Discussion Paper No. 955R

INDETERMINACY OF EQUILIBRIUM IN DYNAMIC MODELS WITH EXTERNALITIES[*]

by

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First Version: May 1991

Revised: January 1992

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[*]This is a much revised version of a paper which has already circulated under the title "Indeterminacy of Equilibria in Models with Infinitely Lived Agents and External Effects." We are grateful to David Cass, David Levine, Joe Ostroy, John Riley and Paolo Siconolfi for useful discussions and criticisms. The present version benefitted from comments received during seminars at Université du Quebec at Montreal, Columbia University, New York University, Brown University, University of Pennsylvania, UCLA, Stanford University, University of Tel Aviv and University of Chicago.

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Abstract

In this paper we study the indeterminacy of equilibria in infinite horizon capital accumulation models with technological externalities. Our investigation encompasses both models with bounded and unbounded accumulation paths, and models with one and two sectors of production. Under reasonable assumptions we find that equilibria are locally unique in the one sector economies, at least as long as cycles are not present and trajectories are therefore monotone. On the other hand we show (by means of an example) that persistent oscillations are possible when the external effect is particularly strong and capital accumulation is bounded. In this case indeterminacy may be present as we are unable to rule out the existence of a continuum of equilibria converging to the cycle. The situation is different in economies with two sectors of production. Here it is very easy to construct analytical examples where a positive external effect induces a two dimensional manifold of equilibria converging to the same steady state (in the bounded case) or to the same constant growth rate (in the unbounded case). For the latter we also point out that the dynamic behavior of these equilibria is quite complicated and that persistent fluctuations in their growth rates are possible.

1. Introduction

Our goal is to clarify the extent to which equilibria are (or are not) indeterminate in infinite horizon capital accumulation models with a representative agent and external effects at the production level.

With indeterminacy we denote a situation in which there exists a continuum of distinct equilibrium paths, all consistent with the same initial condition. In the models we study the latter is typically represented by the initial allocation of the capital stock.

Two articles by Lucas [1988] and Romer [1986] have been particularly instrumental in spreading the idea that models with technological externalities are appropriate to describe the endogenous nature of growth phenomena. While a variety of different implementations have since been proposed the basic intuition is quite simple. It is assumed that, due either to the lack of appropriate markets or to the intrinsic nature of the production process, the productivity of an individual firm's input(s) is affected by the aggregate level of utilization of the same or other input(s). In the simplest aggregate world with a single commodity and production process one writes the production function of the individual firm as $f(x, K)$ where x is the firms's own capital stock and K is the aggregate one. The latter is assumed to affect both the average and marginal productivities of the former. In certain instances the external effect is assumed to be strong enough to induce aggregate increasing returns even if individual decision makers still face decreasing payoffs from their own inputs.

Beside the obvious effect of rendering the associated competitive equilibrium inefficient the introduction of such an externality has other two, important, implications.

It allows to retain the notion of competitive equilibrium when studying economies in which unbounded growth is fueled by some form of non–convexity in the aggregate production set. Secondly it induces a positive complementarity between individual efforts, the full implications of which cannot be captured by market prices. When private returns from the investment of capital are affected by its aggregate level, multiple expectations-driven equilibria become a possibility. Societies with distinct institutional mechanisms may coordinate private beliefs in different ways, thereby generating different publicly held expectations about future economic events. In such circumstances it would not be surprising if the otherwise similar private agents ended up formulating very different investment plans. This can take place in spite of totally identical technologies, preferences and initial economic conditions.

From a theoretical viewpoint this situation is commonly described by means of dynamic models in which competitive equilibrium is indeterminate. If the given initial condition, supposedly a description of all the "fundamentals" of the economy, is not enough to pin down the future evolution of the system then other, extra-economic, factors need to be brought onto the stage. While this needs not be the only compelling explanation for the factual diversity in the growth patterns of various countries it certainly appears as one worth investigating.

The relevance of this point of view is reinforced by the apparent pervasiveness of indeterminacy in dynamic economic models, something of which we have started to become aware since the work of Kehoe and Levine on the Overlapping Generations Model (Kehoe-Levine [1985]).

This same form of indeterminacy has been found, for example, in the area of dynamic search and matching, Diamond [1982], Mortensen [1982]. In these models the proportion of agents involved in search is the relevant state variable; it has a positive external influence on individual efforts because it increases the probability with which matches occur. In such circumstances the competitive equilibrium outcome is affected by the "pessimism" or "optimism" of individual expectations. When the agents come to believe that tomorrow "is a good day" the individual, and therefore the aggregate, search effort increases and the ensuing equilibrium will be one with a high number of matches: the optimist expectations have been fulfilled. Similarly for the case in which everybody believe that tomorrow is "a bad day". A continuum of equilibrium trajectories, parameterized by different expectations, may therefore depart from a common initial condition. This was intuitively transparent in the early contributions to this literature and has been rigorously proved, for various specifications of the basic models, by a number of authors, e.g. Diamond-Fudenberg [1989], Howitt-McAfee [1988], Boldrin-Kiyotaki-Wright [1991], Mortensen [1991].

Similar results emerge in dynamic models of production and accumulation when market incompleteness is introduced. In a context quite close to the one studied here, i.e. a one sector OLG model, Boldrin [1990] has shown that, with the kind od external effects discussed earlier, a weak form of indeterminacy is possible as a countable infinity of accumulation paths may depart from the same initial aggregate stock. Matsuyama [1991] also shows that equilibria may be indeterminate in a dynamic model of the industrialization process in which there are two sectors (agriculture and manufacturing) and increasing returns prevail in the manufacturing sector.

Quite surprisingly no example[1] of similar phenomena has been provided for the representative agent model of capital accumulation. On logical grounds nothing seems to prevent the kind of expectations-driven indeterminacy we described before from occuring also in this context. Given the extent to which models of this form are now used for the purposes of empirically assessing the economic sources of growth, it seems important to clarify the matter. If indeterminacy is present the interpretation of many simple estimations, obtained by pooling together data from a variety of different countries, can be questioned as there is no reason to believe that these countries should be moving along the same equilibrium path. On the other hand if a set of hypotheses can be found under which equilibria are locally unique, then one would rest assured that a minimal theoretical framework exists within which comparative statics and dynamics exercises can be carried out.

With this purpose in mind we set out to investigate both one- and two-sector models of capital accumulation with externalities. For both specifications we consider separately the case of bounded and unbounded trajectories as they require different analytical instruments. The results we are able to derive are mixed and open to different interpretations.

In the one sector model indeterminacy can be ruled out under fairly weak assumptions, in particular under a set of assumptions that seems "reasonable" to us and that is consistent with those adopted in the more applied literature. We show that monotone trajectories, either bounded or unbounded, are locally unique and that the unbounded ones display a unique asymptotic constant growth rate. This is a positive result insofar as one believes that fluctuations are not caused by the presence of externalities and that a simple one good model is appropriate for the study of aggregate growth. Notice, in particular, that under the assumptions adopted here the old neoclassical prediction of "convergence" obtains once again, albeit in a different form. Even if the relative difference in income levels is maintained, countries starting from different capital stocks should eventually grow at the same constant rate.

On the other hand we are unable to rule out the possibility that a continuum of equilibria converging to a cycle may exist. Example 2.2 below shows that cycles do appear quite easily in this kind of models, a fact that we believe should be taken into account in the study of the interplay between growth and business cycles phenomena.

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[1] A qualification is needed: two examples (Kehoe-Levine-Romer [1991] and Spear [1991]) do exist. But they use specifications of the external effect that are quite different from the one we are studying here and, more generally, from those adopted in the literature on endogenous growth. They will be discussed below.

The two–sector models we examine have only one capital good, which can be interpreted either as human or physical capital. They do not include, therefore, the models with both physical and human capital stocks that were suggested in Lucas [1988] and more recently in Romer [1990]. In any case even in this simpler world the comforting results of the one–sector framework are easily turned upside-down. Examples of indeterminate equilibria abound and they can be derived from very standard utility and production functions such as CES, Cobb-Douglas and linear ones. Furthermore, in the case of unbounded growth, the very same examples can exhibit indeterminate and perpetually oscillating (i.e. chaotic) asymptotic growth rates for certain set of parameters. Quite naturally an issue of "realism" can be made with regard to the parameter values at which these more complicated phenomena arise. While they do not appear as far away from reality as those previously encountered in the optimal growth brand of the chaotic dynamics literature (e.g. Boldrin-Deneckere [1990]) they do rely on particularly strong externalities. For this reason and for the lack of reliable empirical evidence about the external effects consistent with this type of technology, we refrain from speculating on the positive implication of our findings.

The differences between our results and those presented in Kehoe-Levine-Romer [1991] and Spear [1991] should be mentioned at this point. In both papers a one–sector growth model is studied, the difference laying in the type of external effect considered. The first group of authors specify the individual production function as $f(x, C)$, where C is the aggregate consumption level and x is the individual stock of capital. They show by means of an example that such an economy may have a locally stable steady state around which equilibria are therefore indeterminate. In the paper by Spear a different type of external effect is introduced: the production function is written as $f(x, K')$, where K' is tomorrow's aggregate capital stock which is assumed to have a positive effect on today's productivity. In this case the author derives a set of sufficient conditions under which stationary sunspot equilibria exist in a neighborhood of a stationary state. Neither article consider the case of unbounded growth nor the kind of external effects we are studying here. The results are, in this sense, non comparable.

This paper contains two more sections and the conclusions. Next section is dedicated to the one–sector model whereas section 3 will discuss the case of two sectors. Most of the formal proofs are collected in the final appendix.

2. The One Sector Model

We begin with a general description of the model under a set of assumptions encompassing both the case of bounded (Subsection 2.1) and unbounded (Subsection 2.2) accumulation paths. The economy is composed of two continua of agents: consumers indexed by $i \in [0,1]$ and firms indexed by $j \in [0,1]$. There is only one good which is used both as consumption and capital input. Each consumer i is infinitely lived and owns a firm j and an initial stock of capital k_0^i. He maximizes total discounted utility by choosing a consumption stream $\{c_t^i\}_{t=0}^{\infty}$ that solves:

$$\max \sum_{t=0}^{\infty} u(c_t^i)\delta^t \tag{$P(i)$}$$

$$\text{s.to}: \sum_{t=0}^{\infty} p_t c_t^i \leq \sum_{t=0}^{\infty} \pi_t^i + q_0 k_0^i$$

for given sequences of prices $\{p_t\}_{t=0}^{\infty}$, and incomes $\{\pi_t^i\}_{t=0}^{\infty}$. All consumers are identical in the sense that they have the same discount factor $0 \leq \delta < 1$, utility function $u(\cdot)$ and initial capital stock k_0.

Assumption 2.1 *The utility function $u : \Re_+ \to \Re$ is C^2, increasing and strictly concave.*

Each firm is described by a production function $G(k^j, k, \ell)$ which depends on the private amount of capital stock k^j, the aggregate capital stock $k = \int_0^1 k^j dj$, and labor ℓ. The latter is inelastically supplied by the consumers and will be normalized to one. Except for the external factor, k, the production function G is standard.

Assumption 2.2 *$G : \Re_+^3 \to \Re_+$ is of class C^2. For any given $k \geq 0$ it exhibits the following properties:*

i) $G(\lambda k^j, k, \lambda\ell) = \lambda G(k^j, k, \ell),\ \forall \lambda \geq 0$;

ii) $G(\cdot, k, \cdot)$ *is increasing and concave;*

iii) $G_{11}(\cdot, k, \ell) < 0$ *for all* $\ell > 0$.

Denote with $0 \leq \mu \leq 1$ the capital depreciation rate. We define $f : \Re_+^2 \to \Re_+$ as $f(k^j, k) = G(k^j, k, 1) + (1-\mu)k^j$. Firms buy their initial stock of capital k_0^j from the consumers at price q_0. In each subsequent period $t = 1, 2, \ldots$ they sell their output at a price p_t, buy the future capital stock also at a price p_t and pay a dividend income $\pi_t^j = p_t[f(k_t^j, k_t) - k_{t+1}^j]$ to their owners. At $t = 0$ the dividend paid is instead: $\pi_0^j = p_0[f(k_0^j, k_0) - k_1] - q_0 k_0^j$. Given the initial capital stock k_0^j, the two sequences $\{p_t, k_t\}_{t=0}^{\infty}$ of prices and aggregate capital stocks and the initial price q_0, every firm j maximizes total discounted cash-flow by choosing an accumulation sequence which solves:

$$\max\left\{\sum_{t=0}^{\infty} p_t[f(k_t^j, k_t) - k_{t+1}^j] - q_0 k_0^j\right\} \tag{$P(j)$}$$

under a non negativity constraint on the k_t^j's.

Recall that initial wealth $k_0 = \int_0^1 k_0^i di = k_0^i$ is equally distributed among consumers and that each one of them owns one of the identical firms.

Definition 2.1 *A competitive equilibrium is given by a scalar q_0 and a set of sequences $\{p_t, k_t, \pi_t, c_t\}_{t=0}^{\infty}$ such that:*

a) given $\{p_t, \pi_t\}_{t=0}^{\infty}$ and q_0, the sequence $\{c_t\}_{t=0}^{\infty}$ solves $P(i)$ for all $i \in [0, 1]$;

b) given $\{p_t, k_t\}_{t=0}^{\infty}$ and q_0, the sequence $\{k_t\}_{t=0}^{\infty}$ solves $P(j)$ for all $j \in [0, 1]$;

c) markets clear:

- $f(k_t, k_t) = y_t = c_t + k_{t+1}$ *for all* $t = 0, 1, 2, \ldots$, *and*
- $\pi_t = p_t[f(k_t, k_t) - k_{t+1}]$ *for* $t = 1, 2, 3, \ldots$, *and*
- $\pi_0 = p_0[f(k_0, k_0) - k_1] - q_0 k_0$.

With a few more technical assumptions existence of an equilibrium is not difficult to obtain. The problem becomes more difficult if one seeks a representation of the equilibrium sequences $\{k_t\}_{t=0}^{\infty}$ by means of a pair of continuous functions $\theta : \Re_+^2 \to \Re_+$ and $\tau : \Re_+ \to \Re_+$ such that $k_{t+1}^i = \theta(k_t^i, k_t)$ for all $i \in [0, 1]$ and $k_{t+1} = \tau(k_t) = \theta(k_t, k_t)$. Here we proceed under the assumption that an equilibrium satisfying Definition 2.1 exists from every initial condition k_0.

Proposition 2.1 *Let Assumptions 2.1 and 2.2 hold and let $\{x_t\}_{t=0}^{\infty}$ be a sequence satisfying $x_0 = k_0$ and $0 < x_{t+1} < f(x_t, x_t)$ for all $t \geq 0$. Then $(q_0, \{p_t, x_t, \pi_t, c_t\}_{t=0}^{\infty})$ with:*

$$\begin{cases} c_t = f(x_t, x_t) - x_{t+1}, \\ p_0 = 1, \text{ and } p_{t-1}/p_t = f_1(x_t, x_t) \text{ for } t = 1, 2, \ldots, \\ \pi_0 = p_0[f(x_0, x_0) - x_1] - q_0 x_0, \text{ and } \pi_t = p_t[f(x_t, x_t) - x_{t+1}] \text{ for } t = 1, 2, \ldots, \text{ and} \\ q_0 = p_0 f_1(x_0, x_0). \end{cases}$$

is an equilibrium for our economy if and only if $\{x_t\}_{t=0}^{\infty}$ *satisfies:*

$$u'(f(x_t, x_t) - x_{t+1}) = \delta u'(f(x_{t+1}, x_{t+1}) - x_{t+2}) f_1(x_{t+1}, x_{t+1}), \tag{EE}$$

and

$$\lim_{t \to \infty} \delta^t x_t u'(f(x_t, x_t) - x_{t+1}) f_1(x_t, x_t) = 0. \tag{TC}$$

Proof: See Appendix.

Proposition 2.1 is useful because it allows us to search for equilibria by looking at those solutions of the dynamical system (EE) that satisfy (TC). In the sequel of this paper equilibria will therefore be sequences $\{x_t\}_{t=0}^{\infty}$ that, given a sequence $\{k_t\}_{t=0}^{\infty}$, solve the "parametric" programming problem:

$$\max\Big\{\sum_{t=0}^{\infty} u(f(x_t, k_t) - x_{t+1})\delta^t\Big\} \tag{PP}$$

$$\text{subject to} : 0 \leq x_{t+1} \leq f(x_t, k_t)$$

and also the "fixed point problem" $x_t(\{k_t\}_{t=0}^{\infty}) = k_t$ for all t. A more detailed discussion of the equivalence between a competitive equilibrium with externalities and the programming cum fixed point problem (PP) can be found in Kehoe-Levine-Romer [1991].

Before proceeding with our analysis we need to make our notion of indeterminacy more precise.

Definition 2.2 *Let* $\{x_t\}_{t=0}^{\infty}$ *denote an equilibrium for an economy with initial condition* $x_0 = k_0$. *We say that it is an* **indeterminate equilibrium** *if for every* $\epsilon > 0$ *there exists another sequence* $\{y_t\}_{t=0}^{\infty}$, *with* $0 <| y_1 - x_1 |< \epsilon$ *and* $y_0 = x_0 = k_0$, *which is also an equilibrium according to Definition 2.1.*

The intuitive notion is that an equilibrium is indeterminate when there exists a whole interval of equilibrium paths starting off from its same initial condition.

2.1 Bounded Accumulation Paths

We consider first the case in which the aggregate production function $F(x) = f(x, x)$ does not allow for persistent growth. We retain Assumptions 2.1 and 2.2 and add the following:

Assumption 2.3 *The production function $F(x) = f(x, x)$ has the properties:*
(i) There exists an $\overline{x} > 0$ such that $F(x) > x$ for $0 < x < \overline{x}$ and $F(x) < x$ for $x > \overline{x}$.
(ii) The partial derivative f_1 satisfies: $f_1(\overline{x}, \overline{x}) < 1$ and $\lim_{x \to 0} f_1(x, x) > 1/\delta$.

It follows from very standard arguments that all equilibria are bounded and an interior stationary state exists.

Proposition 2.2 *Under Assumption 2.1, 2.2 and 2.3 all equilibria $\{x_t\}_{t=0}^{\infty}$ satisfy the following properties:*
(•) $0 \leq x_t \leq \max\{\overline{y}, x_0\}$, where $\overline{y} = \max\,\{F(x); x \in [0, \overline{x}]\}$.
(•) There exists a value $x^* \in (0, \overline{x})$ such that $x_t = x^*$ for all t, is an equilibrium.

Proof: See Appendix.

Without loss of generality, we can assume for the remainder of this subsection that $0 \leq x_0 \leq \overline{y} = \max\,\{F(x); x \in [0, \overline{x}]\}$.

We have not yet specified the "sign" of the external effect: it could be either negative or positive. Our claim is that with positive externalities monotone equilibria cannot be indeterminate. It is easy to see that indeterminacy arises when negative external effects are present, as we show in example 2.1 below. The qualification "monotone" in the first part of our claim is needed because when the externality is positive and very strong we cannot rule out a continuum of equilibria converging to a periodic orbit. In fact we conjecture that such a type of indeterminacy should arise in a neighborhood of period two cycles like the one we derive below in example 2.2. We will now proceed to show that in the absence of oscillations equilibria are always locally unique.

Theorem 2.1 *Under Assumptions 2.1, 2.2 and 2.3 all monotone equilibria are locally unique.*

Proof: See Appendix.

By slightly strengthening our conditions a more general result can be proved.

Theorem 2.2 *Let Assumption 2.1, 2.2 and 2.3 be true and assume furthermore that the private return on capital $f_1(x,x)$ is a non increasing function of the capital stock. Then all interior equilibria are locally unique. Moreover, there exists a unique value $x^* \in (0,\overline{x})$ such that if $x_0 \leq x^*$ then $\{x_t\}_{t=0}^{\infty}$ satisfies $x_t \leq x_{t+1} \leq x^*$ and if $x_0 \geq x^*$ then $\{x_t\}_{t=0}^{\infty}$ satisfies $x^* \leq x_{t+1} \leq x_t$ for every t.*

Proof: See Appendix.

An implication of Theorem 2.2 is that when the private rate of return on capital is non–increasing the Turnpike Theorem applies also to one–sector models with externalities. One could extend the theorem to the case of noninterior equilibria, but this would require introducing a cumbersome amount of notation.

A simple example will show that everything unravels when the externality is negative, i.e. when $f_2(x,x) < 0$ around the stationary state x^*.

Example 2.1 Set $u(c) = c^{1-\gamma}/(1-\gamma)$ and $f(x,k) = ax + bx^{\alpha}k^{\beta} + dk^{\rho}$. Adopting the notation introduced in the proof to theorem 2.1 one can verify that:

$$\begin{cases} f_1 = a + \alpha b x^{\alpha+\beta-1}, \\ f_2 = \beta b x^{\alpha+\beta-1} + d\rho x^{\rho-1}, \\ f_1' = \alpha b(\alpha+\beta-1)x^{\alpha+\beta-2}, \\ F' = a + d\rho x^{\rho-1} + b(\alpha+\beta)x^{\alpha+\beta-1}, \\ \sigma = u'/u'' = -c/\gamma; \end{cases}$$

where the equilibrium condition $x = k$ has been substituted everywhere. A unique interior steady state $x^* = 1$ exists if we choose $\delta^{-1} = a + \alpha b$. The algebra can be simplified by picking appropriate parameter values, such as: $\delta = 8/9, a = 7/8,\ d = -1/8,\ \alpha = 1-\epsilon$ and $b = 1/(4(1-\epsilon))$, for some $0 < \epsilon < 1$. The associated consumption level will be $c^* = \epsilon/(4(1-\epsilon)) > 0$ and $f_2(x^*,x^*) < 0$ if we choose, for example, $\beta = 2\epsilon$ and $\rho > 4\epsilon/(1-\epsilon)$. We can then linearize around x^* to obtain the characteristic equation: $\lambda^2 + a_1\lambda + a_2 = 0$, where a_1 and a_2 are again defined as in the proof to theorem 2.1. Both eigenvalues will be inside the unit circle if and only if: (i) $1-a_2 > 0$, (ii) $1+a_1+a_2 > 0$, and (iii) $1-a_1+a_2 > 0$ simultaneously hold. To get (i) set, for example, $\epsilon < 1/5$ and $\rho \geq 2$. Then (ii) is always true and (iii) obtains for any $\gamma > \epsilon/9$. Equilibria near x^* are therefore indeterminate.

We shall now show, again by means of an example, how cycles may emerge when the positive external effect is strong enough to make the private return on investments $f_1(x,x)$ an increasing function of the capital stock. We conjecture that such cycles can become locally asymptotically stable and therefore originate indeterminate equilibria.

Example 2.2 We need a production function f for which $f_1(x,x) = \delta^{-1}$ has a solution x^* such that f_1 is increasing in a neighborhood of x^*. The choice $f(x,k) = axk - \frac{b}{2}x^2k$ would do, at least as a local representation. We can normalize $x^* = 1$ by choosing $a = \delta^{-1} + b$ and we can assure that f_1 is increasing around x^* by imposing further that $\delta^{-1} > b$. Now consider the linearization of (EE) around this steady state and use the flip bifurcation theorem (see Guckenheimer and Holmes [1983] for the technical details) to generate a period two cycles. The crucial requirement is that the characteristic equation $\lambda^2 + a_1\lambda + a_2 = 0$ has a solution at $\lambda = -1$. In general this is equivalent to $2(1+F') = -\sigma f_1'/f_1$, where we have set again $\sigma = -u'/u''$ and all functions are evaluated at x^*. One can see by inspection that f_1 increasing is necessary for the bifurcation condition to be realized. Choose $u(c) = c^{1-\gamma}/(1-\gamma)$, so that $\sigma = -\gamma/c$ and use the parameter γ as a bifurcation parameter. The bifurcation value is then $\gamma^* = 2cf_1(1+F')/f_1'$. Again one can verify that all the remaining technical conditions are satisfied and so a cycle of period two will exist for values of γ near γ^*. A back of the envelope calculation can give an idea of the magnitudes of γ which are consistent with the presence of equilibrium cycles. We will choose the other parameters consistently with what the Real Business Cycles literature deems to be realistic values (a recent paper by Baxter and King [1991] is used as a source in this case). Accordingly we set $\delta = (1.065)^{-1} = .938$ and the depreciation rate equal to 6% per period. This together with the restriction of the steady state value to one gives a net output equal to $.125 + b/2$ per period and a net capital income equal to .125 also per period. To make this consistent with a capital share in national income equal to 42% we need to set $b = .35$. Then γ^* can be computed to be approximately 2.4.

2.2 Unbounded Accumulation Paths

In this subsection we show that similar conclusions hold also in the presence of persistent growth if the one sector model is retained as a description of the aggregate technology. More precisely we will prove that, under reasonable hypotheses, equilibria are locally unique in the following sense: given an initial condition x_0 there exists at most one sequence $\{x_t\}_{t=0}^{\infty}$ satisfying (EE) and (TC) and growing asymptotically at some constant rate. The latter requirement implies that those models in which the asymptotic growth rate is not bounded and in which the stock of capital grows infinitely big infinitely fast are not captured by our analysis. Also excluded from our consideration are cyclic growth paths, i.e. sequences of capital stocks growing at an oscillatory rate. While they certainly are a logical possibility we fail to see how a non farfetched example could be built within the one–sector context.

We remind the reader that the regularity Assumptions 2.1 and 2.2 are maintained throughout this section and only positive external effects will be considered. Our argument will proceed along these steps: first we show that (under only the extra assumptions required to guarantee unbounded accumulation) equilibrium orbits are locally unstable, thereby preventing nearby equilibria from merging into each other asymptotically. Then we introduce a set of additional assumptions about the behavior of the utility and production functions "at infinity". This allows us to prove there exists a unique constant growth rate and that the latter is dynamically unstable, thereby implying the existence of at most one equilibrium path growing asymptotically at a constant rate.

We begin by assuming that unbounded growth at a bounded rate is possible:

Assumption 2.4 *The aggregate production function $F(x) = f(x, x)$ satisfies:*

(•) $\liminf_{x \to +\infty}[F(x) - x] > 0$;

(•) $\liminf_{x \to +\infty} f_1(x, x) > \delta^{-1}$;

(•) $\lim_{x \to +\infty} F(x)/x = L < +\infty$.

It is simple to verify that the first two parts of Assumption 2.4 together with strict concavity of the utility function imply that equilibrium consumption sequences are monotone increasing. This, together with feasibility considerations of the type we already used in the proof to Theorem 2.1, implies that also the capital stock sequence is monotone increasing along an equilibrium trajectory. Notice also that the third part of Assumption 2.4 effectively bounds the capital growth rate by L and, for x large, it implies $F(x) = Lx + g(x)$ with $\lim_{x \to \infty} g(x)/x = 0$.

To see why orbits satisfying (EE) cannot converge to each other pick any one of them $\{x_t\}_{t=0}^{\infty}$ and compute the linear approximation to (EE) in a neighborhood of it. The associated jacobian matrix is time dependent and with some algebra one can check that its two real roots, at any regular point of the trajectory $\{x_t\}_{t=0}^{\infty}$, are given by:

$$\lambda_t^1 = \frac{u''(c_t)}{u'(c_t)} \frac{u'(c_{t+1})}{u''(c_{t+1})}; \qquad \lambda_t^2 = F'(x_t)$$

Then we have:

Proposition 2.3 *Under Assumptions 2.1, 2.2 and 2.4 all equilibrium trajectories are locally unstable at least along one direction. Under the further assumption that the utility function displays non–decreasing elasticity of substitution in consumption, they are saddle–points.*

Proof: By assumption 2.4 and the hypothesis that the external effect is positive, $\lambda_t^2 > 1$ for all t. The first part is then a simple application of well known results from dynamical system theory (see e.g. Irwin [1980, page 114]). The second part follows from the same results and the fact that consumption is monotone increasing along equilibrium trajectories which implies $\lambda_t^1 \leq 1$ for all t. **Q.E.D.**

With some additional efforts one can in fact show that the eigenvalue we denoted with λ_t^1 is associated to the eigenspace which lies in the direction of the orbit around which the linearization takes place whereas the second one is transversal to it.

As in the previous subsection our attention concentrates upon those equilibria that are characterized by Proposition 2.1, i.e. by (EE) and (TC). If in (EE) we write $x_t = x$, $x_{t+1} = \lambda_t x$, $x_{t+2} = \lambda_{t+1}\lambda_t x$, we obtain a parameterized implicit function θ_x mapping the growth rate of capital during time t, λ_t into λ_{t+1} the growth rate in the subsequent period. In general the map θ_x depends on the value of x, the current stock of capital, and the latter changes in each period. We are therefore facing a sequence of such maps θ_x. On the other hand we are interested only in the behavior of θ_x at "large" values of x. One then needs to assure that a function $\theta_\infty = \lim_{x\to\infty} \theta_x$ exists which is well defined from the interval $(0, L]$ into itself. The analysis of the asymptotic behavior of equilibria then reduces to the study of the fixed points of such a function and of their dynamic stability under iteration of θ_∞ itself. Under reasonable hypotheses one can see that (apart from L) only one such fixed point exists, the instability of which is easily verified. The fact that the only feasible asymptotic growth rate is dynamically unstable is enough to imply local uniqueness of the equilibrium sequence at least when the capital stock is "large enough". In fact, for given x_0 the economy cannot pick x_1 arbitrarily as this would almost always imply a value of λ_0 that leads either to $-\infty$ or to L asymptotically, neither of which is admissible in equilibrium. By adding this finding to the results contained in proposition 2.3 our conclusions will follow.

We now proceed to introduce our assumptions on the asymtotic behavior of the utility and production functions.

Assumption 2.5 *The private rate of return is aymptotically constant, i.e.*

$$\lim_{x \to \infty} f_1(x, x) = \pi > \delta^{-1}$$

Assumption 2.6 *The utility function is such that:*

$$c > c' \quad \textit{implies} \quad \frac{u''(c)c}{u'(c)} \leq \frac{u''(c')c'}{u'(c')}$$

Assumption 2.5 prevents the private rate of return from continuously oscillating between a lower and an upper bound. This condition is necessary for the existence of a constant growth rate equilibrium. Along such equilibrium the stock of capital and the level of consumption must be growing at the same constant rate: this follows from assumption 2.4 on the asymptotic linearity of the production function. Assumption 2.6 instead requires the utility function to display a non-decreasing elasticity of substitution in consumption (or, which is the same in this context, non-decreasing relative risk aversion). Uniqueness of the constant growth rate is mostly a consequence of this condition.

Let us begin by rewriting the Euler Equation as an implicit function of $x, \lambda_t, \lambda_{t+1}$:

$$\psi(x, \lambda_t, \lambda_{t+1}) = -u'(F(x) - \lambda_t x) + \delta u'(F(\lambda_t x) - \lambda_{t+1}\lambda_t x) f_1(\lambda_t x, \lambda_t x) = 0 \qquad (EE)$$

Strict concavity of u guarantees the existence of a continuous function $\theta_x : \Re_+ \to \Re$ satisfying

$$\psi(x, \lambda, \theta_x(\lambda)) = 0 \qquad (2.1)$$

for all finite values of x. Next we define the asymptotic functions

$$\Psi(\lambda_t, \lambda_{t+1}) = \lim_{x \to \infty} \psi(x, \lambda_t, \lambda_{t+1}) \qquad (2.2)$$

and

$$\theta_\infty(\lambda) = \lim_{x\to\infty} \theta_x(\lambda), \text{ i.e. } \Psi(\lambda, \theta_\infty(\lambda)) = 0. \tag{2.3}$$

While both Ψ and θ_∞ are well defined limits (finite or infinite), they are not necessarily continuous under our hypotheses. We must therefore introduce the technical assumption:

Assumption 2.7 *The function $\theta_\infty : \Re_+ \to \Re$ defined in (2.3) is continuous on the interior of its domain.*

Continuity of θ_∞ depends in a complicated form on the properties of the production and utility functions. It can be derived from a number of different special hypotheses about the behavior of the two "fundamentals" u and F. We do not see any advantage in pursuing this more general approach here. Under the set of assumptions we have collected one can prove the following theorem.

Theorem 2.3 *Under Assumptions 2.1, 2.2 and 2.4-2.7 and given an initial condition x_0 there exists a unique equilibrium path for our economy. Along such path the growth rate of the capital stock $\lambda_t = x_{t+1}/x_t$ converges to a constant growth rate $\lambda^* = \lim_{c\to\infty} \lambda^*(c)$, with the latter solving*

$$\frac{u'(c)}{u'(\lambda c)} = \delta\pi$$

Proof: See Appendix.

The theorem will be now illustrated by means of some examples. We begin with the simplest one.

Example 2.3 Let $u(c) = \frac{c^{1-\gamma}}{(1-\gamma)}$, $f(x,k) = ax + bx^\alpha k^{1-\alpha}$ with $a, b > 0, \alpha \in (0,1)$. It is immediate to verify that when $\delta(a + \alpha b) > 1$ all of our assumptions hold. The asymptotic function θ_∞ in this case can be computed directly and is given by:

$$\theta_\infty(\lambda) = L - [\delta(a + \alpha b)]^{1/\gamma} \frac{L - \lambda}{\lambda} \tag{2.4}$$

The two asymptotic roots are therefore

$$\begin{cases} \lambda_1 = [\delta(a + \alpha b)]^{1/\gamma}; \\ \lambda_2 = L = a + b. \end{cases}$$

In our theoretical treatment we have assumed an equilibrium always exists and purposely ignored the fact that in certain instances no growth rate can be found that satisfies the transversality condition (TC). Here two different cases are still possible:

Case 1: $\lambda_2 < \lambda_1$, then no equilibrium exists that satisfies our hypotheses, because both growth rates conflict with the transversality condition.

Case 2: $\lambda_1 < \lambda_2$, then there is a unique equilibrium growth path if the transversality condition is satisfied. The latter requires $\delta(a + \alpha b)^{1-\gamma} < 1$. In these circumstances it is easy to verify that the asymptotic map (2.4) is unstable at the fixed point λ_1.

Consider now the effect that a linear utility function (violating assumption 2.1) has in an economy which is otherwise identical to the previous one. If we set $u(c) = c$, the optimal individual capital stock x_{t+1} for given (x_t, k_t, k_{t+1}), needs no longer be interior. The constraint $x_{t+1} \leq ax_t + bx_t^\alpha k_t^{1-\alpha}$ should therefore be imposed explicitely for all $t = 0, 1, 2, \ldots$. By manipulating the first order conditions for the individual optimization problem one can derive the optimal accumulation policy of the representative agent:

$$x_{t+1} = \min \left\{ax_t + bx_t^\alpha k_t^{1-\alpha};\ \left(\frac{ab\delta}{1 - a\delta}\right)^{\frac{1}{1-\alpha}} k_{t+1}\right\}.$$

Noting that $ab\delta/(1 - a\delta) > 1$ if and only if $\delta(a + \alpha b) > 1$ we have three possible cases:

1. $\delta(a + \alpha b) < 1$; then the equilibrium condition $x_t = k_t$ imposes $x_t = 0$ for all t;
2. $\delta(a + \alpha b) > 1$; then at $x_t = k_t$ one has $x_{t+1} = (a + b)x_t$ or $x_t = (a + b)^t x_0$, which violates the transversality condition and cannot therefore be an equilibrium;
3. $\delta(a + \alpha b) = 1$; then $x_{t+1} = \min \{ax_t + bx_t^\alpha k_t^{1-\alpha}; k_{t+1}\}$. So any path that satisfies $x_{t+1} \leq ax_t + bx_t^\alpha k_t^{1-\alpha}$ and the transversality condition is an equilibrium path. The transversality condition here is equivalent to $\lim_{t\to\infty} \delta^t x_t = 0$, so any growth rate $\lambda \leq \min \{a + b, 1/\delta\}$ is an equilibrium growth rate.

The next example shows how the sequence of functions θ_x converges to the map θ_∞ as $x \to \infty$.

Example 2.4 Let the utility function be $u(c) = -\exp(-c)$ and take a general production function. The Euler Equation $\psi(x, \lambda_t, \lambda_{t+1})$ becomes

$$\exp[-(F(x) - \lambda_t x)] = \delta \exp[-(F(\lambda_t x) - \lambda_t\lambda_{t+1}x)]f_1(\lambda_t x, \lambda_t x) \qquad (2.5)$$

which is clearly not continuous in λ when $x \to +\infty$. Nevertheless one can verify the following. The (EE) (2.5) can be reduced to

$$F(x) - F(\lambda_t x) + \lambda_t \lambda_{t+1} x - \lambda_t x + k(x, \lambda_t) = 0 \tag{2.6}$$

where $k(x, \lambda_t) = \log(\delta f_1(\lambda_t x, \lambda_t x))$. Dividing both sides of (2.6) by x and rearranging we have:

$$\theta_x(\lambda) = \frac{F(\lambda x)}{\lambda x} - \frac{F(x)}{\lambda x} + 1 - \frac{k(x, \lambda)}{\lambda x}$$

which satisfies all the general properties derived in the proof to Theorem 2.3 . Taking limits as $x \to \infty$ one finally obtains the asymptotic function θ_∞, which is

$$\theta_\infty(\lambda) = L - \frac{L}{\lambda} + 1$$

The unique asymptotic equilibrium growth rate is therefore $\lambda^* = 1$ to which the economy converges as the stock of capital goes to infinity. Note that the asymptotic Euler Equation is not verified as an equality here, at least as long as $\delta\pi > 1$ holds. The equilibrium sequence is one along which capital stock and consumption grow unbounded at an ever decreasing rate and become constant only "at infinity".

Finally our last example shows how a violation of assumption 2.6 may come around and falsify our result.

Example 2.5 Again we need not restrict the production function to any particular form. Assume the marginal utility of consumption is given by $u'(c) = 1/\log(c+1)$. The latter does not satisfy assumption 2.6.

By rearranging the Euler Equation for finite values of x one obtains:

$$\theta_x(\lambda) = \frac{F(\lambda x)}{\lambda x} - \frac{[F(x) - \lambda x]^{\delta f_1(\lambda x, \lambda x)}}{\lambda x} \tag{2.7}$$

By graphical inspection one can observe that (because of assumption 2.5) the sequence of functions implied by (2.7) moves outward as $x \to \infty$. Its limit being a discontinuous function equal to $-\infty$ for $\lambda < L$ and to $+\infty$ for $\lambda > L$. The growth rate $\lambda = L$ is a fixed point of such a function but it is not an equilibrium for obvious reasons. Therefore there is no asymptotic equilibrium satisfying Theorem 2.3

3. The Two–Sector Model

In this section we make the assumption that consumption and capital are different commodities and that as such they might be produced by different combinations of labor and capital inputs. We will show that this is enough to generate robust examples of indeterminate equilibria. The first example (subsection 3.1) displays a continuum of equilibria converging to a stationary state, whereas the second (subsection 3.2) has a continuum of equilibria growing at a common asymptotic growth rate. For this second economy we also point out that at certain parameter values there exists a continuum of equilibria converging (in growth rates) to a chaotic attractor. A detailed study of this more complicated case can be found in Boldrin [1992].

We retain here the market and demographic structures of the economies studied in Section 2. There is a continuum of consumers $i \in [0,1]$ and two continua of firms $j \in [0,1]$, one for each sector. Within each sector firms are identical and each consumer owns the same initial amount k_0 of capital stock and supplies a fixed unitary amount of labor in each period. Capital can be freely shifted from one sector to the other at the beginning of each production period. There is an external effect in production, which may affect either one or both production processes.

Such external effect comes from the aggregate stock of capital and can be given any of the many interpretations found in the recent literature. We do not believe this is the place for us to argue in favor or against the empirical relevance of these different sources of externalities. Their implications are not yet well understood and their 'measurement' is far from being accomplished. We only would like to stress that the state variable "capital stock" in this context can be interpreted either as physical or as human capital.

An obvious modification of definition 2.1 provides the appropriate notion of Competitive Equilibrium to be used in what follows.

Let the production function of a typical firm in either sector be denoted as $F^i(x_t^i, \ell_t^i, k_t)$, with $i = 1$ for consumption and $i = 2$ for investment. We assume that, given the aggregate stock of capital k_t, both $F^i(\cdot,\cdot,k)$'s satisfy Assumption 2.2. Assuming that markets are fully competitive in every other respect one can define the Production Possibility Frontier (PPF) faced by a representative individual as:

$$T(x_t, x_{t+1}, k_t) = \max_{x_t^1, \ell_t^1} F^1(x_t^1, \ell_t^1, k_t)$$

$$\text{s.to}: x_{t+1} \leq F^2(x_t^2, \ell_t^2, k_t) + (1-\mu)x_t$$

$$x_t^1 + x_t^2 \leq x_t$$

$$\ell_t^1 + \ell_t^2 \leq 1,$$

where x_t denotes the private and k_t the aggregate stock of capital. The parameter $\mu \in [0,1]$ is the capital depreciation factor and one is the total amount of labor available to an individual in each period.

Now denote with $u(c)$ the representative individual utility function and with $V(x, x', k)$ the composition $u(T(x, x', k))$. Then, as in the one sector model above, interior equilibria can be characterized by means of a variational equation (EE) and a transversality condition (TC). In the notation just introduced they are:

$$V_2(x_t, x_{t+1}, x_t) + \delta V_1(x_{t+1}, x_{t+2}, x_{t+1}) = 0 \qquad (EE)$$

and

$$\lim_{t \to \infty} \delta^t x_t V_1(x_t, x_{t+1}, x_t) = 0 \qquad (TC)$$

respectively. The proof of this statement is omitted and we proceed to study (EE) and (TC) directly. For more details on the mechanics of the two sector model the reader is referred to Boldrin [1989].

3.1 Bounded Accumulation Paths

Before producing the analytical example we should illustrate our intuition by means of the general model.

Linearization of (EE) around a steady state x^* gives the characteristic equation

$$\lambda^2 + \lambda\left\{\frac{V_{22}}{\delta V_{21}} + \frac{V_{11} + V_{13}}{V_{21}}\right\} + \left\{\frac{1}{\delta} + \frac{V_{23}}{\delta V_{12}}\right\} = 0 \qquad (3.1)$$

where it should be understood that the functions $V_{ij}, i, j = 1, 2, 3$ are evaluated at the steady state. Our contention is that there exists an admissible set of parameter values at which both roots of (3.1) are inside the unit circle. In such circumstances equilibria are indeterminate, as x_0 near x^* implies that for all x_1 in an ϵ-ball around x_0 the path $(x_0, x_1, \ldots)$ is an equilibrium converging to x^*.

Once again the necessary and sufficient conditions for both roots of a quadratic equation of the type $\lambda^2 + a_1\lambda + a_2 = 0$ to be inside the unit circle are:

$$(1 - a_2) > 0; \quad (1 + a_1 + a_2) > 0; \quad (1 - a_1 + a_2) > 0.$$

For equation (3.1) they translate into:

$$\begin{cases} \frac{1}{\delta} + \frac{V_{23}}{\delta V_{12}} < 1; \\ 1 + \frac{1}{\delta} + \frac{V_{23}+V_{22}}{\delta V_{21}} + \frac{V_{11}+V_{13}}{V_{21}} > 0; \\ 1 + \frac{1}{\delta} + \frac{V_{23}-V_{22}}{\delta V_{21}} - \frac{V_{11}+V_{13}}{\delta V_{21}} > 0. \end{cases} \tag{3.2}$$

A careful examination of (3.2) shows that, contrary to the one-sector model, there exists economic conditions under which the three inequalities are simultaneously satisfied. In fact if V_{12} and V_{23} have opposite signs the first condition can be obtained. Of the other two, only one is really binding: if $0 < a_2 < 1$, then $a_1 > 0$ implies the second inequality in (3.2) is always satisfied, whereas $a_1 < 0$ implies that the third is automatically satisfied. Notice also that whatever sign a_1 may have, its magnitude can be made quite small by forcing V_{11} and V_{13} to cancel each other.

More intuitively our economy has to display these three properties.

1. A steady state value such that the consumption sector has a higher capital–labor ratio than the investment sector ($T_{12} < 0$) and a relatively inelastic marginal utility of consumption ($V_{12} = u'T_{12} + u''T_2T_1 < 0$).
2. A positive externality that also reduces the cost (in utils) of producing additional capital stock ($V_{23} = u'T_{13} + u''T_2T_3 > 0$).
3. An external effect that increases the marginal value of the current stock of capital together with a moderately concave utility function ($V_{13} = u'T_{13} + u''T_1T_3 > 0$).

Neither of these conditions appear economically unreasonable nor they are very difficult to formalize. The example we provide next is just the simplest we could come up with. Others, more "realistic" ones can be derived from more elaborated and better specified two–sector economies.

To simplify matters we begin by choosing a linear utility function $u(c) = c$, so that $V(x, x', k) = T(x, x', k)$. In light of the second part of example 2.3 above it is worth stressing that the same results would carry through with, say, a CES utility function. Only the algebra would be messier. The output of the consumption good is given by $c = (\ell^1)^\alpha (x^1)^{1-\alpha}$ and output of the investment good is given by $y = \min\{\ell^2, x^2/\gamma\}$, with

$\alpha, \gamma \in (0,1)$. The aggregate stock of capital k has also the effect of increasing the efficiency level of the otherwise exogenous unitary labour supply. In other words the external effect is assumed to be observationally equivalent to labour-augmenting technological progress. Denoting with ℓ_t the total number of efficiency units of labor at time t we represent the externality as $\ell_t = k_t^\eta$. The allocational constraint is then $\ell_t^1 + \ell_t^2 \leq \ell_t$, for each t. To simplify further we will also assume istantaneous depreciation. The PPF for the representative agent is then given by:

$$T(x, x', k) = (k^\eta - x')^\alpha (x - \gamma x')^{1-\alpha}.$$

Equilibria are those sequences $\{x_t\}_{t=0}^\infty$ that, given a sequence $\{k_t\}_{t=0}^\infty$ solve the parametric programming problem

$$\max \sum_{t=0}^{\infty} \delta^t (k_t^\eta - x_{t+1})^\alpha (x_t - \gamma x_{t+1})^{1-\alpha} \tag{3.3}$$

$$\text{subject to} : 0 \leq x_{t+1} \leq \min\left\{k_t^\eta, \frac{x_t}{\gamma}\right\}$$

and that also satisfy $x_t = k_t$ for all $t = 0, 1, 2, \ldots$.

The unique interior steady state solution to (3.3) is computed by solving the equation $T_2(x^*, x^*, x^*) + \delta T_1(x^*, x^*, x^*) = 0$, which gives:

$$x^* = \left\{ \frac{(\delta - \gamma)(1-\alpha)}{(\delta - \gamma)(1-\alpha) + \alpha(1-\gamma)} \right\}^{\frac{1}{1-\eta}}$$

Some tedious but nevertheless straightforward algebra will now prove the following theorem.

Theorem 3.1 *There exists an open set of values in the parameter space $(\alpha, \delta, \eta, \gamma)$, such that the equilibria of the growth model (3.3) are indeterminate.*

Proof: In light of the previous discussion it suffices to show the existence of some combinations of parameters at which the inequalities (3.2) are satisfied. The constants a_1 and a_2 can be computed as:

$$a_1 = \frac{z^{-1} - \gamma}{\delta} + \frac{1 - \eta (x^*)^{\eta - 1} z^{-1}}{z^{-1} - \gamma},$$

$$a_2 = \frac{1 - \eta (x^*)^{\eta - 1} z^{-1}}{\delta}.$$

where:

$$z = \frac{(x^*)^{\eta-1} - 1}{1-\gamma}.$$

It is then a simple numerical matter to verify that, for example, in a neighborhood of the parameter values $\alpha = .5$, $\delta = .5$, $\eta = .5$ and $\gamma = .2$, the inequalities (3.2) are all satisfied. The statement then follows from the continuity of the functions in (3.2). **Q.E.D.**

We like to stress that this is not meant to be a realistic model. The point of our exercises is simply that of pointing out the extreme qualitative difference between the one–sector and the two–sector formulation and the fact that indeterminate equilibria are pervasive in the latter.

3.2 Unbounded Accumulation Paths

As mentioned in the introduction indeterminacy is also possible for the two-sector model in the presence of endogenous growth. Again we will be satisfied with making our point by means of a very simple, almost trivial, example.

In order to better llustrate the equilibrium behavior in the presence of externalities we will begin this subsection with a brief analysis of the standard case. Once again there are two goods: a consumption good produced with a Cobb-Douglas technology $c = (x^1)^\alpha(\ell^1)^{1-\alpha}$, and an investment good produced with a linear one, $i = bx^2$. The aggregate capital stock x_t induces the constraint $x_t \geq x_t^1 + x_t^2$, and evolves according to the law of motion $x_{t+1} = (1-\mu)x_t + i_t$. Also in this case we will introduce a few innocuous simplifications: the utility function will be chosen to be linear and the exogenous labor supply ℓ will be set equal to one in every period.

One can write the PPF as $T(x, x') = (\gamma x - ax')^\alpha$, with $\gamma = 1 + (1-\mu)/b > 1$, and $a = 1/b$. The Euler Equation associated to this simple optimization problem can be easily manipulated to yeld a one dimensional map from current to future growth rates of the stock of capital:

$$\lambda_{t+1} = \tau(\lambda_t) \equiv \theta + (\delta\theta)^{\frac{1}{1-\alpha}} - \theta(\delta\theta)^{\frac{1}{1-\alpha}}\lambda_t^{-1} \tag{3.4}$$

where $\theta = b + (1-\mu) > 1$ is necessary to make persistent growth feasible. The function τ has two fixed points,

$$\lambda_1 = \theta, \text{ and } \lambda_2 = (\delta\theta)^{\frac{1}{1-\alpha}}.$$

The first root, $\lambda_1 = \theta$, should be ruled out as a possible equilibrium with constant growth as consumption is forever zero along such an accumulation path. For the second root to be an equilibrium we need to verify that the transversality condition is satisfied. At λ_2, (TC) requires $\delta\theta^\alpha < 1$. The latter inequality also guarantees that $\lambda_2 < \lambda_1$ and that λ_2 is an unstable fixed point of τ.

As we should have expected, in an optimal growth model without any external effect if an equilibrium exists it is also determinate.

We shall now proceed to modify this model by appending an external effect to the production function of the consumption good. Set $c = k^\eta (x^1)^\alpha$. Then the PPF faced by a representative consumer-producer becomes:

$$c_t = k_t^\eta (\gamma x_t - a x_{t+1})^\alpha \tag{3.5}$$

where, as usual, k_t denotes the aggregate capital stock which is treated parametrically by the representative agent. Given a $\{k_t\}_{t=0}^\infty$ equilibria are sequences $\{x_t\}_{t=0}^\infty$ solving

$$\max \sum_{t=0}^{\infty} k_t^\eta (\gamma x_t - a x_{t+1})^\alpha \delta^t \tag{3.6}$$

$$\text{subject to} : 0 \leq x_{t+1} \leq \theta x_t .$$

and satisfying $x_t = k_t$ for all t.

As in our previous treatment of the one–sector model we will restrict ourselves to the study of sequences with bounded growth rate. In this example it is always true that:

$$\limsup_{t \to \infty} \frac{x_{t+1}}{x_t} \leq \theta$$

Furthermore the functional forms have been chosen to guarantee that the Euler Equation associated to (3.6) can be written in the form $\psi(x, \lambda_t, \lambda_{t+1})$ and that by simple manipulation a map $\tau(\lambda_t) = \lambda_{t+1}$ can be derived that satisfies $\psi(x, \lambda, \tau(\lambda))$ independently of x. The latter is:

$$\lambda_{t+1} = \tau(\lambda_t) \equiv \theta - (\delta\theta)^{\frac{1}{1-\alpha}} \lambda_t^\beta (\theta - \lambda_t) \tag{3.7}$$

where $\beta = \frac{\alpha + \eta - 1}{1 - \alpha}$. Given an initial condition $\lambda_0 > 0$ every uniformly bounded trajectory of the dynamical system τ is candidate to be an equilibrium. In order to be one it has also to satisfy the appropriate transversality condition. Among the bounded trajectories a special role is played by the fixed points and the closed orbits of τ and our analysis will concentrate on them. Nevertheless, as we will briefly point out later, there are other more complicated orbits of τ that also satisfy (3.8) and therefore are equilibria. Some of them can be chaotic.

Along a balanced growth path with constant growth rate equal to λ the transversality condition reads as:

$$\lim_{t\to\infty} \alpha\gamma\delta^t x_t^{\eta+1}(\gamma x_t - a x_{t+1})^{\alpha-1} = \lim_{t\to\infty} \text{const}\cdot(\delta\lambda^{\alpha+\eta})^t = 0 \tag{3.8}$$

To prove our claim we only need to show that there exists a fixed point of τ that satisfy (3.8) and is asymptotically stable for the dynamics $\lambda_{t+1} = \tau(\lambda_t)$. This is spelled out in our last theorem. More generally, though, indeterminacy can also arise in the following more complicated fashion : there exists a subset $\Lambda \subset [0,\theta]$, which is an attractor for $\lambda_{t+1} = \tau(\lambda_t)$ and which contains a more than countable number of points. As the analysis of this case would lead us astray we prefer to bypass it here. We refer the reader to Boldrin [1982] for a more detailed study.

Theorem 3.2 *In the model of growth with externalities described by the programming problem* (3.6) *equilibria are indeterminate when the following restrictions are satisfied.*

- $\alpha + \eta > 1$,
- $\delta\theta < 1 < \delta\theta^{\alpha+\eta}$,
- $\lambda_2 > \theta - 1/\beta$.

Then $\lambda_2 = (\delta\theta)^{\frac{1}{(1-\alpha-\eta)}}$ *is the only constant growth rate that satisfies the transversality condition. It is also asymptotically stable under iterations of* (3.7).

Proof: See Appendix.

The form of indeterminacy described in our theorem is the familiar one in which for a given initial condition x_0 there exist an open interval of values of x_1 that are all consistent with equilibrium. These distinct trajectories grow asymptotically at a common rate λ_2 but need not converge to each other, i.e. they typically grow "parallel" forever. It is difficult to say if the parameter values at which this phenomenon occurs may be considered "realistic" or otherwise, mainly because the model we are using is rather simplified. To get an idea of the range of values we are considering let us play the parameterization game one more time. Choose a depreciation rate of about 10% and a capital/output ratio around 3.4 in the investment sector to obtain a value of θ equal to 1.2. With a relatively low discount factor, say $\delta = .80$ one needs $\alpha = .5$, $\eta = 1.$ to bring λ_2 around the "credible" value of 1.08. Then, as it can be easily verified, also the stability condition is satisfied and equilibria are indeed indeterminate. Everything clearly relies on the magnitude of the externalities and on their pervasiveness: a matter about which very little empirical evidence is available.

The indeterminate and chaotic equilibria we mentioned above arise at about the same parameter values when $\lambda_2 < \theta - 1/\beta$. τ is then a non-monotone mapping of the interval $[0, \theta]$ into itself for which both stationary states λ_1 and λ_2 are dynamically unstable.

One final comment on the interpretation to be given to the last theorem and to the case of "chaotic indeterminacy" we just outlined. According to this model two countries that start from the same initial stock and follow different equilibria from then on will display a *common average growth rate* in the long run and, while their capital stocks may persistently be different, (because different values of x_1 were chosen) we should not observe them growing apart in their relative conditions. In other words models of the type discussed here can account for the fact that certain countries never catch-up with the leader and for the fact that growth rates may be out of phase. On the other hand they cannot account for the fact that countries that started in almost similar conditions have been growing very differently, some of them reaching full economic development while other remained at the underdevelopment level. To explain "poverty traps" we have to look somewhere else, most probably to the positive interactions between the accumulation of human and physical capital stocks.

4. Conclusions

We have studied the determinacy of competitive equilibrium in infinite horizon models of capital accumulation with productive externalities.

In the standard one-sector model we have proved that equilibria converging to a steady state are always locally unique and that unbounded equilibria converging to a stationary growth rate are also locally unique under reasonably mild conditions. In such models indeterminacy may still arise around cyclic paths (which we have proved to be possible) but it seems quite difficult to obtain without substantially complicating the model.

The presence of endogenous oscillations is of interest by itself. It underlies the fact that external effects in production and the market incompleteness they imply may be useful in explaining the self–sustaining nature of the business cycles. The example we provide is not meant to be taken seriously: it is simply meant to remind that endogenous oscillations obtain at much more realistic parameter values once the complete markets assumptions is dropped. The relevance of this fact for business cycle theory should be investigated in the future.

We have also addressed the problem of indeterminacy within the context of a two–sector growth model again in the presence of an aggregate externality. In this case indeterminacy of equilibrium seems to be always possible and indeed appears quite easily even in the simplest model. For very standard functional forms of the utility and production functions and for parameter values that appear altogether not unreasonable there exists a continuum of distinct equilibria departing from a common initial stock of capital and either converging to the same steady state or growing asymptotically at a common rate.

The practical implications of these results cannot be fully evaluated given the simplified models adopted here. Further research along these lines should clarify if the phenomenon we have pointed out is robust with regards to a number of empirically relevant perturbations of the stylized models we have studied here. From the point of view of the theory of economic development an important extension is to models with more than one stock of capital (physical and human) and to models of technological change and/or industrialization. From the point of view of business cycle theory one would be curious as to what implications an endogenous labor supply and more realistic production functions would have on the model's dynamics. From a general perspective it seems that the study of multisector growth models with external effects is a promising avenue for the long overdue reconciliation between the theory of economic growth and the theory of the business cycle.

Appendix

Proof of Proposition 2.1 The proof is standard and will only be sketched here. Interiority of $\{x_t\}_{t=0}^{\infty}$ together with the (strict) concavity of $P(i)$ and $P(j)$ for given $\{p_t, \pi_t, k_t\}_{t=0}^{\infty}$ imply that the unique solution to each maximization problem is characterized by the following necessary and sufficient conditions:

$$\begin{cases} \delta^t u'(c_t) = \lambda p_t, \ \lambda > 0, \\ \sum_{t=0}^{\infty} p_t c_t = \sum_{t=0}^{\infty} \pi_t + q_0 x_0, \\ \lim_{t\to\infty} p_t c_t = 0. \end{cases}$$

for $P(i)$ and,

$$\begin{cases} p_0 f_1(x_0, k_0) = q_0, \\ p_t f_1(x_t, k_t) = p_{t-1}, \\ \lim_{t\to\infty} p_t x_t = 0. \end{cases}$$

for $P(j)$. Now set $k_t = x_t$ for all t and assume (EE) and (TC) are satisfied. By substitution one verifies that $c_t = f(x_t, x_t) - x_{t+1}$ satisfies a) and that x_t satisfies b) of definition 2.1. As c) is satisfied by construction the candidate sequence is an equilibrium. The only if part can be obtained similarly by manipulating the necessary conditions given above to show that (EE) and (TC) are satisfied. **Q.E.D.**

Proof of Proposition 2.2 Equilibrium paths are nonnegative by construction. If $x_0 > \overline{y}$, $x_t < x_{t-1}$ for all $t \geq 1$, and if $x_0 \leq \overline{y}$ then $x_t \leq \overline{y}$ for all t. A stationary equilibrium point $x^* \in (0, \overline{x})$ exists if, setting $c^* = f(x^*, x^*)$, the two conditions (EE) and (TC) are satisfied by the pair x^*, c^*. This is easily verified. **Q.E.D.**

Proof of Theorem 2.1 A monotone equilibrium satisfies either $x_{t+1} \geq x_t$ or $x_{t+1} \leq x_t$ for all t. It will therefore converge to some x^* because of Proposition 2.2. Local uniqueness follows if all values of x^* that solve $f_1(x, x) = \delta^{-1}$ are either saddle points or sources for the dynamical system induced by (EE) on $[0, \overline{y}] \times [0, \overline{y}]$. This can be verified by linearizing (EE) around a stationary point and computing the associated eigenvalues. They are the two roots of a quadratic polynomial of the type: $\lambda^2 + a_1\lambda + a_2 = 0$, with $a_1 = -(1 + F' + \sigma f_1'/f_1)$, and $a_2 = F'$, where all functions are evaluated at x^* and the symbols $F' = f_1 + f_2$, $f_1' = f_{11} + f_{12}$, $\sigma = u'/u''$ have been introduced. Then x^* is a sink if both roots are less than one in modulus. This requires, among other things, that $a_2 < 1$. But: $a_2 = f_1 + f_2 = \delta^{-1} > 1$ at x^*. **Q.E.D.**

Proof of Theorem 2.2 First notice that $x^* \in (0, \overline{x})$. We need only to prove that all equilibria are monotone. The rest follows from Theorem 2.1. We will articulate the proof in a lemmata.

Lemma 1. If $x_t \leq x^*$, then $c_t \geq c_{t-1}$ and if $x_t \geq x^*$ then $c_t \leq c_{t-1}$, (strict inequality in x implies strict inequality in c).

Proof: If $x_t \leq x^*$,$\delta f_1(x_t, x_t) \geq 1$ will hold, which implies $u'(c_{t-1})/u'(c_t) = \delta f_1(x_t, x_t) \geq 1$ and so $c_t \geq c_{t-1}$ because u is concave. Similarly when $x_t \geq x^*$.

Lemma 2. If $x_t \leq x^*$ then $x_{t+1} \geq x_t$.

Proof: Lemma 1 implies already $c_t \geq c_{t-1}$. Assume that $x_{t+1} < x_t$. Then (EE) implies:

$$\frac{u'(c_{t-1})}{u'(c_t)} = \frac{u'(c_t) f_1(x_t, x_t)}{u'(c_{t+1}) f_1(x_{t+1}, x_{t+1})} \geq 1 \tag{$*$}$$

We will show that a contradiction with (*) arises. To do this, notice first that $x_{t+1} < x_t$ implies $c_{t+1} < c_t$. In fact, if $c_{t+1} \geq c_t$ and $x_{t+1} < x_t \leq x^*$, then $x_{t+2} = F(x_{t+1}) - c_{t+1} < F(x_t) - c_t = x_{t+1}$ and so $x_{t+2} < x^*$, which implies (by Lemma 1) that $c_{t+2} \geq c_{t+1}$. This in turn gives $x_{t+3} = F(x_{t+2}) < F(x_{t+1}) - c_{t+1} = x_{t+2}$. By iteration the sequence $\{x_{t+i}\}_{i=0}^{\infty}$ satisfies $x_{t+i} < x_{t+i-1} \leq x^*$ for all $i \geq 1$ and the sequence $\{c_{t+i}\}_{i=0}^{\infty}$ satisfies $c_{t+i} \geq c_{t+i-1} > 0$ for all $i \geq 1$. Let $\underline{x} < x^* = \lim_{i \to \infty} x_{t+i}$ and $\overline{c} = \lim_{i \to \infty} c_{t+i}$. Then $\overline{c} > 0$, and $\overline{c}$ is finite because $\underline{x} < x^*$ implies $f(\underline{x}, \underline{x})$ is bounded. Hence, $u'(\overline{c}) \in (0, \infty)$ and $\delta f_1(\underline{x}, \underline{x}) = 1$ has to hold, which contradicts $\underline{x} < x^*$. So $x_{t+1} < x_t$ implies $c_{t+1} < c_t$.

Now recall that f_1 is non-increasing and u' is decreasing, then $x_{t+1} < x_t$ implies:

$$u'(c_t) f_1(x_t, x_t) \leq u'(c_t) f_1(x_{t+1}, x_{t+1}) < u'(c_{t+1}) f_1(x_{t+1}, x_{t+1}),$$

which contradicts (*). Therefore, $x_t \leq x^*$ implies $x_{t+1} \geq x_t$.

Lemma 3. If $x_t \leq x^*$ then $x_t \leq x_{t+1} \leq x^*$.

Proof: Only the part $x_{t+1} \leq x^*$ needs to be proved. Again, pretend $x_{t+1} > x^*$. Then (by Lemma 1) $c_{t+1} < c_t$ will hold and $x_{t+2} = F(x_{t+1}) - c_{t+1} > F(x_t) - c_t = x_{t+1}$ and, as in Lemma 2, iterations will give two sequences, $\{x_{t+i}, c_{t+i}\}_{i=0}^{\infty}$ with $x_{t+i+1} > x_{t+i} > x^*$ and $c_{t+i} < c_{t+i-1}$. Once again set $\lim_{i \to \infty} x_{t+i} = \overline{x} > x^*$ and $\lim_{i \to \infty} c_{t+i} = \underline{c}$. If $\underline{c} > 0$, then $u'(\underline{c})$ is finite and $\delta f_1(\overline{x}, \overline{x}) = 1$ has to hold, which contradicts $\overline{x} > x^*$. If $\underline{c} = 0$ and $u'(\underline{c})$ is not finite then, for i large enough, $f_1(x_{t+i}, x_{t+i}) \leq \gamma < 1$ must hold. Hence: $u'(c_{t+i+1}) = [\delta f_1(x_{t+i+1}, x_{t+i+1})]^{-1} u'(c_{t+i}) \geq (\delta\gamma)^{-1} u'(c_{t+i})$, which implies: $u'(c_{t+i}) \geq$

$(\delta\gamma)^{-i}\underline{u}$ (for some constant $\underline{u}$ and i large). The latter gives: $\lim_{t\to\infty} x_t f_1(x_t, x_t)\delta^t u'(c_t) \geq \lim_{t\to\infty} x_t f_1(x_t, x_t)(\gamma)^{-t}\underline{u} = +\infty$. This contradicts (TC) and proves the Lemma.

Lemma 4. If $x_t \geq x^*$ then $x_t \geq x_{t+1} \geq x^*$.

Proof: One needs only to replicate the proofs to lemmata 1-3, with the appropriate changes in the inequalities. Now lemmata 3 and 4, together with theorem 2.1 are equivalente to the statement of theorem 2.2. **Q.E.D.**

Proof of Theorem 2.3 Begin by noticing that Proposition 2.3 together with monotonicity of the accumulation paths leaves only the following two possible scenarios under which equilibria are not unique:

1) Given x_0 there exists a nontrivial set of values for x_1 giving origin to orbits that are parallel to each other from a certain period onward.
2) Given x_0 there exists a nontrivial set of values for x_1 giving origin to distinct orbits, i.e. to orbits that have different asymptotic behavior.

Case 2) will be ruled out by proving that there exists a unique constant growth rate. We will also show that the latter is unstable (under iterations of the map θ_∞) which rules out case 1). In fact stricy concavity of u implies that the functions θ_x and θ_∞ are never constant. Hence equilibria cannot become parallel "in one period" but they can do so only asymptotically. This requires convergence to the unique growth rate λ^* that we will show instead to be dynamically unstable.

Begin by using the continuity of u' to write the Euler Equation (at large enough values of x) as

$$-u'[(L-\lambda)x] + u'[(L-\theta_x(\lambda))\lambda x]\delta\pi = 0 \qquad (EE_1)$$

Notice further that the fixed points of θ_x that are balanced growth equilibria are solutions to:

$$\frac{u'(c)}{u'(\lambda c)} = \delta\pi \qquad (EE_2)$$

that satisfy the (TC) condition. Due to the strict concavity of u the latter has a unique solution $\lambda^*(c)$ for every given c (or equivalently x). Inspection of (EE_1) and use of the implicit function theorem also shows that for large enough values of x the functions θ_x satisfy the following properties:

a) θ_x is monotone increasing;
b) $x > x'$ implies $\theta_x(\lambda) > \theta_{x'}(\lambda)$;

c) $\theta_x(\lambda_i) = \lambda_i$, for $i = 1, 2$, and $\lambda_1 > \lambda_2$ imply that $\theta_x'(\lambda_1) < \theta_x'(\lambda_2)$;

d) $\theta_x(1) < 1$, and $\theta_x(L) < L$.

Applying the implicit function theorem to (EE_2) finally proves that:

f) if $\lambda^*(c)$ solves (EE_2) and $c > c'$ then $\lambda^*(c) < \lambda^*(c')$.

Property a) follows from strict concavity of the utility function while properties b), c), d) and f) follow from the assumption that u displays non–decreasing elasticity of substitution. All together they imply that for large values of x the function θ_x is as depicted in figure 1. More precisely θ_x has two interior fixed points, $1 < \lambda_1(x) < \lambda_2(x) < L$, the smallest of which is also the unique solution to (EE_2), i.e. $\lambda_1(x) = \lambda^*(c)$, $c = (L - \lambda_1)x$, (this last fact being a consequence of b) and f) together). Clearly $\lambda_1(x)$ is unstable and $\lambda_2(x)$ stable under iteration of θ_x.

Assumption 2.7 then implies that the fixed points of θ_∞ are the (uniform) limits (in the interval $[1, L]$) of the fixed points of the sequence of functions θ_x. The balanced growth equilibrium is then given by the unique limit $\lambda^* \geq 1$ of the sequence $\lambda_1(x)$ $(\lambda^*(c))$ as $x \to \infty$ $(c \to \infty)$. **Q.E.D.**

Proof of Theorem 3.2 Derivation of (3.7) from (EE) is a simple matter of algebra. Similarly it is straightforward to verify that when $\alpha + \eta = 1$ the function τ has only one fixed point equal to θ. When $\alpha + \eta \neq 1, \tau$ has the two fixed points $\lambda_1 = \theta$, $\lambda_2 = (\delta\theta)^{\frac{1}{(1-\alpha-\eta)}}$. The transversality condition reduces to $\delta\lambda_i^{\alpha+\eta} < 1$. The case $\alpha + \eta < 1$ is similar to the model without externality. It is easy to see that the root λ_2 is the unique equilibrium and that it is unstable.

The case $\alpha + \eta > 1$ requires a few extra computations. Here $\beta > 0$, so that $\tau(0) = \theta > 1$, $\tau(\theta) = \theta$, and $\tau'(\lambda) = \left(\delta\theta\right)^{\frac{1}{1-\alpha}} \lambda^\beta \left(1 - \beta(\theta - \lambda)\right)$. This implies, in particular that $\tau'(\lambda_1) > 0$ whereas $\tau'(\lambda_2)$ may be of either sign. The condition $\delta\theta^{\alpha+\eta} > 1$ guarantees at once that $\lambda_1 > \lambda_2$, and that λ_2 satisfies the trasversality condition. To check that λ_2 is stable one has only to notice that τ has a minimum at $\lambda^* = \theta - 1/\beta$ and that our last condition is equivalent to $\lambda^* < \lambda_2$. **Q.E.D.**

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Figure 1: Shape of the Asymptotic Map