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DURATION OF DEBT OVERHANG WITH TWO LENDER BANKS

by

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Abstract:

This paper discusses the duration of the debt overhang with two lender banks. We model the problem as an infinite horizon game with two banks as players. In every period, each bank decides either to sell its loan exposure to the debtor country at the present secondary market price, or to wait and keep its exposure to the next period. Under the assumptions of homogeneous price function and short length of periods, we show that the expected duration in the equilibrium becomes large when the degree of homogeneity is low, and tends to $\ln 2/ \ln \beta^2$ (β is the annual interest factor) as the degree of homogeneity approaches zero. This result implies that the lower bound for the duration is 4 years. We interpret it as a tendency for the debt overhang to last for a somewhat long time.

1. Introduction

Since 1982 the problem of debt overhang has been present in the financial relations between the less developed countries and the foreign commercial banks. The debt overhang is a serious problem for the debtor countries as well as for the banks. On one hand, it keeps the debtor country in a bad economic situation and prevents from growing, and on the other hand, it deteriorates the financial situations of the lender banks. The problem of debt overhang has been accompanied with the secondary market for debts where loan exposures of banks are traded at a discounted price. The existence of the secondary market creates a possibility for the debtor country to buy back its debts at a discounted price, and it gives an opportunity for the lender banks to sell their "bad" loans and recoup at least part of their money. However, it is observed that the debt overhang continues to exist even in the presence of the secondary market.

To understand the problem of debt overhang, Kaneko and Prokop (1991) and Prokop (1991) give game theoretical models of debt overhang where lender banks are players and decide whether or not to sell their loan exposures to the debtor country at the discounted price on the secondary market. Their analyses show that there is a great tendency for the present situation of the debt overhang to remain almost unchanged. It remains to investigate the duration of the debt overhang, i.e., how long it takes for the problem of debt overhang to disappear. The present paper seeks to answer this question.

We work on the problem formulated in Prokop (1991), which is an infinite horizon game with two lender banks as players. Since lender banks are very cautious and simultaneously instanteneous decision makers, the length of a period of the game should be regarded as short or almost zero. To describe this feature of a short period and to consider the effect of the length of a period on the duration of debt overhang, we devide each annual period of the game into n subperiods in which the decision making takes place. In every subperiod of the game each bank decides either to sell its loan exposure to the debtor country at the current secondary market price or to wait and postpone this decision to the next subperiod. By increasing the number of subperiods n, we capture the situation with very short periods of decision making. This treatment of the varying length of a period can be done in the framework of Prokop (1991).

First, under the assumption of the homogeneity of the price function, we show that the equilibrium probability of a bank waiting in every subperiod is close to one, when the degree of homogeneity is high. When the degree of homogeneity is low, the equilibrium probability of a bank waiting in every subperiod is bounded away from one, but it still remains quite high.

Second, we focus directly on the duration of debt overhang in equilibrium. Under the assumptions of homogeneous price function and short length of subperiods, we show that the expected duration of debt overhang in the central equilibrium becomes large when the degree of homogeneity is low. When the degree of homogeneity approaches zero , the expected duration tends to $\ln 2/\ln \beta^2$, where β is the annual interest factor. This result implies that the lower bound for the duration is 4 years. We interpret it as a tendency for the debt overhang to last for a long time.

Next, we calculate the average ending price, i.e., the average level of the secondary market price at which banks sell their loan exposures. We show that if the price function is homogeneous of degree k, then the expected ending price becomes close to $P(D^0)\ \dfrac{2}{1+2^{-k}}$ when the legth of a subperiod tends to zero. This ending price tends to zero when the degree of homogeneity of the price function is low, and remains almost unchanged when the degree of homogeneity approaches zero.

Finally, we consider the duration of debt overhang in an example with a price function which is almost homgeneous for the large values of debt. In the example, the expected equilibrium duration of debt overhang still remains quite long when the length of a subperiod tends to zero. This may suggest that our conclusions about the long duration of debt overhang could be extended on a class of price functions larger than the homogeneous one.

2. Dynamic Endurance Game and Its Equilibria

We consider a country who has debt obligations of equal size to two foreign banks. The country has been unable to maintain the service payments on the outstanding debts for some periods. We assume the existence of the secondary market for debts, where the country is ready to buy its debts at the secondary market price. The price of debt on the secondary market is assumed to depend on the current total outstanding debts. This price is expressed by the function $P(D)\colon R_+ \to R_+$ with the property:

(2.1) P(D) is a decreasing function of the total outstanding debt D and P(D) $\to$ 0 as $D \to \infty$.

This P(D) is country-specific and also depends upon the choice of the present period.

The present period is called 0 and the length of each original period is one year. We devide each period into n subperiods of equal length in which decisions of banks are made. In every subperiod each bank has two possible choices either to sell its loan exposure or to wait and keep it to the next subperiod. Each bank's pure strategies in every subperiod are s and w, where s denotes selling the loan exposure to the debtor country at the current secondary market price, and w denotes waiting and postponing the decision to the next subperiod.

Each bank discounts the future revenues by the interest rate ($r > 0$). We denote the annual interest factor $1 + r$ by β. We assume that

(2.2) $$\beta^2 < 2.$$

Since the banks make their decisions in each subperiod, we need the interest factor for a subperiod. The interest factor for a subperiod $(1 + r_n) = \beta_n$ must give the accumulated interest from n subperiods equal to the annual interest rate, i.e., $(1 + r_n)^n = 1 + r$, or

(2.3) $$\beta_n = \beta^{1/n}.$$

Consider the situation when bank $i = 1,2$ keeps its loan exposure $D^0/2$ until subperiod t ($t \geq 0$). In this case, its exposure increases by the accrued interests to $\beta_n{}^t D^0/2$. The total outstanding debt D^t in subperiod t becomes $\beta_n{}^t D^0/2$ when the other bank has already sold, or $\beta_n{}^t D^0$ otherwise. The secondary market price in subperiod t is given as $P(D^t)$. If bank i sells in subperiod t, then the present value of repayment is given as

$$\frac{1}{\beta_n{}^t}\, \beta_n{}^t \frac{1}{2}\, D^0 P(D^t) = \frac{1}{2}\, D^0 P(D^t).$$

If bank i and the other bank wait in subperiod t, then bank i does not get any payoff in this subperiod, but it will face the same decision problem in the next subperiod.

We assume that

(2.4) after one bank sells its loan exposure, the other bank sells its exposure immediately in the next subperiod.

Although one bank could keep its exposure for several subperiods after the other sells its exposure, this behavior of keeping the loan exposure is not optimal, since the secondary market price of debt will necessarily decrease. Thus we can assume (2.4) without loss of generality.

There are two cases in which the game terminates. The first case is that both banks wait until subperiod $t-1$ and both sell in subperiod t. The second case is that both banks wait until subperiod $t-1$ and one bank sells its loan exposure in subperiod t, and the other bank waits in subperiod t and sells its exposure in subperiod $t+1$. The payoff to bank i = 1, 2 in the game $\Gamma_n(0, D^0)$ is defined by

$$
(2.5)\qquad \begin{cases} \dfrac{1}{2} D^0 P(\beta_n{}^t D^0) & \text{if both banks wait until subperiod } t-1 \text{ and bank } i \text{ sells in subperiod } t, \\[2ex] \dfrac{1}{D^0} P(\beta_n{}^{t+1} D^0/2) & \text{if both banks wait until subperiod } t-1 \text{ and bank } j\ (j \neq i) \text{ sells in subperiod } t \text{ and bank } i \text{ waits in subperiod } t. \end{cases}
$$

Our game $\Gamma_n(0, D^0)$ with the interest factor β_n is described in Figure 1. In the game tree, the payoffs to the banks are given in three branches. In the first branch both banks sell their exposures in subperiod 0. In the second, bank 1 sells in subperiod 1 and bank 2 sells in subperiod 2. The third case is that bank 2 sells its exposure in subperiod 2 and bank 1 sells its exposure in subperiod 3.

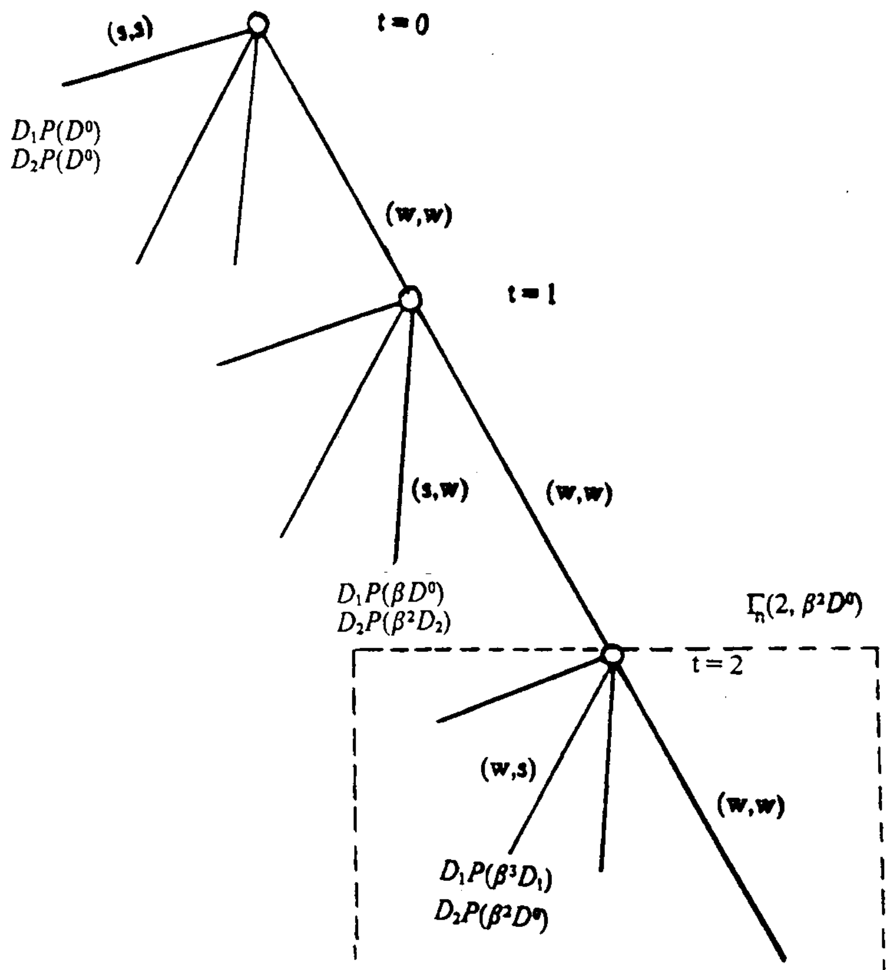

Figure 1.

We allow each bank to use behavior strategies. Since each bank's decision is made in subperiod t only when both banks have waited until subperiod t by assumption (2.4), a behavior strategy of bank $i = 1, 2$ in the game $\Gamma_n(0, D^0)$ is represented as a sequence $b_i^0 = (p_i^0, p_i^1, ...)$, where p_i^t is a probability of bank i's waiting in subperiod t $(t = 0, 1, ...)$ if both banks keep their loan exposures until subperiod t. Denote the set of all behavior strategies of bank i by B_i^0. A behavior strategy combination for the game $\Gamma(0, D^0)$ is a vector $b^0 = (b_1^0, b_2^0) = ((p_1^0, p_1^1, ...), (p_2^0, p_2^1, ...))$.

The expected payoff to bank $i = 1, 2$ for a behavior strategy combination b^0 is the sum of

(i) the expected payoff from selling the exposure in subperiod t $(t = 0, 1, ...)$ under the assumption that the other bank does not sell earlier – $\frac{1}{2} D^0 P(\beta_n{}^t D^0)(1 - p_i^t) \prod_{k=0}^{t-1} p_i^k p_j^k$; and

(ii) the expected payoff from selling its loan exposure in subperiod $t + 1$ under the assumption that the other bank sells in period t – $\frac{1}{2} D_i P(\beta_n{}^{t+1} D^0/2) p_i^t (1 - p_j^t) \prod_{k=0}^{t-1} p_i^k p_j^k$.

Thus the expected payoff from the game $\Gamma_n(0, D^0)$ under the strategy combination b^0 is given by

$$H_i^0(b^0) = \sum_{t=0}^{\infty} \frac{1}{2} D^0 P(\beta_n{}^t D^0)(1 - p_i^t) \prod_{k=0}^{t-1} p_i^k p_j^k + \sum_{t=0}^{\infty} \frac{1}{2} D^0 P(\beta_n{}^{t+1} D^0/2) p_i^t (1 - p_j^t) \prod_{k=0}^{t-1} p_i^k p_j^k. \tag{2.6}$$

We use the convention $\prod_{k=0}^{-1} p_i^k p_j^k = 1$.

We have described the dynamic endurance game of lender banks $\Gamma_n(0, D^0)$. To investigate the decisions of banks in the game $\Gamma_n(0, D^0)$, we adopt the concept of the subgame perfect equilibrium of the extensive game (Selten (1975)).

In addition to the subgame perfection, we require the equilibrium to have the time continuation property. To define the time continuation property, we introduce a <u>retrospective extension</u> of the game $\Gamma_n(0, D^0)$ as a game $\Gamma_n(t, \beta^t D^0)$ for $t = -1, -2, ...$, so that the game $\Gamma_n(0, D^0)$ is a subgame of the game $\Gamma_n(t, \beta^t D^0)$. A subgame perfect equlibrium $\hat{b}^0$ is said to have the <u>time continuation property</u> iff for any $t = -1, -2, ...$ there is a subgame perfect equilibrium $\hat{b}^t$ in the retrospective extension $\Gamma_n(t, \beta_n{}^t D^0)$ of $\Gamma_n(0, D^0)$ such that $\hat{b}^t$ induces $\hat{b}^0$ and the realization probability of the subgame $\Gamma_n(0, D^0)$ is positive.

The time continuation property states that the present game situation results as a continuation of the past history. The game $\Gamma_n(0, D^0)$ is a result of previous decisions of banks. Therefore the present game is a subgame of the game of any preceding period. If the realization probability of the game $\Gamma_n(0, D^0)$ is zero in an equilibrium for $\Gamma_n(t, \beta_n{}^t D^0)$ then the present situation would be different from $\Gamma_n(0, D^0)$. However, we assume that the game $\Gamma_n(0, D^0)$ is reached. Therefore it is compatible with the consideration of $\Gamma_n(0, D^0)$ to assume that the realization probability is positive.[1]

Technically speaking our game is the same as the game of Prokop (1991) except for the different interest factor. Thus applying Theorem 1 of Prokop (1991) to the game $\Gamma_n(0, D^0)$ with interest factor β_n, we obtain that our game has three types of subgame perfect equilibria satisfying the time continuation property: central, alternating, and mutating. However, as observed in Prokop (1991), each equilibrium is similar to the unique central equilibrium in the sense that the probabilities of a bank waiting are determined by similar formulas and take similar values. Therefore we focus our considerations on the characterization of the central equilibrium.

In the central equilibrium $\hat{b}^0 = (\hat{b}_1^0, \hat{b}_2^0) = ((\hat{p}_1^0, \hat{p}_1^1, ...), (\hat{p}_2^0, \hat{p}_2^1, ...))$, bank i = 1, 2 waits in every period t $(t = 0, 1, ...)$ with probability

$$(2.7) \qquad \hat{p}_i^t = \hat{p}^t = \frac{P(\beta_n^{t+1} D^0/2) - P(\beta_n^t D^0)}{P(\beta_n^{t+1} D^0/2) - P(\beta_n^{t+1} D^0)}.$$

Prokop (1991) showed under the assumption of continuity of the price function that in every equilibrium of the dynamic game the probability of a bank waiting in each subperiod is close to 1 when the interest factor is low. Here, we look at the changes in the probability of a bank waiting when the shape of the price function changes. Throughout Section 3, we assume that

(2.8) the price function $P(\cdot)$ is homogeneous of degree k,

i.e., $P(\lambda D) = \lambda^k P(D)$, where $\lambda \in R$ and $k < 0$. Since the degree of homogeneity k determines the shape of the price function, we investigate the changes of the probabilities $\hat{p}^t$ and $\tilde{p}^t$ of bank waiting in subperiod t as the degree of homogeneity changes.

Under the assumption (2.8), the probabilities of a bank waiting in each subperiod t in the central equilibrium are independent of t, and equal

$$(2.9) \qquad \hat{p}^t = \frac{P(\beta_n D^0/2) - P(D^0)}{P(\beta_n D^0/2) - P(\beta_n D^0)} = \frac{(\beta_n D^0/2)^k - (D^0)^k}{(\beta_n D^0/2)^k - (\beta_n D^0)^k} = \frac{1 - (\beta_n/2)^{-k}}{1 - (1/2)^{-k}}.$$

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[1] The time continuation property is a concept independent from the time consistency in the macroeconomic literature. The time consistency property is, instead, implied by the subgame perfection.

We have the following global behavior of the central equilibrium probabilities of waiting in each subperiod.

Proposition 1. If the price function is homogeneous of degree $k < 0$, then for every t = 0, 1, ...,

(i) $\hat{p}^t \to 1$ as $k \to -\infty$;

(ii) $\hat{p}^t \to 1 - (\ln \beta^{1/n} / \ln 2)$ as $k \to 0$.

Proof. (i) It follows directly from (2.9) and the assumption (2.2);

(ii) From (2.9) by the de L'Hospital rule, we have

$$\lim_{k \to 0} \hat{p}^t = \lim_{k \to 0} \frac{1 - (\beta_n/2)^{-k}}{1 - (1/2)^{-k}} = \lim_{k \to 0} \frac{(\beta_n/2)^{-k} \ln(\beta_n/2)}{(1/2)^{-k} \ln(1/2)} = \frac{\ln(\beta_n/2)}{\ln(1/2)} = 1 - (\ln \beta_n / \ln 2).$$

Using equality (2.3), we obtain claim (ii). //

Claim (i) of the proposition says that the central equilibrium probabilities of a bank waiting in each subperiod are close to 1 as the degree of homogeneity of the price function becomes high. Claim (ii) says that when the degree of homogeneity becomes low the probability of a bank waiting in the central equilibrium is close to $1 - (\ln \beta^{1/n} / \ln 2)$. Since $1 < \beta < 2^{1/2}$, the central equilibrium probability of a bank waiting in each subperiod is greater than 1/2.

The relatively high probability of a bank waiting in each subperiod in the central equilibrium of the game $\Gamma_n(0, D^0)$ suggests that the situation of debt overhang may continue to exist for some time. In the next section, we estimate the duration of the debt overhang.

3. The Duration of Debt Overhang

We consider the space of events that the problem of debt overhang disappears in subperiod $t = 0, 1, \ldots$. On this space, we define a random variable $\hat{X}_n$ by

(3.1) $\hat{X}_n = \frac{1}{n} t$ if the debt overhang disappears in subperiod $t = 0, 1, \ldots$ in the central equilibrium.

There are three ways in which the debt overhang disappears in subperiod t in the central equilibrium. First is when both banks wait until subperiod $t - 1$ and both sell their exposures to the debtor country in subperiod t. Second and third way is when both banks wait until subperiod

$t-1$ and one bank sells its loan exposure to the debtor country in subperiod $t-1$ and the other bank waits in subperiod $t-1$ and sells its loan exposure in subperiod t. That is,

$$(3.2)\qquad Pr(\hat{X}_n = \tfrac{1}{n}\, t) = \begin{cases} (1-\hat{p}^0)^2 & \text{when } t = 0 \\ \prod_{\tau=0}^{t-1}(\hat{p}^\tau)^2\,(1-\hat{p}^t)^2 + 2\prod_{\tau=0}^{t-2}(\hat{p}^\tau)^2\,(1-\hat{p}^{t-1})\,\hat{p}^{t-1} & \text{when } t \geq 1. \end{cases}$$

We use the convention that $\prod_{\tau=0}^{-1}(\hat{p}^\tau)^2 = 1$.

Since $\hat{p}^\tau$ is the same for $\tau = 0, 1, \ldots$, it is not difficult to check that $\sum_{t=0}^{\infty} Pr(\hat{X}_n = \frac{1}{n}\, t) = 1$.

Now we can state the main results of our paper.

Theorem 1. If the price function is homogeneous of degree $k < 0$ and the length of a subperiod is $\frac{1}{n}$, then the expected duration of debt overhang in the central equilibrium $E(\hat{X}_n)$ tends to $\dfrac{2^{-k}-1}{2\,(-k)\ln\beta}$ as $n \to \infty$.

Theorem 1 gives an explicit formula for the expected duration of debt overhang. For example, an annual interest rate of 10% and a degree of homogeneity $k = -1$ give the duration of 3.8 years. Denote the expected duration by $E(X_\infty)$. It has the following properties.

Proposition 2. If the price function is homogeneous of degree $k < 0$, then

(i) $E(X_\infty)$ is a decreasing function of the degree of homogeneity k;

(ii) $E(X_\infty) \to \infty$ as $k \to -\infty$;

(iii) $E(X_\infty) \to \dfrac{\ln 2}{\ln \beta^2}$ as $k \to 0$.

From the above proposition, we can conclude that the expected duration of debt overhang has a lower bound equal to $\ln 2/\ln\beta^2$. For example, an annual interest rate smaller than 10% gives the expected duration at least four years. The duration becomes even longer when the degree of homogeneity takes large negative values. This is a long duration for only two lender banks.

Now we consider the average ending price, defined as the level of the secondary market price at which the debt overhang is reduced in the central equilibrium. We define a random variable $\hat{P}_n$ by

$$\hat{P}_n = P(\beta_n{}^t D^0) \qquad \text{if the debt overhang is reduced in subperiod } t = 0, 1, \ldots .$$

The debt overhang is reduced in subperiod t when at least one bank sells its loan exposure subperiod t. That is,

$$Pr\left(\hat{P}_n = P(\beta_n{}^t D^0)\right) \;=\; \prod_{\tau=0}^{t-1} (\hat{p}^\tau)^2 \left(1 - (\hat{p}^t)^2\right) \qquad \text{for } t \geq 0.$$

Since $\hat{p}^\tau$ is the same for $\tau = 0, 1, \ldots$, it is not difficult to check that $\sum_{t=0}^{\infty} Pr\left(\hat{P}_n = P(\beta_n{}^t D^0)\right) = 1$.

We have the following theorem.

Theorem 2. If the price function is homogeneous of degree $k < 0$ and the length of a subperiod is $\frac{1}{n}$, then the expected ending price $E(\hat{P}_n)$ tends to $P(D^0)\ \dfrac{2}{1 + 2^{-k}}$ as $n \to \infty$.

Theorem 2 gives an explicit formula for the average ending price. For example, a degree of homogeneity $k = -1$ gives the expected ending price equal to $\frac{2}{3} P(D^0)$. Denote the expected ending price by $E(P_\infty)$. It has the following properties.

Proposition 3. If the price function is homogeneous of degree $k < 0$, then

(i) $E(P_\infty) \to 0$ as $k \to -\infty$;

(ii) $E(P_\infty) \to P(D^0)$ as $k \to 0$.

The above proposition says that the expected ending price tends to zero when the degree of homogeneity decreases and remains almost unchanged when the degree of homogeneity tends to zero. The prediction that the expected ending price tends to zero when the degree of homogeneity takes large negative values is compatible in this case with the infinite duration of debt overhang following from Proposition 2.

Before proving the above claims, we first give the following lemma.

Lemma 1. For $x \in (-1, 1)$, it holds $\sum_{t=1}^{\infty} t\,(x)^{t-1} \;=\; \dfrac{1}{(1-x)^2}$ (Leja (1979, p. 149)).

Proof of Theorem 1. We have

$$E(\hat{X}_n) = \frac{1}{n}\, 0\, (1 - \hat{p}^0)^2 + \frac{1}{n} \sum_{t=1}^{\infty} t \left[\prod_{\tau=0}^{t-1} (\hat{p}^\tau)^2 (1 - \hat{p}^t)^2 + 2 \prod_{\tau=0}^{t-2} (\hat{p}^\tau)^2 (1 - \hat{p}^{t-1})\, \hat{p}^{t-1} \right].$$

Since $\hat{p}^\tau$ is the same for $\tau = 0, 1, \ldots$, we may drop the superscript and denote $\hat{p}^\tau = \hat{p}$. Now, we obtain

$$E(\hat{X}_n) = \frac{1}{n} \sum_{t=1}^{\infty} t \left[((\hat{p})^2)^t (1 - \hat{p})^2 + 2\left((\hat{p})^2\right)^{t-1} (1 - \hat{p})\, \hat{p} \right] = \frac{1}{n} \left[(\hat{p})^2 (1 - \hat{p}^t)^2 + 2\,\hat{p}\,(1 - \hat{p}) \right] \sum_{t=1}^{\infty} t\, ((\hat{p})^2)^{t-1}.$$

It follows from Lemma 1 for $x = (\hat{p})^2 \in (0, 1)$ that $\sum_{t=1}^{\infty} t\,((\hat{p})^2)^{t-1} = \dfrac{1}{(1-(\hat{p})^2)^2}$. Thus

$$E(\hat{X}_n) = \frac{1}{n}\left[(\hat{p})^2 (1-\hat{p})^2 + 2\hat{p}\,(1-\hat{p})\right]\frac{1}{(1-(\hat{p})^2)^2} = \frac{1}{n}\,\frac{\hat{p}^2}{1+(\hat{p})^2} + \frac{2\hat{p}}{n\,(1-\hat{p})\,(1+\hat{p})^2}.$$

We take the limit of the last expression when $n \to \infty$. Since $\hat{p} \to 1$ as $n \to \infty$, we obtain

$$\text{(3.3)} \qquad \lim_{n\to\infty} E(\hat{X}_n) = 0 + \frac{1}{2\lim_{n\to\infty} n(1-\hat{p})}.$$

We have the following lemma.

Lemma 2. $n(1-\hat{p}) \to \dfrac{(-k)\ \ln\beta}{2^{-k}-1}$ as $n \to \infty$.

Proof. From (2.9) and (2.3), we have

$$\lim_{n\to\infty} n(1-\hat{p}) = \lim_{n\to\infty} n\frac{(\beta^{1/n})^{-k}-1}{2^{-k}-1} = \frac{1}{2^{-k}-1}\lim_{n\to\infty}\frac{(\beta^{1/n})^{-k}-1}{1/n}.$$

Applying the de L'Hospital rule, we obtain

$$\lim_{n\to\infty} n(1-\hat{p}) = \frac{1}{2^{-k}-1}\lim_{n\to\infty}\frac{\beta^{-k/n}\,(-1/n^2)\,(-k)\ \ln\beta}{(-1/n^2)} = \frac{(-k)\ \ln\beta}{2^{-k}-1}. \qquad //$$

From (3.3) and Lemma 2, we have the claim of Theorem 1. //

Proof of Proposition 2. (i) To prove that $E(X_\infty)$ is a decreasing function of $k$for $k\in(-\infty, 0)$, we have to check that the derivative of $E(X_\infty)$ w.r.t. k is negative for $k\in(-\infty, 0)$. Indeed,

$$\frac{d}{dk}\left(\frac{2^{-k}-1}{2\,(-k)\ln\beta}\right) = -\frac{2^{-k}\,(-k)\ln 2 \,-\, 2^{-k}+1}{2\,k^2\ \ln\beta}$$

Since the denominator of the last expression is always positive, we have to show that the numerator is positive for $k\in(-\infty, 0)$. The numerator is a function of k. Denote it by $f(x)$. We obtain

$$\frac{df(k)}{dk} = -\,2^{-k}\,(-k)\,(\ln 2)^2 < 0 \ \text{ for } \ k\in(-\infty, 0).$$

It means that $f(x)$ is a decreasing function of k. Thus, it takes the lowest value on the interval $(-\infty, 0/$ when $k = 0$. Since $f(0) = 0$, we have that $f(k) > 0$ for $k\in(-\infty, 0)$.

(ii) By the de L'Hospital rule, we have

$$\lim_{k\to-\infty}\frac{2^{-k}-1}{2\,(-k)\ln\beta} = \frac{1}{2\ln\beta}\lim_{k\to-\infty}\frac{(-2^{-k})\ln 2}{(-1)} = \infty.$$

Thus we obtain claim (ii) of Proposition 2. Similarily,

$$\lim_{k\to 0} \frac{2^{-k}-1}{2(-k)\ln\beta} = \frac{1}{2\ln\beta}\lim_{k\to 0}\frac{(-2^{-k})\ln 2}{(-1)} = \frac{\ln 2}{\ln\beta^2}.$$

Thus we have claim (iii) of Proposition 2. //

Proof of Theorem 2. We have

$$E(\hat{P}_n) = \sum_{t=0}^{\infty} P(\beta_n{}^t D^0)\prod_{\tau=0}^{t-1}(\hat{p}^\tau)^2\,(1-(\hat{p}^t)^2).$$

Since $\hat{p}^\tau$ is the same for $\tau = 0, 1, \ldots$, we may drop the superscript and denote $\hat{p}^\tau = \hat{p}$. Now, we obtain

$$E(\hat{P}_n) = \sum_{t=0}^{\infty} P(\beta_n{}^t D^0)\,(\hat{p})^{2t}\,(1-(\hat{p})^2).$$

From the homogeneity of $P(\bullet)$, it follows that $P(\beta_n{}^t D^0) = \beta_n{}^{kt}P(D^0)$. Thus we have

$$E(\hat{P}_n) = P(D^0)\,(1-(\hat{p})^2)\sum_{t=0}^{\infty}(\beta_n{}^k(\hat{p})^2)^t.$$

Applying Lemma 1 for $x = \beta_n{}^k(\hat{p})^2 \in (0, 1)$, we obtain

$$E(\hat{P}_n) = P(D^0)\,\frac{1-(\hat{p})^2}{1-\beta_n{}^k(\hat{p})^2}.$$

From (2.9) and (2.3), it follows

$$E(\hat{P}_n) = P(D^0)\,\frac{1-\left(\dfrac{1-(\beta^{1/n}/2)^{-k}}{1-(1/2)^{-k}}\right)^2}{1-\left(\dfrac{1-(\beta^{1/n}/2)^{-k}}{1-(1/2)^{-k}}\right)^2\beta^{k/n}}.$$

Multiplying numerator and denominator of the last expression by $(1-(1/2)^{-k})^2$, we obtain

$$E(\hat{P}_n) = P(D^0)\,\frac{(1-(1/2)^{-k})^2-(1-(\beta^{1/n}/2)^{-k})^2}{(1-(1/2)^{-k})^2-(1-(\beta^{1/n}/2)^{-k})^2\,\beta^{k/n}}.$$

Applying the de L'Hospital rule, we have

$$E(\hat{P}_\infty) \equiv \lim_{n\to\infty} E(\hat{P}_n)$$

$$= P(D^0) \lim_{n\to\infty} \frac{-2\,(1-(\beta^{1/n}/2)^{-k})\,(\beta^{1/n}/2)^{-k}\,(-k/n^2)\,\ln\beta}{-2\,(1-(\beta^{1/n}/2)^{-k})\,(\beta^{1/n}/2)^{-k}\,(-k/n^2)\,\ln\beta\,\beta^{k/n} - (1-(\beta^{1/n}/2)^{-k})^2\,\beta^{k/n}\,\ln\beta\,(-k/n^2)}.$$

Thus

$$E(\hat{P}_\infty) = P(D^0) \lim_{n\to\infty} \frac{2}{\beta^{k/n} + \beta^{2k/n}2^{-k}} = P(D^0)\,\frac{2}{1+2^{-k}}\,. \quad //$$

Proof of Proposition 3. It follows immediately from Theorem 2. //

4. An Example with a Price Function Homogeneous in the Limit

A natural step in the investigation of the duration of debt overhang is to consider a larger class of price functions. Unfortunately, we did not succeed in obtaining complete results for any other class of functions. Instead, we present the duration of debt overhang in an example with a price function homogeneous in the limit.

Example 4.1. Let $D^0 = 1$ and $\beta = 1.1$. Assume that the price function is given by

$$P(D) = \frac{90}{D+1}\ .$$

The central equilibrium strategy of each bank is to wait in subperiod $t = 0, 1, \ldots$ with probability

$$\hat{p}^t = \left(\frac{2}{\beta^{1/n}} - 1\right) \frac{(\beta^{1/n})^{t-1} + 1}{(\beta^{1/n})^t + 1}. \tag{4.1}$$

From (3.2) and (4.1), we calculate the limit expected duration of the debt overhang in the central equilibrium when the frequency of banks' decision making becomes high,

$$E(\hat{X}_\infty) \equiv \lim_{n\to\infty} E(\hat{X}_n) = \frac{17}{36\,\ln\beta}.$$

For $\beta = 1.1$, $E(\hat{X}_\infty) = 4.95$, i.e., the expected duration of the debt overhang in the central equilibrium is close to 5 as the frequency of the banks' decision making is high.

Now, we can make a comparison between the case of the homogeneous price function and the price function which is almost homogeneous for the large values of debt. Observe that for the large values of D, the price function P(D) is close to the function $F(D) = \frac{90}{D}$, which is homogeneous of degree -1. By Theorem 2, for $k = -1$, we obtain

$$E(\hat{X}_\infty) = \frac{1}{2 \ln \beta},$$

which for $\beta = 1.1$ equals to 5.25. Thus in our example with the approximately homogeneous price function, the limit duration of the debt overhang is similar to the limit duration of the debt overhang when the price function is homogeneous. In all presented cases, under often decision making by banks, the limit duration of the debt overhang with two banks is quite long.

5. Conclusions

In the present paper, we investigated the duration of debt overhang using the dynamic framework for the banks behavior given in Prokop (1991). We modified the game of Prokop (1991) to allow the banks to make many decisions within the period of one year. We assumed that each annual period of the game is devided into n subperiods in which the decision making takes place. The modified game has the same types of equilibria as the game of Prokop (1991). We showed that if the price function is homogeneous, then the expected duration of the debt overhang in the central equilibrium becomes almost constant when the frequency of the banks' decision making is high. In this case, the constant limit duration of debt overhang is long when the degree of homogeneity of the price function is high. When the degree of homogeneity is low, the constant is close to $\ln 2/ \ln \beta^2$, where β is the annual interest factor. We interpret these results as a possibility for the debt overhang to exist for a relatively long time.

We attempted to relax the assumption of the homogeneity of the price function, but without a major success. Instead, we presented an example of the debt overhang with a price function which becomes almost homogeneous for the large values of debt. In the example, the expected equilibrium duration of the debt overhang still remains quite long when the banks make the decisions about selling or waiting frequently. This suggests that our conclusions about the long duration of debt overhang could be extended on the cases of the non-homogeneous price functions. To make a definite conclusion, however, further research is necessary.

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