) \quad p^s_1\ (\bar{x}^s - \Sigma\ \bar{y}^s_{1k} - \omega_1) = 0. \end{aligned} \tag{12}$$

These prices, p^s_1, are proportional to dual variables of the resource contraints in (7).

<u>Lemma 3</u> There exists prices, $p_1^s \in \mathscr{P}$, satisfying (12).

Proof:

Since $\mathscr{T}^s$ is convex by (8), Γ^s is a convex set. Since P^s solves (7), Γ^s does not meet the relative interior of the level set $\{x \in X \mid U(x) \geqq U(\bar{x}^s)\}$. This level set is convex by A.6. Thus there is at least one hyperplane through $\bar{x}^s$ separating Γ^s from the level set. p_1^s is the vector normal to one of these hyperplanes. qed

The "paper" plan at stage s is the solution to (7), P^s. The central price message to the firms is the vector p_1^s.

<u>Firms</u>

Taking the central price messages as constant parameters, the firms find market allocations y_{2k}^s, proposals y_{1k}^s, and prices of marketed goods, p_2^s, such that:

(i) for each firm k, $y_k^s = (y_{1k}^s, y_{2k}^s)$ maximizes firm profits at the combined prices $p^s = (p_1^s, p_2^s)$. That is $p^s y_k^s \geqq p^s y_k$ for all $y_k \in Y_k$. (13)

(ii) Markets clear: $\Sigma\, y_{2k}^s \geqq 0$

Walras law holds: $p_2^s \,\Sigma\, y_{2k}^s = 0$.

<u>Theorem 1</u> If A.1, 2, 3, 4, 5 and 6 hold, then there is an equilibrium $(\Pi y_k^s,\ p_2^s/p_1^s)$ satisfying the rules (13).

Proof:

Consider the problem:

$$\text{Max} \quad p_1^s \,\Sigma\, y_{1k} \quad \text{subject to} \tag{14}$$

$$(y_{1k}, y_{2k}) \in Y_k, \quad \forall\, k \in \mathscr{K}$$

$$\Sigma\, y_{2k} \geqq 0,$$

or

$$\text{Max} \quad p_1^s\, \Sigma\, y_{1k} \quad \text{s.t.} \quad \Pi y_{1k} \in \mathcal{A}\,. \tag{15}$$

By Lemma 1, $\mathcal{A}$ is compact. Thus (15) and (14) have a solution $y_k^s = (y_{1k}^s, y_{2k}^s)$. Associated with the constraint $\Sigma\, y_{2k} \geqq 0$ in (14) are the dual variables p_2^*. We will show that there is a price vector p_2^s proportional to p_2^* such that $(\Pi y_k^s, p^s)$ satisfies (13).

Let $y_1^s = \Sigma\, y_{1k}^s$ and $y_2^s = \Sigma\, y_{2k}^s$. Since $\text{int}\, \mathcal{Y} \neq \emptyset$ by A.2, the interior of $\mathcal{Y}_2(y_1^s)$ relative to $\mathcal{R}^{n_2}$ is not empty. Thus, if $\mathcal{Y}_2(y_1^s)$ meets $\text{int}\, \Omega$ there are vectors $\tilde{y}_2, \tilde{\tilde{y}}_2 \in \mathcal{Y}_2(y_1^s) \cap \Omega$ such that $\tilde{\tilde{y}}_2 < \tilde{y}_2$. By A.5, one can find a $\tilde{\tilde{y}}_1 > y_1^s$ such that $(\tilde{\tilde{y}}_1, \tilde{\tilde{y}}_2) \in \mathcal{Y}$. Since, by Lemma 3, $p_1^s \geqq 0$, $p_1^s \tilde{\tilde{y}}_1 > p_1^s y_1^s$ contrary to "(y_1^s, y_2^s) solves (14)." Thus $\mathcal{Y}_2(y_1^s)$ does not meet the interior of Ω.

By A.1, $\mathcal{Y}$ is a convex set. Its projection $\mathcal{Y}_2(y_1^s)$ is also convex. Since Ω is convex and $\mathcal{Y}_2(y_1^s) \cap \text{int}\, \Omega = \emptyset$, there is a hyperplane in $\mathcal{R}^{n_2}$ through y_2^s separating $\mathcal{Y}_2(y_1^s)$ and Ω. Let p_2^* be a vector orthogonal to the hyperplane. Since Ω lies in the upper half space defined by the hyperplane,

$$p_2^* \geqq 0, \quad p_2^* y_2^s = 0. \tag{16}$$

$\mathcal{Y}_2(y_1^s)$ contained in the lower half space implies

$$p_2^* y_2^s \geqq p_2^* y_2 \quad \text{for all} \quad y_2 \in \mathcal{Y}_2(y_1^s). \tag{17}$$

Furthermore, since (y_1^s, y_2^s) solves (14),

$$p_1^s y_1^s \geqq p_1^s y_1 \quad \text{for all} \quad y_1 \in \mathcal{Y}_1(y_2^s). \tag{18}$$

From (17) and (18), from convexity and smoothness of $\mathcal{Y}$ (A.1 and A.2) and from Lemma B.3, there is a hyperplane in $\mathcal{R}^n$ supporting $\mathcal{Y}$ at (y_1^s, y_2^s) containing hyperplanes in $\mathcal{R}^{n_1}$ with normalization p_1^s and in $\mathcal{R}^{n_2}$ with normalization p_2^*. Let p_2^s be proportional to p_2^* so that $p^s = (p_1^s, p_2^s)$ is a vector orthogonal to this hyperplane in $\mathcal{R}^n$. Since $\mathcal{Y}$ is the vector sum of the sets Y_k, condition (i) of (13) holds. Condition (ii) of (13) follows from (16) and (14) because p_2^s is proportional to p_2^*.

qed

An example of a market exchange process through which the equilibrium allocations and proposals can be found in a decentralized manner is provided in section five. The production programmes y_{1k}^s determined in (13i) constitute the firms' messages to the center. Sending these proposals completes stage s. At stage s+1, these messages are added to the memory $\mathcal{T}^s$ to form $\mathcal{T}^{s+1}$. The center then proceeds as in the previous iteration, computing a new plan P^{s+1} and new prices p_1^{s+1}. This process continues until convergence occurs (convergence will occur when the total value of the firms proposals $p_1^s \Sigma y_{1k}^s$ is the same as the value of the current plan $p_1^s \Sigma \bar{y}_{1k}^s$) or until the central agent decides to stop the procedure and implement the current plan.

Plan Implementation

To implement the plan $P^S = \{\bar{x}^S, \Pi \bar{y}^S_{1k}\}$ at stage S, firms again enter markets, this time taking both the production levels of planned goods, $\bar{y}^S_{1k}$, and the prices p^S_1, as given. The market equilibrium $(\Pi y^S_k, p^S_2)$ satisfies:

(i) for each $k \in \mathcal{K}$, (y^S_{1k}, y^S_{2k}) Max $(p^S_1, p^S_2)(y_{1k}, y_{2k})$ subject to $(y_{1k}, y_{2k}) \in Y_k$ and $y_{1k} \geqq \bar{y}^S_{1k}$. (19)

(ii) $p^S_2 \geqq 0$, $\Sigma\, y^S_{2k} \geqq 0$ and $p^S_2\, \Sigma\, y^S_{2k} = 0$.

Firms are required to produce the quotas $\bar{y}^S_{1k}$. For this and any excess production of planned goods they are rewarded by the center according to the prices p^S_1.

The existence of an equilibrium satisfying (19) follows as a corollary of Theorem 1. To see this, define the sets $Z_k \equiv \{ y_k \in Y_k \mid y_{1k} \geqq \bar{y}^S_{1k}\}$ and $\mathcal{Z} \equiv \Sigma\, Z_k$. If $\mathcal{Y}$ and the Y_k satisfy A.1, 3, 4 and 5, then the Z_k and $\mathcal{Z}$ will also. If A.2 holds, then $\mathcal{Z}$ is smooth on the interior of the half space $\{ y_1 \geqq \Sigma\, \bar{y}^S_{1k} \} \times \mathcal{R}^{n_2}$. Hence

Corollary 1 There is a solution to the plan implementation rules (19) if A.1, 2, 3, 4, 5 and 6 hold.

III Properties of the Planning Subsystem

In this section, the mixed planning - marketing process defined by (7), (12), (13) and (19) is considered as a planning procedure. This mixed procedure has properties suggested by Malinvaud in [22] as minimal properties of any practical planning procedure. These are:

Definition 1 A planning procedure is well defined if at each stage there are solutions to the operations by which the prospective indices (prices), the firm proposals and the plan are determined.

For the mixed procedure considered here to be well defined, there must exist solutions to the rules (7), (12) and (13). This was demonstrated in Lemma 2, Lemma 3 and Theorem 1.

Definition 2 A procedure is feasible if the plan generated at each iteration is both

(a) attainable--the plan satisfies the resource and technological constraints of the economy, and

(b) implementable--there is a solution to the rules for plan implementation.

From (2), (13) and assumption A.7, the previous joint proposals at stage s, Πy_{1k}^{τ} (τ=0,...,s-1), are elements of the feasible planned goods production set, $\mathcal{J}$. By Lemma 1, $\mathcal{J}$ is convex. The memory set, $\mathcal{J}^s$, defined in (8) is formed from convex combinations of these proposals. Hence

$$\mathcal{J}^s \subset \mathcal{J} \quad \text{for all s.} \tag{20}$$

By (20), (7) and (2), the plan at stage s, $P^s = \{\bar{x}^s, \Pi\bar{y}^s_{1k}\}$, satisfies the resource and technology constraints of the economy as outlined in (1). Thus, P^s is attainable at each s. By Corollary 1, P^s is also implementable at each stage s. Therefore, the procedure is feasible.

Definition 3 The procedure is monotonic if the utility value of the plan is nondecreasing as the planning process progresses.

From (8) it is clear that

$$\mathcal{A}^s \subset \mathcal{A}^{s+1} \quad \text{for all s.} \tag{21}$$

By the rule (7), therefore,

$$U(\bar{x}^s) \leq U(\bar{x}^{s+1}) \quad \text{for all s.} \tag{22}$$

Satisfaction of feasibility and monotonicity will guarantee that the actions specified by the plan can be carried out and that the current plan is at least as good as any plan considered previously. Malinvaud argued that these characteristics are important if it happens that the central planner decides to stop the planning procedure after a finite number of iterations but before convergence has occurred. It is also desirable that the procedure is not prevented from eventually finding a solution to the planning problem (1).

Definition 4 The procedure is convergent if the utility value of the plan converges to the highest value achievable, given complete information concerning resource and technology constraints.

<u>Theorem 2</u> Under assumptions A.1 - A.7, the planning procedure defined in (7), (12), (13) and (19) is well defined, feasible and monotonically convergent.

Proof:

From the previous remarks, all that remains to be demonstrated is that the sequence $\{U(\bar{x}^s)\}_{s=1}^{\infty}$ converges to $\bar{U}$, where $\bar{U}$ is the maximum value of $U(x)$ in the problem (1).

Notice that the problem (1) is identical to (7) when the set $\mathcal{O}^s$ in (7) is replaced by the set $\mathcal{O}$. By Lemma 1, $\mathcal{O}$ is compact. Thus, by (7), (20) and an argument identical to that used in the proof of Lemma 2, the sequence $\{\bar{x}^s\}$ belongs to a bounded set. Therefore, the sequence $\{U(\bar{x}^s)\}$ has a least upper bound, U^*, by continuity of U. Furthermore, by the monotone property, (22), U^* is the accumulation point of $\{U(\bar{x}^s)\}$. The proof of the theorem is by contradiction of $U^* < \bar{U}$.

The structure of proof is as follows: (i) we first consider certain bounded sequences generated by the procedure; next it is shown (ii) that any joint firm proposal satisfying the rules (13) maximizes joint firm revenue from planned goods on the set $\mathcal{O}$; (iii) the limit point of the sequence of price values of the production plans $\{p_1^s \Sigma y_{1k}^s\}$, is the maximum value of joint firm revenue from planned goods on the feasible set $\mathcal{O}$ valued at the limit prices p_1^*; (iv) Walras law for planned goods holds at these limit points; and (v) in the limit, any final consumption programme yielding higher utility than the least upper bound U^* will have a higher value in terms of the limit prices p_1^*; (vi) finally, the contradiction is established.

(i) The mixed procedure generates the following bounded sequences:

sequence	reason why bounded	limit point
$\{\bar{x}^s\}$	from previous remarks above	
$\{U(\bar{x}^s)\}$	continuity of U, $\{\bar{x}^s\}$ is bounded	U^*
$\{p_1^s\}$	$p_1^s \in \mathscr{S}$; the simplex $\mathscr{S}$ is bounded	p_1^*
$\{p_1^s\bar{x}^s\}$	$\{p_1^s\}$, $\{\bar{x}^s\}$ are bounded, px is continuous	a^*
$\{p_1^s \Sigma \bar{y}_{1k}^s\}$	$\Pi\bar{y}_{1k}^s \in \mathscr{A}^s \subset \mathscr{A}$, by (20), (21) and Lemma 1	$\bar{b}^*$
$\{p_1^s \Sigma y_{1k}^s\}$	$\Pi y_{1k}^s \in \mathscr{A}$, by (12) and Lemma 1	b^*

Since these are bounded sequences, subsequences that converge simultaneously to the limit points indicated above can be chosen. Further consideration is restricted to these subsequences.

(ii) $p_1^s \Sigma y_{1k}^s \geqq p_1^s \Sigma y_{1k}$ for all $\Pi y_{1k} \in \mathscr{A}$.

Suppose false. Then there is a programme $\Pi\hat{y}_{1k} \in \mathscr{A}$ such that

$$p_1^s \Sigma \hat{y}_{1k} > p_1^s \Sigma y_{1k}^s \tag{23}$$

and $\Pi\hat{y}_{1k}$ solves Max $p_1^s \Sigma y_{1k}$ subject to $\Pi y_{1k} \in \mathscr{A}$. Associated with $\Pi\hat{y}_{1k}$ is a programme of marketed goods production, $\Pi\hat{y}_{2k}$, and dual variables $\hat{p}_2$ associated with the constraint $\Sigma y_{2k} \geqq 0$ in the definition of $\mathscr{A}$ (see (2)). Then, by duality theory, $\hat{p}_2 \Sigma \hat{y}_{2k} \leqq p_2 \Sigma \hat{y}_{2k}$ for all $p_2 \in \Omega$. This implies

$$0 = \hat{p}_2 \Sigma \hat{y}_{2k} \leqq p_2^s \Sigma \hat{y}_{2k}. \tag{24}$$

From (i) of (13), $p_1^s \Sigma y_{1k}^s + p_2^s \Sigma y_{2k}^s \geqq p_1^s \Sigma \hat{y}_{1k} + p_2^s \Sigma \hat{y}_{2k}$. But (23) then implies $p_2^s \Sigma y_{2k}^s > p_2^s \Sigma \hat{y}_{2k}$. By (ii) of (13), $p_2^s \Sigma \hat{y}_{2k} < 0$ which contradicts (24).

(iii) $\bar{b}^* \geqq p_1^* \Sigma y_{1k}$ for all $\Pi y_{1k} \in \mathcal{Y}$.

By compactness of $\mathcal{Y}$ (Lemma 1), by (ii) above and by the Maximum Theorem (Berge [4], p. 116)

$$b^* \geqq p_1^* \Sigma y_{1k} \quad \text{for all} \quad \Pi y_{1k} \in \mathcal{Y}. \tag{25}$$

From (β) of (12), $p_1^s \Sigma y_{1k}^q \leqq p_1^s \Sigma \bar{y}_{1k}^s$ for all $q < s$. Since Πy_{1k}^q belongs to the bounded set $\mathcal{Y}$, in the limit, $b^* \leq \bar{b}^*$. (iii) then follows from (25).

(iv) $a^* - \bar{b}^* - p_1^* \omega_1 = 0.$

This follows from (γ) of (12).

(v) $p_1^* x > a^*$ for all $x \in X$ such that $U(x) > U^*$.

From (iii) and A.7, $\bar{b}^* \geqq p_1^* \Sigma y_{1k}^o$. (iv) then implies that $a^* \geqq p_1^*(\Sigma y_{1k}^o + \omega_1)$. But, again by A.7, $\Sigma y_{1k}^o + \omega_1 \geqq x^o$. Thus, $a^* \geqq p_1^* x^o$, where $x^o \in \text{int } X$. (v) follows from A.6, A.7, (α) of (12) and Lemma B.4.

(vi) Now suppose there is a feasible programme $\tilde{p} = \{\tilde{x}, \Pi \tilde{y}_{1k}\}$ such that $U(\tilde{x}) > U^*$. But then, from (iii), (iv) and (v), $p_1^*(\tilde{x} - \Sigma \tilde{y}_{1k} - \omega_1) > 0$, which contradicts $p_1^* \geqq 0$ and $\tilde{x} - \Sigma \tilde{y}_{1k} \leqq \omega_1$. qed

The Mixed System Considered as a Reform of Pure Planning Procedures Utilizing Central Memory.

The mixed market planning procedure is a member of a family of planning procedures that utilize central memory and operate within a discrete time framework. Other members of this family include the Malinvaud (section 5 of [22]) and the Younes [33] procedures. It is interesting to compare the mixed procedure with other members of this class. In order to compare two procedures, it is necessary to compare them within the same class of economies. In Theorem 2, it was shown that the mixed procedure performs well by one set of criteria within the class of economies for which commodities can be partitioned into final consumption goods and for which A.1 - A.8 hold.

Remark The mixed procedure performs as well as the Malinvaud procedure, by the criteria defined in Definitions 1 - 4, when the hypothesis that commodities can be partitioned into final and non-final goods and the property of smoothness of the aggregate production set, $\mathcal{Y}$, (A.2) are added to Hypotheses 6, 7 and 8 in [22] (A.3, 4 and 5 are not required by the mixed procedure when individual production sets are bounded, as in Malinvaud). The mixed procedure performs as well as the Younes procedure when goods can be partitioned, smoothness of $\mathcal{Y}$ (A.2), necessary input (A.4) and positive marginal product (A.5) are added to the Younes assumptions.

Does the mixed procedure have better performance than other memory procedures such as Malinvaud and Younes, under some other criteria? A natural criterion to consider is efficiency. Efficiency of the information exchange process can be considered from two points of view -- from the standpoint of the central planning agency and from that of the entire economy.

Computational Burden at the Central Planning Agency

The Malinvaud - Younes Procedures both require the solution, at each stage s, of a mathematical programming problem, similar to (7). The Malinvaud and Younes problems both have $(n_1 + n_2 + K)$ constraints and $(n_1 \neq sK)$ variables. The problem (7), however, only has $(n_1 + 1)$ constraints and $(n_1 + s)$ variables. 10/ Thus, the introduction of markets into the Malinvaud - Younes planning framework reduces the computational burden placed on the central agent by a factor of $(n_2 + K - 1)$ constraints and $(K-1)$ variables.

How large is this reduction? This depends on the size of n_2 and the structure of the firms. If the firms are large and vertically integrated, then the size of n_2 (the number of "marketed" goods, or the number of primary and intermediate goods) is not likely to be very large. On the other hand, if the firms are small and not vertically integrated, then n_2 may be

---

10/ (7) can be written as Max $U(x)$ s.t.

$$x_i - \sum_{k \in \mathcal{V}} \sum_{\tau=0}^{s-1} \lambda^\tau Y^\tau_{1ki} \leqq \omega_{1i}, \quad i=1,\ldots,n_1$$

$$\sum_{\tau=0}^{s-1} \lambda^\tau = 1$$

$$x \in X, \ \lambda^\tau \geqq 0, \ \tau = 0,\ldots,s-1.$$

The x_i and the λ^τ are the variables in this problem.

quite large. For example, marketed goods may include all the different varieties of steel products (ingots, rods, sheets of various dimensions and alloys). The planned goods, on the other hand, will include a relatively small number of final products (automobiles, tanks and missiles).

Furthermore, a substantial saving in communication costs resulting from the integration of markets is not reflected in the size of n_2. The actual number of firms in communication with the center may be reduced as well. The mixed system does not require any communication between the central agent and firms specializing in the production of marketed goods and using only marketed goods as inputs. For example, the central government may communicate with major defense contractors for planned goods, such as missiles. There are, however, a large number of firms specializing in different electrical and chemical components with whom the major defense contractors must negotiate. The central government need not be involved in this subcontracting process.

Hence, there can be substantial savings for the central agent in terms of computation and communication resulting from the utilization of markets. However, this savings results from a shift in computation from the central planning agency to the firms themselves. Whether the net impact of markets on a memory procedure such as the Malinvaud system is a reduction in computation and communication costs depends on the nature of the market adjustment mechanism.

Informational Efficiency of the Mixed System

Without considering specific technologies used in communicating and computing, systems can be compared using the concept of informational efficiency introduced by Hurwicz in [16]. Recent work by Hurwicz [17], Mount and Reiter [25] and Reiter [26] has shown that, when the message space utilized by an adjustment process is Euclidean, a comparison of adjustment process according to the Hurwicz notion of informational efficiency is equivalent to a comparison based on the dimensionality of the message space.

Thus, when individual production sets are bounded, the Malinvaud and the mixed procedures perform equally well in terms of informational efficiency. The message space of the Malinvaud procedure consists of vectors of n prices and n proposed production levels by each of K firms. The dimension of the message space is thus $(K + 1)\,n$. Similarly, the message space of the mixed procedure consists of vectors of n prices (n_1 centrally computed and n_2 computed by markets) and n proposed production levels (n_1 sent to the center, n_2 demands in the markets) by each of K firms. When individual production sets are not bounded, but A.3 - A.5 hold, the mixed procedure still has a message space of size $(K + 1)\,n$. The Younes procedure, however, requires that the center communicate both prices and production quotas for each firm. The size of the message space is then $(2K + 1)\,n$. Hence, the utilization of markets and the inclusion of labor (or some other esssential input) among the marketed goods results in some informational efficiency over the Younes planning system when the class of environments includes unbounded production sets.

This result should be qualified, however. Whether there is in fact a market adjustment mechanism whose message space has dimension $(K + 1)\,n_2$ to complement the planning adjustment mechanism whose message space has dimension $(K + 1)\,n_1$ has not as yet been shown. Indeed, for the market mechanism we are about to consider, the class of environments described by assumptions A.1 - A.7 will be restricted to a smaller class. While this smaller family will include unbounded production sets, it will not cover constant returns to scale production functions.

In any case, the utilization of markets as a reform of the Malinvaud - Younes planning systems does considerably reduce the computational burden on the central agent, and does allow unbounded production sets without the requirement of central quantity directives. As a guide for reformers, these markets must include markets for labor (or some essential input). Furthermore, when markets are allowed to operate during the planning process, it must be recognized that the responses, y^s_{1k}, of the firms are not independent. Thus, rather than forming individual memory sets for each firm (as is done in Malinvaud and Younes) the reformed center must keep track of this dependence by forming a memory set of joint proposals, as the set $\mathcal{J}^s$ in (8).

V The Mixed Procedure Considered as an Economic System Combining Vertical and Horizontal Information Flows.

In the presentation of the mixed procedure of section 2, the market subsystem was assumed to be self equilibrating. In Theorem 1, the existence of that equilibrium was demonstrated. Two important questions remain. The first concerns the existence of a market adjustment process. The second asks whether there is an adjustment process that relies primarily on horizontal information flows. If this second question is answered affirmatively, then the mixed market planning procedure is a description of the mixed system Kornai suggested, with vertical information flows in the planning subsystem (central price indices, plans and firm proposals of planned goods) and horizontal information flows in the market subsector (rates of exchange and offers).

The answer to the first question, whether there is an adjustment mechanism for the market subsystem, is not obvious. Walras law does not automatically hold on a subspace of the commodity space. Yet Walras law is important for stability of the Walrasian tatonnement process. Thus, the common description of market adjustment, tatonnement, is by no means an obvious answer to the question of existence of an adjustment process in the market subsystem. Nor is the tatonnement process an answer to the second question. The tatonnement mechanism is not a description of a market process based on horizontal information flows. Information flows are vertical, between a firm and the auctioneer, but not between two firms.

The process investigated here is one of bilateral exchange. Given the central prices, p_1^s, each firm has an induced preference ordering of marketed goods. Two bundles of marketed goods are compared according to the revenue from planned goods attainable. Clearly, if all marketed goods are primary inputs, then, given the initial distribution of resources ω_{2k}, the pure production model collapses into a pure exchange model. By exchanging contracts to deliver goods rather than the goods themselves, intermediate products can be exchanged as well in a bilateral process. 11/

The Restricted Class of Environments

In order to make use of the Feldman [8] model of bilateral exchange, the induced preferences must exist, must be strictly convex and monotone in some common component and must be representable by a continuously differentiable function. In order that these conditions hold the following restrictions are placed on the individual production sets.

A.9 Each Y_k is a strictly convex and smooth set.

By this assumption, constant returns to scale production functions are not considered, though unbounded strictly decreasing returns are.

A.10 There is a marketed good, ℓ, such that for each firm $k \in \mathcal{K}$, if y'_{2k}, $y''_{2k} \in Y_{2k}$ stand in the relationship $y'_{2k} \leqq y''_{2k}$ and $y'_{2k\ell} < y''_{2k\ell}$, then for each $y''_{1k} \in Y_{1k}(y''_{2k})$ there is a $y'_{1k} \in Y_{1k}(y'_{2k})$ such that $y'_{1k} \leq y''_{1k}$.

11/ These latter intermediate products, as pointed out earlier, are responsible for a substantial reduction in central computation and communication achieved by integrating markets into a planning procedure such as Malinvaud or Younes. However, in order that every firm possess induced preferences over marketed goods, no firm can specialize in marketed goods production and consumption. Hence, bilateral exchange is not an ideal model of market adjustment.

By A.10, there is a positive "marginal product" relationship between levels of input of a marketed good ℓ and levels of output of some planned good. This market good ℓ for which there is a positive marginal product must be the same commodity for all firms (An example is labor). Implicitly, by A.10, we require that each firm either produces a planned good or uses a planned good as an input in the production of a marketed good. No firm is allowed to specialize in marketed goods only. This assumption is necessary for each firm to have an induced preference ordering on marketed commodities.

A.11 $\quad p_1^s > 0$ for all s.

This condition will be satisfied if the utility function $U(x)$ is strictly monotone increasing in all arguments.

The Induced Preference Orderings

The k^{th} firm's preferences on marketed goods are determined by the revenue that can be obtained from planned goods, given the central prices p_1^s.

Definition 5 The k^{th} firm's revenue function is

$$\pi^k(y_{2k}) \equiv \underset{Y_{1k}(y_{2k})}{\text{Maximum}} \quad p_1^s y_{1k}.$$

π^k is defined on the projection of the production set, $Y_{2k} = \mathcal{P}_2 Y_k$. From A.9, $Y_{1k}(y_{2k})$ is strictly convex and from A.11, $p_1^s > 0$. Thus π^k is well defined.

<u>Definition 6</u> The k^{th} firm's induced preference ordering $\succcurlyeq_k$ is such that, for any pair $y'_{2k}, y''_{2k} \in Y_{2k}$, $y'_{2k} \succcurlyeq_k y''_{2k}$ $(y'_{2k} \succ_k y''_{2k})$ if and only if $\pi^k(y'_{2k}) \geqq \pi^k(y''_{2k})$ $(\pi^k(y'_{2k}) > \pi^k(y''_{2k}))$.

These preferences have properties, given A.9 - 11, that insure the convergence of a bilateral exchange process to the equilibrium defined in (13).

<u>Lemma 4</u> Under 4.1, π^k is a continuous function on the set Y_{2k}.

proof:

Let $\{y^q_{2k}\}$ be a sequence in Y_{2k} that converges to y^o_{2k}. Since the sequence converges, there is a $0 \leqq \lambda < \infty$ such that $y^q_{2k} \in B_\lambda$ for each q. (B_λ is the closed Euclidean ball centered on the origin with radius λ.) Define the truncated production set $Y_k(\lambda) \equiv Y_k \cap B_\lambda$. Since, by A.1, Y_k is closed and convex, $Y_k(\lambda)$ is a compact and convex set. By Lemma B.5,

$$Y_{1k}(y_{2k}/\lambda) \equiv \{y_{1k} \mid (y_{1k}, y_{2k}) \in Y_k(\lambda)\} \tag{26}$$

is a continuous correspondence on the projection $\mathcal{P}_2 Y_k(\lambda)$. Then, by the Maximum Theorem (Berge [4], p 116), π^k is a continuous function on the set $\mathcal{P}_2 Y_k(\lambda)$. Hence $\pi^k(y^q_{2k}) \to \pi^k(y^o_{2k})$. Thus π^k is continuous on the set Y_{2k}. qed

<u>Lemma 5</u> Under A.9, $\succcurlyeq_k$ is strictly convex.

proof:

If, for every pair $y'_{2k}, y''_{2k} \in Y_{2k}$ such that $y'_{2k} \succcurlyeq_k y''_{2k}$, $y_{2k}(\sigma) \equiv \sigma y'_{2k} + (1-\sigma)y''_{2k} \succ_k y''_{2k}$ for every $\sigma \in (0,1)$, then $\succcurlyeq_k$ is strictly convex.

Define y'_{1k}, y''_{1k} so that (y'_{1k}, y'_{2k}) and (y''_{1k}, y''_{2k}) belong to Y_k and $p^s_1 y'_{1k} = \pi^k(y'_{2k})$, $p^s_1 y''_{1k} = \pi^k(y''_{2k})$. Since Y_k is strictly convex, $(y_{1k}(\sigma), y_{2k}(\sigma)) \in \text{int } Y_k$ for each $\sigma \in (0,1)$. Thus there is a $\hat{y}_{1k} > y_{1k}(\sigma)$ such that $(\hat{y}_{1k}, y_{2k}(\sigma)) \in Y_k$. Hence, $p^s_1\hat{y}_{1k} > p^s_1 y_{1k}(\sigma) \geqq p^s_1 y''_{1k}$ since $p^s_1 y'_{1k} \geqq p^s_1 y''_{1k}$. This implies $\pi^k(y_{2k}(\sigma)) \geqq p^s_1\hat{y}_{1k} > \pi^k(y''_{2k})$. Hence, by Definition 6, $\succcurlyeq_k$ is strictly convex. qed

Lemma 6 Under A.10 and A.11, $\succcurlyeq_k$ is monotone decreasing in the component ℓ.

proof:

Choose $y'_{2k}, y''_{2k} \in Y_{2k}$ so that $y'_{2k} \leqq y''_{2k}$ and $y'_{2k\ell} < y''_{2k\ell}$. Take $y''_{1k} \in Y_{1k}(y''_{2k})$ so that $p^s_1 y''_{1k} = \pi^k(y''_{2k})$. Then, by A.10, there is a $y'_{1k} \in Y_{1k}(y'_{2k})$ such that $y'_{1k} \geq y''_{1k}$. Since $p^s_1 > 0$ by A.11, $p^s_1 y'_{1k} > p^s_1 y''_{1k}$. $y'_{1k} \in Y_{1k}(y'_{2k})$ implies $\pi^k(y'_{2k}) \geqq p^s_1 y'_{1k} > p^s_1 y''_{1k} = \pi^k(y''_{2k})$. Hence $y'_{2k} \succ_k y''_{2k}$. qed

Lemma 7 If A.1 and A.9 hold, then π^k has continuous first partial derivatives on the set Y_{2k}.

proof:

Let $f: W \to \mathcal{R}^1$ be a proper, closed, convex function. As in Rockafellar [29, pp 214 - 215] a subgradient vector of f at $x \in W$ is a vector v such that

$$f(z) \geq f(x) + v(x-z), \quad \text{for all} \quad z \in W. \tag{27}$$

The subdifferential of f at x, $\partial f(x)$, is the set of all such subradients. From Theorem 25.1 in [29], if f has a unique subgradient vector at x, then f is differentiable at x. In this case, the unique subgradient is also the gradient of f, where the gradient ∇f is the vector of first partial derivatives. That is

$$\partial f(x) = \{\nabla f(x)\} \tag{28}$$

where

$$\nabla f(x) \equiv (\partial f/\partial x_1, \ldots, \partial f/\partial x_m),$$

$$\partial f/\partial x_j \equiv \lim_{\lambda \to 0} \frac{f(x + \lambda e_j) - f(x)}{\lambda} \quad .$$

and e_j is the characteristic vector.

Furthermore, from Theorem 24.4 in [29], $\partial f(x)$ is an upper semi-continuous correspondence on the set W. Thus, if f has a unique sub-gradient at every point $x \in W$, then $\partial f(x)$ is a function and $\nabla f(x) = \partial f(x)$ is continuous on the set W. Now consider the function $g(x)$. If $g(x)$ is a concave, continuous function that is finite for every $x \in W$, then $-g(x)$ is a proper, closed, convex function. Moreover, if $g(x)$ is such that $-g(x)$ has a unique subgradient at every $x \in W$, then $\nabla g(x) = -(\nabla - g(x))$ is continuous on W.

From Lemma 4, π^k is continuous on Y_k and, from Lemma 5, π^k is concave. Since $p_1^s > 0$ and Y_k is strictly convex, π^k is finite on Y_{2k}. Thus, for π^k to have continuous first partial derivatives, it suffices to show that $-\pi^k$ has a unique subgradient at every $y_{2k} \in Y_{2k}$. This will follow from A.9 (Y_k is smooth).

By (27), for every $v \in \partial(-\pi^k)(\hat{y}_{2k})$,

$$\pi^k(y_{2k}) - vy_{2k} \leqq \pi^k(\hat{y}_{2k}) - v\hat{y}_{2k} \quad \text{for all} \quad y_{2k} \in Y_{2k}. \tag{29}$$

From (29), if $y_{2k} \in Y_{2k}$ is such that $\pi^k(y_{2k}) \geqq \pi^k(\hat{y}_{2k})$ then $vy_{2k} \geqq v\hat{y}_{2k}$. Hence $vy_{2k} \geqq v\hat{y}_{2k}$ for all $y_{2k} \in U(\hat{y}_{2k})$, where $U(\hat{y}_{2k})$ is the level set $\{y_{2k} \in Y_{2k} \mid \pi^k(y_{2k}) \geqq \pi^k(\hat{y}_{2k})\}$.

Since Y_k is convex, closed and smooth (A.1 and A.9), $U(\hat{y}_{2k})$ is smooth by Lemma B.6 and by Definition 5. Therefore, the subgradient v is unique and $\nabla f(y_{2k})$ is a continuous function at every $\hat{y}_{2k} \in Y_{2k}$.

qed

A Market Adjustment Process

The market adjustment process we utilize consists of sequences of bilateral trade moves between pairs of firms. These trade moves are exchanges of contracts to deliver goods. The k^{th} enterprise manager promises to deliver marketed goods to the manager of k'. This promise is based on his current net position due to accumulated contracts from past trade moves, his knowledge of the production set Y_k and the revenue from planned goods that can be obtained by exchanging marketed goods. Feldman [8] has shown that if we restrict attention to sequences of bilateral trade moves such that each pair of traders meet every so often and each trade cannot be blocked by the pair trading and if certain other conditions hold, then every limit point of such a sequence is the market equilibrium described in (13).

An allocation is a vector $z = (y_{21}, \ldots, y_{2K})$ of marketed goods. A trade is a move from an allocation z to an allocation w. The set of feasible allocations is $A \equiv \{z = (y_{21}, \ldots, y_{2K}) \mid y_{2k} \in Y_{2k};\ \Sigma\, y_{2k} = 0\}$. It is contained in a projection of the set $\mathcal{J}^*$ of (4). Since $\mathcal{J}^*$ was shown to be bounded in the proof to Lemma 1, A is bounded and, since Y_k is closed (A.1), A is compact. The set A corresponds with the set $A(\omega)$ in Feldman [8], page 464.

A bilateral trade between $\ddot{k}$ and $\tilde{k}$ is a movement from an allocation $z' \in A$ to an allocation $z'' \in A$ such that $y'_{2k} = y''_{2k}$ for all but at most two firms (i.e., for all $k \in \mathcal{K} \sim \{\tilde{k}, \ddot{k}\}$) and such that $\pi^k(y'_{2k}) \leq \pi^k(y''_{2k})$ for all k. The trade is optimizing if there is no allocation $z^* \in A$ such that $y''_{2k} = y^*_{2k}$ for all $k \in \mathcal{K} \sim \{\tilde{k}, \ddot{k}\}$ and $\pi^k(y^*_{2k}) > \pi^k(y''_{2k})$ for some $k \in \{\tilde{k}, \ddot{k}\}$.

A sequence of trade moves is described by the sequence of allocations $\{z^q\}$. A round of trade moves is the interval $\{q', q' + 1, \ldots, q' + m\}$. The sequence is a rotating sequence if for every q' and every pair of firms $\{\tilde{k}, \ddot{k}\}$ there is a $q \in \{q', \ldots, q' + m\}$ at which time the pair $\{\ddot{k}, \tilde{k}\}$ meet. In order that every possible pair meets in each round, $m \geq K(K-1)/2$.

The following Theorem gives sufficient conditions under which sequences of bilateral trade moves lead to an equilibrium satisfying (13).

Theorem 3 Suppose A.1, 3, 4, 9, 10, 11 hold and each firm has a strictly positive initial endowment of the good ℓ, $\omega_{2k\ell}$, described in A.10. Also, suppose $\{z^q\}$ is a sequence of allocations resulting from a rotating sequence of optimizing bilateral trade moves and suppose that for some q, the set $\{z \in A \mid \pi^k(z_k) \geqq \pi^k(z_k^q) \ \forall\ k \in \mathscr{K}\}$ is contained in the interior of A. Then any limit point of $\{z^q\}$ is an equilibrium satisfying the conditions of (13).

proof:

As noted above, from A.1, 3, 4, the set A is compact and convex. Lemmas 4, 5, 6 and 7 state that each revenue function π^k has continuous first partial derivatives, represents preferences $\succcurlyeq_k$ that are continuous, strictly convex and are monotone decreasing in at least one common component, ℓ. From Theorem 3 in Feldman [8] any limit point z^* of $\{z^q\}$ is Pareto Optimal (and hence all such limit points are valued equally by each firm since the sequence $\{\pi^k(z_k^q)\}$ is non-decreasing by the definition of an optimizing bilateral move). That is, there is no allocation $\tilde{z} \in A$ such that

$$\pi^k(\tilde{z}_k) \geqq \pi^k(z_k^*), \ > \ \text{for some} \ \ k \in \mathscr{K}. \tag{30}$$

Since $z^* \in A$, $\Sigma\, z_k^* = 0$. Define the level set

$$U_k^* \equiv \{z_k \in Y_{2k} \mid \pi^k(z_k) \geqq \pi^k(z_k^*)\} \ \ \text{for each} \ \ k \in \mathscr{K}. \tag{31}$$

From (30), $0 \in \mathrm{Bd}\ \Sigma\, U_k^*$. Since each π^k is continuous and concave by Lemmas 4 and 5, U_k^* is convex and closed. Thus there is a hyperplane $\mathcal{H}$ supporting $\Sigma\, U_k^*$ at $\Sigma\, z_k^* = 0$. Let the vector p_2^* be normal to $\mathcal{H}$. Since π^k is decreasing, it follows that, for each $k \in \mathcal{K}$,

$$p_2^* z_k^* \geqq p_2^* z_k \quad \text{for all} \quad z_k \in U_k^*. \tag{32}$$

Let $y_{1k}^* \in Y_{1k}$ so that $p_1^s y_{1k}^* = \pi^k(z_k^*)$. Then

$$p_1^s y_{1k}^* \geqq p_1^s y_{1k} \quad \text{for all} \quad y_{1k} \in Y_{1k}(z_k^*). \tag{33}$$

Now suppose that $y_{2k} \in Y_{2k}(y_{1k}^*)$. Then, by definition of π^k, $\pi^k(y_{2k}) \geqq \pi^k(z_k^*)$. Thus

$$Y_{2k}(y_{1k}^*) \subset U_k^* \quad \text{for all} \quad k \in \mathcal{K}. \tag{34}$$

From (32) and (34)

$$p_2^* z_k^* \geqq p_2^* y_{2k} \quad \text{for all} \quad y_{2k} \in Y_{2k}(y_{1k}^*). \tag{35}$$

(33) and (35) imply

$$p_2^* \,\Sigma\, z_k^* \geqq p_2^* y_2 \quad \text{for all} \quad y_2 \in \mathcal{Y}_2\ (\Sigma\, y_{1k}^*) \tag{36}$$

and

$$p_1^s \,\Sigma\, y_{1k}^* \geqq p_1^s y_1 \quad \text{for all} \quad y_1 \in \mathcal{Y}_1\ (\Sigma\, z_k^*).$$

Since $\mathcal{Y}$ is convex and smooth by A.9 there is a hyperplane $\mathcal{H}^*$ supporting $\mathcal{Y}$ at $(\Sigma\, y_{1k}^*, \Sigma\, z_k^*)$ with normalizations $p_1^s \in \mathcal{R}^{n_1}$ and $p_2^* \in \mathcal{R}^{n_2}$ by Lemma B.3. Define p_2^s to be proportional to p_2^* so that

(p_1^s, p_2^s) is normal to $\mathcal{N}^*$, and define $y_{1k}^s \equiv y_{1k}^*$, $y_{2k}^s \equiv z_k^*$ for each $k \in \mathcal{K}$. $(p_2^s, y_{1k}^s, y_{2k}^s)$ is an equilibrium satisfying (13). qed.

In the same manner, a bilateral trade process will lead to the equilibrium defined in (19) by substituting the sets Z_k defined in the Corollary to Theorem 1 for Y_k in Lemmas 4 - 7 above and in the above Theorem.

Corollary Every limit point of a rotating sequence of optimizing bilateral trade moves, in which firms take the plan quotas $\bar{y}_{1k}^s$ as given, is a solution to the plan implementation rules (19).

Conclusion

Combining Section 5 with Section 2 we have an economic system consisting of planning and market subsystems. The planning and market subsystems are distinguished by the nature of the information flows. In the planning subsystem information flows vertically, between a firm and the center. Information flows horizontally in the market subsystem between individual firms. The planning subsystem satisfies the Malinvaud criteria for evaluating planning procedures, and involves less computation and communication than other memory based planning procedures. The market subsubsystem, consisting of a bilateral exchange process among the firms converges to a market equilibrium. The mixed system enjoys some informational efficiency over a pure planning system when production sets are not bounded, provided labor (or some other essential input) in included among the marketed commodities.

The model is limited in that it does not take into account final consumption by private individuals. Lange, in his discussion of the role of markets in a socialist economy in [21] emphasized the need for the inclusion of labor among marketed goods in order to insure freedom of consumer choice. Our result is that there is a technological reason, based on communication costs, for labor to be allocated by markets, without considering private consumption of individuals.

APPENDIX A

Summary of the Model

n the number of commodities

n_1 the number of planned goods

n_2 the number of marketed goods; $n_2 = n - n_1$

x n_1 vector of final consumption of planned goods

X admissible set of final consumption vectors: $X \subset \mathcal{R}^{n_1}$

U Central agent's utility function; $U: X \to \mathcal{R}$

$\mathcal{K}$ index set of firms

K cardinality of the set $\mathcal{K}$

y_k an n vector of net output by firm k

y_{1k} the planned goods component subvector of y_k: $y_{1k} = \theta_1 y_k$

y_{2k} the marketed goods component subvector of y_k: $y_{2k} = \theta_2 y_k$

Y_k the set of producible vectors y_k by firm k_1

Y_{1k} the set of firm k producible planned goods vectors: $Y_{1k} = \theta_1 Y_k$

Y_{2k} $= \theta_2 Y_k$

$Y_{1k}(y_{2k})$ The set of planned goods vectors producible by firm k

when marketed goods are fixed at the level y_{2k}:

$$Y_{1k}(y_{2k}) \equiv \{y_{1k} \mid (y_{1k}, y_{2k}) \in Y_k\} = \theta_1(Y_k \cap \mathcal{R}^{n_1} \times \{y_{2k}\}).$$

$$Y_{2k}(y_{1k}) \equiv \theta_2(Y_k \cap \{y_{1k}\} \times \mathcal{R}^{n_2})$$

$\mathcal{Y}$ the aggregate production set: $\mathcal{Y} \equiv \sum_{k \in K} Y_k$

$$\mathcal{Y}_1 \equiv \theta_1 \mathcal{Y};\ \mathcal{Y}_1(y_2) \equiv \theta_1(\mathcal{Y} \cap \mathcal{R}^{n_1} \times \{y_2\})$$

$\mathcal{Y}_2 \equiv \theta_2\mathcal{Y}$; $\mathcal{Y}_2(y_1) \equiv \theta_2(\mathcal{Y} \cap \{y_1\} \times \mathcal{R}^{n_2})$

ω_1 initial endowment of planned goods

ω_{2k} $k^{\underline{th}}$ firm's ownership of marketed goods endowments.

Mathematical Notation

$\mathcal{R}^n$ the n dimensional Euclidean space

Let x and y be vectors in $\mathcal{R}^n$. Define the relations $>$, $\geqq$, and $\geq$ as:

$$x > y \text{ if } x_i > y_i \text{ for all } i = 1,\ldots,n$$

$$x \geqq y \text{ iff } x_i \geqq y_i \text{ for all } i = 1,\ldots,n$$

$$x \geq y \text{ iff } x \geqq y \text{ but } x \neq y.$$

$x \quad y$ the Cartesian product of the vectors x and y

$\prod_{j \in \mathcal{J}} y_j$ the Cartesian product of the vectors y_j for all j in the index set $\mathcal{J}$.

$\prod y_j$ an abbreviation of $\prod_{j \in \mathcal{J}} y_j$.

θ_1 the projection mapping of $\mathcal{R}^{n_1 + n_2}$ into $\mathcal{R}^{n_1}$; $\theta_1(y_1,y_2) = y_1$

$\theta_2 : \mathcal{R}^{n_1+n_2} \to \mathcal{R}^{n_2}$; $\theta_2(y_1,y_2) = y_2$

$\mathcal{P}_1^*: \mathcal{R}^{K(n_1+n_2)} \to \mathcal{R}^{Kn_1};\ \mathcal{P}_1^*(\prod_{k\in K}(y_{1k},y_{2k})) = \prod_{k\in K} y_{1k}$

$\mathcal{P}_2^*: \mathcal{R}^{K(n_1+n_2)} \to \mathcal{R}^{Kn_2};\ \mathcal{P}_2^*(\prod_{k\in K}(y_{1k},y_{2k})) = \prod_{k\in K} y_{2k}$

$\mathcal{R}_+^n$ the nonnegative orthant of $\mathcal{R}^n$

Ω the nonnegative orthant of a Euclidean Space whose dimension can be determined from context

$\mathcal{S} \equiv \{x \in \mathcal{R} \mid \sum_i x_i = 1\}$; the unit simplex

$X + Y \equiv \{z \mid z = x + y$ for all $x \in X$ and all $y \in Y\}$

$X - Y \equiv \{z \mid z = x - y$ for all $x \in X$ and all $y \in Y\}$

$X \sim Y \equiv \{x \in X \mid x \notin Y\}$

$\|x\|$ the Euclidean norm (distance)

$B_\delta \equiv \{x \mid \|x\| < \delta\}$; the Euclidean ball of radius δ

$\mathrm{cl}\ C \equiv \bigcap_{\delta>0} (C + B_\delta)$; the closure of the set C

$\mathrm{int}\ C \equiv \{x \in C \mid \exists\, \delta > 0 \ni \{x\} + B_\delta \subset C\}$; the interior of C

$\mathrm{Bd}\ C \equiv \mathrm{cl}\ C \sim \mathrm{int}\ C$; the boundary of C

ri C the interior of C relative to the affine hull of C

$0^+C \equiv \{y \mid (x + \lambda y) \in C$ for every $\lambda \geqq 0$ and $x \in C\}$; the recession cone of C

smooth The convex set C is smooth if, for every $c \in \mathrm{Bd}\ C$, there is one and only one hyperplane through c supporting C.

APPENDIX B

<u>Lemma B.1</u> Let $\{\prod_{\ell\in\mathcal{L}} a_\ell^q\}$ be a sequence such that $a_\ell^q \in \mathcal{A}_\ell$ for all q and for each ℓ in $\mathcal{L}$, where $\mathcal{A}_\ell$ is a set in $\mathcal{R}^m$ and $\mathcal{L}$ is a finite set of integers. Suppose $(0^+ \text{ cl } \sum_\ell \mathcal{A}_\ell) \cap (0^+ \text{ cl } - \sum_\ell \mathcal{A}_\ell) = \{0\}$ and the sequence $\{\sum_\ell a_\ell^q\}$ is bounded. Then the sequence $\{\prod a_\ell^q\}$ is also bounded.

Proof: (From Hurwicz and Reiter [18], Theorem 1.)

Suppose that, for some $\hat{\ell} \in \mathcal{L}$, $\{\|a_{\hat{\ell}}^q\| \to \infty$ as $q \to \infty$. Define

$$u^q \equiv a_{\hat{\ell}}^q + \sum_{\ell\in\mathcal{L}\sim\{\hat{\ell}\}} a_\ell^1, \text{ and}$$

$$v^q \equiv a_{\hat{\ell}}^1 + \sum_{\ell\in\mathcal{L}\sim\{\hat{\ell}\}} a_\ell^q .$$

Since $a_\ell^q \in \mathcal{A}_\ell$ for all $\ell \in \mathcal{L}$, $u^q \in \Sigma\mathcal{A}_\ell$ and $v^q \in \Sigma\mathcal{A}_\ell$. Let $U \equiv \{u^1, u^2, \ldots\}$ and $V \equiv \{v^1, v^2, \ldots\}$. Then U and V are contained in $\Sigma\mathcal{A}_\ell$. Hurwicz and Reiter show that $\|a_\ell^q\| \to \infty$ implies that there is an $a^* \neq 0$ such that $-a^* \in 0^+\text{cl } V$ and $a^* \in 0^+\text{cl } U$. This implies that $a^* \in (0^+ \text{ cl } \Sigma\mathcal{A}_\ell) \cap (0^+\text{cl } - \Sigma\mathcal{A}_\ell)$, which contradicts the hypotheses of the Lemma.

Q.E.D.

<u>Lemma B.2</u> Let Z be a convex set in $\mathcal{R}^{m+p}$ with non-empty interior and let $\bar{z} = (\bar{x},\bar{y}) \in \text{Bd } Z$. If $u \in \mathcal{R}^m$ is a vector such that $u\bar{x} \geqq ux$ for all $(x,\bar{y}) \in Z$,

then there is a hyperplane in $\mathcal{R}^{m+p}$ with normalization u in $\mathcal{R}^m$ such that the hyperplane supports Z at $\bar{z}$.

Proof:

By hypothesis, int Z is relatively open in $\mathcal{R}^{m+p}$. The set $M \equiv \{(x,\bar{y}) \mid ux = u\bar{x}\}$ is a nonempty affine set in $\mathcal{R}^{m+p}$ that does not meet int Z. By Theorem 11.2 in [29], there is a hyperplane $\mathcal{H}$ containing M such that Z is bounded by $\mathcal{H}$. Since $\mathcal{H}$ contains M, $\mathcal{H}$ must pass through $\bar{z}$. Thus $\mathcal{H}$ is a supporting hyperplane for Z at $\bar{z}$. Furthermore, since $\mathcal{H}$ contains M, $\mathcal{H}$ must have the normalization u in $\mathcal{R}^m$.

Q.E.D.

<u>Lemma B.3</u> Let Z be a smooth convex set in $\mathcal{R}^{m+p}$ with nonempty interior, and let $\bar{z} = (\bar{x},\bar{y}) \in$ Bd Z. If $u \in \mathcal{R}^m$ is a vector such that $u\bar{x} \geqq ux$ for all $(x,\bar{y}) \in Z$ and if $v \in \mathcal{R}^p$ is a vector such that $v\bar{y} \geqq vy$ for all $(\bar{x},y) \in Z$, then there is a hyperplane in $\mathcal{R}^{m+p}$ with normalization u in $\mathcal{R}^m$ and v in $\mathcal{R}^p$ such that the hyperplane supports Z at $\bar{z}$.

proof:

By Lemma B.2, there are two hyperplanes supporting Z at $\bar{z}$, one with normalization u in $\mathcal{R}^m$, the other with normalization v in $\mathcal{R}^p$. But Z is smooth. The two hyperplanes must be identical.

Q.E.D.

<u>Lemma B.4</u> Let $\{p^s\}$ be a convergent sequence in the simplex $\mathcal{S}$ with limit

point p^* and let $\{p^s x^s\}$ and $\{U(x^s)\}$ be convergent sequences with limit points a^* and U^* such that for each s $x^s \in X$,

(i) $p^s x > p^s x^s$ for all $x \in X$ such that $U(x) > U(x^s)$, and

(ii) $p^s x^s \geqq p^s x^o$ where $x^o \in \text{int } X$.

If $U: X \to \mathcal{R}^1$ is a continuous function on the closed, convex set X, then (iii) $p^* x > a^*$ for all $x \in X$ such that $U(x) > U^*$.

proof:

Suppose $\tilde{x}$ is an element of X such that $U(\tilde{x}) > U^*$. Then, by continuity of U, there is an $s^o > 0$ such that $U(\tilde{x}) > U(x^s)$ for all $s > s^o$. From (i), $p^s \tilde{x} > p^s x^s$ for all $s > s^o$. In the limit, $p^* \tilde{x} \geqq a^*$. Hence

$$p^* x \geqq a^* \text{ for all } x \in X \text{ such that } U(x) > U^*. \tag{1}$$

Since $p^s \in \mathcal{S}$ and $\mathcal{S}$ is compact, $p^* \in \mathcal{S}$. Thus

$$p^* \geq 0. \tag{2}$$

Since $p^s x^s \geqq p^s x^o$ (by (ii)), $a^* \geqq p^* x^o$. Since $x^o \in$ int X and $p^* \geq 0$,

$$a^* \neq \underset{X}{\text{minimum}}\ p^* x. \tag{3}$$

From (1)

$$U(x) \leqq U^* \text{ for all } x \in X \text{ such that } p^* x < a^*. \tag{4}$$

By (3), there is an $\hat{x} \in X$ such that $p^*\hat{x} < a^*$. Suppose $\tilde{x} \in X$ is such that $p^*\tilde{x} = a^*$. Let $\{\theta^q\}$ be a sequence such that $\theta^q \in (0,1]$ for all q and $\theta^q \to 0$. Take $x^q \equiv \theta^q\hat{x} + (1 - \theta^q)\tilde{x}$. For each q, $p^*x^q < a^*$ so that $U(x^q) \leqq U^*$ (by (4)). By continuity of U, $U(x^q) \to U(\tilde{x})$. Hence $U(\tilde{x}) \leqq U^*$. Therefore,

$$U(x) \leqq U^* \text{ for all } x \in X \text{ such that } p^*x = a^*. \tag{5}$$

(iii) follows from (4) and (5).

Q.E.D.

<u>Lemma B.5</u> Let $Z \subset \mathcal{R}^{m+p}$ be a convex, compact set with elements of the form (y,x) where $y \in \mathcal{R}^m$ and $x \in \mathcal{R}^p$. Then the correspondence F from $\mathcal{P}_x A$ to $\mathcal{P}_y Z$ such that

$$F(x) = \{y \in \mathcal{R}^m \mid (y,x) \in Z\}$$

is continuous, where $\mathcal{P}_x$ and $\mathcal{P}_y$ are the projection mappings $\mathcal{P}_x(y,x) = x$, $\mathcal{P}_y(y,x) = y$.

Proof:

The graph of $F(x)$ is the compact set Z. Thus $F(x)$ is upper semi-continuous. It remains to show that F is lower semi-continuous.

Let C be an open set in $\mathcal{P}_y Z$. Then

$$F^{-1}C = \{x \in \mathcal{R}^p \mid F(x) \cap C \neq \emptyset\}$$

$$= \mathcal{P}_x[(\mathcal{P}_y^{-1}C) \cap Z].$$

Suppose C is also convex. Since C is open, ri $C = C$. Clearly, $\mathcal{P}_y^{-1}C \neq \emptyset$ and $\mathcal{P}_y$ is a linear transformation. Hence, by Theorem 6.7 of [29], ri $\mathcal{P}_y^{-1} C = \mathcal{P}_y^{-1} C$ and $\mathcal{P}_y^{-1} C$ is an open set. By Theorem 3.4 of [29], $\mathcal{P}_y^{-1}C$ is also convex. Therefore, since $C \subset \mathcal{P}_y Z$, $\mathcal{P}_y^{-1}C \cap Z$ is a nonempty, convex open set in Z. Thus, by Theorem 6.6 in [29], ri $\mathcal{P}_x[(\mathcal{P}_y^{-1}C) \cap Z] = \mathcal{P}_x[(\mathcal{P}_y^{-1}C) \cap Z]$. Thus $F^{-1}C$ is an open set. If C is not convex, then there is a covering of open convex sets C_i such that $\bigcup C_i = C$ (namely, the open Euclidean balls of selected rational points in C (Berge [4], Theorem 1, page 93). For each of these open convex sets, $F^{-1}C_i$ is open. It is easy to show that $F^{-1}C = \bigcup F^{-1}C_i$. Thus, $F^{-1}C$ is an open set. Hence F is lower semi-continuous, by Berge [4], Theorem 1, page 109.

Q.E.D.

<u>Lemma B.6</u> Suppose Z is a smooth, closed, convex set in $\mathcal{R}^{m+n}$, and f is a function on $\mathcal{P}_y Z$, the projection of Z into $\mathcal{R}^n$, such that

$$\hat{f}(y) \equiv \underset{X(\hat{y})}{\text{Maximum}}\ px$$

where $X(\hat{y})$ is the mapping $X(\hat{y}) \equiv \{x \in \mathcal{R}^m \mid (x,\hat{y}) \in Z\}$. Then the level set $U(\hat{y}) \equiv \{y \in \mathcal{P}_y Z \mid f(y) \geqq f(\hat{y}))\}$ is smooth at each $\hat{y} \in \mathcal{P}_y Z$.

Proof:

Suppose false. Then for some $\hat{y} \in \mathcal{P}_y Z$ there are vectors u, $v \in \mathcal{S}$ such that u is not proportional to v and

$$vy \geqq v\hat{y} \quad \text{for all} \quad y \in U(\hat{y}), \text{ and} \tag{i}$$

$$uy \geqq u\hat{y} \quad \text{for all} \quad y \in U(\hat{y}). \tag{ii}$$

Define $\hat{x} \in \mathcal{R}^m$ so that $\hat{x} \in X(\hat{y})$ and $p\hat{x} = f(\hat{y})$. Then

$$\hat{x} \text{ maximises } px \text{ subject to } x \in X(\hat{y}). \tag{iii}$$

Consider the set $Y(\hat{x}) \equiv \{y \in \mathcal{R}^n \mid (\hat{x},y) \in Z\}$. For every $y \in Y(\hat{x})$, there is an $x \in X(y)$ such that $px \geqq p\hat{x}$, since $\hat{x} \in X(y)$. Thus, for every $y \in Y(\hat{x})$, $f(y) \geqq f(\hat{y})$. Hence $Y(\hat{x}) \subset U(\hat{y})$. But, from (i) and (ii),

$$vy \geqq v\hat{y} \quad \text{for all} \quad y \in Y(\hat{x}), \text{ and} \tag{iv}$$

$$uy \geqq u\hat{y} \quad \text{for all} \quad y \in Y(\hat{x}). \tag{v}$$

From smoothness of Z, from (iii), (iv) and (v) and from Lemma B.3, there are distinct hyperplanes $\mathcal{H}^*$ with normalizations p and v in $\mathcal{R}^m$ and in $\mathcal{R}^n$, and $\mathcal{H}^{**}$ with normalizations p and u in $\mathcal{R}^m$ and in $\mathcal{R}^n$ supporting Z at $(\hat{x},\hat{y})$. This is a contradiction, since, by the smoothness of Z, there can only be one such hyperplane. Thus $U(\hat{y})$ is smooth at all $\hat{y} \in \mathcal{P}_y Z$.

Q.E.D.

REFERENCES

[1] Aoki, M., "An Investment Planning Process for an Economy with Increasing Returns," Review of Economic Studies, July, 1971.

[2] Arrow, K. J., "Toward a Theory of Price Adjustment," in P. A. Baran, T. Scitovsky and E. S. Shaw (eds), The Allocation of Economic Resources (Stanford, California; 1958).

[3] Arrow, K. J., and L. Hurwicz, "Decentralization and Computation in Resource Allocation," in R. W. Pfouts (ed), Essays in Economics and Econometrics, (Chapel Hill; 1960).

[4] Berge, C., Topological Spaces, (New York; 1963).

[5] Debreu, G., Theory of Value, (New Haven; 1959).

[6] Diamond, P. A., "A Model of Price Adjustment," Journal of Economic Theory, 3, (June, 1971).

[7] Dreze, J. H., and D. de La Vallee Poussin, "A Tatonnement Process for Public Goods," Review of Economic Studies, April, 1971 (38,2).

[8] Feldman, A. M., "Bilateral Trading Processes, Pairwise Optimality and Pareto Optimality," Review of Economic Studies, October, 1973 (40,4).

[9] Fisher, F. M., "Quasi-competitive Price Adjustment by Individual Firms," Journal of Economic Theory, October, 1973.

[10] Foley, D. K., "Resource Allocation and the Public Sector," Yale Economic Essays, 1967.

[11] Fouregard, C., "Contribution a l'Etude du Role des Administrations dans la Theorie Mathematique de l'Equilibre et de l'Optimum," Econometrica, April, 1969 (37,2).

[12] Hayek, F. A. von, "The Present State of the Debate," in F. A. von Hayek (ed), Collectivist Economic Planning (London, 1935).

[13] Heal, G. M., "Planning Without Prices," Review of Economic Studies, 1969.

[14] Heal, G. M., "Planning Prices and Increasing Returns," Review of Economic Studies, 1971.

[15] Hurwicz, L., "Optimality and Informational Efficiency in Resource Allocation Processes," in K. J. Arrow, S. Karlin and P. Suppes (eds) Mathematical Methods in the Social Sciences, (Stanford, 1960).

[16] Hurwicz, L., "On the Dimensional Requirements of Informationally Decentralized Pareto-Satisfactory Processes," (mimeo) presented at the CEME Conference on Decentralization, Northwestern University, Evanston, Illinois, February, 1972.

[17] Hurwicz, L., "The Design of Mechanisms for Resource Allocation," Proceedings of the American Economics Association, 1973.

[18] Hurwicz, L., and S. Reiter, "On the Boundedness of the Feasible Set Without Convexity Assumptions," International Economic Review, Vol. 14, No. 3, 1973.

[19] Kornai, J., Anti - Equilibirum, (Amsterdam, 1971).

[20] Kornai, J., and B. Martos, "Autonomous Control of the Economic System," Econometrica, 41, 3 (May, 1973).

[21] Lange, O., "On the Economic Theory of Socialism," Review of Economic Studies, 1936, reproduced in B. Lippincott (ed) On the Economic Theory of Socialism, (Minneapolis, 1938).

[22] Malinvaud, E., "Decentralized Procedures for Planning," in E. Malinvaud and M.O.L. Bacharach, Activity Analysis in the Theory of Growth and Planning, (New York, 1967).

[23] Malinvaud, E., "Prices for Individual Consumption, Quantity Indicators for Collective Consumption," Review of Economic Studies, 1973.

[24] Marshak, T., "Decentralizing the Command Economy: The Study of a Pragmatic Strategy for Reformers," Technical Report No. 8, Collaborative Research on Economic Systems and Organizations, Center for Research in Management Science, (Berkeley, 1970).