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MILLIONS OF ELECTION OUTCOMES FROM A SINGLE PROFILE

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ABSTRACT. By changing the choice of a positional voting method, different election rankings can result from a fixed profile. A geometric theory is developed to explain why this occurs, to completely characterize all possible sets of rankings that can arise in this manner, to determine the number of rankings and other properties of these sets of rankings, to design profiles that cause the different conclusions, to develop elementary tools to analyze actual data, and to compare new types of social choice solutions that are based on the set of rankings admitted by a profile. A secondary theme is to indicate how results for voting theory can be obtained with (relative) ease when they are analyzed with a geometric approach.

1. INTRODUCTION

It can be discomforting to discover that a single, fixed profile can lead to different sincere election rankings of the $n \geq 3$ candidates when the choice of the positional voting procedure varies. The voters' preferences remain fixed, but different election outcomes emerge with changes in the choice of the tallying method. Whenever this happens, we must wonder which election outcome is the group's "true" ranking of the alternatives. It is this kind of concern that motivates, to a large extent, the search for an election procedure that offers some degree of integrity for the associated election outcomes.

Is this issue of multiple outcomes a serious one? How badly can the election outcomes vary with a fixed profile? Fishburn [3] showed with two different positional voting methods that there exists a profile so that the election ranking determined by each method is the reversal of the other. But much more can happen! In (Saari [4]) I showed for $n \geq 3$ alternatives that anything can occur when up to $n-1$ completely different positional voting methods[1] are used. Namely, choose any $n-1$ rankings of the n candidates. No matter what are these rankings, the theorem ensures the existence of a profile whereby the election outcome for the jth positional voting method is the jth selected election ranking, $j = 1, \ldots, n-1$.[2] Moreover, this assertion is best possible; there are many ways to choose n rankings of the n candidates so that it is impossible for all of them to be supported by a single profile.

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[1] See Saari [4] for the technical definition.

[2] This conclusion extends to all subsets of candidates; see Saari [4, 7] and the references for more information.

This theorem imposes no restrictions on the $n-1$ rankings, so there need not be any relationship among them. This proves that reversals and other kinds of election behavior can coexist. For example, with $n = 5$ candidates and any four completely different positional voting methods, there exists a profile so that the outcome is $c_1 \succ c_2 \succ c_3 \succ c_4 \succ c_5$, or its reversal $c_5 \succ c_4 \succ c_3 \succ c_2 \succ c_1$, or the mixture $c_2 \succ c_5 \succ c_1 \succ c_4 \succ c_3$, or the reversal of the mixture $c_3 \succ c_4 \succ c_1 \succ c_5 \succ c_2$ depending on which of the four tallying methods is used. Moreover, these assertions are not restricted to examples concocted to illustrate theoretical possibilities; they can and do arise in practice. To illustrate, I call attention to a recent entertaining paper where J-P Benoit [1] analyzes data involving the choice of the baseball's Most Valuable Player (MVP) to show how different rankings and different choices of the MVP could occur with changes in the voting method.[3]

While this theorem asserting the arbitrariness of election outcomes underscores an intriguing, disturbing aspect of voting procedures, it does not even begin to suggest the true magnitude of the difficulty. This is corrected here. As asserted, the above conclusion is sharp in that one cannot expect a profile to support more than $n-1$ rankings selected *in an arbitrary fashion*. But this value of $n-1$ does not limit the number of election rankings emerging from a single profile; there are many other election outcomes, related to the original $n-1$ choices, that can surface. So, if

$$Sup_n(\mathbf{p}) = \{ \text{ all election rankings of the } n \text{ candidates that can arise from profile } \mathbf{p} \text{ with changes in positional voting methods } \}$$

is the set of election rankings supported by profile $\mathbf{p}$, then it is of distinct interest to understand the properties of $Sup_n(\mathbf{p})$. For instance, what are the entries in $Sup_n(\mathbf{p})$? Can an upper bound be found for $|Sup_n(\mathbf{p})|$? How does $Sup_n(\mathbf{p})$ change with $\mathbf{p}$? Which voting methods yield which ranking? These are the types of questions answered here.

To give a flavor of the new results, notice that my earlier theorem ensures the existence of a ten-candidate profile to support nine different election rankings. What I now show is the existence of a ten-candidate profile that supports *millions* of different rankings of these ten candidates! More precisely, there exists a profile $\mathbf{p}$ where $Sup_{10}(\mathbf{p})$ has more than 80 million different rankings.[4] In fact, I show for any $n \geq 3$ that $Sup_n(\mathbf{p})$ can contain at least $\frac{2}{3}$ of the $n!$ possible rankings without tie votes. Moreover, the rankings from $Sup_{10}(\mathbf{p})$ can offer distinctly conflicting information -- for each of the ten candidates, there exist rankings in $Sup_{10}(\mathbf{p})$ where that candidate is top-ranked, and other rankings in $Sup_{10}(\mathbf{p})$ where that candidate is bottom ranked. One might expect – or at least hope – that the profiles leading to

[3]Indeed, Benoit's paper and personal correspondence between us about the merits of our favorite baseball teams provided the initial motivation for this current article. I resurrected the results given here from an (1987) unpublished manuscript entitled "The Geometry of Voting" to see if there is a method that would crown a Chicago player as the MVP.

[4]So, which one of these 80 million different rankings of the candidates reflects the true views of these voters? If discomfort is caused by the earlier conclusion that nine arbitrarily different rankings can emerge from a single profile, then these new assertions must introduce serious concern about the choice of a voting method and the meaning of an election outcome.

such astronomical numbers of rankings that offer conflicting information about the candidates all coming from a single profile must constitute a rare, highly unlikely event; indeed, one might conjecture that the tame situation where a single ranking holds for all choices of the voting methods is the dominating, more probable situation. While I defer a careful study of the probabilistic issues to a later article, I do suggest – by showing how to construct profiles to demonstrate such outcomes – why we must believe that just the opposite is true! Indeed, for a large class of quite reasonable probability distributions over the space of profiles and with a sufficient number of voters, there are strong reasons why we should expect it to be significantly more likely for a ten-candidate profile to support millions of different election outcomes rather than just one, or just two, ..., or just fifty, ...

The purpose of this current article, then, is to exploit the recently developed geometric theory for voting (see, for example, Saari [5, 6, 7] and the references) to develop an approach for $n \geq 3$ candidates allowing the following kinds of results:

1. There exists a profile whereby, when different choices of positional voting methods are used, $Sup_n(\mathbf{p})$ contains more than 53% of all possible rankings of the n candidates, and this proportion increases with values of n. For each $n \geq 2$, the proportion of all rankings without ties that can be in $Sup_n(\mathbf{p})$ is $\frac{n-1}{n}$.
2. For a given $\mathbf{p}$, all rankings in $Sup_n(\mathbf{p})$ can be determined. Moreover, for each ranking in $Sup_n(\mathbf{p})$, one can determine the set of all possible positional voting methods that yield this ranking with profile $\mathbf{p}$.
3. One can characterize all possible sets of rankings that become a $Sup_n(\mathbf{p})$ for some choice of $\mathbf{p}$.
4. One of the goals of this article is to indicate how to develop easily used tools to completely analyze problems and issues resulting from actual election data.
5. An approach is outlined to design profiles that illustrate the various assertions. This approach is then used to provide intuition about the probability of the various assertions.
6. As a single profile can support very large numbers of rankings, it is not clear which one of these rankings represent the "true" views of the voters. On the other hand, suppose all rankings in $Sup_n(\mathbf{p})$ have a certain property; e.g., the same candidate always is top-ranked. The fact that this property is preserved by all possible ways there are to tally the ballots forms a compelling argument to accept this property as reflecting the actual views of voters. It is indicated how to develop a theory about these commonly held properties.

It is worth commenting that this list includes several types of long-standing issues from voting theory that are notoriously difficult to analyze by use of standard approaches from choice theory. On the other hand, these questions become conceptually simpler to understand, analyze, and extend by using a geometric approach toward voting. This approach, as indicated here, is to translate a given issue about positional voting into a geometric construct. In this manner, it becomes conceptually apparent from the geometry how the issue should and could be studied. What difficulties remain are the technical details needed to extract and analyze the associated geometry. Thus, a secondary theme of this article is to outline this geometric approach. To help develop geometric intuition, the ideas behind each of the major conclusions is demonstrated first for $n = 3$ with its elementary geometry. For a first reading the reader may wish to consider only these intuitive proofs.

Positional voting methods and the geometry.

For the remainder of this section, I introduce the necessary notation and basic geometry. The interested reader can find added details and illustrating examples in the above noted references.

A voting vector for n candidates,

$$\overrightarrow{w}^n = (w_1, w_2, \ldots, w_n), \quad w_{i+1} \geq w_i,\ i = 1, \ldots n-1, \quad w_1 > w_n,$$

is used to tally elections by assigning w_i points to the ith ranked candidate on a ballot. In the standard manner, the total number of points assigned to each candidate determines the election ranking where "more is better."

It is immediate that if two voting vectors $\overrightarrow{w}^n_1, \overrightarrow{w}^n_2$ are related by scalars a, b according to the relationship

$$\overrightarrow{w}^n_1 = a\overrightarrow{w}^n_2 + b(\overbrace{1, \ldots, 1}^{n \text{ terms}}), \tag{1.1}$$

then the election rankings must always be the same. So, by use of Eq. 1.1, we can and do assume that all voting vectors are in the *normalized form* where $w_n = 0$, $\sum_{i=1}^n w_i = 1$. This assumptions leads to the introduction of *the space of normalized voting vectors for n candidates;* it is the portion of a $n-2$ dimensional simplex given by

$$VV^n = \{\overrightarrow{w}^n \,|w_i \geq w_{i+1},\ i = 1, \ldots, n-1,\ w_n = 0,\ \sum_{i=1}^n w_i = 1\}. \tag{1.2}$$

Certain voting vectors represent instructions to the voters "to vote for s of the candidates;" they play a critical role in our analysis. The normalized form of such a vector is

$$\overrightarrow{E}^n_s = (\overbrace{\frac{1}{s}, \ldots, \frac{1}{s}}^{s \text{ terms}}, 0, \ldots, 0), \quad s = 1, \ldots, n-1.$$

The importance of these vectors derives from the geometric fact that they are the vertices of VV^n. Namely, the vectors $\{\overrightarrow{E}^n_s\}_{s=1}^{n-1}$ form a convex basis for the convex set VV^n, so each voting vector $\overrightarrow{w}^n \in VV^n$ has a unique convex representation

$$\overrightarrow{w}^n = \sum_{s=1}^{n-1} \lambda_s \overrightarrow{E}^n_s, \quad \lambda_s \geq 0, \quad \sum_{s=1}^{n-1} \lambda_s = 1. \tag{1.3}$$

Indeed, the vectors of convex weights in the associated $n-2$ dimensional simplex

$$\Lambda^n = \{\lambda = (\lambda_1, \ldots, \lambda_{n-1}) \,|\, \lambda_s \geq 0,\ \sum_{s=1}^{n-1} \lambda_s = 1\}$$

are identified in the natural one-to-one fashion with the voting vectors from VV^n.

A profile indicates what portion of all voters have each of the $n!$ possible rankings of the candidates without ties. To find a geometric representation for the profiles, list the $n!$ rankings in some order and assign the ith ranking to the ith axis of $R^{n!}$. In this manner the ith component of a point in $R^{n!}$ indicates the number, or the portion of voters with the ith ranking of the candidates. More specifically, the simplex

$$Si(n!) = \{\mathbf{p} = (p_1, \dots, p_{n!}) \in R^{n!} \,|\, p_i \geq 0, \sum_{i=1}^{n!} p_i = 1\}, \tag{1.4}$$

is the space of normalized profiles; the value of p_i indicates the portion of all voters with the ith ranking of the candidates. As an example, there are only $3! = 6$ rankings for three candidates, and these rankings are denoted by

(1.5)

p component	**Ranking**	**p component**	**Ranking**
p_1	$c_1 \succ c_2 \succ c_3$	p_4	$c_3 \succ c_2 \succ c_1$
p_2	$c_1 \succ c_3 \succ c_2$	p_4	$c_2 \succ c_3 \succ c_1$
p_3	$c_3 \succ c_1 \succ c_2$	p_6	$c_2 \succ c_1 \succ c_3$

The profile $\mathbf{i}_{n!} = (\frac{1}{n!}, \dots, \frac{1}{n!})$ is the baricentric point of the simplex $Si(n!)$; it corresponds to where there is an equal number of voters of each voter type.

A similar geometric construction defines the space of election outcomes in R^n. For the n candidates $\{c_1, \dots, c_n\}$, identify c_i with the ith coordinate axis of R^n where the magnitude of the component indicates the strength of support for c_i. So, a comparison of the magnitudes of the components of a vector $\mathbf{v} = (v_1, \dots, v_n) \in R^n$, using the "more is better" binary relationship "$\geq$", determines a ranking of the candidates. In this manner $\mathbf{v} = (4, 1, 9) \in R^3$ defines the ranking $c_3 \succ c_1 \succ c_2$ as $v_3 > v_1 > v_2$, and $(2, 5, 2)$ defines the ranking $c_2 \succ c_1 \sim c_3$. The set of all vectors leading to a specified election ranking defines its *ranking region.* It follows from the inequalities that a ranking region is an open set if and only if it represents a strict ranking (i.e., a ranking without ties).

Because of the inequalities imposed on the weights of a voting vector, $\overrightarrow{w}^n$ is either in the $\mathcal{A}_n = c_1 \succ c_2 \succ \dots \succ c_n$ ranking region, or in a ranking region in the boundary of this region. The last situation occurs only for voting vectors where some of the weights have the same value, so the equal weights represent tie values in the ranking.

Each of the $n!$ rankings of the candidates (without ties) can be viewed as a permutation of $\mathcal{A}_n$, where $\rho_i(\mathcal{A}_n)$ denotes the permutation corresponding to the ith listed ranking of the candidates, $i = 1, \dots, n!$. For a ballot representing the ranking $\rho_i(\mathcal{A}_n)$, there is a unique permutation of $\overrightarrow{w}^n$, denoted by $\rho_i(\overrightarrow{w}^n)$, that corresponds to how points are assigned to the candidates. For example, the vector $(w_2, 0, w_1)$ denotes the tally of a ballot for the ranking $c_3 \succ c_1 \succ c_2$ because it reflects the second place ranking of c_1 (she is assigned w_2 points), the third place ranking of c_2 (who is assigned $w_3 = 0$ points), and the top ranking of c_3. More generally we have

$$(1.6)\qquad \begin{array}{|cccc|} \rho_i(\overrightarrow{w}^3) & \rho_i(\mathcal{A}_3) & \rho_i(\overrightarrow{w}^3) & \rho_i(\mathcal{A}_3) \\ (w_1,w_2,0) & c_1 \succ c_2 \succ c_3 & (0,w_2,w_1) & c_3 \succ c_2 \succ c_1 \\ (w_1,0,w_2) & c_1 \succ c_3 \succ c_2 & (0,w_1,w_2) & c_2 \succ c_3 \succ c_1 \\ (w_2,0,w_1) & c_3 \succ c_1 \succ c_2 & (w_2,w_1,0) & c_2 \succ c_1 \succ c_3 \end{array}.$$

We now use this notation to represent the election. The points assigned to the candidates with a ballot representing the ith ranking is $\rho_i(\overrightarrow{w}^n)$, so if p_i of the voters have this ranking, then they contribute the portion $p_i\rho_i(\overrightarrow{w}^n)$ of points toward the final tally. By summing over all $n!$ voter types, the final tally is

$$(1.7)\qquad f(\mathbf{p}, \overrightarrow{w}^n) = \sum_{i=1}^{n!} p_i\rho_i(\overrightarrow{w}^n).$$

The election ranking corresponding to this tally is determined by the ranking region that contains $f(\mathbf{p}, \overrightarrow{w}^n)$; in other words, the election ranking is determined in the familiar fashion by comparing how many points each candidate received as given by the magnitudes of the components of the vector $f(\mathbf{p}, \overrightarrow{w}^n)$.

Notice that Eq. 1.7 defines a convex combination of the vectors $\{\rho_i(\overrightarrow{w}^n)\}_{i=1}^{n!}$. As each of these permutation vectors is in

$$Si(n) = \{\mathbf{v} \in R^n \mid \sum_{i=1}^{n} v_i = 1; \quad v_i \geq 0\},$$

it must be that $f(\mathbf{p}, \overrightarrow{w}^n) \in Si(n)$. Thus, an election can be interpreted as defining a mapping

$$(1.8)\qquad f(-, \overrightarrow{w}^n) : Si(n!) \to Si(n).$$

The baricentric point of $Si(n)$ is $\mathbf{i}_n = (\frac{1}{n}, \dots, \frac{1}{n})$ which represents the ranking region of complete indifference $c_1 \sim c_2 \sim \cdots \sim c_n$. As one might expect, $f(\mathbf{i}_{n!}, \overrightarrow{w}^n) = \mathbf{i}_n$ for all $\overrightarrow{w}^n$.

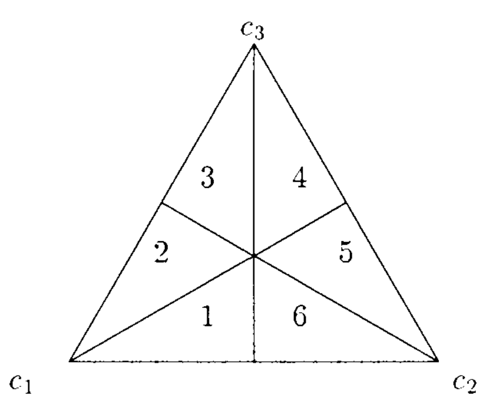

Figure 1

As all outcomes of Eq. 1.7 are in $Si(n)$, $Si(n)$ is the space of (normalized) election outcomes. This $n-1$ dimensional simplex is the convex hull defined by the n vertices $\mathbf{e}_1 = (1,0,\ldots,0), \mathbf{e}_2 = (0,1,0,\ldots,0),\ldots,\mathbf{e}_n = (0,\ldots,0,1)$. The jth vertex, $\mathbf{e}_j$, represents an unanimity situation where all possible points are assigned to c_j. In general, the components of $\mathbf{v} \in Si(n)$ represents the division of the points assigned to all candidates. From the geometry, we see that $Si(3)$ is the equilateral triangle in Figure 1. The vertex labelled with c_1, then, is at $\mathbf{e}_1$ in R^3 while the vertex for c_3 is anchored at $\mathbf{e}_3$. The numbers in the regions correspond to the 3! rankings of candidates given in the above listing. Thus, if a point $f(\mathbf{p}, \overrightarrow{w}^3)$ is in the region labelled 5, the election ranking must be $c_2 \succ c_3 \succ c_1$. The boundary, or *indifference lines* separating the six open triangles correspond to where there is a tie vote between (or among) candidates.

We now turn to the geometric approach. Issues about election outcomes often involve questions of "Who beats whom?" As the election rankings are determined by ranking regions, such issues are converted to geometric constructs involving the ranking regions. For instance, an issue involving the ranking of the candidates concerns the individual ranking regions of $Si(n)$. On the other hand, for an issue involving the top-ranked (or the bottom-ranked candidate), we are interested only in who is top-ranked – not in how the other candidates are ranked. This means that such an issue concerns the geometry of certain unions of ranking regions. Using Figure 1, the region where c_1 is top-ranked is the union of region 1, region 2, and the line segment between them –representing a tie vote between c_2 and c_3. If c_1 is bottom-ranked, then the relevant construct is the union of regions 4 and 5 along with the associated line segment. If c_1 is middle ranked, then we get the non-convex region given by the union of regions 3 and 6.

Issues about the likelihood of an election property concerns the set of profiles; thus an understanding of such issues comes from the geometry of $Si(n!)$. The relevant geometric constructs are imposed upon $Si(n!)$ via the connecting link f (the election mapping) and the geometric constructs of $Si(n)$ developed to capture the issues being studied. Thus, once an issue about elections is posed, the first step is to translate it into the language of a geometric construct of $Si(n)$. By using the inverse image with respect to the election mapping f of appropriate geometric regions of $Si(n)$, the appropriate geometry is imposed upon $Si(n)$. These are the relevant geometric entities to be studied.

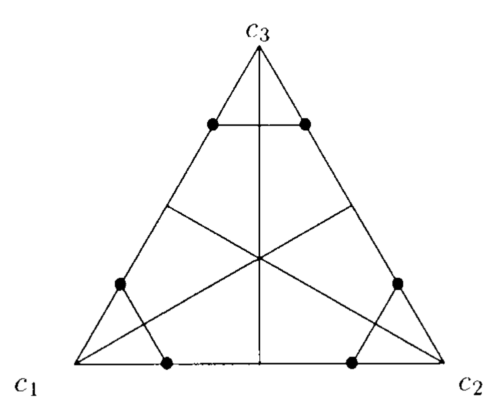

Figure 2

One difficulty is that the space of profiles $Si(3!)$ is five-dimensional, so it cannot be represented as a standard Cartesian figure. Nevertheless, there are crude ways to represent $Si(3!)$ based on the interpretation that a component of $\mathbf{p} \in Si(3!)$ designates the portion of voters with a particular ranking. Thus, the jth component of $\mathbf{p}$ is listed in the ranking region of Figure 1 corresponding to the jth ranking. This is done in Figure 2. Also in Figure 2, the vectors $\{\rho_i(\overrightarrow{w}^3)\}_{i=1}^6$ are represented by the six dots on the boundary of the simplex for the BC. The election outcome, then, is a convex combination of the vectors with weights $\{p_i\}$. Versions of these two figures play a role in our subsequent analysis for the special case of three candidates.

2. PROPERTIES OF $Sup_n(\mathbf{p})$

In this section we provide answers to issues of the following kind.

1. For a given profile $\mathbf{p}$, how can one determine in a simple fashion whether $|Sup_n(\mathbf{p})| > 1$?
2. How does one determine the rankings in $Sup_n(\mathbf{p})$?
3. For each ranking $\mathcal{R} \in Sup_n(\mathbf{p})$, how can one determine which positional voting methods have the election outcome $\mathcal{R}$ for profile $\mathbf{p}$?

The following two facts form basic tools for our investigation.

Theorem 1. *Assume there are $n \geq 3$ candidates.*

a. *A normalized election tally can be represented as*

$$f(\mathbf{p}, \overrightarrow{w}^n) = \sum_{s=1}^{n-1} \lambda_s f(\mathbf{p}, \overrightarrow{E}_s^n) \tag{2.1}$$

where the vector of convex weights $\lambda = (\lambda_1, \dots, \lambda_{n-1}) \in \Lambda^n$ determines the coefficients for the convex combination of $\overrightarrow{w}^n$. (Equation 1.7.)

b. *In the simplex $Si(n)$ there exists a ball $B(\mathbf{i}_n, r)$ with radius $r > 0$ centered on the baricentric point $\mathbf{i}_n$ with the following property. Choose $n-1$ points $\mathbf{v}_s \in B(\mathbf{i}_n, r), s = 1, \dots, n-1$. There exists $\mathbf{p} \in Si(n!)$ so that*

$$f(\mathbf{p}, \overrightarrow{E}_s^n) = \mathbf{v}_s, \quad s = 1, \dots, n-1.$$

Proof. The first assertion follows from the linearity of f in the $\overrightarrow{w}^n$ variable. We have that

$$f(\mathbf{p}, \overrightarrow{w}^n) = f(\mathbf{p}, \sum_{s=1}^{n-1} \lambda_s \overrightarrow{E}_s^n) = \sum_{s=1}^{n-1} \lambda_s f(\mathbf{p}, \overrightarrow{E}_s^n). \tag{2.2}$$

The second conclusion is a restriction of results from (Saari [4, 7]) to the voting vectors $\{\overrightarrow{E}_s^n\}$. □

Let $Co_n(\mathbf{p})$ be the convex hull defined by the $n-1$ points $\{f(\mathbf{p}, \overrightarrow{E}_s^n\}_{s=1}^{n-1}$. The importance of $Co_n(\mathbf{p})$ is that it is the geometric construct corresponding to $Sup_n(\mathbf{p})$. As such, issues raised about $Sup_n(\mathbf{p})$ are resolved by studying the geometric properties of $Co_n(\mathbf{p})$. The purpose of the next statement is to justify this identification between $Co_n(\mathbf{p})$ and $Sup_n(\mathbf{p})$ while introducing certain basic properties of $Co_n(\mathbf{p})$. In this statement, I use the fact that, by definition, the set $Co_n(\mathbf{p})$ is a correspondence (a set valued mapping) from $Si(n!)$ to $Si(n)$.

Corollary 1.1. *Assume given $n \geq 3$ candidates and a profile* $\mathbf{p}$.

a. *Let $\mathbf{v} \in Co_n(\mathbf{p})$. There exists a voting vector $\overrightarrow{w}^s$ so that*

$$f(\mathbf{p}, \overrightarrow{w}^s) = \mathbf{v}.$$

In particular, the vector of convex weights $\lambda \in \Lambda^n$ that identifies the position of $\mathbf{v}$ within the convex hull $Co_n(\mathbf{p})$ is the same vector of convex weights that defines the associated $\overrightarrow{w}^n \in VV^n$.

b. *An election ranking $\mathcal{R}$ is in $Sup_n(\mathbf{p})$ if and only if $Co_n(\mathbf{p})$ has a nonempty intersection with the $\mathcal{R}$ ranking region.*

c. *The correspondence $Co_n(\mathbf{p})$ is continuous.*

Proof. Part a: This assertion is the converse of the one implied by Eq. 2.2; the difference is that the convex weights are determined by the election outcome rather than the voting vector. If $\mathbf{v} \in Co_n(\mathbf{p})$, then there exists $\lambda = (\lambda_1, \ldots, \lambda_{n-1}) \in \Lambda^n$ so that

$$\mathbf{v} = \sum_{s=1}^{n-1} \lambda_s f(\mathbf{p}, \overrightarrow{E}^n_s) = \sum_{s=1}^{n-1} f(\mathbf{p}, \lambda_s \overrightarrow{E}^n_s) = f(\mathbf{p}, \sum_{s=1}^{n-1} \lambda_s \overrightarrow{E}^n_s), \tag{2.3}$$

so the appropriate choice of a voting vector is $\overrightarrow{w}^n = \sum_{s=1}^{n-1} \lambda_s \overrightarrow{E}^n_s$.

Part b: According to part a, each point in $Co_n(\mathbf{p})$ corresponds to an election outcome for $\mathbf{p}$, so the conclusion follows from the definition of the ranking regions.

Part c: This is an immediate consequence of the linear form and smoothness of f. □

For a fixed $\mathbf{p}$, $f(\mathbf{p}, -)$ is a linear mapping from the space of voting vectors VV^n to $Si(n)$. Therefore, we must expect the image set, $Co_n(\mathbf{p})$, to inherit many of the geometric properties of VV^n. This basic intuition guides the development given in this article.

We now start providing answers for the above questions. To determine the rankings in $Sup_n(\mathbf{p})$, just compute the $n-1$ election outcomes $f(\mathbf{p}, \overrightarrow{E}^n_s)$ and find the set $Co_n(\mathbf{p})$. Then, according to Corollary 1.1b, we only need determine which ranking regions intersect $Co_n(\mathbf{p})$. An equivalent analytic approach is to vary the values of λ from Eq. 2.1 to find the admissible rankings. Also, several immediate properties of $Sup_n(\mathbf{p})$ can be determined. For instance, it now is immediate that *a necessary and sufficient condition for only one election ranking to be supported by* $\mathbf{p}$ *is that each of the $n-1$ tallies $\{f(\mathbf{p}, \overrightarrow{E}^n_s)\}$ leads to this same, single ranking.*

To determine what positional voting methods yield each entry of $Sup_n(\mathbf{p})$, we use Eq. 2.1 and the geometry. For example, each ranking region $\mathcal{R}$ of $Si(n)$ is a convex set. So, if $\mathcal{R} \cap Co_n(\mathbf{p}) \neq \emptyset$, then $\mathcal{R} \cap Co_n(\mathbf{p})$ is a non-empty convex subset of $Co_n(\mathbf{p})$. By using Corollary 1.1a, each point in $\mathcal{R} \cap Co_n(\mathbf{p})$ defines a vector of convex weights from Λ^n, and the convexity of $\mathcal{R} \cap Co_n(\mathbf{p})$ ensures that the set of these vectors defines a convex set $\Lambda_{\mathcal{R}} \subset \Lambda^n$. Using the identification of the convex weights for election outcomes (Corollary 1.1a) with the convex representation of a voting vector, it follows that the set of voting vectors causing the ranking $\mathcal{R}$ with profile $\mathbf{p}$ is

$$\{\overrightarrow{w}^n = \sum_{s=1}^{n-1} \lambda_s \overrightarrow{E}^n_s \mid \lambda = (\lambda_1, \ldots, \lambda_{n-1}) \in \Lambda_{\mathcal{R}}\}.$$

It is of significance that after the geometry is used to determine what can happen, the subsequent analysis needed to carry out each of these steps is reduced to elementary algebra and vector analysis. As such, they constitute realistic tools based on elementary techniques to use to analyze real data. This is illustrated in the following three candidate example.

Example: For $n = 3$, consider the 15 voter profile where

Number of voters	Ranking
6	$c_1 \succ c_3 \succ c_2$
5	$c_2 \succ c_3 \succ c_1$
4	$c_3 \succ c_2 \succ c_1$

With this example, the normalized plurality outcome is

$$f(\mathbf{p}, \overrightarrow{E}^3_1) = \frac{6}{15}(1,0,0) + \frac{5}{15}(0,1,0) + \frac{4}{15}(0,0,1) = (\frac{6}{15}, \frac{5}{15}, \frac{4}{15})$$

with the ranking $c_1 \succ c_2 \succ c_3$. The second required computation is the anti-plurality outcome

$$f(\mathbf{p}, \overrightarrow{E}^3_2) = \frac{6}{15}(\frac{1}{2}, 0, \frac{1}{2}) + \frac{5}{15}(0, \frac{1}{2}, \frac{1}{2}) + \frac{4}{15}(0, \frac{1}{2}, \frac{1}{2}) = (\frac{6}{30}, \frac{9}{30}, \frac{15}{30})$$

with the election ranking $c_3 \succ c_2 \succ c_1$. As $f(\mathbf{p}, \overrightarrow{E}^3_1)$ and $f(\mathbf{p}, \overrightarrow{E}^3_2)$ do not have the same ranking, it is an immediate consequence of Corollary 1.1 that this profile supports several different election rankings.

To determine $Sup_3(\mathbf{p})$, we only need to compute $Co_3(\mathbf{p})$. This is the line segment joining $f(\mathbf{p}, \overrightarrow{E}^3_1)$ and $f(\mathbf{p}, \overrightarrow{E}^3_2)$ given by

$$Co_3(\mathbf{p}) = \{s(\frac{6}{15}, \frac{5}{15}, \frac{4}{15}) + (1-s)(\frac{6}{30}, \frac{9}{30}, \frac{15}{30}) : 0 \leq s \leq 1\}. \tag{2.4}$$

It follows immediately from the geometric representation of this line segment, depicted in Figure 3, that this profile supports seven different election rankings. These seven rankings can be determined from the figure.

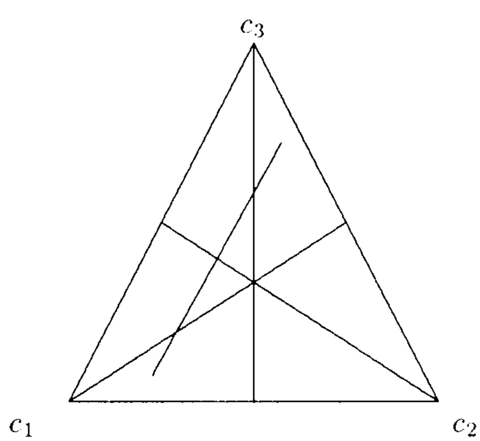

Figure 3

To find which positional voting methods cause each of the rankings in $Sup_3(\mathbf{p})$, define

$$\vec{w}^{\,3}_s = s\vec{E}^{\,3}_1 + (1-s)\vec{E}^{\,3}_2 = (\frac{1+s}{2}, \frac{1-s}{2}, 0).$$

According to Corollary 1.1a, we have that

$$f(\mathbf{p}, \vec{w}^{\,3}_s) = (\frac{6+6s}{30}, \frac{9+s}{30}, \frac{15-7s}{30}). \tag{2.5}$$

Thus, the values of s from Eq. 2.4 or Eq. 2.5 that preserve a given ranking are the same values of s that define the associated voting vectors. Therefore, the following table is derived by solving three trivial algebra problems.

Ranking	**Values of** s **for** $\vec{w}^{\,3}_s$
$c_3 \succ c_2 \succ c_1$	$s \in [0, \frac{3}{5})$
$c_3 \succ c_2 \sim c_1$	$s = \frac{3}{5}$
$c_3 \succ c_1 \succ c_2$	$s \in (\frac{3}{5}, \frac{9}{13})$
$c_3 \sim c_1 \succ c_2$	$s = \frac{9}{13}$
$c_1 \succ c_3 \succ c_2$	$s \in (\frac{9}{13}, \frac{3}{4})$
$c_1 \succ c_3 \sim c_2$	$s = \frac{3}{4}$
$c_1 \succ c_2 \succ c_3$	$s \in (\frac{3}{4}, 1]$

For example, as the Borda Count (BC) vector $(2, 1, 0)$ has the normalized form $\vec{w}^{\,3}_{\frac{1}{3}} = (\frac{2}{3}, \frac{1}{3}, 0)$, it follows from the above table and Eq. 2.5 that the BC outcome for this profile is $c_3 \succ c_2 \succ c_1$ with the normalized tally $(\frac{12}{45}, \frac{14}{45}, \frac{19}{45})$. □

Robustness.

A related issue is robustness. Namely, will slight changes in a given profile $\mathbf{p}$ preserve $Sup_n(\mathbf{p})$? If a ranking $\mathcal{R} \in Sup_n(\mathbf{p})$ is realized with $\vec{w}^{\,n}$, does the ranking remain intact with slight changes in the voting vector? These kinds of questions are answered by using part c of the above corollary. The geometry proves that robust conclusions are to be expected.

Corollary 1.2. *Let $n \geq 3$ candidates be given.*

a. *(Robustness with respect to profiles.) Suppose there are no ties in any of the $n-1$ election rankings for $\{f(\mathbf{p}, \vec{E}^{\,n}_s)\}$. There exists an open set of normalized profiles in $Si(n!)$ that support the same set of election rankings.*

b. *(Robustness with respect to voting vectors.) For a given profile $\mathbf{p}$, if $\mathcal{R} \in Sup_n(\mathbf{p})$ has no ties, then there exists an open set of normalized voting vectors $\vec{w}^{\,n}$ (of dimension $n-2$) that lead to this election ranking.*

c. *(Dimension count for voting vectors.) More generally, if $Co_n(\mathbf{p})$ meets a ranking region $\mathcal{R} \subset Si(n)$ so that the codimension of $\mathcal{R} \cap Co_n(\mathbf{p})$ in $Co_n(\mathbf{p})$ is k, then there is a convex subset of VV^n with codimension k that yield the election ranking $\mathcal{R}$.*

Recall, the codimension of $\mathcal{R} \cap Co_n(\mathbf{p})$ in $Co_n(\mathbf{p})$ is the difference between the dimensions of $Co_n(\mathbf{p})$ and $\mathcal{R} \cap Co_n(\mathbf{p})$. Part c shows how the geometric orientation of $Co_n(\mathbf{p})$ plays a critical role in determining the size of the set of voting vectors leading to different outcomes. For instance, with $\mathbf{p}$ from the above three-candidate example, when $Co_n(\mathbf{p})$ meets a ranking regions with a tie vote, the intersection is a point, so the codimension is unity. This means that the sets of voting vectors realizing this outcome must be a convex subset of Λ^3 with dimension $dim(\Lambda^3) - 1 = 1 - 1 = 0$, or a point. On the other hand, it is easy to find profiles so that $f(\mathbf{p}, \overrightarrow{E}^3_1)$ is on the ranking region $c_1 \succ c_2 \sim c_3$ while $f(\mathbf{p}, \overrightarrow{E}^3_2)$ has the ranking $c_2 \sim c_3 \succ c_1$. Here $Co_n(\mathbf{p})$ meets the two regions with a single tie vote with codimension zero; by the corollary, this means there is an open set of voting vectors that gives rise to the same outcome.

As an extreme example suppose all $n-1$ vertices of $Co_n(\mathbf{p})$ agree. (According to Theorem 1b, such situations exist.) Thus, the codimension of $Co_n(\mathbf{p}) \cap f(\mathbf{p}, \overrightarrow{E}^n_1)$ is zero; this means that $f(\mathbf{p}, \overrightarrow{w}^n) = f(\mathbf{p}, \overrightarrow{E}^n_1)$. In other words, there exist profiles where the normalized election tally is the same for all choices of voting vectors.

Proof. Part a: This is an immediate consequence of the continuity of the correspondence $Co_n(\mathbf{p})$.

Part b and c: As $Co_n(\mathbf{p})$ and each ranking regions are convex sets, so is $\mathcal{R} \cap Co_n(\mathbf{p})$. For fixed $\mathbf{p}$, the mapping $f(\mathbf{p}, -)$ is linear in the voting vector variable. Thus, $f^{-1}(\mathbf{p}, -)(\mathcal{R} \cap Co_n(\mathbf{p}))$ must be a convex subset of VV^n. If $\mathcal{R}$ is an open set (so, the ranking has no ties), then $\mathcal{R} \cap Co_n(\mathbf{p}))$ is a (relatively) open subset of $Co_n(\mathbf{p})$. It now follows from he continuity of f that $f^{-1}(\mathbf{p}, -)(\mathcal{R} \cap Co_n(\mathbf{p}))$ is a convex subset of VV^n. This completes part b. The dimension statement is an immediate consequence of the implicit function theorem. □

As the above illustrates, the various issues about $Sup_n(\mathbf{p})$ can be answered in terms of the possible positioning of $Co_n(\mathbf{p})$ within the simplex $Si(n)$. With a given profile $\mathbf{p}$, these computations are straightforward. Alternatively, if we don't have the profile, but we do have the tallies $\{f(\mathbf{p}, \overrightarrow{w}^n_j)\}$ for a set of vectors $\{\overrightarrow{w}^n_j\}$, then a similar analysis is possible. For instance, by knowing $f(\mathbf{p}, \overrightarrow{w}^3_{s_1})$ and $f(\mathbf{p}, \overrightarrow{w}^3_{s_2})$ $s_1 \neq s_2$, an obvious application of Eq. 2.1, leads to the values of $f(\mathbf{p}, \overrightarrow{E}^3_1)$ and $f(\mathbf{p}, \overrightarrow{E}^3_2)$. Because these last two values determine $Co_3(\mathbf{p})$, the above analysis applies.

3. p-SPECIFIC PROPERTIES AND OTHER PROPERTIES OF $Sup_n(\mathbf{p})$

What remains is to determine the admissible placements of $Co_n(\mathbf{p})$ without using specific profiles. In this way, general properties of $Sup_n(\mathbf{p})$, including a listing of all possible values for $|Sup_n(\mathbf{p})|$, can be found. The basic tool is Theorem 1-b.

According to Theorem 1-b, one can choose *any* $n-1$ points in a ball in $Si(n)$ centered about the point of complete indifference $(\frac{1}{n}, \dots, \frac{1}{n})$, specify which point is to be identified with which voting vectors $\overrightarrow{E}^n_s$, and there is a profile so that the selected points are the election outcome of the associated voting vector. In other words, one can choose any desired convex hull of $n-1$ points inside of $B(\mathbf{i}_n, r)$, and Theorem 1b ensures the existence of a profile $\mathbf{p}$ so that $Co_n(\mathbf{p})$ is this hull! Consequently, we can ignore profiles and concentrate on the simpler problem of determining the geometric properties of all possible choices of convex hulls determined by $n-1$ points in $Si(n)$. In this manner, we can characterize the properties of $\{Sup_n(\mathbf{p})\}_{\mathbf{p} \in Si(n!)}$.

To illustrate with $n = 3$ candidates, it follows from Theorem 1b that any line segment (or point) in the ball given in Figure 4 is $Co_3(\mathbf{p})$ for some choice of $\mathbf{p} \in Si(3!)$. Armed with this conclusion, all of the general properties of $Sup_3(\mathbf{p})$ and $Co_3(\mathbf{p})$ can be determined just by examining the admissible geometric properties of such lines in $Si(3)$.

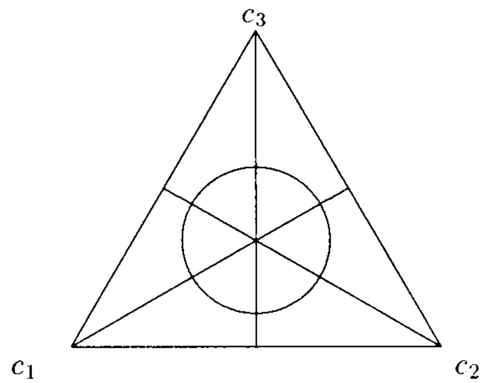

Figure 4

The kinds of geometric properties that should be examined are dictated by the voting issues. For instance, because a profile can support a large number of election rankings, it is not clear which ranking most accurately reflects the views of the voters. On the other hand, there are situations which offer a natural answer for questions of this type. For instance, if a particular property holds for all rankings in $Sup_n(\mathbf{p})$ – say, the same candidate always is top-ranked – then this forms a compelling argument to adopt this property as reflecting the true views of the voters.

Definition. A property that holds for all rankings in $Sup_n(\mathbf{p})$ is called a **p**-*specific* property.

a. If a candidate is the top-ranked for all rankings in $Sup_n(\mathbf{p})$, then she is the **p**-preferred candidate.

b. If a candidate is bottom-ranked for all rankings in $Sup_n(\mathbf{p})$, then she is the **p**-denied candidate.

c. If a candidate is not the **p**-denied candidate and never is top-ranked for all rankings in $Sup_n(\mathbf{p})$, then she is a **p**-indifferent candidate.

It is tempting to accept the **p**-preferred candidate as the voters' top- choice. After all, she is top-ranked independent of the choice of a voting method, so what more could be demanded of her? Similarly, the evidence indicates that a **p**-denied candidate is the voters' bottom-choice. These appealing arguments justify a further study of **p**-specific properties. Results about **p**-specific properties are obtained by determining the various ways a convex hull can be positioned within $B(\mathbf{i}_n, r)$; this geometric approach is used to prove the next theorem. Before stating the result, recall that c_i is a *Condorcet winner* if she wins all of the majority vote pairwise elections with the other candidates; she is a *Condorcet loser* if she loses all of these elections.

Theorem 2. *Let $n \geq 3$ candidates be given.*

a. *(Existence.) There exist open sets of profiles defining a* **p***-preferred (a* **p***-denied) candidate. There exist open sets of profiles where both a* **p***-preferred and a* **p***-denied candidate are defined.*
b. *(Non-existence.) There exist open sets of profiles where a* **p***-preferred candidate and/or a* **p***-denied candidate do not exist.*
c. *(Comparison.) When they exist, the* **p***-preferred candidate can never be a Condorcet loser, but she need not be the Condorcet winner. Indeed, for any integer* k *between 1 and* $n-1$*, there exists an open set of profiles with the* **p***-specific property that both a* **p***-preferred winner and Condorcet winner exist and the Condorcet winner is ranked in* k*th place. Similarly, the* **p***-denied candidate never can be a Condorcet winner, but she need not be the Condorcet loser; for any integer* k *between 2 and* n*, there exists an open set of profiles with the* **p***-specific property that a* **p***-denied candidate and a Condorcet loser exist; the Condorcet loser is ranked in* k*th position.*
d. *(Comparison of existence.) There exist open sets of profiles where a* **p***-preferred winner exists, but a Condorcet winner does not, and vice versa. A similar statement holds for the existence of* **p***-denied candidates and the Condorcet loser.*

Some readers may view these assertions showing that the **p**-preferred (denied) candidate need not be the Condorcet winner (loser) as a criticism of **p**-specific properties. Perhaps; but perhaps a more accurate interpretation is to treat these assertions as underscoring failings of the Condorcet winner (loser). After all, a **p**-preferred candidate survives *all* of the infinite different possible ways there are to aggregate the voters' preferences with positional voting methods – she is the winner no matter what weights are invoked to measure intercandidate comparisons. This constitutes a significantly more stringent test of voters' beliefs – one that involves comparisons among candidates – than just the pairwise comparisons used to define a Condorcet winner.

The proof of this theorem uses the following statement that is of independent interest. Of theoretical importance is the simpler approach used in the proof to establish the existence of rankings.

Proposition 1. *Let* $n \geq 3$ *candidates be given. Let* $\mathbf{N}^n_j$ *be the vector from* $\mathbf{i}_n$ *to the vertex of* $Si(n)$ *identified with* c_j*. The plane in* $Si(n)$ *passing through* $\mathbf{i}_n$ *with normal vector* $\mathbf{N}^n_j$ *assigns the constant value of* $\frac{1}{n}$ *for* c_j*, so any convex hull on this plane with* $\mathbf{i}_n$ *as an interior point passes through all ranking regions except those* c_j *extreme situations where* c_j *is top-ranked, bottom ranked, or tied with* $k < n-1$ *other candidates for top- ranked or bottom ranked.*

To see the geometry for $n = 3$ with Figure 1, draw a vector from the baricentric point to the $\mathbf{e}_3$. The plane described in the proposition becomes a horizontal line passing through $\mathbf{i}_3$. This line does not meet any region where c_3 is top-ranked, bottom-ranked, or tied with another candidate for being either top or bottom ranked. On this horizontal line, any line segment with $\mathbf{i}_3$ in the interior has the same properties.

Proof of Proposition 1. This proposition involves establishing the existence of a hull with the asserted properties. To simplify the analysis, note that $\mathbf{i}_n$ is a boundary point for each of the ranking regions. This means that if $\mathbf{i}_n$ is an interior point of a hull $Co_n(\mathbf{p})$, then $Co_n(\mathbf{p})$ meets the same ranking regions as the plane spanned by $Co_n(\mathbf{p})$. Conversely, if a plane passes through $\mathbf{i}_n$, a hull always can be found on this plane with $\mathbf{i}_n$ as an interior point. In fact, this hull can be chosen so that its vertices are in open ranking regions. Consequently, to establish the existence of certain properties, we only need to find a plane passing through $\mathbf{i}_n$ that meets the appropriate ranking regions.

Without loss of generality, consider $\mathbf{N}_1^n$ and candidate c_1. The defined plane passes through $\mathbf{i}_n = (\frac{1}{n}, \dots, \frac{1}{n})$ and the coordinate value for c_1 is fixed everywhere along this plane. Thus this fixed coordinate value for c_1 must be $\frac{1}{n}$ - the average value for a coordinate of $\mathbf{x} \in Si(n)$. By virtue of being the average value, x_1 cannot be the largest or the smallest value of the coordinates of $\mathbf{x} \in Si(n)$ (unless all n coordinates equal $\frac{1}{n}$). This is the only constraint that can be imposed upon the coordinates and the average value, so the conclusion follows. □

Proof of Theorem 2: Part a: This follows from Theorem 1-b. To show that c_1 can be a $\mathbf{p}$-preferred candidate, just place the convex hull into an open ranking region (or union of such ranking regions) where c_1 is top-ranked. The open set assertion about the profiles follows from the robustness assertion. A similar argument proves the rest of the claims.

Part b: Choose a convex hull that intersects ranking regions where more than one candidate is top-ranked, or bottom ranked. The assertion follows.

Part c: As it has been well known (at least for $n = 3$ candidates) since the time of Condorcet and Borda, a Condorcet winner (loser) need not be Borda top (bottom) ranked. This statement holds for all n; indeed, a result from Saari [6] characterizes all possible ways the Condorcet winners and losers must be positioned within the BC rankings. There is considerable flexibility; for any BC ranking where candidate c_j receives more than $\frac{1}{n}$ of all points cast, there are profiles so that c_j also is the Condorcet winner. (Thus, from Proposition 1, the Condorcet winner could be BC ranked next to the bottom.) So, the first step is to choose a BC ranking where c_1 is BC top- ranked and c_2, the Condorcet winner, is kth ranked where $1 \leq k \leq n-1$; such profiles exist.

The second step also depends upon results from Saari [5, 6]. Expressing an assertion from these papers in terms used in this current article, we have that for any BC strict ranking of the n candidates and associated majority vote rankings of the $\binom{n}{2}$ pairs of candidates there exists an open set of profiles so that $Sup_n(\mathbf{p})$ consists of a single ranking that is this BC ranking and the majority vote rankings of the pairs also remain the same. Use this result for the ranking from step one. This leads to the conclusion that the convex hull is positioned in such a manner that c_1 is the $\mathbf{p}$-preferred candidate while c_2, the Condorcet winner, is kth ranked.

The remaining parts of the assertion follow in a similar manner by using the properties of the Borda Count. For instance, as a Condorcet loser can never be BC top-ranked, but she can be kth ranked for $2 \leq k \leq n$; thus a $\mathbf{p}$-preferred candidate never can be a Condorcet loser but the kth ranking of the Condorcet loser is an admissible $\mathbf{p}$-specific property .

It is worth describing the positioning of $Co_n(\mathbf{p})$ that denies the Condorcet winner the honor of being $\mathbf{p}$-preferred because she is kth ranked. Suppose c_2

is the Condorcet winner. The plane $\mathcal{P}$ passing through $\mathbf{i}_n$ with normal vector $\mathbf{N}_2^n = (-\frac{1}{n}, \frac{n-1}{n}, -\frac{1}{n}, \dots, -\frac{1}{n})$ divides $Si(n)$ into two parts; the part containing $\mathbf{N}_2^n$ (and the c_2 vertex of $Si(n)$) is where c_2 receives at least $\frac{1}{n}$ of all points cast in a BC election. (See Figure 5 for the case $n = 3$.) Now, as described above, a Condorcet winner must receive at least $\frac{1}{n}$ of all points cast in a Borda election. This means that if c_2 is the Condorcet winner, then the point of $Co_n(\mathbf{p})$ corresponding to the BC election outcome must be on this side of the plane $\mathcal{P}$. In fact, as also described above, this is essentially the only restriction that can be imposed upon the Condorcet winner. As long as c_2 satisfies this condition, it can be the Condorcet winner.

So, to have c_2 the Condorcet winner while c_1 is the $\mathbf{p}$ preferred candidate, the hull must be positioned in the interior of the union of regions where c_1 is top-ranked and the BC point of the hull is on the c_2 side of the $\mathcal{P}$ plane. For more specific positioning (where c_2 is kth ranked) the sector is chosen to keep the BC ranking in the regions where c_2 is kth ranked. This is possible from Proposition 1 and its proof and the robustness assertions. The intersection of these regions forms an open sector. The geometry imposes restrictions on the positioning of the hull; there remain ample opportunities for this construction.[5] (For $n = 3$ this is the small, shaded triangle in Figure 5.) The only difference in the geometry when comparing the Condorcet loser and the $\mathbf{p}$-preferred candidate is that if c_2 is the Condorcet loser then the BC point now must be on the other side of the $\mathcal{P}$ plane. In this situation, it is geometrically impossible for the hull to have the BC point on the correct side of $\mathcal{P}$ while being in the interior of the regions where c_1 is the winner or tied for being in top place. However, by use of the proposition, anything else is possible. This is a geometric explanation of the assertion.

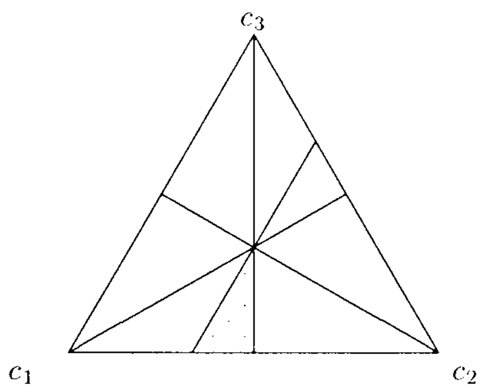

Figure 5

Part d: Choose a ranking where c_1 is top-ranked but there is no Condorcet winner, and then follow the lead of the proof of part c. Geometrically, there is no restriction on the choice of the convex hull. To show that a Condorcet winner can exist where a $\mathbf{p}$-preferred candidate does not, recall from Saari [4, 5] that one can choose the plurality and the anti-plurality election rankings of the n candidates in any desired way and still have the freedom to choose the majority vote rankings of

[5] From the restrictions on the construction, one might conjecture that when there exists a $\mathbf{p}$ preferred candidate and a Condorcet winner, it is more probable that they agree than disagree. With wide classes of probability distributions on the profiles, this can be shown to be the case.

the $\binom{n}{2}$ pairs of candidates. So, if the rankings are chosen so that c_1 is plurality top-ranked, c_2 is anti-plurality top-ranked, and c_3 is the Condorcet winner, the assertion follows. Notice, however, that the hull must be placed in such a manner that the BC point is on the correct side of the $\mathcal{P}$ plane. The other assertions follow in a similar manner. □

A similar approach can be used to understand other $Sup_n(\mathbf{p})$ and $\mathbf{p}$-specific properties.

Corollary 2.1. *Assume there are $n \geq 3$ candidates.*

a. *(Comparison) There exist situations where a $\mathbf{p}$ indifferent candidate is a Condorcet winner, and other situations when it is a Condorcet loser.*
b. *(Nonexistence) A $\mathbf{p}$-indifferent candidate need not exist. Indeed, for any n there exists open sets of profiles where each candidate is top-ranked for at least one ranking in $Sup_n(\mathbf{p})$.*
c. *(Extreme non-existence) If $n \geq 4$ is even, then there exists a $\mathbf{p}$ so that for each candidate c_j there exists a strict ranking in $Sup_n(\mathbf{p})$ where c_j is top-ranked and another strict ranking in $Sup_n(\mathbf{p})$ where c_j is bottom-ranked.*

The profiles described in the last part of part c can inspire electoral debate. It asserts that for the same profile and for each candidate c_j, there exist choices of voting vectors that appoint c_j as the top-ranked candidate, but then there are other choices of voting vectors that can be used to cast serious doubt on this outcome because they end up with c_j being bottom-ranked!

Proof. Part a. This is a simple exercise using the above ideas and the geometry of the ranking regions, so it is left to the reader.

Part b. Consider the plane described in Proposition 1 for the normal vector $\mathbf{N}_1^n$. According to the proposition, for every candidate c_j, $j = 2, \ldots, n$, the plane must meet at least one open ranking region where c_j is top-ranked. The boundaries of all ranking regions where c_1 is top-ranked meet this plane at $\mathbf{i}_n$. Therefore, with a small parallel translation of the plane along $\mathbf{N}_1^n$, the translated plane meets all regions where c_1 is top-ranked, or tied with up to $n-2$ other candidates for being top-ranked. Moreover, because the original regions where c_j, $j = 2, \ldots, n$, was top-ranked are open regions, if the translation is sufficiently small (yet positive), then the plane still meets all of these regions. The conclusion now follows. Incidentally, a similar argument proves the existence of open sets of profiles so that each candidate is bottom ranked for some ranking in $Sup_n(\mathbf{p})$.

Part c. To prove the last assertion, suppose $n \geq 4$ is even. There exists a plane passing through $\mathbf{i}_n$ so that half of the vertices are on each side of the plane, say $c_1, c_2, \ldots, c_{n/2}$ are on one side and the remaining vertices are on the other. Each pair of vertices, where one is chosen from each side of the plane, defines an edge of $Si(n)$. One such plane, with normal $(1, 1, \ldots, 1, -1, -1, \ldots, -1)$, must pass through all edges of this type and only through edges of this type. The point of intersection on each edge is the midpoint (by the choice of the normal vector).

Consider a two-dimensional face of $Si(n)$ defined by the three candidates c_i, c_j, c_k where (c_i, c_j) and (c_k, c_j) are two of the above edges. In this face (an equilateral triangle), the plane is a line segment that must pass through two open regions where c_j is top-ranked. (To see this, connect the midpoints of two edges of $Si(3)$ from Figure 1. Notice that the line must pass through the regions where the candidate identified with the common vertex of these two edges is top-ranked.) Near this face, all other candidates are ranked below the specified three candidates, thus this plane passes through a region where c_j is top-ranked. By construction, this is true for any index c_j.

Using the symmetry of ranking regions with respect to $\mathbf{i}_n$ and the fact that this constructed plane passes through $\mathbf{i}_n$, it follows that any ranking represented on this plane has its reversed ranking also represented on the plane. Thus, for each candidate c_j there exist open ranking regions that meet this plane where c_j is bottom-ranked. As any hull $Co(\mathbf{p})$ on this plane with $\mathbf{i}_n$ as an interior point meets the same ranking regions as the plane, the conclusion follows. Notice (for future reference) that any small translation of this hull up or down along the normal vector still meets the ranking regions with these properties.

To see why this assertion need not hold for odd values of $n \geq 3$, just examine the properties of line segments in $Si(3)$; it is impossible to have half of the vertices on each side of the plane. However, the assertion does hold for $n-1$ of the candidates; this can be seen with the five candidate example described in the introductory section and from the results in Saari [4]. I leave further refinements to the interested reader. □

The above constructions can create the impression that these conclusions rely upon delicate placements of the hull. While such delicacy may simplify the proof of an assertion, it is not needed to support the basic phenomenon. The next "stability" assertion shows that once the existence of a set $Co_n(\mathbf{p})$ with certain properties is established, there are large degrees of freedom to move this hull while retaining the same properties. Thus, in the last construction showing that each candidate can be top and bottom ranked for some strict ranking in $Sup_n(\mathbf{p})$, there is considerable room for moving the vertices. As another consequence of this statement, which plays an important role in the analysis of the likelihood of the various events, it shows that the restriction of points to $B(\mathbf{i}_n, r)$ imposes no restrictions on our analysis of election rankings.

Proposition 2. *Suppose each of the vertices of $Co_n(\mathbf{p})$ are in open ranking regions. If $Co_n(\mathbf{p}')$ is another hull where each vertex remains in the same open ranking region and where $\mathbf{i}_n \notin Co(s\mathbf{p} + (1-s)\mathbf{p}')\, \forall s \in [0,1]$, then $Sup_n(\mathbf{p}) = Sup_n(\mathbf{p}')$.*

Alternatively, consider two hulls in $Si(n)$ with vertices in open ranking regions. Suppose if one hull has a vertex in a given ranking region, then so does the other – this defines pairs of vertices with one from each hull. Connect each pair of vertices with a straight line segment; this creates $n-1$ line segments. Parameterize each line segment in the standard manner by $s \in [0,1]$ where $s = 0$ represents the vertex of the first hull, and $s = 1$ is the corresponding vertex of the second hull. If $\mathbf{i}_n$ is not in any hull with a vertices on the line segments corresponding to $s \in [0,1]$, then both of the original hulls meet the same ranking regions.

In other words, there is no change in $Sup_n(\mathbf{p})$ for changes in $\mathbf{p}$ as long as during the transformation of the hull, each vertex remains in the same open ranking region and, in transforming from one position to another, the hull does not pass through $\mathbf{i}_n$. The alternative description releases any need to consider profiles; only the geometry of the hulls is needed. Thus, although the construction proving the assertions of

Corollary 2.1b involved "sufficiently small" translates of a plane, according to the proposition, this requirement can be dropped.

The geometric support of this assertion is based on the observation that if a ranking is added or dropped from the set of supported rankings, then, by continuity, the hull must first pass through a ranking region with a tie vote; the hypothesis makes this geometrically impossible. For instance, with $n = 3$ and Figure 1, if one vertex of $Co_3(\mathbf{p})$ is in region 3 and the other in region 1, then $|Sup_n(\mathbf{p})| = 5$ independent of the positioning of these vertices. On the other hand, if the vertices are in regions 1 and 4, then the choice of rankings and the number of rankings change if the line segments are on different sides of $\mathbf{i}_3$. Incidentally, it is clear from the proof that this assertion extends to the situation whereby the vertices remain in the same ranking region whether or not these are open ranking regions.

Proof: The only way the set of rankings can change (a new ranking is added or an original ranking is dropped) is if for some s in $Co_n(s\mathbf{p}+(1-s)\mathbf{p}')$, the associated convex hull passes through an indifference region. One such region is the point $\mathbf{i}_n$, but, according to the assumptions, $\mathbf{i}_n \notin Co(s\mathbf{p} + (1 - s)\mathbf{p}')\,\forall s \in [0, 1]$. Therefore, the hull must pass through an indifference region that is at least one-dimensional.

There are two ways a hull can pass through an indifference region with positive dimension. The first is that a vertex passes through an indifference hyperplane; a possibility outlawed by assumption. To analyze the second possibility, note that if the normal vector for an indifference plane is not collinear with the normal vector for the hull, then they meet transversely – the intersection persists for changes in the profile, so the same rankings occur. Therefore, should the second possibility happen, it must be that a normal vector for the hull becomes collinear with a normal vector of some indifference plane. This means there is a pair of candidates (defining the indifference plane), $\{c_k, c_t\}$ whereby for all points in $Co(s\mathbf{p} + (1 - s)\mathbf{p}')$, the relative ranking is $c_k \succ c_t$ for $s < s'$, but $c_t \succ c_k$ for $s > s'$. This requires $x_k > x_t$ for $s < s'$ and $x_t > x_k$ for $s > s'$ for all points in $Co(s\mathbf{p} + (1 - s)\mathbf{p}')$. Using Eq. 2.1, the first situation requires the relative ranking $c_k \succ c_t$ for each vertex of $Co(s\mathbf{p} + (1 - s)\mathbf{p})$ where $s < s$ and the reversed relative ranking of this pair for each vertex where $s > s'$. This contradicts the assumption that the vertices remain in the same open ranking regions.

The alternative description follows from the above, Theorem 1b, and the smoothness of f in both variables. From these results, we have that the line segments joining vertices of the two hulls defines a parameterized line segment of profiles. By use of this line segment, we now have the above proof. □

There are other related behaviors and $\mathbf{p}$-specific properties that can be analyzed as indicated above. But a couple of features are beginning to emerge. The first is that while one can identify certain attractive properties to investigate, the existence of $\mathbf{p}$-specific properties can be restricted. Secondly, in the analysis of the $\mathbf{p}$-specific properties, there is a critical reliance upon the properties of the Borda Count. This is no accident; we now know (Saari [6]) that the BC plays a central role in the analysis of the properties of positional voting processes. This strongly suggests (but does not prove) that the $\mathbf{p}$-specific properties that are viewed as being favorable are inherited from the BC properties, while undesirable properties can be attributed to the properties of other choices of positional voting methods.[6] In turn, this leads to the natural suggestion that instead of using $\mathbf{p}$- specific properties, the BC is the appropriate choice to use for voting processes. The BC must be a serious candidate as the method to determine the "true" wishes of the voters.

The Admissible Values for $|Sup_n(\mathbf{p})|$.

The remaining topic of this section is to find all admissible values for $|Sup_n(\mathbf{p})|$.

Theorem 3. *Assume $n \geq 3$ candidates are given.*

a. *Let k be an integer satisfying*

$$0 \leq k \leq n! - (n-1)!. \tag{3.1}$$

There exists a profile $\mathbf{p}$ where $Sup_n(\mathbf{p})$ contains precisely k strict rankings. Conversely, if there are k strict rankings in $Sup_n(\mathbf{p})$, then k satisfies Eq. 3.1.

b. *Let $C(n)$ be the number of ranking regions in $Si(n)$; that is, $C(n)$ is the number of n-candidate rankings with and without ties among candidates. For any k satisfying*

$$1 \leq k \leq C(n) - [1 + C(n-1) + \sum_{i=1}^{n-1} \binom{n-1}{i} C(n-i-1)], \tag{3.2}$$

there exists a profile $\mathbf{p}$ so that $|Sup_n(\mathbf{p})| = k$. Conversely, if $|Sup_n(\mathbf{p})| = k$, then k must satisfy Eq. 3.2.

The value of $C(n)$ grows rapidly with increasing values of n. To see this, define $\tau(k) = max(k-1, 0)$ and $\theta(k_1, k_2, \dots, k_n)$ to be the multiple of the factorials of the number of times each non-zero value occurs in the n tuple. For instance, $\theta(4,3,4,5,5,4,2,2,0,0) = 3!1!2!2!$ because 4 appears three times, 3 appears once, 5 appears twice, 2 appears twice, and the 0 terms are ignored. By using standard combinatoric arguments, we have that

(3.3)

$$C(n) = n! + \sum_{\substack{j_1 \geq j_2 \geq \cdots \geq j_{n/2} \\ j_{n/2} \geq 0, j_1 \geq 2 \\ j_i \neq 1 \forall i \geq 2}} \frac{\{\binom{n}{j_1} \times \dots \binom{n - \sum_{k=1}^{t-1} j_k}{j_t} \times \binom{n - \sum_{k=1}^{n/2} j_k}{j_{n/2}} (n - \sum_{i=1}^{n/2} \tau(j_i))!}{\theta(j_1, \dots, j_{n/2}\}}$$

It follows from this equation that $C(n)/n! \to \infty$ as $n \to \infty$.

For all values of n, it is possible for $Sup_n(\mathbf{p})$ to contain a significant portion (at least $\frac{2}{3}$)) of all rankings without ties. This fraction increases with the limit of unity with an increase in the value of n.

Corollary 3.1. *For $n \geq 2$ candidates $Sup_n(\mathbf{p})$ can include up to but not exceeding $1 - \frac{1}{n}$ of the $n!$ possible rankings without ties.*

Proof of the corollary. There are $n!$ open ranking regions, so, according to Theorem 3, it is possible for up to $\frac{n! - (n-1)!}{n!} = \frac{n-1}{n}$ of the strict rankings to be obtained with a single profile. □

[6] Other geometric arguments can be developed to support this assertion, but the point is made.

Theorem 3 indicates that incredibly large number of rankings can be supported by a single profile. An appreciation for the magnitudes of these numbers, how fast they grow, and how the proportion of rankings in $Sup_n(\mathbf{p})$ approaches unity as $n \to \infty$ is gained from the following table.

n	**Strict Rankings in** $Sup_n(\mathbf{p})$	**Max** $\vert Sup_n(\mathbf{p})\vert$	$\frac{\textbf{Max } \vert Sup_n(\mathbf{p})\vert}{C(n)}$
2	1	1	0.3333
3	4	7	0.5385
4	18	45	0.6338
5	96	371	0.7176
6	600	3645	0.7783
7	4320	38,131	0.8063
8	35,280	451,893	0.8279
9	322,560	5,997,341	0.8462
10	3,625,920	84,830,767	0.8569

The geometry leading to the proof of the theorem is indicated in Figure 6. The convex hulls $Co_3(\mathbf{p})$ are line segments. A line segment in $Si(3)$ is uniquely determined by its orientation, as computed by a parallel line passing through $\mathbf{i}_3$, and the required translation to return the orientation segment to the original line segment. Conversely, the behavior of all possible line segments can be determined by first considering all possible orientations through $\mathbf{i}_3$ and then all possible parallel translations.

Orientation. Depending on the orientation of a line passing through $\mathbf{i}_3$, the orientation line intersects either two or zero of the open regions. The situation depicted in Figure 6 has the orientation line meeting two open regions and missing $3! - 2!$ open regions. Half of these missed regions are on one side of the orientation line, half on the other.

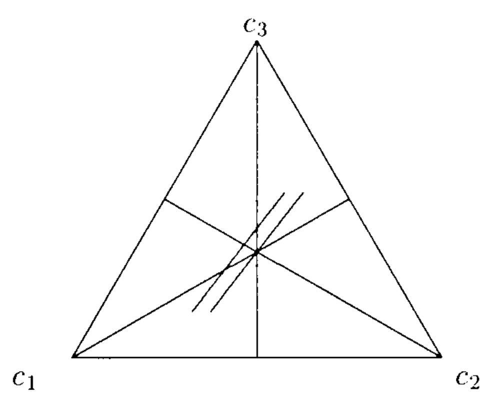

Figure 6

Translation. To maximizes the value of $|Sup_3(\mathbf{p})|$, the optimal situation corresponds to a parallel translation of the orientation line off of $\mathbf{i}_3$ but where the end points stay in the same open regions. It is clear that a small translation meets all of the original open regions and half of the missing ones; it follows from Proposition 2 that any translation satisfying the specified conditions has this property. Thus, this choice of $Co_3(\mathbf{p})$ meets $\frac{3!-2!}{2} + 2! = \frac{3!+2!}{2} = 4$ regions. This is the bound on rankings without ties.

When all ranking regions are considered, it is geometrically obvious that the orientation line meets three regions. We give a different, more complicated argument with only one virtue that it explains Eq. 3.2 for $n = 3$. First, use the fact that the orientation line divides $Si(3)$ into two regions. It follows from the geometry of $Si(3)$ that there exists a candidate, say c_2, where all ranking regions corresponding to where c_2 is top-ranked are on in one of these half regions.[7] This means that the orientation line misses all of these regions. To count these regions, note that there are $C(2)$ of them where c_2 is top-ranked (corresponding to the number of rankings of the remaining two candidates) and $\binom{2}{1}C(1)$ regions where c_2 is tied for being top-ranked with another candidate. Therefore, the orientation line misses all these $C(2) + \binom{2}{1}C(1) = 5$ regions. By symmetry of ranking regions (with respect to $\mathbf{i}_3$), the orientation line also misses the corresponding $C(2) + \binom{2}{1}C(1) = 5$ regions where c_2 is bottom, or tied with another candidate for being bottom-ranked. Thus, the orientation line meets $C(3)-2[C(2)+\binom{2}{1}C(1)]$ regions, and it misses $2[C(2)+\binom{2}{1}C(1)]$ of them. A translation of the orientation line that keeps the endpoints in the same open regions meets $C(3)-2[C(2)+\binom{2}{1}C(1)]+[C(2)+\binom{2}{1}C(1)]-1 = C(3)-[C(2)+\binom{2}{1}C(1)+1] = 7$ regions. The "-1" term corresponds to the fact that the translated line loses contact with the boundary ranking region $\mathbf{i}_n$. By truncating the length of the translated line or using a parallel translate where an endpoint leaves an original open region, one can get examples of hulls meeting any number of regions between 1 and the indicated upper bound.

Other orientations. It remains to consider the orientation lines that meet no open regions. (Such lines prove the existence of $k = 0$ in Eq. 3.1.) For this to occur, the orientation line must be in a line of binary indifference. Therefore, it misses 2α regions, α of them on each side of the orientation line, so it meets $C(3) - 2\alpha = 3$ regions. For the same reasons given above, the translated line can meet no more than $C(3) - [\alpha + 1]$ regions. In fact, the translated line meets a fewer number of regions; it must meet no more than $\alpha = 5 < C(3) - [\alpha + 1] = 7$ ranking regions because the translated line must lose contact with all of the original regions corresponding to rankings with ties. This completes the proof for $n = 3$. □

Proof of Theorem 3. The proof for part a follows that of part b, so only part b is proved. The geometric properties of a convex hull are determined by the orientation (of the plane it spans) through $\mathbf{i}_n$ and its parallel translation – the main concern is attaining the asserted maximum value. For the orientation, choose a $n-2$ dimensional plane passing through $\mathbf{i}_n$, or a hull on this plane with $\mathbf{i}_n$ as an interior point and vertices in open ranking regions. This plane misses 2α ranking regions where, as shown below,

$$(3.4) \qquad \alpha \le [C(n-1) + \sum_{i=1}^{n-1} \binom{n-1}{i} C(n-i-1)],$$

[7] This is not true for all values of $n \ge 4$. For instance, consider the plane used for the proof of Corollary 2.1c – no candidate can be found with the indicated property.

so the plane meets $C(n) - 2\alpha$ regions. By symmetry (of ranking regions with respect to $\mathbf{i}_n$), there are α ranking regions are on each side of the orientation plane that do not meet the plane. Consequently, a parallel translate of this plane, or the hull, can meet no more than $[C(n) - 2\alpha] + \alpha - 1 = C(n) - [\alpha + 1]$ of the ranking regions where the "-1" term represents the ranking region $\mathbf{i}_n$ that cannot be on the translated region. The translated region meets precisely this number of regions if only the ranking region $\mathbf{i}_n$ is totally contained in the original ranking region; here, the translation loses contact only with the region of complete indifference $\mathbf{i}_n$. If the plane starts with this property and if the vertices all vertices start in open ranking regions, then the properties remain as long as the vertices remain in these regions. Thus, an upper bound for the number of ranking regions is $C(n) - [\alpha + 1]$. According to inequality 3.4, an upper bound for $|Sup_n(\mathbf{p})|$ is $[C(n) - [1 + C(n-1) + \sum_{i=1}^{n-1} \binom{n-1}{i} C(n-i-1)]$. The fact that all values can be obtained is achieved by changing the vertices so that they now meet and pass through different indifference ranking regions.

Inequality 3.4. It remains to prove Inequality 3.4. If a ranking region is on one side of of a plane, then the ranking region corresponding to the reversal of the original ranking is on the other side. (This is due to the symmetry of ranking regions of $Si(n)$ with respect to $\mathbf{i}_n$. Therefore, it suffices to minimize the number of ranking regions that are on one side of a plane P. According to Proposition 1, there are orientations of P where the only ranking regions in one half space are defined by a particular candidate, say c_1. These ranking regions are where c_1 is top-ranked, or tied for top-ranked with up to $n-2$ other candidates. In other words, there are $C(n-1) + \sum_{i=1}^{n-1} \binom{n-1}{i} C(n-i-1)$ ranking regions on this side of P, and the same number on the other. Already, by use of this value of α, this establishes that $Max(|Sup_n(\mathbf{p})| \geq [C(n) - [1 + C(n-1) + \sum_{i=1}^{n-1} \binom{n-1}{i} C(n-i-1)]$. What needs to be shown is that $Max_{\mathbf{p} \in Si(n!)}(|Sup_n(\mathbf{p})|)$ cannot be larger. That is, inequality 3.4 needs to be proved.

By symmetry (and the corresponding fact following from the scalar product that the angle between any two vectors from $\{\mathbf{N}_k^n\}$ is $90^o < \arccos(\frac{-1}{n-1}) \leq 120^o$) at least one vertex must be on each side of the plane. The particular geometric setting described by Proposition 1 has a single vertex on one side of the plane, and this vertex is placed so that all ranking regions where c_1 is top-ranked or tied for top-ranked are in this half space. It follows from Proposition 2 that whenever this geometry is observed, the indicated value of α occurs.

In changing the geometry, the plane needs to be moved so that it finally comes in contact with one of these ranking regions. The first possibility is with a tie vote, so suppose at least one region representing a top-ranked tie vote with c_1 now is in the plane. Let β represent the ranking region with largest dimension and where c_1 is tied for top-ranked that first meets P. If part of β is in P, then $\beta \subset P$. (This uses the fact that the ranking regions are linear objects obtained from the intersection of planes and that P is a plane.) Clearly, this geometry still requires the c_1 vertex to be the only vertex on this side of P. (Before another vertex can enter this half space, it must pass through P; i.e., the vector $\mathbf{N}_j^n \in P$. But the line defined by this plane includes the ranking region where c_j is bottom ranked, and all candidates are tied for top- ranked with c_1.) As $\beta \subset P$, rather than being on the c_1 vertex side of

P, this subtracts from α the value $[C(n-1)+\sum_{i=1}^{n-1}\binom{n-1}{i}C(n-i-1)]$.

Much more information follows from the α value. After all, if β represents a tie vote with fewer than $n-2$ other candidates, then P must also include all of the boundary regions of β. (Again, this uses the fact that the ranking regions are linear objects and that the boundaries must be in the same linear space as the ranking region. For instance, if the region is $c_1 \sim c_2 \sim c_3 \succ c_4 \cdots \succ c_n$, then, P also meets the regions $c_1 \sim c_2 \sim c_3 \sim c_4 \succ \ldots c_n)$, etc.) On the other hand, this orientation forces several regions that previously met P to be on the c_1 side of P; each such region adds to the count of α.

Above P are the regions where c_1 is top-ranked. Thus, all regions where the top-ranked tie with c_1, indicated by β, is broken to the advantage of c_1 must be above P. Similarly, the more numerous number of regions (if $n \geq 4$) where the ties are broken to the disadvantage of c_1 are on the other side of P. By the symmetry of the ranking regions, it follows that the reversal of each of these rankings now are on the c_1 side of P. In total, the value of α has increased. Notice, since the one of the boundary rankings in P has $n-1$ candidates tied for first and one candidate in bottom place, and since the plane passes through $\mathbf{i}_n$, the reversal of this ranking is also on P. But, this one-dimensional line includes the vertex. Thus, if the orientation of P is changed so that an open region with c_1 top-ranked meets P, then there are at least two vertices – c_1 and another one – that are on the same side of P.

What remains, then, is to consider what happens when more than one vertex is on the same side of P. By symmetry, attention can be restricted to the side that has no more than $\frac{n}{2}$ vertices. (So, for anything new to occur, $n \geq 4$.) The basic idea can be seen with two vertices c_1, c_2. Here the geometry dictates that all open regions where one candidate is top ranked and the other second ranked are on this c_1 side of P. If a boundary region of one of the open regions with this top two ranking is in P, then, by use of the above argument, it follows that more regions are on this c_1 side of the plane than if no boundary regions were included. But since any open ranking region is equivalent to another, it follows that there are at least as many ranking regions in on this side of P as in the one vertex setting. (A more careful count, using the possible orientations of the vertices and the angle between $\mathbf{N}_j^n$ proves that more rankings are admitted.) This completes the proof. □

4. Space of Profiles

To complete the story, it is necessary to suggest how likely are the various events described in the previous sections.[8] To provide intuition about these likelihoods, I first outline how profiles are designed to obtain the different kinds of outcomes, and then I use this discussion to indicate what we should expect from probability statements.

A basis of voter types.

Designing profiles can be a difficult task. The prime complication revolves around the number of variables – as the design of an n-candidate profile potentially involves

[8]As noted above, I defer to elsewhere a rigorous discussion about the likelihood of the various outcomes. This discussion is based a geometric approach developed to avoid the serious combinatoric complications.

$n!$ independent variables (the number of voter types), even linear programming approaches are not adept at handling the millions of variables.(See, for instance, Chamberlain [2].) However, by use of the geometry, the number of voter types can be significantly reduced.

Theorem 4. *Assume a profile* $\mathbf{p}$ *is to be constructed so that the vertices of* $Co_n(\mathbf{p})$ *are in regions representing specified strict rankings of the* n *candidates. There exists such a profile requiring no more than* $(n-1)^2$ *voter types.*

Proof. The proof of this assertion is immediate. The hull $Co_n(\mathbf{p})$ is uniquely defined by its $n-1$ vertices. In turn, the ranking region of each vertex is determined by $n-1$ inequalities of the coordinates of $\mathbf{x} \in Si(n)$. Through the election mapping, this leads to $(n-1)^2$ linear algebraic inequalities in the profile variables. From elementary algebra – "k equations and k unknowns" – these equations can be solved in the designated number of variables provided these $(n-1)^2$ equations are independent and do not define contradictory relationships. That this is true is one of the assertions from Saari [4, 7]. □

Example: Theorem 4 indicates that four voter types are required for $n = 3$. However, if strict rankings are involved, the geometry can be exploited to obtain a further reduction – only three voter types are needed. To see this and to see how to select the three voter types, start with c_j top-ranked for r_j, $j = 1, 2, 3$ and n_j the number of voters of this type. So, if the plurality ranking is $c_1 \succ c_2 \succ c_3$, then $n_1 > n_2 > n_3$..

It remains to choose the voter types so that a profile can be found to obtain any given anti-plurality ranking. The anti-plurality outcome is in the hull defined by the midpoints of the three edges of $Si(3)$. (See, for example, Figure 1.) However, if c_j is the top-ranked candidate of a specified anti-plurality ranking, then only the edge of the anti-plurality hull closest to c_j need be considered. For instance, if $c_j = c_3$, then this is the edge of the anti-plurality hull connecting the midpoints of the two side edges of $Si(3)$. To force the anti- plurality outcome to be on this edge, choose the second candidate for each r_j so that $\rho_j(\overrightarrow{E}^3_2)$ is one of the two midpoints on the side edges of $Si(3)$. This requires c_2 to be the second ranked candidate for both r_1 and r_3 but it imposes no restrictions on r_2: the ranking for r_2 is chosen to obtain the designated anti-plurality outcome. For instance, if the anti-plurality ranking is to be $c_3 \succ c_2 \succ c_1$, then the outcome must be on the right-hand side of this line forming an edge of the anti-plurality hull. Thus the second ranked candidate for r_3 must be c_2, and it leads to the inequality $n_1 < n_2 + n_3$. The operative set of inequalities is

$$n_1 > n_2 > n_3,\ n_2 + n_3 > n_1,$$

where the smallest integer solution is $n_1 = 4, n_2 = 3, n_3 = 2$. Another solution, $n_1 = 6, n_2 = 5, n_3 = 4$, defines the example in Section 2. Of course, there are many other solutions involving more than three voter types. However, it is not difficult to show (via this geometric construction) that if $|Sup_3(\mathbf{p})| = 7$, then there must be at least nine voters. □

From the geometry it follows that any pair of strict rankings can be obtained with 3 voter types. (For $n \geq 3$ candidates and $n-1$ vertices in regions with strict rankings, $n(n-2)$ voter types are required.) On the other hand, similar geometric arguments prove that not all pairs of strict rankings for $n = 3$ can be obtained with profiles involving only two voter types; the following assertion uses this kind of analysis to show the number required to obtain a single ranking. I leave to the interested reader the generalization of this assertion to arbitrary values of n.

Proposition 3. *For $n = 3$ candidates, if $Sup_3(\mathbf{p})$ contains only one ranking, and this is a strict ranking, then $\mathbf{p}$ involves at least three types of voters. Any such profile requires at least three voters.*

What we see from these arguments and results is that the geometry can be used to determine the number of voters and the number of voter types required to have $Sup_n(\mathbf{p})$ satisfy some specified properties. In turn, this gives information about the likelihood of such profiles occurring.

Proof. Without loss of generality, assume that the ranking is $c_1 \succ c_2 \succ c_3$ and that there is a two voter type profile with this outcome for both the plurality and anti-plurality outcomes. Of the two voter types r_1, r_2, c_j must be the top-ranked candidate for r_j and $n_1 > n_2 > 0$ to realize the plurality outcome. If the second ranked candidate for r_2 is c_3, then the anti-plurality outcome must be on the line connecting the midpoint of the right edge of $Si(3)$ with either the midpoint of the bottom or the left edge of $Si(3)$. As either line misses the ranking region for $c_1 \succ c_2 \succ c_3$, a contradiction arises. Thus, $r_2 = c_2 \succ c_1 \succ c_3$. If c_2 is the second ranked candidate for r_1, then the anti-plurality outcome is forced to be the midpoint of the bottom edge of $Si(3)$, a ranking that is not the desired one. Thus $r_1 = c_1 \succ c_3 \succ c_2$ and the anti-plurality outcome is on the line connecting the midpoints of the bottom and left edges of $Si(3)$. But, as $n_1 > n_2$, this point must be in the region $c_1 \succ c_3 \succ c_2$ rather than the designated one. This contradiction completes the proof.

To obtain the minimal number of voters, use the above geometric analysis of three voter types to obtain that $r_1 = c_1 \succ c_2 \succ c_3, r_2 = c_2 \succ c_1 \succ c_3, r_3 = c_1 \succ c_3 \succ c_2$ with the inequalities $n_1 + n_3 > n_2 > 0, n_1 + n_2 > n_3$ where the minimal solution is, of course, $n_1 = n_2 = n_3 = 1$. □

What emerges are dual sets of constraints for the design of profiles. The first is the set of rankings for the vertices of $Co_n(\mathbf{p})$ needed in order to realize a certain kind of outcome; the second are the restrictions on voter types and the combinatorics to obtain these vertices as an outcome. It is not overly difficult to extend this reasoning to show that with a large class of probability distributions, that it is less likely for $Sup_3(\mathbf{p})$ to consists of a single strict ranking than it is for $Sup_3(\mathbf{p})$ to have at least two rankings. This is because in order for $Sup_3(\mathbf{p})$ to consist of a single strict ranking, all vertices of $Co_n(\mathbf{p})$ must be in the same (open) ranking region - this introduces a serious constraint on the combinatorics. Then, the geometry of $Si(3)$ imposes a further constraint on the minimum number of voter types – hence on the minimum number of voters. For instance, using the above geometric analysis it takes a simple computation to arrive at the following table where it is assumed that the voters are uniformly distributed among the voter types. Notice the small likelihood of $Sup_3(\mathbf{p})$ consisting of a single strict ranking (the last column).

# voters	values for $\lvert Sup_3(\mathbf{p})\rvert$	$Prob(\lvert Sup_3(\mathbf{p})\rvert = 1$ strict rank)
1	2	0.0000
2	1, 3	0.000
3	1, 2, 3, 5	0.0556
4	1 − 5	0.0417

Indeed, the construction in the above example even suggests that with a sufficient number of voters, we shouldn't expect a great difference between the probability of $Sup_3(\mathbf{p})$ consisting of a single strict ranking and the probability that $|Sup_3(\mathbf{p})| \geq 5$. This intuition is correct. Furthermore, as indicated in the example, these ideas can be used to determine the minimum number of voters needed to achieve the various results.

The difficulty with Theorem 4 is that the choice of the $(n-1)^2$ voter types is dependent upon the rankings. Instead, we might prefer a basis of voter types that serves for all problems.

Definition. A listing of $n(n-1)$ voter types, VT^n, is a *basis of voter types* for the design of profiles if

1. for each $s = 1, \ldots, n-1$, there exists n linearly independent vectors in

$$\{\rho_i(\overrightarrow{E}^n_s)\}_{i \in VT^n};$$

indeed, the convex hull formed by these vectors has $\mathbf{i}_n$ as an interior point, and

2. the vectors

$$\{\rho_i(\overrightarrow{E}^n_1), \ldots, \rho_i(\overrightarrow{E}^n_{n-1})\}_{i \in VT^n} \tag{4.1}$$

are linearly independent.

The vectors in Eq. 4.1 are used in the design of the desired outcomes. These vectors provide no savings with $n = 3$ as it requires all six voter types. Where savings are obtained is with the design of a profile for $n \geq 4$. To see how to design a basis with $n = 4$, for each c_j, let c_j be top- ranked for $n-1$ of the rankings. From the c_j vertex, there are $n-1$ edges; the midpoint on the edge represents $\rho_i(\overrightarrow{E}^n_2)$ which is determined by the two top-ranked candidate – the candidates identified with the two vertices defining this edge. So, choose a second ranked candidate so that each edge from a vertex is represented.

What remains is the choice of the third ranked candidate. The point $\rho_i(\overrightarrow{E}^n_3)$ is determined by the three top-ranked candidates; it is the baricentric point in the equilateral triangle defined by any three vertices. Each edge is the edge for two triangles, and corresponding to the midpoint of each edge, there are two rankings. So, for each pair, choose the third candidate for each ranking so that the corresponding $\rho_i(\overrightarrow{E}^n_3)$ points are in different triangles.

By construction, both the convex hull conditions and linear independence are satisfied.

Choice of vertices.

The above scheme to design profiles can be even further simplified by exploiting the flexibility of the geometry of $Si(n)$ to select the vertices for $Co_n(\mathbf{p})$ – an issue that plays an important role in the determination of likelihoods. To illustrate how this is done, suppose the goal is to find profiles where $Sup_n(\mathbf{p})$ has the maximum number of rankings. The first step is to determine the positioning of the orientation plane so that it intersects $(n-1)!$ strict ranking regions and $C(n-1)$ regions. Using Proposition 1, select a normal vector, say $\mathbf{N}_2$. Now, a hull needs to be selected on this plane; namely, it remains is to choose the vertices so that $\mathbf{i}_n$ is an interior point of the orientation hull and the vertices are in open ranking regions. This is done by choosing $n-1$ vectors $\mathbf{v}_i$ so that

1. each $\mathbf{v}_i$ is orthogonal to $\mathbf{N}$,
2. the vectors $\mathbf{i}_n + \mathbf{v}_i$ are in different open ranking regions of $Si(n)$ (and are the vertices of the orientation convex hull), and
3. the zero vector is an interior point of the convex hull defined by the vectors $\{\mathbf{v}_i\}$ (to ensure that $\mathbf{i}_n$ is an interior point of the orientation convex hull).

Once such a hull is selected, it is translated off of $\mathbf{i}_n$ but so that the translated vertices (and edges) remain in the same ranking regions. According to Proposition 2, we only need worry about rankings of these vertices. It is this geometry we use to find the rankings for the vertices of the hull.

Example: To indicate the ideas, reconsider the problem of finding a $\mathbf{p}$ so that $|Sup_3(\mathbf{p})| = 7$. The normal vector is $\mathbf{N}_2 = (0,1,0) - \mathbf{i}_3 = (-\frac{1}{3}, \frac{2}{3}, -\frac{1}{3})$. A vector in $Si(3)$ must be orthogonal to $(1,1,1)$, so the sum of its components must equal zero. Thus such a vector can be expressed as $\mathbf{v} = (v_1, v_2, -(v_1 + v_2))$. The condition that $\mathbf{v}$ is orthogonal to $\mathbf{N}$ requires $v_2 = 0$, so the vector must be $\mathbf{v} = (v_1, 0, -v_1)$. The second condition is satisfied if $v_1 \neq 0$, so only the third need be considered. This condition is satisfied should $\mathbf{v}_1 = (1,0,-1) = -\mathbf{v}_2$. Thus, the vertices of the translated hull must be in $c_1 \succ c_2 \succ c_3$ and in $c_3 \succ c_2 \succ c_1$. This is the above construction.

For $n = 4$ candidates, the normal vector becomes $\mathbf{N}_2 = (-\frac{1}{4}, \frac{3}{4}, -\frac{1}{4}, -\frac{1}{4})$. Again, the sum of the components of the $\mathbf{v}_i$ vectors must be zero; again, the orthogonality condition forces the second component to equal zero. Thus, the vectors can be expressed as $\mathbf{v}_i = (v_1^i, 0, v_3^i, v_4^i)$, $v_1^i + v_3^i + v_4^i = 0$. To satisfy the of strict rankings, the three components of $\mathbf{v}_i$ are chosen to have distinct non-zero values. The final requirement is satisfied if three positive scalars λ_i can be found so that $\sum \lambda_i \mathbf{v}_i = \mathbf{0}$. As a simple example, choose $v_1^1 = 3, v_3^1 = -1, v_4^2 = -2$ to define $\mathbf{v}_1 = (3,0,-2,-1)$. By using a cyclic permutation, we obtain $\mathbf{v}_2 = (-1,0,3,-2), \mathbf{v}_3 = (-2,0,-1,3)$. Thus, one (of many) choice of rankings for this problem is $r_1 = c_1 \succ c_2 \succ c_3 \succ c_4, r_2 = c_3 \succ c_2 \succ c_1 \succ c_4, r_3 = c_4 \succ c_2 \succ c_3 \succ c_1$. □

This construction plays an important role in understanding how the probabilities of the events change with n. As already noted, if $Sup_n(\mathbf{p})$ is a single ranking, then all $n-1$ vertices of $Co_n(\mathbf{p})$ must be in the same ranking region. If there are a sufficient number of voters uniformly distributed over voter types, then this restriction on the vertices of $Co_n(\mathbf{p})$ imposes a restriction on the likelihood the single outcome will occur. A similar constraint is imposed upon the vertices of $Co_3(\mathbf{p})$ to obtain $|Sup_3(\mathbf{p})| = 7$; as derived above, the vertices must be on opposite ends of the line segment. Therefore, this restriction on the rankings of the vertices imposes a similar restriction on the likelihood of the event occurring. Now notice the significant flexibility in the choice of the rankings of the vertices obtained in going from $n = 3$ to $n = 4$. In the first case, with $\mathbf{N}_2$ there are only two choices – given by the opposite ends of the line segments. In the second case with $\mathbf{N}_2$ there are more than a hundred different choices. This significant relaxation in the choice of the vertices is marked by a corresponding relaxation in the combinatorics of voters needed to satisfy these conditions. In turn, we should expect a higher likelihood of the event occurring.

As n increases in value, we then see that a single or a small number of outcomes in $Sup_n(\mathbf{p})$ corresponds to severe restrictions on the vertices of $Co_n(\mathbf{p})$; this lessens the likelihood of occurrence. On the other hand, there is considerable flexibility in choosing the vertices so that $|Sup_n(\mathbf{p})|$ attains reasonably large values; this should increase the likelihood of this event. Thus, we should not be surprised with a sufficient number of voters and large enough values of n to discover that it is quite likely for $|Sup_n(\mathbf{p})|$ to have large values. All of this is verified with a different geometric development given elsewhere.

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