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THE AFTERNOON EFFECT [1]

by

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ABSTRACT

Data from wine auctions indicates that identical products sold sequentially typically follow a decreasing pattern of prices, known as the *afternoon effect*. This is explained, for both first and second price auctions, by appealing to risk averse bidders. Earlier bids are then equal to expected later prices plus a risk premium associated with the risky future price. This logic rests on the assumption of *nondecreasing absolute risk aversion*, which is necessary for pure strategy equilibrium bidding functions to exist. This, *decreasing absolute risk aversion* implies *ex post* inefficiency with positive probability. Data from wine auctions is used to confirm the existence of the afternoon effect.

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Introduction

It is common in the theory of auctions to analyze the sale of a single object, even though many copies of the same good are often sold at actual auctions. For example, in the June 23, 1990 sale of fine wines at Christie's of Chicago, of 1355 total lots, the sale of a wine was followed immediately by the offering of an identical wine 119 times. The fact that similar goods may be sold sequentially has generally been ignored in the auction literature (some exceptions are Milgrom and Weber (1982b) and Weber (1983)) and yet results from the empirical study of sequential auctions have posed an intriguing puzzle. If two similar objects are to be sold one immediately after the other to risk neutral traders, equilibrium arguments suggest that on average they should generally sell at the same price. Otherwise, agents bidding in the high-price period would do better on average to participate only in the low-price period. Ashenfelter (1989) finds a definite pattern in the prices of objects sold at auctions sequentially. Objects sold at later periods more frequently sell at lower prices than higher prices than identical wines sold in earlier periods. These periods are frequently just minutes apart.

This result is exactly opposite to that predicted by the standard affiliated values, risk neutral model. Milgrom and Weber (1982b) show that, with independent private values, expected prices should remain constant and with affiliation the expected price should rise over time. Expected prices rise because early auctions release information about the value of the good, thereby reducing concerns about the *winner's curse* in subsequent auctions, a phenomenon that Milgrom and Weber (1982a) call the *Linkage Principle*. Thus, the pattern of prices found by Ashenfelter is inconsistent with the received theory.

This paper analyzes the independent, private values model but investigates the effects of risk aversion on the path of prices. Ashenfelter (1989) suggests that the pattern of prices is consistent with risk averse bidders, because then the expected first period price will equal the expected second period price plus a risk premium for the randomness in the second period. This intuition turns out to require an assumption. Consider a twice repeated second price auction. The price in the second period is clearly random, and the behavior of the bidders in the second price auction has the dominant strategy of bidding the value, as proved by Vickrey (1961). Now consider the first auction. From a bidder's perspective, the first period price is also a random variable, and a change in the bid changes the distribution of the first period price, and thus, in computing the best response bid, the bidder is comparing the change in the expected utility associated with changing the first period bid, which includes an effect of the distribution of the prices that bidder obtains if he wins, to the random second period price. Thus, it is not immediately clear that a bidder bids the expected second period price (conditional on the bidder winning) plus a risk premium.

In fact, Ashenfelter's intuition is correct for some utility functions and not for others. It is shown that only in the case of nondecreasing absolute risk aversion (Pratt (1964)) do pure strategy, monotonic equilibrium bidding functions exist for two-period repeated first price or second price auctions. In this case, the path of expected prices follow the pattern exhibited in the data. The expected winning price in the second period is lower than that of the first, and the difference is a risk premium.

The intuition for the declining path of prices can be seen by noting that any auction represents a gamble for a bidder. A player submitting a bid in the first period of two period auctions uses the expected utility of the second auction to assess the cost of losing in the first period. For a risk neutral bidder, the fact that the utility generated by the second period auction is a random variable is irrelevant. For a risk averse bidder, though, the randomness of utility from the final auction reduces its value and therefore increases the bid he is willing to make in the first period.

This intuition also indicates why nondecreasing absolute risk aversion (NDARA) is needed for the result. In second price auctions, for example, the first period bid of a bidder with valuation x is the expected value of the third order statistic conditional on all other valuations being lower than x plus a risk premium associated with the gamble of the second period auction. The first component is clearly increasing in x, however, the second is only increasing in x in the case of NDARA. The possibility of constructing monotonic, pure strategy bidding functions may then be frustrated by the opposing relative attitudes to risk of agents with higher valuations.

The fact that NDARA is necessary for the existence of pure strategy, monotonic bidding functions is important for two reasons. First, there is a general acceptance that at least *increasing* absolute risk aversion is an unsatisfactory characterization of attitudes to risk (see for example, Stiglitz (1991)). Second, if equilibrium bidding functions are not monotonic or are not in pure strategies, then with positive probability the sequential auction will result in an allocation which does not give the objects to those who value them the most. With positive probability, the ex post allocation is inefficient.

The structure of the paper is as follows. Section 2 describes the environment and Sections 3 and 4 characterize equilibrium bidding functions in the case of NDARA for the second-price and first-price auctions respectively. Section Five presents an example in which agents exhibit strictly decreasing absolute risk aversion (DARA) and characterizes a mixed strategy equilibrium. Section Six provides an analysis of data from wine auctions from Christie's of Chicago to confirm Ashenfelter's (1989) finding. The last section offers conclusions.

2. The Environment

There are $n \geq 3$ potential buyers for two identical items. Each buyer i has a value x_i, known only to buyer i, for one unit. Values are identically and independently distributed with cumulative distribution function F, which is assumed to have a continuous density f, and f has support $[0,x_H]$. A buyer i who purchases a single item at price p receives Von Neumann utility $u(x_i-p)$, where x_i is that buyer's value. There is no increase in utility associated with obtaining a second unit, so that all buyers have zero utility of a second unit. Note that this structure forces values to be monetary, in that the utility depends only on the difference of value and payment, but allows for risk aversion through the function u. We assume that u has a continuous nonpositive second derivative and positive first derivative, and set $u(0)=0$ without loss of generality.

We will consider two distinct games. In the first, the goods are sold sequentially by sealed-bid second price auction[4]. In the second, the goods are sold sequentially by first price sealed bid auctions. Both auctions have a zero reserve price. A useful benchmark is the expected price when the two goods are sold simultaneously. The method analogous to a second price auction for a single good is a third price auction, where the highest two bidders receive the goods at a price equal to the third highest bid. Similar to Vickrey's (1961) proof for the second price auction, bidders have a dominant strategy to report their valuations honestly. Thus a third price auction produces an expected price equal to the third highest valuation. A first price auction allows the two highest bidders to obtain the item at their bid. We include the following result, proven in Weber (1983). Denote by $X_{(i)}$, and $x_{(i)}$, the random variable, and its realization, that is the i^{th} highest of n *i.i.d.* draws from F. Thus, with two goods to sell, the realized price in the third price auction is $x_{(3)}$. When the sample size is not n, we will represent the i^{th} order statistic as $X_{(i:m)}$, where m is the sample size; thus we suppress the sample size when it is n and not otherwise.

Proposition 1 (Revenue Equivalence; Weber(1983)): Consider the sale of k objects to n > k bidders with i.i.d. private valuations. Then the $k+1^{st}$ price auction produces a price $x_{(k+1)}$. The bid of a risk neutral buyer with valuation x in a first price auction is $E\{X_{(k:n-1)} | X_{(k:n-1)} \leq x\}$. Thus the expected revenue is the same in the two auctions under risk neutrality.

Remark 1: This result generalizes the usual bidding result which shows that in a sealed bid auction the symmetric equilibrium bidding function is the expectation of the highest of the other bidders' values, conditional on those all being less than the given bidder's value. In effect, Proposition 1 shows that each bidder bids his estimate of $X_{(k+1)}$ given that his value is at least $X_{(k)}$.

3. Twice Repeated Second Price Auctions

The second price auction is simpler to analyze than the first price auction because the bidders continue to have a dominant strategy, to bid their true valuation, in the second auction. A pure strategy

[4] See McAfee and McMillan, 1987, for a description of auction games.

symmetric equilibrium in this environment is a bidding function B_1, so that a bidder with value x_i bids $B_1(x_i)$ in the first of the two auctions, and then bids x_i in the second.

Suppose B_1 is increasing. Fix a buyer and let $Y_1 \geq Y_2 \geq ... \geq Y_{n-1}$ be the order statistics of the other buyers' values. The payoff to a buyer who bids $b = B_1(r)$ when his value is x is:

(1) $v(r,x) = \mathrm{E}\{u(x-B_1(Y_1)) | Y_1 \leq r\}\mathrm{P}(Y_1 \leq r) + \mathrm{E}\{u(x-Y_2) | Y_1 \geq r \,\&\, Y_2 \leq x\}\mathrm{P}(Y_1 \geq r \,\&\, Y_2 \leq x)$.

The two terms in equation (1) represent the events of winning the first and second auction, respectively. The buyer wins the first auction if his bid $B_1(r)=b$ exceeds the bid $B_1(Y_1)$ of the highest value competitor. The buyer wins the second auction provided he loses the first auction, in which case the Y_1 value buyer wins the first, and has the highest value in the second auction. The next proposition shows that following the pure strategy bidding function B_1 comprises an equilibrium if u displays nondecreasing absolute risk aversion.

Proposition 2 (Existence): There exists a symmetric increasing pure strategy equilibrium bidding function B_1 for every distribution F if and only if u displays nondecreasing absolute risk aversion. In this case, B_1 satisfies

(2) $$u(x-B_1(x)) = \int_0^x u(x-y)\frac{(n-2)F(y)^{n-3}f(y)}{F(x)^{n-2}}\,dy.$$

All proofs are provided in the Appendix.

Remark 2: There appears to be a consensus that individuals display decreasing absolute risk aversion (this is suggested by Pratt (1964)), i.e. that risk premia decline, for a fixed gamble, as wealth increases. In this event, Proposition 2 indicates that either nonmonotonic strategies are used, or bidders must randomize. However, there is a pure strategy equilibrium for the case of constant absolute risk aversion.

Remark 3: An intuition for the necessity of NDARA is as follows. In the second auction, the price will be the third highest value. Thus, a bidder with value x expects to pay $\mathrm{E}\{X_{(3)} | X_{(3)} \leq x\}$. This is random, so the bidder also associates a risk premium $R(x)$ to this amount; i.e. the certainty equivalent of competing in the second auction is $\mathrm{E}\{X_{(3)} | X_{(3)} \leq x\}-R(x)$, which represents his expected profits minus his risk premium. Now consider a slight decrease in the bid, from $B_1(x) = \mathrm{E}\{X_{(3)} | X_{(3)} \le x\} + R(x)$ to $B_1(r)$. The only event in which this has an effect on the bidder's utility is when $x > Y_1 > r$, in which case he loses the first auction and wins the second, where bidding $B_1(x)$ would have him win the first. Therefore, necessarily,

$$(3) \qquad \mathrm{E}\{u(x - B_1(Y_1)) | x > Y_1 > r\} \ge \mathrm{E}\{u(x - Y_2) | x > Y_1 > r\}.$$

Since these are equal as $r \to x$, this inequality says that the risk premium of the left hand side increases with a slight increase in x, i.e. increasing absolute risk aversion.

Remark 3 also provides an intuition for the afternoon effect, because a bidder's bid equals the expected price in the second auction, plus a risk premium.

Proposition 3 (Afternoon Effect): $EB_1(X_{(2)}) \ge X_{(3)}$, that is, the expected price obtained in the first auction exceeds the price obtained in the second auction. Moreover, if u is strictly concave, this inequality is strict.

4. Twice Repeated First Price Auctions

Repeated first price auctions are significantly more difficult than the second price case because bidders lack a dominant strategy in the last period. As a result, it may matter whether the price obtained in the first auction is announced to the remaining buyers or not. In particular, a buyer who bids less than his equilibrium bid in the first auction may learn that he has the highest valuation if the winning bid is announced. On the other hand, if the winning bid is not announced, then losing bidders know only that the winning bid exceeded their bid, which produces different information for the different bidders. We will assume that the winning bid in the first auction is announced prior to the second. This is in accord with government procurement statutes[5] and with practice in some auctions.

In the second round of bidding, bidders will know their own value and the bid of the first period winner. We will use $b_1(x)$ to represent the bid of a buyer with value x in the first auction, and $b_2(x, Y_1)$ to represent the second period bid when the first period bidder bid $b_1(Y_1)$.

[5] See McAfee and McMillan, 1988.

Proposition 4: Increasing equilibrium bidding functions b_1 and b_2 exist if u displays nondecreasing absolute risk aversion, and do not exist if u displays decreasing absolute risk aversion. If u displays nondecreasing absolute risk aversion, then $b_2(x,Y_1)$ does not depend on Y_1, and we suppress Y_1. b_2 is given by $b_2(0) = 0$ and

$$(4) \qquad b_2'(x) = \frac{(n-2)f(x)}{F(x)} \frac{u(x-b_2(x))}{u'(x-b_2(x))}.$$

b_1 is given by $b_1(0) = 0$ and

$$(5) \qquad b_1'(x) = \frac{(n-1)f(x)}{F(x)} \frac{u(x-b_1(x)) - u(x-b_2(x))}{u'(x-b_1(x))}.$$

Remark 4: The characterization for existence is not quite as tight as in the second price case. It appears possible for buyers to have increasing absolute risk aversion with low values, and decreasing with high values, and for equilibrium bidding functions to exist. However, we see that everywhere decreasing risk aversion is inconsistent with the existence of pure strategy symmetric equilibria. As before, the afternoon effect exists.

Proposition 5 (Afternoon Effect): If u displays nondecreasing absolute risk aversion, then $Eb_1(X_{(1)}) \geq Eb_2(X_{(2)})$.

There is an interesting relationship between the bid B_1 in the first of two second price auctions and the bid b_2 second of two first price auctions, which is developed in the following result.

Proposition 6: If u displays constant absolute risk aversion, then $B_1(x) = b_2(x)$, for all x. If u displays nondecreasing absolute risk aversion, then $B_1(x) \leq b_2(x)$, for all x.

The final result of this section ranks the auction types with respect to the seller's revenue. It presumes nondecreasing absolute risk aversion, so that the bidding functions represent equilibria.

Proposition 7: $Eb_1(X_{(1)}) \geq Eb_2(X_{(2)}) \geq EB_1(X_{(2)}) \geq EX_{(3)}$. Thus, in both periods, the sequential first price auction produces a higher expected price than the sequential second price auction, which in turn produces higher prices than the simultaneous third price auction.

5. Mixed Strategy Equilibria – An Example

If bidders do not exhibit increasing absolute risk aversion, then pure strategy equilibria with monotonic bidding functions may not exist. This section characterizes the mixed strategy equilibrium of an auction with bidders displaying decreasing absolute risk aversion.

Consider a repeated second price auction. As before, equilibrium behavior in the last period is simply to submit a bid equal to the bidder's valuation. Suppose, though, that bidders follow a mixed strategy for their bids in the first period. Let the strategy of a bidder of type z be such that $\phi(b{:}z)$ is the probability that type z submits a bid of b or lower. The joint distribution of x and b then is

$$(6)\qquad G(b,x) = \int_0^x \phi(b{:}z)f(z)\,dz.$$

Lemma 8: If all bidders follow a symmetric mixed strategy given by $\phi(b{:}z)$, then the expected utility to a bidder of type x from a bid b is

$$(7)\qquad V(b,x) = (n-1)\int_0^b u(x-\beta)G(\beta,x_H)^{n-2}G_b(\beta,x_H)\,d\beta + (n-1)\int_b^\infty G_b(\beta,x_H)\int_0^x (n-2)u(x-y)G_x(\beta,y)G(\beta,y)^{n-3}\,dy\,d\beta\,.$$

The proof is omitted, but to understand the expression, note that $G(\beta,x_H)^{n-1}$ is the unconditional probability that only bids less than β are made in the first round by the n-1 other bidders and $[G(\beta,y)/G(\beta,x_H)]^{n-2}$ is the probability that all n-2 remaining types have valuation less than y given that a bid β won in the first round. Thus the expression is just the sums of expected value of the first round and the second round.

For it to be a best response for a bidder of type x to submit a bid b, then the first order conditions from (7) must be satisfied. We must have

$$(8)\qquad u(x-b)G(b,x_H)^{n-2} = \int_0^x u(x-y)\,(n-2)\,G(b,y)^{n-3}\,G_x(b,y)\,dy\,.$$

Let $\beta(x) = \inf\{b\colon \phi(b{:}x) = 1\}$ denote the supremum of the support of type x's mixed strategy.

Lemma 9: Suppose $\beta(x)$ is increasing and $\phi(b{:}x)$ is a nonatomic distribution, increasing for all $b \in [0,\beta(x)]$, then $\beta(x)$ must satisfy

$$(9) \qquad \frac{u(x-\beta(x))}{u'(x-\beta(x))} = \frac{\int_o^x u(x-y)F^{n-3}(y)f(y)\,dy}{\int_o^x u'(x-y)F^{n-3}(y)f(y)\,dy}.$$

Note that (9) provides a simple characterization of the upper end of the support of the mixed strategy bidding function. In particular, if bidders use only pure strategies, (9) yields equation (2) in Section 3.

For the remainder of the section, consider the special case, $f(y) = 1$, $n = 3$ and $u(w) = w^{\alpha}$, that is, bidders utilities exhibit constant relative risk aversion but decreasing absolute risk aversion. The unique $\beta(\cdot)$ which satisfies (9) is $\beta(x) = \frac{x}{1+\alpha}$. However, it can be shown using Lemma A2 in the appendix that, if all bidders were to follow this pure strategy bidding function, any one bidder of type x does strictly better by lowering his bid. Nevertheless, there exists a symmetric, mixed strategy equilibrium bidding function with upper end of the support, $\beta(x)$.

Proposition 10: Let

$$\phi(\beta(z){:}x) = \frac{2}{\pi}\left[\arcsin\left(\left(\frac{z}{x}\right)^{1/2}\right) - \frac{3(zx-z^2)^{1/2}}{3x-2z}\right]$$

for $z \in [0,x]$ and one for $z > x$. For the twice repeated second price auction game with $n = 3$, $u(w) = w^{\alpha}$, $\alpha = 1/2$, $f(y) = 1$, a strategy profile in which a bidder of type x submits a bid less than or equal to $\beta(z) = 2z/3$ with probability $\phi(\beta(z){:}x)$ forms a Nash equilibrium.

6. The Empirical Significance of the Afternoon Effect

We obtained data from Christie's wine auctions in Chicago in 1987. This data represents four distinct auctions[6]. There were 411 instances where the same wine was sold more than once in the same auction. In 177 instances, we obtain three prices for the same wine in the same auction. We treat the same wine sold at a different auction as a distinct product. It is important to understand that the products we are treating as homogeneous are indeed homogeneous; they represent cases of the same vintage of the same wine sold on the same day in the same city.

[6] The auctions occurred on February 7, April 11, October 27 and December 5, 1987.

Table 1 provides summary statistics for the last two sales of each wine. Note that the afternoon effect is present; a wine sells for an average of $724.66 in the first auction, and $714.35 in the second, a difference of $10.31, or approximately 1.4%. This is a reasonable magnitude to be attributed to risk aversion.

Table I: Christie's Chicago 1987 Repeated Wine Sales Statistics

Auction:	First		Second	
Means:	$724.66		$714.35	
	Mean		Std. dev. of mean	
2nd/1st	0.9922		0.000276	
Direction		Rose	Fell	No change
1st vs. 2nd:		15%	31%	54%
1st,2nd,3rd:*		12%	33%	55%**

* Total of 177 sales.
** includes up then down and down then up patterns.

To formally test that the afternoon effect is present, we follow Ashenfelter (1989) and examine the ratio of prices, and the proportion of times prices rose, fell and remained constant. As shown in Table I, the ratio of the second price to the first is 0.9922, which is different than 1 with a t-statistic of 28.3. These numbers are similar to those found by Ashenfelter. Now look at the instances where the price rose, fell, and stayed the same. In 127 instances, or 31%, the price fell. In 62 instances, or 15%, the price rose, and in the remaining instances, it stayed the same. These numbers are also similar to those found by Ashenfelter.[7] Consider the trinomial variable which is 1 with probability p and -1 with probability p and 0 with probability $1\text{-}2p$, where 1, -1 and 0 refer to prices rising, falling and remaining the same. The probability of observing data as extreme as that observed (i.e. at least 127 instances of

[7] Data from auctions in 1990 exhibited similar behavior although there is some indication that the absolute value of the fall in price is no longer so large.

falling prices and no more than 62 instances of rising prices) is

$$\sum_{k=0}^{62} \sum_{j=127}^{411-k} \frac{411!}{k!\,j!\,(411-k-j)!} p^{k+j} (1-2p)^{411-k-j} \approx 0.00003.$$

Thus, the probability of observing this kind of split between the number of falling prices and the number of rising prices is much less than one percent.[8] Thus we overwhelmingly reject the hypothesis that prices are equally likely to rise as to fall, in favor of the existence of an afternoon effect, that prices are more likely to fall than to rise.

A similar outcome arises when we look at the data for three sales. The first auction had a higher price in 33%, a lower price in 12%, and the same price in 55% (this case includes those situations where the two price changes were in opposite directions) of the 177 cases where three sales of each wine occurred. Because of the reduced sample size, the effect is not as significant.

Figure 1 provides a scatterplot of the second price as a function of the first price. The curves through the data represents the forty-five degree line and the result of a regression of the second price on a quadratic function of the first price.[9]

7. Concluding Remarks

The necessity of nondecreasing absolute risk aversion for the existence of pure strategy monotonic bidding functions suggests that in sequential auctions, at least, it is not always the case that objects end up in the hands of those who value them the most. In a way, this result may not be so surprising since with risk aversion, an auction really offers two types of 'goods' -- the object to be traded and risk. An

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[8] This estimate of the probability was computed using a normal approximation. Note that the mean of the trinomial is 0 and the variance is $2p$. The sample mean is 65/411 and the variance of the sample mean is $2p/411$. Since $p \leq 127/411$, we have $\Pr(X>65/411) \approx \text{Prob}(Z>(65/411)/(2p/411)^{1/2}) \leq \text{Prob}(Z>4.08) \approx 0.0000292$.

[9] This regression produces an R^2 of .989 and the following estimates.

Term	Coefficient	t-statistic
Constant	-30.0	5.1
Linear	1.09	101
Squared	-0.00006	15

Thus the data indicates that the size of the afternoon effect is increasing and concave in the range of the data.

object cannot be traded without some imposition of risk on the bidders. The ex ante welfare consequences of the allocation generated by a sequential auction have to take into account the relative allocation of risk among agents with differing attitudes to risk. The fact that a sequential auction may lead to ownership of a good by someone who values the good less than another losing bidder is not in itself evidence of *ex ante* inefficiency. Nevertheless, it is the case that in single period auctions, goods are allocated to those who value them the most so there is never an incentive for a buyer to attempt to resell the object. This is not true in sequential auctions with DARA and it is an open question what might occur in sequential auctions if retrading were allowed.

One would like to have a characterization of equilibrium bidding strategies for general n-period auctions. While the characterization of bidding functions via the first order conditions is easily extended to this case, the verification that these functions satisfy sufficient conditions for a maximum becomes more complicated and we have not been able to come up with a clear generalization. Finally, it would also be desirable to be able to characterize equilibria for general utility functions. The difficulty of computing the equilibrium mixed strategies just in the simple example of Section 5 suggests that such an exercise would be a daunting one.

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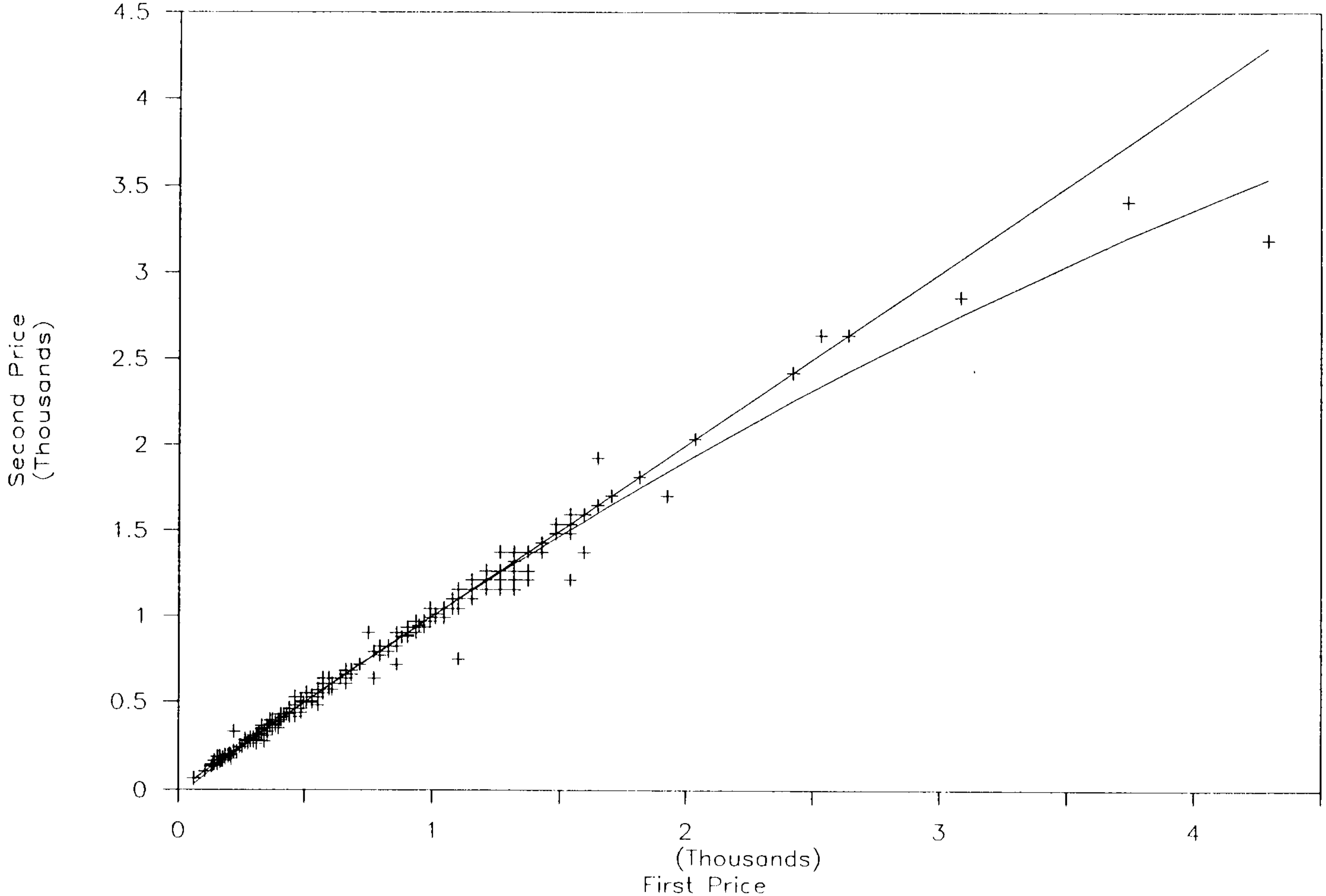
Figure One

Second Price on First Price



Appendix

Two mathematical lemmas are used several times below. The first was proved by Guesnerie and Laffont (1984) in the generality used here; however, special cases were used by several authors, notably Myerson (1981) prior to this. Subscripts are used to denote partial derivatives.

*Lemma A1: Suppose v:[a,b] → **R** is twice continuously differentiable. Then*

(A1) $(\forall r)(\forall x)\ v(r,x) \leq v(x,x)$ *implies*

(A2) $(\forall x)\ v_1(x,x) = 0$ *and*

(A3) $(\forall x)\ v_{12}(x,x) \geq 0$. *Moreover,* (A2) *and*

(A4) $(\forall r)(\forall x)\ v_{12}(r,x) \geq 0$ *imply* (A1).

A version of Lemma A2 appears in McAfee (1991).

*Lemma A2: Suppose u: **R** → **R** is thrice continuously differentiable, increasing and concave. Then* $u(c) = \text{E}u(X)$ *implies* $u'(c) \geq (>, \leq, <)\ \text{E}u'(X)$ *for all real valued random variables, X if and only if u satisfies nondecreasing (increasing, nonincreasing, decreasing) absolute risk aversion.*

Proof: Let $Y = u(X)$, and $c = u^{-1}(\text{E}[Y])$. Then

$$u'(c) \geq \text{E}[u'(X)] \text{ if and only if } u'(u^{-1}(\text{E}[Y])) \geq \text{E}u'(u^{-1}(Y)).$$

This holds for all random variables X if and only if $u'(u^{-1}(\cdot))$ is concave, or if $\frac{u''(u^{-1}(\cdot))}{u'(u^{-1}(\cdot))}$ is nonincreasing. Since u^{-1} is increasing, this is equivalent to $-\frac{u''(\cdot)}{u'(\cdot)}$ is nondecreasing, or NDARA. The other cases are similar. ■

Remark A1: Define the risk premium $R(W)$ for a mean zero gamble X by $\text{E}u(W+X) = u(W-R(W))$.

Differentiation yields

$$\text{(A5)} \qquad R'(W) = -\frac{Eu'(W+X) - u'(W-R(W))}{u'(W-R(W))}.$$

Lemma A2 is equivalent to stating that $R'(W)$ is positive (negative) if and only if u displays increasing (decreasing) absolute risk aversion.

Proof of Proposition 2:

Necessary conditions: Suppose that $B_1(\cdot)$ is a monotonic symmetric bidding function for period one. By (1), if bidder one of type x chooses to bid as type r, he receives, if $r \geq x$,

$$\text{(A6)} \qquad V(r;x) = (n-1)\int_0^r u(x - B_1(y))F^{n-2}(y)f(y)\,dy$$

$$+ (n-1)(1 - F(r))\int_0^x u(x-y)(n-2)F^{n-3}(y)f(y)\,dy,$$

and if $r < x$,

(A7) $$V(r;x) = \int_0^r u(x-B_1(y))\,(n-1)F^{n-2}(y)f(y)\,dy$$
$$+ (n-1)(1-F(x))\int_0^x u(x-y)\,(n-2)F^{n-3}f(y)\,dy$$
$$+ (n-1)\int_r^x (n-2)\int_0^{y_1} u(x-y)F^{n-3}(y)f(y)\,dy\,f(y_1)\,dy_1 .$$

Differentiating either (A6) or (A7) and setting $V_1(x;x) = 0$ yields (2).

The necessary second order condition for $B_1(x)$ to be an optimal response for bidder type x is, from Lemma A1, $V_{12}(x,x) \geq 0$. Both (A6) and (A7) yield:

$$u(x-B_1(x)) = \int_0^x u(x-y)\frac{(n-2)F^{n-3}(y)f(y)}{F^{n-2}(x)}\,dy \text{ implies}$$
$$u'(x-B_1(x)) \geq \int_0^x u'(x-y)\frac{(n-2)F^{n-3}(y)f(y)}{F^{n-2}(x)}\,dy.$$

Lemma A2 and the definition of $B_1(\cdot)$ in (2) then yields that NDARA is necessary for the first order condition definition of $B_1(\cdot)$ to be an optimal response for all random variables.

Sufficient Conditions: From (A6), if $r \geq x$,

(A8) $$V_{12}(r;x) = f(r)(n-1)\left(u'(x-B_1(r))F^{n-2}(r) - (n-2)\int_0^x u'(x-y)F^{n-3}(y)f(y)\,dy\right)$$
$$\geq f(r)(n-1)\left(u'(x-B_1(x))F^{n-2}(r) - (n-2)\int_0^x u'(x-y)F^{n-3}(y)f(y)\,dy\right) \geq 0.$$

From (A7), for $r \leq x$, we need to show $V_1(r,x) \geq 0$, or,

(A9) $$u(x-B_1(r)) \geq \int_0^r u(x-y)\,\frac{(n-2)F^{n-3}(y)f(y)}{F^{n-2}(r)}\,dy.$$

Fix r and define $\beta(x)$ by

(A10) $$u(x-\beta(x)) = \int_0^r u(x-y)\,\frac{(n-2)F^{n-3}(y)f(y)}{F^{n-2}(r)}\,dy.$$

Note that $\beta(r) = B_1(r)$ and that $\beta(x)$ is the expected value of y_2 plus the risk premium associated with "income" x (the support of y_2 ranges from 0 to r). NDARA then gives us $\beta(r) \leq \beta(x)$ so

(A11) $$u(x-B_1(r)) = u(x-\beta(r)) \geq u(x-\beta(x)) = \int_0^r u(x-y)\,\frac{(n-2)F^{n-3}(y)f(y)}{F^{n-2}(r)}\,dy,$$

which yields the result. ■

Proof of Proposition 3: This follows from (2) since

$$x - B_1(x) = \mathrm{E}\{x - X_{(1:n-2)} \mid X_{(1:n-2)} \leq x\} \text{ - Risk premium,}$$

or,

$$B_1(x) \geq \mathrm{E}\{X_{(1:n-2)} : X_{(1:n-2)} \leq x\}, \text{ and thus,}$$

$$\mathrm{E}B_1(X_2) \geq \mathrm{E}\{X_{(1:n-2)} : X_{(1:n-2)} \leq X_{(2)}\} = \mathrm{E}\{X_{(3)}\}. \qquad \blacksquare$$

Proof of Proposition 4:

Assume that there exist monotonic bidding functions, $b_1(\cdot)$ and $b_2(\cdot,\cdot)$. $b_2(\cdot, y_1)$ defines a symmetric monotonic and differentiable equilibrium bidding function in the second auction when all bidders know the value of Y_1.

I. Period Two Strategies: Suppose that bidder 1 bids $b = b_2(r_2, y_1)$ in period two. His final period expected return is

$$\text{(A12)} \quad V^2(r_2;x,y_1) = \begin{cases} u(x-b_2(r_2,y_1))\left(\dfrac{F(r_2)}{F(y_1)}\right)^{n-2} & \text{if } r_2 \leq y_1 \\ u(x-b_2(r_2,y_1)) & \text{if } r_2 > y_1 \end{cases}.$$

The first order condition is $0 = \dfrac{\partial V^2(x;x,y_1)}{\partial r_2}$, which yields, for $r_2 < y_1$,

$$\text{(A13)} \qquad 0 = -u'(x-b_2(r_2,y_1))\frac{\partial b_2(r-2,y_1)}{\partial r_2}F^{n-2}(r_2) + u(x-b_2(r_2,y_1))(n-2)F^{n-3}(r_2)f(r_2)\Big|_{r_2=x}.$$

This yields

$$\text{(A14)} \quad \frac{\partial b_2(x,y_1)}{\partial r_2} = \begin{cases} \dfrac{(n-2)f(x)}{F(x)}\dfrac{u(x-b_2(x,y_1))}{u'(x-b_2(x,y_1))} & \text{if } x \leq y_1 \\ 0 & \text{if } x > y_1 \end{cases}.$$

with the boundary condition, $b_2(0,y_1) = 0$. $b_2(\cdot, y_1)$ is increasing and is independent of y_1 for $x \leq y_1$, $b_2(x,y_1) = b_2(x)$. If $x > y_1$, (A14) implies that $b_2(x,y_1) = b_2(y_1,y_1)$. Note that

$$\text{(A15)} \quad \frac{\partial^2 V^2(r;x,y_1)}{\partial r \partial x} = -u''(x-b_2(r))b_2'(r)\left(\frac{F(r)}{F(y_1)}\right)^{n-2} + u'(x-b_2(r))\frac{(n-2)f(r)F(r)^{n-3}}{F(y_1)^{n-2}} \geq 0,$$

so the necessary conditions for an equilibrium strategy defined in (A14) are sufficient as well, by Lemma A1.

II. Period One Strategies:

Necessary Conditions: Now fix the equilibrium bidding function in the second auction $b_2(\cdot)$, defined in (A14), and fix a candidate bidding function in the first period $b_1(\cdot)$.

Subgame perfection requires that whatever bidder 1's behavior in the first auction, if $b_2(\cdot)$ is the equilibrium bidding function in the second auction, he will bid $b_2(x)$. Recall that b_2 is independent of y_1 whenever $x \leq y_1$, but if $x > y_1$, then the agent bids $b_2(y_1)$. Therefore, if $r_1 \geq x$, the agent's expected utility is:

$$\text{(A16)} \quad V^1(r_1,x) = u(x-b_1(r_1))F^{n-1}(r_1) + (n-1)(1-F(r_1))u(x-b_2(x))F^{n-2}(x).$$

And if $r_1 < x$,

$$\text{(A17)} \quad V^1(r_1,x) = u(x-b_1(r_1))F^{n-1}(r_1) + u(x-b_2(x))(n-1)(1-F(x))F^{n-2}(x)$$
$$+ \int_{r_1}^{x} u(x-b_2(y_1))(n-1)F^{n-2}(y_1)f(y_1)\,dy_1.$$

The first term in both (A16) and (A17) represents the event of winning the first auction with a bid $b_1(r_1)$. The second term in (A16) represents the event of losing the first auction and winning the second, with a bid of $b_2(x)$, since $y_1 \geq r_1 \geq x$. The second term in (A17) represents the event of losing the first auction and winning the second because exactly one bidder had a value greater than x, while the third term represents the case of losing the first auction and winning the second because the highest value of another bidder fell in $[r_1,x]$, so that $y_1 < x$, and the bidder bids $b_2(y_1)$ in this instance. The second term is independent of r_1. The first order conditions are slightly different depending on $r_1 > x$ or $r_1 < x$ but they are continuous at $r_1 = x$. If $r_1 \geq x$,

$$\text{(A18)} \quad \frac{\partial V^1(r_1,x)}{\partial r_1} = -u'(x-b_1(r_1))b_1'(r_1)F^{n-1}(r_1) + u(x-b_1(r_1))(n-1)F^{n-2}(r_1)f(r_1)$$
$$- (n-1)u(x-b_2(x))F^{n-2}(x)f(r_1).$$

and if $r_1 < x$,

$$\text{(A19)} \quad \frac{\partial V^1(r_1,x)}{\partial r_1} = -u'(x-b_1(r_1))b_1'(r_1)F^{n-1}(r_1) + u(x-b_1(r_1))(n-1)F^{n-2}(r_1)f(r_1)$$
$$- (n-1)u(x-b_2(r_1))F^{n-2}(r_1)f(r_1).$$

As r_1 approaches x from either above or below, (A18) and (A19) approach the same value, so setting either (A18) or (A19) equal to zero at $r_1 = x$ yields the same first order conditions defining a candidate solution of $b_1(x)$:

$$\text{(A20)} \quad b_1'(x) = \frac{(n-1)f(x)}{F(x)} \frac{u(x-b_1(x)) - u(x-b_2(x))}{u'(x-b_1(x))}.$$

Sufficient Conditions: From (A19) and (A20), if $r_1 < x$,

$$\text{(A21)}\quad \frac{\partial V^1(r_1,x)}{\partial r_1} = (n-1)F^{n-2}(r_1)f(r_1)u'(x-b_1(r_1))\left[\frac{u(x-b_1(r_1)) - u(x-b_2(r_1))}{u'(x-b_1(r_1))} - \frac{u(r_1-b_1(r_1)) - u(r_1-b_2(r_1))}{u'(r_1-b_1(r_1))}\right].$$

(A21) can be signed with the use of the following lemma.

Lemma A3: *u exhibits NDARA implies that*

$$\Pi(\beta) = \frac{\partial}{\partial x}\frac{u(x-\alpha) - u(x-\beta)}{u'(x-\alpha)} \geq 0.$$

Proof: $\Pi(\beta) = \dfrac{u'(x-\alpha) - u'(x-\beta)}{u'(x-\alpha)} - \dfrac{(u(x-\alpha) - u(x-\beta))u''(x-\alpha)}{u'(x-\alpha)^2}.$

Therefore

$$\Pi'(\beta) = \frac{u'(x-\beta)}{u'(x-\alpha)}\left(\frac{u''(x-\beta)}{u'(x-\beta)} - \frac{u''(x-\alpha)}{u'(x-\alpha)}\right).$$

NDARA then implies that $\Pi'(\beta)$ is greater than zero if and only if β is greater than α, so $\Pi(\cdot)$ is minimized at $\beta = \alpha$ and $\Pi(\alpha) = 0$. ■

Lemma A3 yields the result that NDARA implies that the bracketed term in (A21) is greater than zero. Thus a bidder of type x can always do better than a bid $b_1(r_1)$, $r_1 < x$, by increasing his bid to $b_1(x)$.

From (A14), (A18) and (A20), if $r_1 \geq x$, we have

$$\text{(A22)}\quad \frac{\partial^2 V^1(r_1,x)}{\partial x \partial r_1} = (n-1)F^{n-2}(r_1)f(r_1)\left[u'(x-b_1(r_1)) - u'(x-b_2(x))\left(\frac{F(x)}{F(r_1)}\right)^{n-2}\right.$$

$$\left. - [u(r_1-b_1(r_1)) - u(r_1-b_2(r_1))]\frac{u''(x-b_1(r_1))}{u'(r_1-b_1(r_1))}\right]$$

$$\text{(A23)}\quad \geq (n-1)F^{n-2}(r_1)f(r_1)\left[u'(x-b_1(r_1)) - u'(x-b_2(r_1)) - [u(x-b_1(r_1)) - u(x-b_2(r_1))]\frac{u''(x-b_1(r_1))}{u'(x-b_1(r_1))}\right].$$

The inequality comes from applying Lemma A3 to the second line of (A22), noting that $-u'' \geq 0$, from the fact that $F(r_1) \geq F(x)$ and from the fact that $u'(x-b_2(x)) < u'(x-b_2(r_1))$ because of concavity.

Lemma A4: *Let* $\rho(\alpha) = -\dfrac{u''(\alpha)}{u'(\alpha)}[u(\alpha) - u(\beta)] + u'(\alpha) - u'(\beta).$

Then $u(\cdot)$ exhibits NDARA implies that $\rho(\alpha) \geq 0$ for $\alpha \geq \beta$.

Proof: $\rho(\beta) = 0$ and $\rho'(\alpha) = -[u(\alpha) - u(\beta)]\dfrac{\partial}{\partial\alpha}\dfrac{u''(\alpha)}{u(\alpha)} \geq 0.$ ■

Lemma A4 along with the fact that $b_1(r_1) \leq b_2(r_1)$, from (A20), implies that (A23) ≥ 0 for $r_1 \geq x$. (A21) ≥ 0 for $r_1 \leq x$ and (A23) ≥ 0 for $r_1 \geq x$ then implies that NDARA is sufficient for a bid $b_1(x)$ to be a best response in period one for a bidder of type x. This establishes sufficiency of NDARA. To see that DARA is inconsistent with the existence of pure strategy bidding functions, note that the inequalities in lemmas A3 and A4 are reversed with DARA. Therefore, the bracketed term in (A21) is less than zero, if DARA holds, which violates the necessary condition that, locally for $r_1 < x$, $\partial V^1/\partial r_1 \geq 0$. ■

Proof of Proposition 5:

Fix a second period equilibrium bidding function, $b_2(\cdot)$ and define a function, $b_1^*(\cdot)$ with $b_1^*(0) = 0$, and

$$\text{(A24)} \quad F^{n-1}(x)u(x-b_1^*(\cdot)) = \int_0^x u(x-b_2(y))(n-1)F^{n-2}(y)f(y)\,dy.$$

Differentiating (A24) with respect to x and rearranging terms gives

$$\text{(A25)} \quad b_1^{*\prime}(x) = \frac{(n-1)f(x)}{F(x)}\frac{u(x-b_1^*(x)) - u(x-b_2(x))}{u'(x-b_1^*(x))} + \left(1 - \int_0^x \frac{u'(x-b_2(y))}{u'(x-b_1^*(x))}\frac{(n-1)F^{n-1}(y)f(y)}{F^{n-1}(x)}dy\right).$$

Consider the term in large braces. By (A24), $b_1^*(\cdot)$ is defined as the certainty equivalent for $u(\cdot)$ of the gamble defined by the right hand side of (A24). Thus, NDARA implies that the term in large braces is negative (Lemma A2). Now consider the definition of $b_1'(\cdot)$ from Proposition 2:

$$\text{(A26)} \quad b_1'(x) = \frac{(n-1)f(x)}{F(x)}\frac{u(x-b_1(x)) - u(x-b_2(x))}{u(x-b_1(x))}.$$

Since $b_1^*(0) = b_1(0)$ and the differential equation defining $b_1^*(\cdot)$ via (A25) implies that b_1^* has a lower slope than $b_1(\cdot)$, then NDARA implies

$$\text{(A27)} \quad b_1(x) \geq b_1^*(x) \text{ for all } x.$$

Therefore, since $b_1^*(x) = E[b_2(X_{(1:n-1)}):X_{(1:n-1)} \leq x]$ + risk premium,

$$E[b_1(X_{(1)})] \geq E[b_1^*(X_{(1)})] \geq E[b_2(X_{(2)})].$$ ■

Proof of Proposition 6:

Differentiate (2) and solve for $B_1'(x)$ to obtain

$$B_1'(x) = \frac{(n-2)f(x)}{F(x)}\frac{u(x-B_1(x))}{u'(x-B_1(X))} + \left(1 - \int_0^x \frac{u'(x-y)}{u'(x-B_1(x))}\frac{(n-2)F(y)^{n-3}f(y)}{F(x)^{n-2}}dy\right).$$

The term in large braces is zero under constant absolute risk aversion, and negative under IARA, by Lemma A2. Comparison with (A14) completes the proposition. ■

Proof of Proposition 7:

The first inequality is Proposition 5. The second follows from Proposition 6, and the third from Proposition 3. ■

Proof of Lemma 9:

If $\beta(x)$ is increasing, then $G_x(b,y) = f(y)$ for y such that $\beta(y) \leq b$ and $G_x(b,y) = \phi(b,y)f(y)$ otherwise. From (8) we have, for all $z < x$, $b = \beta(z)$, by replacing x with z in (8) and eliminating $G(b,x_H)$,

$$\text{(A28)} \quad \frac{u(x-b)}{u(z-b)}\int_0^z u(z-y)F^{n-3}(y)f(y)\,dy = \int_0^x u(x-y)G(b,y)^{n-3}G_x(b,y)\,dy.$$

Differentiating (A28) with respect to x and letting x go to z then yields (9). ■

Proof of Proposition 10:

Note that $\phi(\cdot:x)$ is increasing and that $\phi(0:x) = 0$, $\phi(x:x) = 1$ so $\phi(\cdot:x)$ is a probability distribution function. From (A28) for $x > z$, we have

$$\text{(A29)} \quad \int_z^x u(x-y)\phi(\beta(z):y)\,dy = \frac{u(x-\beta(z))}{u(z-\beta(z))}\int_0^z u(z-y)\,dy - \int_0^z u(x-y)\,dy = T(x:z).$$

Solving for the middle term in (A29), T is given by:

$$T(x:z) = \frac{((1+\alpha)x-z)^\alpha}{\alpha^\alpha}\frac{z}{\alpha+1} - \frac{x^{\alpha+1} - (x-z)^{\alpha+1}}{\alpha+1}.$$

Note that $\frac{\partial T(z:z)}{\partial x} = 0$. Fix z, define $T(x) = T(x:z)$ and $S(y) = \phi(\beta(z):y)$. Multiply equation (A29) by $(a-x)^{1-\alpha}$, integrate to $a \geq z$, using the change of variables $\xi = \frac{x-y}{a-y}$ after rearranging the integrals:

$$\text{(A30)} \quad \int_z^a (a-x)^{1-\alpha}\,T(x)\,dx = \int_z^a (a-x)^{1-\alpha}\int_z^x (x-y)^\alpha S(y)\,dy\,dx = B(\alpha)\int_z^a S(y)(a-y)^2\,dy,$$

where $B(\alpha) = \int_0^1 \xi^\alpha(1-\xi)^{1-\alpha}d\xi$.

Since $\frac{\partial^3}{(\partial a)^3}\int_z^a S(y)(a-y)^2\,dy = 2S(a)$, the solution to (A29) is

$$\text{(A31)} \quad S(a) = \frac{\frac{\partial^3}{(\partial a)^3}\int_z^a (a-x)^{1-\alpha}\,T(x)\,dx}{2B(\alpha)}.$$

Differentiating once with respect to a, integrating by parts using $T(z) = 0$, differentiating with respect to a and integrating by parts using $T'(z) = 0$ then differentiating one last time yields

$$\text{(A32)}\quad \phi(\beta(z):a) = S(a) = \frac{(1-\alpha)}{2B(\alpha)}\int_z^a (a-x)^{-\alpha}\, T''(x)\, dx$$

$$= \frac{1-\alpha}{2B(\alpha)}\int_z^a (a-x)^{-\alpha}\left(\frac{(\alpha^2-1)z((\alpha+1)x-z)^{\alpha-2}}{\alpha^{\alpha-1}} + \alpha(x-z)^{\alpha-1} - \alpha x^{\alpha-1}\right)dx.$$

Setting $\alpha = ½$ and integrating (A32) by Mathematica yields the equation for ϕ in the Proposition. It remains to show that a bidder can not do better by submitting any other bid. Since ϕ satisfies bidder x's first order conditions for all bids less than $x/(1+\alpha)$, we need only check whether a higher bid is a better response. An argument parallelling the proof in Proposition 2 shows that no bid improves on the bids below $x/(1+\alpha)$. ■