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DYNAMICS OF INTERNATIONAL DEBT OVERHANG WITH TWO LENDER BANKS

by

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Abstract:

This paper presents a dynamic formalization of the behavior of creditor banks in the presence of the secondary market for debts. We formulate the problem as an infinite horizon game with two banks as players where each bank decides in every period either to sell its loan exposure to the debtor country at the present secondary market price, or to wait and keep its exposure to the next period. We show that there exist three types of subgame perfect equilibria with the property called the time continuation. We consider the relationships between our equilibria and those of the Kaneko-Prokop (1991) one-period approach to the same problem and show that their one-period approach does not lose much of the dynamic nature of the problem. In every equilibrium, each bank waits in every period with high probability, and the probability is close to 1 when the interest rate is small. If the price function of debt is approximated by some homogeneous function for large values of debt, then the central equilibrium probability becomes stationary in the long run. The stationary probability is relatively high as long as the interest rate is low. These results are interpreted as a tendency for the problem of debt overhang to remain almost unchanged.

1. Introduction

The international debt overhang is a situation of a sovereign country who has borrowed money from foreign banks and has been unable to fulfill the scheduled repayments. The debt overhang is a serious problem for the debtor country, which keeps the country in a bad economic situation and prevents from growing. To understand the problem of debt overhang, Kaneko and Prokop (1991) give a game theoretical model of debt overhang where lender banks are players and decide whether or not to sell their loan exposures to the debtor country at the discounted price on the secondary market. Their analysis shows that there is a great tendency for the present situation of debt overhang to remain unchanged. They focus on the decision making of lender banks in a short period and formulate the situation as a one-shot game. However, the problem of debt overhang is dynamic in nature. In this paper we examine whether their approach captures the dynamics, considering a dynamic formulation of their model and its equilibria.

The secondary market for debts is an important element which accompanies the problem of debt overhang. The appearance of the secondary market for debts has been caused by the country's inability to repay the debts in full. On this market, loan exposures are traded at a discounted price.[1] The existence of the secondary market creates a possibility for the debtor country to buy back its debts at a discounted price. However, there has been some tendency for the debt overhang to persist even in the presence of the secondary market. As in Kaneko and Prokop (1991), we assume that the secondary market is represented as a price function.

We formulate the problem as an infinite horizon game with two banks as players. Since the modelling of the dynamics of debt overhang with many banks is too complex, we consider the two bank case. The game is played as follows. In every period of the game each bank decides either to sell its loan exposure to the debtor country at the current secondary market price or to wait and postpone this decision to the next period. If both banks wait in a period, then they face the same decision problem in the following period. When a bank sells its loan exposure to the debtor country at the current secondary market price, it receives a payoff according to the current secondary market price and leaves the game. After both banks sell their exposures to the debtor country, the game ends.

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[1] See, for example, Sachs and Huizinga (1987), and Hajivassiliou (1989).

In our game, there are three types of subgame perfect equilibria with the time continuation property. By the time continuation property we mean that the game of the present period can be viewed as the result of the game of some previous period so that the extensions of the equilibria of the present game give a positive realization probabilities to the present game. The three types of equilibria are called central, alternating and mutating, respectively. Each equilibrium is very similar to the unique central equilibrium in the sense that the probabilities of a bank waiting are determined by similar formulas and take similar values. Therefore, the average attainable payoffs are almost uniquely determined.

We can compare the equilibria of our dynamic game and the one-period game of Kaneko and Prokop (1991). To make this comparison, we construct a sequence consisting of the mixed strategy equilibria for the one-period games describing the situation of banks in the sequence of the respective periods. This sequence of equilibria coincides with the central equilibrium of our game.[2] That is, we have a decomposition property of the dynamic game into the games of one period. Since all equilibria of the dynamic game are similar to the unique central equilibrium. Thus the one-period approach of Kaneko-Prokop (1991) does not lose much of the dynamic nature of the problem.

Our game may be regarded as a repeated game with a constituent game of the one-period approach. As long as no bank sells its loan exposure until period t, a similar one-period game is played in period $t + 1$. However, the Folk Theorem[3] that almost all payoffs of the one-period game are attainable by an equilibrium of the repeated game, does not hold in our game, since one player's deviation necessarily implies its exit from the game and does not permit a punishment by the other bank. On the contrary to the Folk Theorem, the average attainable payoffs are almost uniquely determined in our game.

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[2] In Kaneko and Prokop (1991), there are also two pure strategy Nash equilibria in the two bank case. The sequences consisting of them become subgame perfect equilibria in our approach. We, however, eliminate them by the time continuation property.

[3] See, for example, Aumann (1982).

We describe some observations on the behavior of banks over time following from our characterization of equilibria. In any equilibrium each bank waits with relatively high probability in every period, and when the interest rate is small, the probability of a bank waiting is close to 1. This can be interpreted as a tendency for the situation of debt overhang to remain almost unchanged.

We also investigate the strategies of the banks when the time passes. We show that if the price function is approximated by some homogenous function for large values of debt, then the probability of waiting in a period in the central equilibrium becomes a stationary probability in the long run. The stationary probability of a bank waiting in every period is relatively high as long as the interest rate is low. This suggests a difficulty of a resolution of the debt overhang solely through the secondary market transactions.

The paper is organized as follows. In Section 2 we formulate the model called the dynamic endurance game. In Section 3 we present the structure of equilibria. The comparative dynamics (statics) and limit results are given in Section 4. In Section 5, we give the proof of the main theorem. Section 6 contains conclusions.

2. Dynamic Endurance Game

We consider the following dynamic situation: a country has debt obligations $D_1 > 0$ and $D_2 > 0$ to foreign banks 1 and 2, respectively. The country has fallen behind with service payments for some periods, and the current situation of the country does not allow for full repayments. The existence of the secondary market for debts is assumed, where the country is ready to buy its debts at the secondary market price. We assume that the price of debt on the secondary market depends upon the current total outstanding debts. This price is expressed by the function $P(D)$: $R_+ \to R_+$ with the property:

(2.1) P(D) is a decreasing function of the total outstanding debt D and P(D) $\to$ 0 as $D \to \infty$.

This P(D) is country-specific and also depends upon the choice of the present period.

We will focus our attention on the decisions of banks over time. The present period is called 0. In every period each bank has two possible choices either to sell its exposure or to wait and keep it to the next period. Each bank's strategies in every period are s and w, where s denotes selling the loan exposure to the debtor country at the current secondary market price, and w denotes waiting and postponing the decision to the next period.

Each bank discounts the future revenues by the interest rate ($r > 0$). We denote the interest factor $1 + r$ by β. We assume that

(2.2) $\beta^2 D_i < D^0$ for $i = 1, 2$,

where $D^0 = D_1 + D_2$. This assumption says that the distribution of loan exposures of banks is relatively equal and the interest rate is relatively small so that the size of one bank's loan exposure with compounded interests from two periods is smaller than the initial debt of the country.

If bank i keeps its loan exposure D_i until period t ($t \geq 0$), then its exposure increases by the accrued interests to $\beta^t D_i$. The total outstanding debt D^t in period t becomes $\beta^t D_i$ if the other bank has already sold, or $\beta^t D^0$ otherwise. The secondary market price in period t is given as $P(D^t)$. If bank i sells in period t, then the present value of repayment is given as

$$\frac{1}{\beta^t}\beta^t D_i P(D^t) = D_i P(D^t).$$

If bank i and the other bank wait in period t, then bank i does not get any payoff in this period, but it will face the same decision problem in the next period.

We assume that

(2.3) after one bank sells its loan exposure, the other bank sells its exposure immediately in the next period.

It is possible that while one bank sells its exposure in some period, the other bank keeps its exposure for several periods after that. In this case, however, keeping the loan exposure for several periods is not an optimal behavior. Indeed, if the bank postpones selling its exposure, the secondary market price of debt will decrease because of the accrued interest. Suppose bank j sells its exposure in period t and bank i does not. Then the secondary market price in period t + 1 is $P(\beta^{t+1} D_i)$, and the present value of loan exposure is $D_i P(\beta^{t+1} D_i)$. If bank i keeps its loan exposure to period t + 2, then the price falls to $P(\beta^{t+2} D_i)$ and the present value is $D_i P(\beta^{t+2} D_i)$. The optimal behavior of bank i is to sell the exposure in period t + 1. Thus we can assume (2.3).

There are two cases in which the game terminates. The first case is that both banks wait until period $t-1$ and both sell in period t. The second case is that both banks wait until period $t-1$ and one bank sells its loan exposure in period t, and the other bank waits in period t and sells its exposure in period $t+1$. The payoff to bank i ($i = 1, 2$) in the game $\Gamma(0, D^0)$ is defined by

(2.4)

$D_i P(\beta^t D^0)$	if both banks wait until period $t-1$ and bank i sells in period t,
$D_i P(\beta^{t+1} D_i)$	if both banks wait until period $t-1$ and bank j ($j \neq i$) sells in period t and bank i waits in period t.

Our game $\Gamma(0, D^0)$ is described in Figure 1. In the game tree, the payoffs to the banks are given in three branches. In the first branch both banks sell their exposures in period 0. In the second, bank 1 sells in period 1 and bank 2 sells in period 2. The third case is that bank 2 sells its exposure in period 2 and bank 1 sells its exposure in period 3.

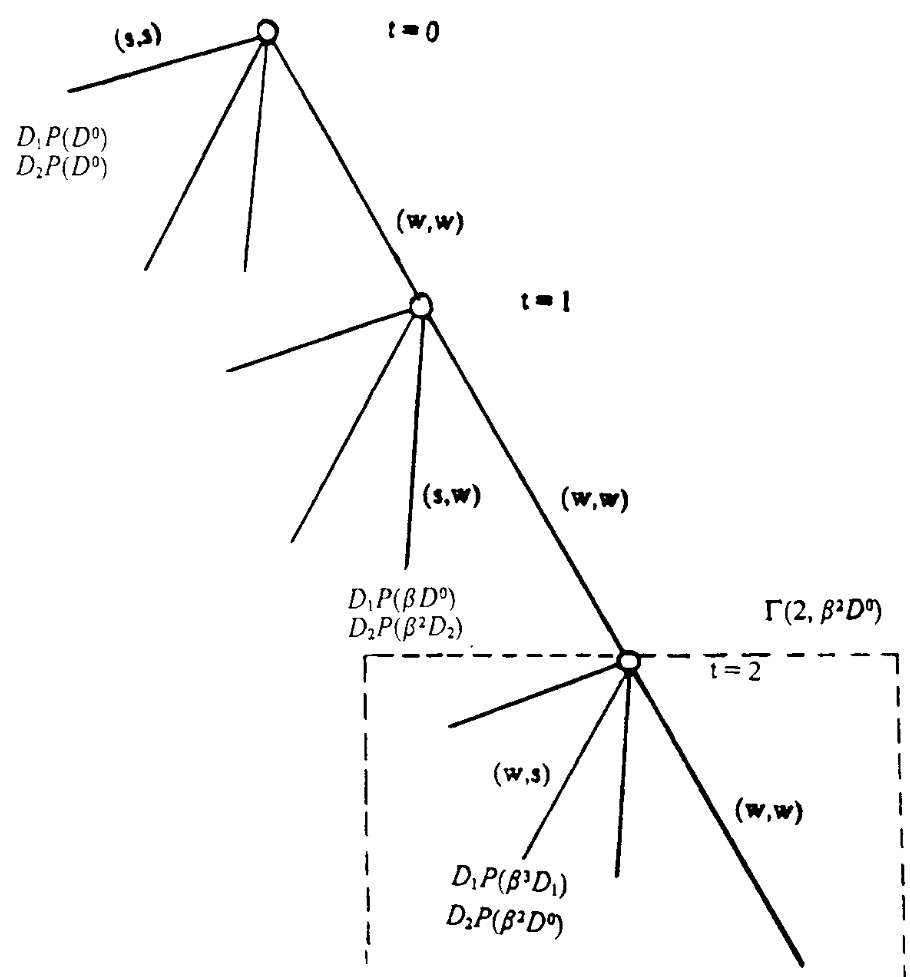

Figure 1.

We allow each bank to use behavior strategies. Since each bank's decision is made in period t only when both banks have waited until period t by assumption (2.3), a behavior strategy of bank $i = 1, 2$ in the game $\Gamma(0, D^0)$ is represented as a sequence $b_i^0 = (p_i^0, p_i^1, ...)$, where p_i^t is a probability of bank i's waiting in period t $(t = 0, 1, ...)$ if both banks keep their loan exposures until period t. Denote the set of all behavior strategies of bank i by B_i^0. A behavior strategy combination for the game $\Gamma(0, D^0)$ is a vector $b^0 = (b_1^0, b_2^0) = ((p_1^0, p_1^1, ...), (p_2^0, p_2^1, ...))$.

The expected payoff to bank $i = 1, 2$ for a behavior strategy combination b^0 is the sum of

(i) the expected payoff from selling the exposure in period t $(t = 0, 1, ...)$ under the assumption that the other bank does not sell earlier – $D_i P(\beta^t D^0)(1 - p_i^t)\prod_{k=0}^{t-1} p_i^k p_j^k$; and

(ii) the expected payoff from selling its loan exposure in period $t + 1$ under the assumption that the other bank sells in period t – $D_i P(\beta^{t+1} D_i) p_i^t (1 - p_j^t)\prod_{k=0}^{t-1} p_i^k p_j^k$.

Thus the expected payoff from the game $\Gamma(0, D^0)$ under the strategy combination b^0 is given by

$$(2.5) \qquad H_i^0(b^0) = \sum_{t=0}^{\infty} D_i P(\beta^t D^0)(1 - p_i^t)\prod_{k=0}^{t-1} p_i^k p_j^k + \sum_{t=0}^{\infty} D_i P(\beta^{t+1} D_i) p_i^t (1 - p_j^t)\prod_{k=0}^{t-1} p_i^k p_j^k.$$

We use the convention $\prod_{k=0}^{-1} p_i^k p_j^k = 1$.

We have described the dynamic endurance game of lender banks $\Gamma(0, D^0)$. In the next section we investigate the decisions of banks in the dynamic endurance game.

3. The Structure of Equilibria

To investigate the decisions of banks in the game $\Gamma(0, D^0)$, we adopt the concept of the subgame perfect equilibrium point of the extensive game (Selten (1975)). To define a subgame perfect equilibrium, we have to consider subgames of the game $\Gamma(0, D^0)$. Here every subtree constitutes a subgame. Thus the subgame which starts at any period t of the game $\Gamma(0, D^0)$ is denoted by $\Gamma(t, \beta^t D^0)$. The strategy for the subgame $\Gamma(t, \beta^t D^0)$ induced by $b_i^0 = (p_i^0, p_i^1, ...)$ is a vector obtained by dropping the first t entries of the vector b_i^0, i.e. $b_i^t = (p_i^t, p_i^{t+1}, ...)$. Let B_i^t be the set of all induced behavior strategies of bank i for the subgame $\Gamma(t, \beta^t D^0)$.

Denote by $H_i^t(b^t)$ the expected payoff to bank i from the subgame $\Gamma(t, \beta^t D^0)$ under the induced behavior strategy combination $b^t = (b_1^t, b_2^t)$.

A behavior strategy combination $\hat{b}^t = (\hat{b}_1^t, \hat{b}_2^t)$ is a Nash equilibrium of the subgame $\Gamma(t, \beta^t D^0)$ iff for i = 1, 2,

$$H_i^t(\hat{b}^t) \geq H_i^t(\hat{b}^t/b_i^t) \quad \text{for all } b_i^t \in B_i^t, \tag{3.1}$$

where $\hat{b}^t/b_i^t$ denotes a strategy combination $\hat{b}^t$ with the replacement of $\hat{b}_i^t$ by b_i^t. A subgame perfect equilibrium of the game $\Gamma(0, D^0)$ is a behavior strategy combination $\hat{b}^0 = (\hat{b}_1^0, \hat{b}_2^0)$ which induces a Nash equilibrium on every subgame of the game $\Gamma(0, D^0)$.

In addition to the subgame perfection, we require the equilibrium to have the time continuation property. To define the time continuation property, we introduce a retrospective extension of the game $\Gamma(0, D^0)$ as a game $\Gamma(t, \beta^t D^0)$ for $t = -1, -2, \ldots$, so that the game $\Gamma(0, D^0)$ is a subgame of the game $\Gamma(t, \beta^t D^0)$. A subgame perfect equlibrium $\hat{b}^0$ is said to have the time continuation property iff for any $t = -1, -2, \ldots$ there is a subgame perfect equilibrium $\hat{b}^t$ in the retrospective extension $\Gamma(t, \beta^t D^0)$ of $\Gamma(0, D^0)$ such that $\hat{b}^t$ induces $\hat{b}^0$ and the realization probability of the subgame $\Gamma(0, D^0)$ is positive.

The time continuation property states that the present game situation results as a continuation of the past history. The game $\Gamma(0, D^0)$ is a result of previous decisions of banks. Therefore the present game is a subgame of the game of any preceding period. If the realization probability of the game $\Gamma(0, D^0)$ is zero in an equilibrium for $\Gamma(t, \beta^t D^0)$ then the present situation would be different from $\Gamma(0, D^0)$. However, we assume that the game $\Gamma(0, D^0)$ is reached. Therefore it is compatible with the consideration of $\Gamma(0, D^0)$ to assume that the realization probability is positive.[4]

The structure of equilibria is described by the following theorem.

Theorem 1. The endurance game of two banks has three types of subgame perfect equilibria satisfying the time continuation property, which are called central, alternating, and mutating. In the unique central equilibrium $\hat{b}^0 = (\hat{b}_1^0, \hat{b}_2^0) = ((\hat{p}_1^0, \hat{p}_1^1, \ldots), (\hat{p}_2^0, \hat{p}_2^1, \ldots))$, bank i waits in every period t $(t = 0, 1, \ldots)$ with probability

$$\hat{p}_i^t = \frac{P(\beta^{t+1} D_j) - P(\beta^t D^0)}{P(\beta^{t+1} D_j) - P(\beta^{t+1} D^0)} \quad \text{for} \quad i, j = 1,2 \text{ and } i \neq j. \tag{3.3}$$

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[4] The time continuation property is a concept independent from the time consistency in the macroeconomic literature. The time consistency property is, instead, implied by the subgame perfection.

There are two alternating equilibria $\tilde{b}^0 = (\tilde{b}_1^0, \tilde{b}_2^0) = ((\tilde{p}_1^0, \tilde{p}_1^1, \ldots), (\tilde{p}_2^0, \tilde{p}_2^1, \ldots))$ in which banks i and j wait with probabilities

$$(3.4) \qquad \tilde{p}_i^t = \frac{P(\beta^{t+1}D_i) - P(\beta^{t-1}D^0)}{P(\beta^{t+1}D_j) - P(\beta^{t+1}D^0)} \text{ and } \tilde{p}_j^t = 1 \text{ if t is even;}$$

and

$$(3.5) \qquad \tilde{p}_i^t = 1 \text{ and } \tilde{p}_j^t = \frac{P(\beta^{t+1}D_i) - P(\beta^{t-1}D^0)}{P(\beta^{t+1}D_i) - P(\beta^{t+1}D^0)} \text{ if t is odd.}$$

In a mutating equilibrium $\bar{b}^0 = (\bar{b}_1^0, \bar{b}_2^0) = ((\bar{p}_1^0, \bar{p}_1^1, \ldots), (\bar{p}_2^0, \bar{p}_2^1, \ldots))$ banks wait in every period t until some period τ $(\tau \geq -2)$[5] with probabilities given by (3.3), in period $\tau+1$ they wait with probabilities

$$(3.6) \quad \bar{p}_i^{\tau+1} = \frac{P(\beta^{\tau+2}D_i) - P(\beta^{\tau+1}D^0)}{P(\beta^{\tau+2}D_j) - P(\beta^{\tau+2}D^0)}, \; \bar{p}_j^{\tau+1} = \frac{P(\beta^{\tau+2}D_i) - P(\beta^{\tau+1}D^0)}{P(\beta^{\tau+2}D_i) - P(\beta^{\tau+3}D^0) + \bar{p}_j^{\tau+2}[P(\beta^{\tau+3}D_i) - P(\beta^{\tau+3}D^0)]},$$

in period $\tau + 2$ they wait with probabilities

$$(3.7) \qquad \bar{p}_i^{\tau+2} = 1, \; \bar{p}_j^{\tau+2} \in \left[\frac{P(\beta^{\tau+3}D_i) - P(\beta^{\tau+1}D^0)}{P(\beta^{\tau+3}D_i) - P(\beta^{\tau+3}D^0)}, \; \frac{P(\beta^{\tau+3}D_i) - P(\beta^{\tau+2}D^0)}{P(\beta^{\tau+3}D_i) - P(\beta^{\tau+3}D^0)} \right],$$

and in period t $(t \geq \tau + 3)$, they wait with probabilities given by (3.4) when $t = \tau + 3, \tau + 5, \ldots$, and with probabilities given by (3.5) when $t = \tau + 4, \tau + 6, \ldots$.

In the central equilibrium, bank i waits in period t with probability given by (3.3). In an alternating equilibrium banks i and j wait with probabilities given by (3.4) in every even period, and with probabilities given by (3.5) in every odd period. Thus each bank alternates its strategy between waiting for sure and waiting with probability $\tilde{p}_i^t$ given in (3.4). In a mutating equilibrium banks behave initially according to the central equilibrium strategies and in some future period their central equilibrium strategies mutate into the alternating equilibrium strategies. The strategies of transitory periods in a mutating equilibrium are given by (3.6) and (3.7).

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[5] When $\tau = -2$, $\bar{p}_1^0$ and $\bar{p}_2^0$ are given by (3.7) and (3.6) is irrelevant. When $\tau = -1$, $\bar{p}_1^0$ and $\bar{p}_2^0$ are given by (3.6).

The central equilibrium differs from the alternating equilibrium in that each bank sells with some probability in the former, and each bank alternates between waiting for sure and selling with some probability in the latter. Nevertheless, the central and alternating equilibria are similar in the sense that the probabilities of waiting in each of them are determined by similar formulas and take similar values as will be shown in the example below. The mutating equilibrium is a combination of the central and the alternating equilibria. The central equlibrium mutates into the alternating equilibrium but not the other way around.

In every equilibrium of our dynamic game the probability of a bank waiting in each period is relatively high as long as the interest factor is low. This suggests that there is a tendency for the situation of the debt overhang to remain unchanged, no matter what equilibrium strategies the banks use.

Although we found three types of equilibria in our dynamic endurance game, the strategies of banks in every equilibrium are close to the strategies of the unique central equilibrium. The central equilibrium of our dynamic game gives the local equilibrium strategies which coincide with the equilibrium of a one-period endurance game investigated in Kaneko and Prokop (1991). In other words, the central equilibrium can be constructed as a sequence of mixed strategy equilibria of the one-shot games of Kaneko-Prokop (1991). This link between the dynamic and one-period formulations allows us to use the one-period approach without losing much of the dynamic nature of the problem.

The following example illustrates the claim of Theorem 1.

Example 3.1. Let $D_1 = D_2 = \frac{1}{2}D^0 = \frac{1}{2}$ and $\beta = 1.1$. Assume that the price function is given by

$$P(D) = \frac{90}{D+1} \quad .$$

The central equilibrium strategy of each bank is to wait in period t with probability

$$\hat{p}_i^t = \frac{9}{11}\,\frac{(1.1)^{t+1}+1}{(1.1)^t+1} \quad .$$

The table below shows some values for the local strategies.

t	0	1	2	3	4	8	16	32
p_i^t	.859	.861	.863	.865	.867	.874	.885	.896

Table 1.

In this example the central equilibrium probability of a bank waiting in period t is high. It increases with the time t and converges to .9 as t becomes large.

The alternating equilibrium strategy of banks i and j is to wait with probabilities

$$\tilde{p}_i^t = \frac{79}{121}\frac{(1.1)^{t+1}+1}{(1.1)^{t-1}+1}, \quad \tilde{p}_j^t = 1 \qquad \text{in every even period;}$$

$$\tilde{p}_i^t = 1, \quad \tilde{p}_j^t = \frac{79}{121}\frac{(1.1)^{t+1}+1}{(1.1)^{t-1}+1} \qquad \text{in every odd period.}$$

The table below shows some values for the local strategies in an alternating equilibrium.

t	0	1	2	3	4	5	8	9
$\tilde{p}_i^t$	.718	1	.725	1	.731	1	.744	1
p_j^t	1	.721	1	.728	1	.734	1	.746

Table 2.

In this example the alternating equilibrium probability of a bank waiting in period t is high, too. The lower probability of waiting increases with the time t and converges to .79 as t becomes large.

An example of a mutating equilibrium is presented in Figure 2.

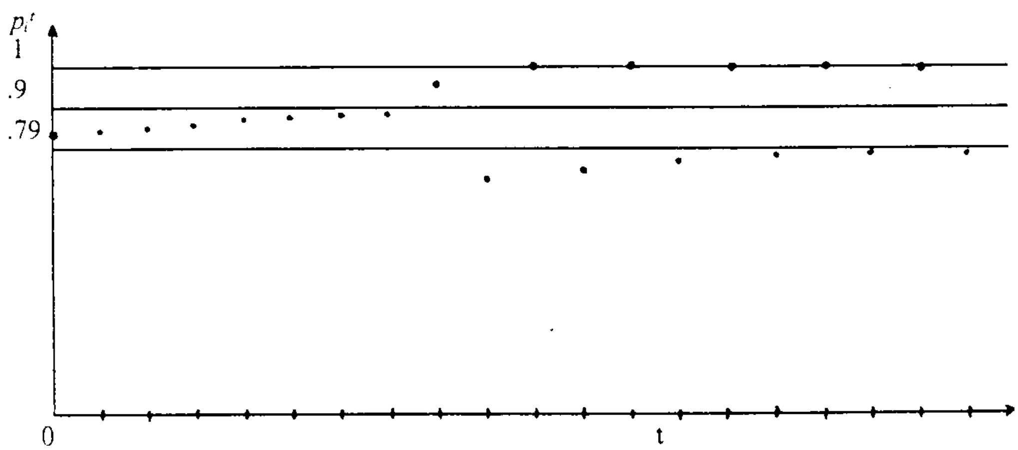

Figure 2.

In a mutating equilibrium a bank waits in every period with probability close to .9, and from some period on starts switching its local strategy either waiting for sure or waiting with probability close to .79. Thus the probability of waiting in each period is high in every equilibrium. We will characterize the behavior of equilibria more precisely in Section 4.

4. The Behavior of Equilibria

The following comparative statics result is true.

Theorem 2. Let $\hat{b}^0 = (\hat{b}_1^0, \hat{b}_2^0) = ((\hat{p}_1^0, \hat{p}_1^1, \ldots), (\hat{p}_2^0, \hat{p}_2^1, \ldots))$ be the central equilibrium. Then for $i,j = 1,2$ $(i \neq j)$ and all $t = 0, 1, \ldots$, it holds

$$\hat{p}_i^t \leq \hat{p}_j^t \quad \text{if and only if} \quad D_i \leq D_j. \tag{4.1}$$

Proof of Theorem 2. From equations given by (3.3), we have that

$$\hat{p}_i^t - \hat{p}_j^t = \frac{[P(\beta^{T-t+1}D_i) - P(\beta^{T-t+1}D_j)][P(\beta^{T-t}D_0) - P(\beta^{T-t+1}D_0)]}{[P(D_j) - P(D_0)][P(D_i) - P(D_0)]} \quad \text{for } j \neq i.$$

From the above equality it follows that

$$\hat{p}_i^t \leq \hat{p}_j^t \quad \text{if and only if} \quad D_i \leq D_j. \qquad //$$

This theorem says that in the central equilibrium a bank with a bigger loan exposure waits longer in every period than the bank with a smaller one. The reasoning for Theorem 2 is as follows. Each bank has in every period two (pure) alternative choices: to wait or to sell. In an equilibrium mixed local strategy, these two choices give the same expected payoffs to each bank, since otherwise a pure local strategy would be chosen. Bank i, evaluating its expected payoff from waiting, takes into account bank j's probability of selling and vice versa. If $D_i \leq D_j$, then bank i's evaluation is more positively affected by D_j than bank j's by D_i. Therefore, to have the same expected values for the two alternative choices, bank i's probability of waiting becomes lower than bank j's.

The result of Theorem 2 gives the same prediction about the behavior of banks in a period as the prediction derived from the one-period model of Kaneko and Prokop (1991). Kaneko and Prokop (1991) showed in a one-period model that a bank with a higher loan exposure has a higher probability of waiting than the other one. Since the sequence of mixed strategy equilibria for the one-period games coincides with the central equilibrium of the dynamic game, our Theorem 2 is obtained immediately. The behavior of lender banks in the Bolivian buyback constitutes the best illustration of these results. All American banks with larger loan exposures have kept their loans but some banks with small exposures have sold theirs.

In the previous section, we observed that the probability of a bank waiting in each period in every equilibrium is relatively high as long as the interest factor is low. Now, we consider the limit behavior of the probability of a bank waiting when the interest factor is close to 1. If the function $P(\bullet)$ is continuous at D^0, then from equations (3.3)-(3.7) we obtain that in every equilibrium of the endurance competition game the probability of a bank waiting in each period converges to 1 as β converges to 1. Thus we have the following property of each equilibrium in the game $\Gamma(0, D^0)$.

Theorem 3. Let $b^0 = (b_1^0, b_2^0) = ((p_1^0, p_1^1, \ldots), (p_2^0, p_2^1, \ldots))$ be any equilibrium for the game $\Gamma(0, D^0)$ with the interest factor β. If $P(\bullet)$ is continuos at D^0, then for each $t \geq 0$ $p_i^t \to 1$ as $\beta \to 1$.

Theorem 3 means that when the interest rate is small (the interest factor is close to 1), each bank has strong incentives to wait and keep its loan exposure in every period along an equilibrium path for our game.

Now, we would like to characterize more precisely the changes of the probability of waiting over time. If the price function $P(\bullet)$ is homogenous, i.e. $P(kD) = k^n P(D)$ where n is a negative number, than from (3.3) the central equilibrium probability of waiting by bank i in period t is constant and equals to

$$\frac{P(\beta D_j) - P(D^0)}{P(\beta D_j) - P(\beta D^0)}.$$

This observation can be generalized in the following way. If the price function P(D) is approximated by a homogenous function F(D) for the large values of D, then the central equilibrium probability of waiting by bank i in period t becomes almost constant as t increases. To see this, we assume that there is a function F(D): $R_+ \to R_+$ homogenous of degree n (i.e. $F(kD) = k^n F(D)$, where n is a negative number), such that $P(D)/F(D) \to 1$ as $D \to \infty$. Then from (3.3),

$$\hat{p}_i^t = \frac{P(\beta^{t+1}D_j) - P(\beta^t D^0)}{P(\beta^{t+1}D_j) - P(\beta^{t+1}D^0)} = \frac{F(\beta^{t+1}D_j)/F(\beta^{t+1}D^0) - P(\beta^t D^0)F(\beta^{t+1}D_j)/\beta^n F(\beta^t D^0)P(\beta^{t+1}D_j)}{F(\beta^{t+1}D_j)/F(\beta^{t+1}D^0) - P(\beta^{t+1}D^0)F(\beta^{t+1}D_j)/F(\beta^{t+1}D^0)P(\beta^{t+1}D_j)}.$$

Thus we obtain

$$\hat{p}_i^t \to \frac{F(D_j)/F(D^0) - 1/\beta^n}{F(D_j)/F(D^0) - 1} = \frac{F(\beta D_j) - F(D^0)}{F(\beta D_j) - F(\beta D^0)} \quad \text{as} \quad t \to \infty.$$

We can summarize the above result in the following theorem.

Theorem 4. Let $\hat{b}^0 = (\hat{b}_1^0, \hat{b}_2^0) = ((\hat{p}_1^0, \hat{p}_1^1, \ldots), (\hat{p}_2^0, \hat{p}_2^1, \ldots))$ be the central equilibrium for the game $\Gamma(0, D^0)$ given by Theorem 1. If there exists a homogenous function $F(D)$ such that $P(D)/F(D) \to 1$ as $D \to \infty$, then

$$\hat{p}_i^t \to \frac{F(\beta D_j) - F(D^0)}{F(\beta D_j) - F(\beta D^0)} \quad \text{as} \quad t \to \infty.$$

This theorem says that a sufficient condition for the central equilibrium probability of a bank waiting in period t to become constant when t increases, is a homogeneity of the price function for the large values of debt. We have not succeeded in finding any interesting necessary conditions for this type of stationarity of banks' behavior to occur.

The following example illustrates the claims of Theorems 2 and 4.

Example 4.1.

Let $D_1 = \frac{1}{3} D^0$, $D_2 = \frac{2}{3} D_0$, $D^0 = 1$, $\beta = 1.1$. Let us also assume that the price function is

$$P(D) = \frac{90}{D+1} \quad .$$

The central equilibrium local strategies are

$$\hat{p}_1^t = \frac{8}{11} \frac{(1.1)^{t-1} + 1}{(1.1)^t + 1} \quad \text{and} \quad \hat{p}_2^t = \frac{19}{22} \frac{(1.1)^{t-1} + 1}{(1.1)^t + 1}.$$

As predicted by Theorem 2, $\hat{p}_1^t < \hat{p}_2^t$ for every t, and as predicted by Theorem 4,

$$\hat{p}_1^t \to .8 \quad \text{and} \quad \hat{p}_2^t \to \frac{19}{20} \quad \text{as} \quad t \to \infty.$$

5. Proof of Theorem 1

For the proof we need a precise definition of expected payoffs from the subgame $\Gamma(t, \beta^t D^0)$. An expected payoff to bank i from the subgame $\Gamma(t, \beta^t D^0)$ under b^0 is defined analogously to (2.3) by

$$\text{(5.1)} \qquad \Pi_i^t(b^t) = \sum_{\tau=t}^{\infty} (1 - p_i^\tau) D_i P(\beta^\tau D^0) \prod_{k=t}^{\tau-1} p_1^k p_2^k + \sum_{\tau=t}^{\infty} p_i^\tau (1 - p_j^\tau) D_i P(\beta^{\tau+1} D_i) \prod_{k=t}^{\tau-1} p_1^k p_2^k$$

for all $t = 0, 1, \ldots$.

We may use the following form of the payoff function

$$\text{(5.2)} \qquad \Pi_i^t(b^t) = (1 - p_i^t) D_i P(\beta^t D^0) + p_i^t (1 - p_j^t) D_i P(\beta^{t+1} D_i) + p_1^t p_2^t \Pi_i^{t+1}(b^{t+1}),$$

which comes from

$$\Pi_i^t(b^t) = (1 - p_i^t) D_i P(\beta^t D^0) + p_i^t (1 - p_j^t) D_i P(\beta^{t+1} D_i)$$
$$+ p_i^t p_j^t \left[\sum_{\tau=t+1}^{\infty} (1 - p_i^\tau) D_i P(\beta^\tau D^0) \prod_{k=t+1}^{\tau-1} p_1^k p_2^k + \sum_{\tau=t+1}^{\infty} p_i^\tau (1 - p_j^\tau) D_i P(\beta^{\tau+1} D_i) \prod_{k=t+1}^{\tau-1} p_1^k p_2^k \right].$$

5.1. The Necessary Conditions for a Subgame Perfect Equilibrium with the Time Continuation Property

In this section, we prove that any subgame perfect equilibrium with the time continuation property must be either central, or alternating, or mutating given by Theorem 1.

We first prove that if in a subgame perfect equilibrium of $\Gamma(t, \beta^t D^0)$ for some $t \leq -1$ a local strategy combination (0, 1) is played in period $m \geq 0$, then the same local strategy combination is played in period $m - 1$.

Lemma 1. Let t and m be integers with $t \leq -1$ and $m \geq 0$, and let $b^t = (b_1^t, b_2^t) = ((p_1^t, p_1^{t+1}, \ldots), (p_2^t, p_2^{t+1}\ldots))$ be a subgame perfect equilibrium for the game $\Gamma(t, \beta^t D^0)$. Then if $(p_i^m, p_j^m) = (0, 1)$, then $(p_i^{m-1}, p_j^{m-1}) = (0, 1)$.

By applying Lemma 1 repeatedly to $m - 1, \ldots, 1, 0$, the local strategy combination (0, 1) is played in every period from $t = -1$ until $t = m$. Thus if a local strategy combination (0, 1) is played in some period $m \geq 0$, then the subgame perfect equilibrium b^t for a retrospective extension $\Gamma(t, \beta^t D^0)$ of $\Gamma(0, D^0)$ gives a realization probability 0 to the game $\Gamma(0, D^0)$. It means that b^0 induced by b^t does not satisfy the time continuation property.

Proof of Lemma 1. Without loss of generality we set $i = 1$ and $j = 2$. Thus the expected payoffs to banks 1 and 2 from the subgame $\Gamma(m, \beta^m D^0)$ under $\hat{b}^t$ are

$$H_1^m((p_1^m, p_1^{m+1}, \ldots), (\hat{p}_2^m, \hat{p}_2^{m+1}, \ldots)) = D_1 P(\beta^m D^0)$$

(5.3) and

$$H_2^m((\hat{p}_1^m, \hat{p}_1^{m+1}, \ldots), (\hat{p}_2^m, \hat{p}_2^{m+1}, \ldots)) = D_2 P(\beta^{m-1} D_2),$$

respectively.

Consider the subgame $\Gamma(m - 1, \beta^{m-1} D^0)$. In this game, we assume that the banks play $b^m = (b_1^m, b_2^m)$ after period m, and then consider optimal behavior of the banks in period $m - 1$. Thus we have one period game of period $m - 1$, whose payoffs are given in the following table.

1 \ 2	s	w
s	$D_1P(\beta^{m-1}D^0)$ $D_2P(\beta^{m-1}D^0)$	$D_1P(\beta^{m-1}D^0)$ $D_2P(\beta^{m}D_2)$
w	$D_1P(\beta^{m}D_1)$ $D_2P(\beta^{m-1}D^0)$	$D_1P(\beta^{m}D^0)$ $D_2P(\beta^{m-1}D_2)$

Table 3.

If the strategy combination $b^{m-1} = ((p_1^{m-1}, b_1^m), (p_2^{m-1}, b_2^m))$ is a Nash equilibrium for the subgame $\Gamma(m-1, \beta^{m-1}D^0)$, then the strategies p_1^{m-1} and p_2^{m-1} constitute a Nash equilibrium of the game discribed by Table 3. We can calculate that the game of Table 3 has the unique Nash equilibrium $(p_1^{m-1}, p_2^{m-1}) = (0, 1)$. //

We look for necessary conditions for a behavior strategy combination to be a subgame perfect equilibrium with the time continuation property. Consider a subgame perfect equilibrium b^0 satisfying the time continuation property. By definition, b^0 induces a Nash equilibrium on every subgame of $\Gamma(0,D^0)$. Take subgames $\Gamma(t, \beta^tD^0)$ and $\Gamma(t+1, \beta^{t+1}D^0)$ for any $t = 0, 1, \ldots$. We want to find a relation between the equilibrium local strategy combination p^t and b^{t+1}. When b^{t+1} is fixed, the payoffs from the subgame $\Gamma(t, \beta^tD^0)$ depend only on the local strategies in period t, and are shown in the following table.

1 \ 2	s	w	
s	$D_1P(\beta^tD^0)$ $D_2P(\beta^tD^0)$	$D_1P(\beta^tD^0)$ $D_2P(\beta^{t+1}D_2)$	(i)
w	$D_1P(\beta^{t+1}D_1)$ $D_2P(\beta^tD^0)$ $\leftarrow (ii)$	$H_1^{t+1}(b^{t+1})$ $\rightarrow$ $H_2^{t+1}(b^{t+1})$	

Table 4.

If the strategy combination $b^t = ((p_1^t, \hat{p}_1^{t-1}), (\hat{p}_2^t, \hat{p}_2^{t-1}))$ is a Nash equilibrium of the subgame $\Gamma(t, \beta^t D^0)$, then the strategies $\hat{p}_1^t$ and $\hat{p}_2^t$ constitute a Nash equilibrium of the matrix game M(t) discribed by Table 4.

We look for all possible equilibria in the matrix game M(t). We have to consider the following seven cases. The classification is made by comparing (i) payoffs for bank 1 in the second column and (ii) payoffs for bank 2 in the lowest row.

1^o $D_iP(\beta^t D^0) > H_i^{t-1}(b^{t-1})$ and $D_jP(\beta^t D^0) < H_j^{t-1}(b^{t-1})$. There exists only one equilibrium $p_i^t = 0$ and $p_j^t = 1$ in the game M(t). The equilibrium payoffs from the game $\Gamma(t, \beta^t D^0)$ are $H_i^t(b^t) = D_iP(\beta^t D^0)$ and $H_j^t(b^t) = D_jP(\beta^{t-1} D_j)$.

2^o $D_1P(\beta^t D^0) > H_1^{t-1}(b^{t-1})$ and $D_2P(\beta^t D^0) > H_2^{t-1}(b^{t-1})$. There exist three equilibria in the game M(t). Two of them coincide with the equilibria of 1^o, and the third one is a mixed strategy equilibrium given by

$$(5.4) \qquad p_i^t = \frac{D_jP(\beta^{t-1}D_j) - D_jP(\beta^t D^0)}{D_jP(\beta^{t-1}D_j) - H_j^{t-1}(b^{t-1})} \quad \text{for } i,j = 1, 2 \ (i \neq j).$$

The equilibrium payoff to bank i from the game $\Gamma(t, \beta^t D^0)$ when the mixed strategies are played is $H_i^t(\hat{b}^t) = D_iP(\beta^t D^0)$.

3^o $D_1P(\beta^t D^0) > H_1^{t-1}(b^{t-1})$ and $D_2P(\beta^t D^0) = H_2^{t-1}(b^{t-1})$. There exist two types of equilibria in the game M(t):

(i) $\quad p_1^t = 0$ and $p_2^t = 1$;

(ii) $\quad p_1^t = 1$ and $p_2^t \in [0, \dfrac{D_1P(\beta^{t-1}D_1) - D_1P(\beta^t D^0)}{D_1P(\beta^{t-1}D_1) - H_1^{t-1}(b^{t-1})}]$.

The equilibrium payoffs from the game $\Gamma(t, \beta^t D^0)$ are, respectively,

(i) $\quad H_1^t(b^t) = D_1P(\beta^t D^0)$ and $H_2^t(b^t) = D_2P(\beta^{t-1}D_2)$;

(ii) $\quad H_1^t(b^t) = (1 - \hat{p}_2^t)D_1P(\beta^{t-1}D_1) + \hat{p}_2^t H_1^{t-1}(b^{t-1})$ and $H_2^t(b^t) = D_2P(\beta^t D^0)$.

4^o If $D_1P(\beta^t D^0) = H_1^{t-1}(b^{t-1})$ and $D_2P(\beta^t D^0) > H_2^{t-1}(b^{t-1})$. There exist two types of equilibria in the game M(t):

(i) $p_1^t = 1$ and $p_2^t = 0$,

(ii) $p_1^t \in [0, \dfrac{D_2P(\beta^{t-1}D_1) - D_2P(\beta^tD^\eta)}{D_2P(\beta^{t-1}D_1) - H_2^{t-1}(b^{t-1})}]$ and $p_2^t = 1$.

The equilibrium payoffs from the game $\Gamma(t, \beta^tD^\eta)$ are, respectively,

(i) $H_1^t(b^t) = D_1P(\beta^{t-1}D_1)$ and $H_2^t(b^t) = D_2P(\beta^tD^\eta)$;

(ii) $H_1^t(b^t) = D_1P(\beta^tD^\eta)$ and $H_2^t(b^t) = (1 - p_1^t)D_2P(\beta^{t-1}D_2) + p_1^tH_2^{t-1}(b^{t-1})$.

5^o $D_iP(\beta^tD^\eta) < H_i^{t-1}(b^{t-1})$ and $D_jP(\beta^tD^\eta) = H_j^{t-1}(b^{t-1})$. There exists only one type of equilibria in the game M(t), namely

$p_i^t = 1$ and $p_j^t \in [0, 1]$.

The equilibrium payoffs from the game $\Gamma(t, \beta^tD^\eta)$ are $H_i^t(b^t) = (1 - p_j^t)D_iP(\beta^{t-1}D_i) + p_j^tH_i^{t-1}(b^{t-1})$ and $H_j^t(b^t) = D_jP(\beta^tD^\eta)$.

6^o If $D_1P(\beta^tD^\eta) = H_1^{t-1}(b^{t-1})$ and $D_2P(\beta^tD^\eta) = H_2^{t-1}(b^{t-1})$. There exist two types of equilibria:

(i) $p_1^t \in [0, 1]$ and $p_2^t = 1$,

(ii) $p_1^t = 1$ and $p_2^t \in [0, 1]$.

The equilibrium payoffs from the game $\Gamma(t, \beta^tD^\eta)$ are, respectively,

(i) $H_1^t(b^t) = D_1P(\beta^tD^\eta)$ and $H_2^t(b^t) = (1 - p_1^t)D_2P(\beta^{t-1}D_2) + p_1^tD_2P(\beta^tD^\eta)$;

(ii) $H_1^t(b^t) = (1 - p_2^t)D_1P(\beta^{t-1}D_1) + p_2^tD_1P(\beta^tD^\eta)$ and $H_2^t(b^t) = D_2P(\beta^tD^\eta)$.

7^o $D_1P(\beta^tD^\eta) < H_1^{t-1}(b^{t-1})$ and $D_2P(\beta^tD^\eta) < H_2^{t-1}(b^{t-1})$. There exists only one equilibrium $p_1^t = 1$ and $p_2^t = 1$ in the game M(t).

The equilibrium payoffs from the game $\Gamma(t, \beta^tD^\eta)$ are $H_1^t(b^t) = H_1^{t-1}(b^{t-1})$ and $H_2^t(b^t) = H_2^{t-1}(b^{t-1})$.

From the conclusions following Lemma 1, we know that the pairs (0, 1) and (1, 0) cannot be local strategies of any subgame perfect equilibrium with the time continuation property for the game $\Gamma(0, D^\eta)$. Thus, we can immediately exclude 1^o as impossible. By the same argument, we can also exclude equilibrium (i) in 3^o and 4^o.

We observe also that 5°, and 6° cannot take place for any t. This is because in any of the above cases the equilibrium payoffs $H_1^t(b^t)$, $H_2^t(b^t)$ do not satisfy the conditions 5°, and 6°.

We show that 7° is not possible, either. Observe that if the payoffs $H_1^{t+1}(b^{t+1})$, $H_2^{t+1}(b^{t+1})$ satisfy one of the conditions 1° – 6°, then the equilibrium payoffs $H_1^t(b^t)$, $H_2^t(b^t)$ do not satisfy the conditions 7°. Thus, if the equilibrium payoffs $H_1^t(b^t)$, $H_2^t(b^t)$ would satisfy assumptions of 7°, then the payoffs $H_1^{t+1}(b^{t+1})$, $H_2^{t+1}(b^{t+1})$ could only satisfy the assumptions of 7°. By induction, the payoffs $H_1^\tau(b^\tau)$, $H_2^\tau(b^\tau)$ for all $\tau \geq t$ could only satisfy the assumptions of 7°. But then, the players would play an equilibrium local strategy combination (1, 1) from the time t on, and it would hold that $H_1^t(b^t) = H_1^\tau(b^\tau)$ and $H_2^t(b^t) = H_2^\tau(b^\tau)$ for any τ $(\tau \geq t)$. However, the strategy combination $b^t = ((1, 1, ...), (1, 1, ...))$ is not a Nash equilibrium for the game $\Gamma(t, \beta^t D^0)$, because for τ sufficiently big $D_1P(\beta^{\tau+1}D_1) < D_1P(\beta^t D^0)$, and because $H_1^t(b^t) = H_1^\tau(b^\tau) < D_1P(\beta^{\tau+1}D_1)$, it is better for the player 1 to sell in period t instead of waiting.

Summarizing, the local strategy combinations which could occur in any subgame perfect equilibrium must satisfy: 2°, 3° (*ii*), or 4° (*ii*).

Observe the following regularities. If the payoffs $H_1^{t+1}(b^{t+1})$, $H_2^{t+1}(b^{t+1})$ satisfy conditions 4°, then the payoffs $H_1^t(b^t)$, $H_2^t(b^t)$ satisfy the conditions

$$2^o \quad \text{when} \quad p_1^t \in \left(\frac{D_2P(\beta^{t+1}D_2) - D_2P(\beta^{t+1}D^0)}{D_2P(\beta^{t+1}D_2) - H_2^{t+1}(b^{t+1})}, \frac{D_2P(\beta^{t+1}D_2) - D_2P(\beta^t D^0)}{D_2P(\beta^{t+1}D_2) - H_2^{t+1}(b^{t+1})} \right]; \text{ and}$$

$$3^o \quad \text{when} \quad p_1^t = \frac{D_2P(\beta^{t+1}D_2) - D_2P(\beta^{t+1}D^0)}{D_2P(\beta^{t+1}D_2) - H_2^{t+1}(b^{t+1})}.$$

If the payoffs $H_1^{t+1}(b^{t+1})$, $H_2^{t+1}(b^{t+1})$ satisfy conditions 3°, then the equilibrium payoffs $H_1^t(b^t)$, $H_2^t(b^t)$ satisfy the conditions

$$2^o \quad \text{when} \quad p_2^t \in \left(\frac{D_1P(\beta^{t+1}D_1) - D_1P(\beta^{t+1}D^0)}{D_1P(\beta^{t+1}D_1) - H_1^{t+1}(b^{t+1})}, \frac{D_1P(\beta^{t+1}D_1) - D_1P(\beta^t D^0)}{D_1P(\beta^{t+1}D_1) - H_1^{t+1}(b^{t+1})} \right]; \text{ and}$$

$$3^o \quad \text{when} \quad p_2^t = \frac{D_1P(\beta^{t+1}D_1) - D_1P(\beta^{t+1}D^0)}{D_1P(\beta^{t+1}D_1) - H_1^{t+1}(b^{t+1})}.$$

If the payoffs $H_1^{t+1}(b^{t+1})$, $H_2^{t+1}(b^{t+1})$ satisfy conditions 2°, then the equilibrium payoffs $H_1^t(b^t)$, $H_2^t(b^t)$ satisfy the conditions 2° as well.

Therefore we can identify exactly three types of candidates for a subgame perfect equilibria of the game $\Gamma(0, D^0)$ satisfying the time continuation property. The first type of candidates are strategy combinations which in all games $\Gamma(t, \beta^t D^0)$ give the payoffs satisfying the conditions 2^o. There is only one such strategy combination and it coincides with the central equilibrium given by (3.3). The second type of candidates are strategy combinations which in the game $\Gamma(t, \beta^t D^0)$ give the payoffs satisfying the assumptions of 3^o when t is even (odd), and the assumptions of 4^o when t is odd (even). There are two such strategy combinations and they coincide with alternating equilibria given by (3.4) and (3.5). The third type of candidates are strategy combinations which in the game $\Gamma(t, \beta^t D^0)$ give the payoffs satisfying the assumptions of 2^o for all t smaller than or equal to $\tau + 2$ ($\tau \geq -2$), and the assumptions of 3^o for $t > \tau + 2$ and t even (odd), and the assumptions of 4^o for $t > \tau + 2$ and t odd (even). This set of strategy combinations coincides with the set of mutating equilibria given by Theorem 1.

Thus we showed that any subgame perfect equilibrium of the game $\Gamma(0, D^0)$ satisfying the time continuation property must be central, alternating or mutating.

5.2. The Sufficient Conditions for a Subgame Perfect Equilibrium with the Time Continuation Property

It remains to show that the strategy combinations called central, alternating and mutating are indeed the subgame perfect equilibria with the time continuation property. First, we prove the following lemmas.

Lemma 2. For any t and any $b_i^t \in B_i^t$ it holds

$$H_i^t(b_i^t, \hat{b}_j^t) - D_iP(\beta^t D^0) = [H_i^{\tau+1}(b_i^{\tau+1}, \hat{b}_j^{\tau+1}) - D_iP(\beta^{\tau+1} D^0)]\prod_{k=t}^{\tau}[p_i^k \hat{p}_j^k \quad \text{for all } \tau \geq t.$$

Proof. From (5.2) and (3.3), we have

$$H_i^t(b_i^t, \hat{b}_j^t) - D_iP(\beta^t D^0) = (1 - p_i^t)D_iP(\beta^t D^0) + p_i^t(1 - \hat{p}_j^t)D_iP(\beta^{t+1} D_i)$$
$$+ p_i^t \hat{p}_j^t H_i^{t+1}(b_i^{t+1}, \hat{b}_j^{t+1}) - D_iP(\beta^t D^0)$$

$$= p_i^t \hat{p}_j^t H_i^{t+1}(b_i^{t+1}, \hat{b}_j^{t+1}) - p_i^t \hat{p}_j^t D_iP(\beta^{t+1} D^0)$$
$$+ p_i^t \hat{p}_j^t D_iP(\beta^{t+1} D^0) - p_i^t D^0 P(\beta^t D^0) + p_i^t D_iP(\beta^{t+1} D_i) - p_i^t \hat{p}_j^t D_iP(\beta^{t+1} D_i)$$

$$= \left[\Pi_i^{t+1}(b_i^{t+1}, \hat{b}_j^{t+1}) - D^0 P(\beta^{t+1} D^0) \right] p_i^t \hat{p}_j^t - p_i^t D^0 P(\beta^t D^0) - p_i^t \hat{p}_j^{t+1} [D_i P(\beta^{t+1} D_i) - D^0 P(\beta^{t+1} D^0)]$$
$$+ p_i^t D_i P(\beta^{t+1} D_i)$$

$$= [\Pi_i^{t+1}(b_i^{t+1}, \hat{b}_j^{t+1}) - D_i P(\beta^{t+1} D^0)] p_i^t \hat{p}_j^t - p_i^t D^0 P(\beta^t D^0)$$
$$- p_i^t \frac{P(\beta^{t+1} D_i) - P(\beta^t D^0)}{P(\beta^{t+1} D_i) - P(\beta^{t+1} D^0)} [D_i P(\beta^{t+1} D_i) - D^0 P(\beta^{t+1} D^0)] + p_i^t D_i P(\beta^{t+1} D_i)$$

$$= [\Pi_i^{t+1}(b_i^{t+1}, \hat{b}_j^{t+1}) - D_i P(\beta^{t+1} D^0)] p_i^t \hat{p}_j^t,$$

i.e., Lemma 2 is true for $\tau = t$ By induction, we obtain the claim of Lemma 2. //

Lemma 3. For any t, any b^t, and $\tau > t$ it holds

1) $$[\Pi_i^{\tau+1}(b^{\tau+1}) - D_i P(\beta^{\tau+1} D^0)] \prod_{k=t}^{\tau} p_1^k p_2^k \to 0 \quad \text{as} \quad \tau \to \infty;$$

2) $$[\Pi_i^{\tau+1}(b^{\tau+1}) - D_i P(\beta^{\tau} D^0)] \prod_{k=t}^{\tau} p_1^k p_2^k \to 0 \quad \text{as} \quad \tau \to \infty.$$

Proof. The expected payoff to bank i from the game $\Gamma(\tau + 1, \beta^{\tau+1} D^0)$ cannot be higher than the payoff when bank j sells in period $\tau + 1$ and bank i waits in period $\tau + 1$ and sells in period $\tau + 2$, i.e., $\Pi_i^{\tau+1}(b^{\tau+1}) \leq D_i P(\beta^{\tau+2} D_i)$ for $\tau = 0, 1, \dots$. By (2.1) $P(\beta^{\tau+2} D_i) \to 0$ as $\tau \to \infty$. Thus, since the payoff to bank i is nonnegative, we have $\Pi_i^{\tau+1}(b^{\tau+1}) \to 0$ as $\tau \to \infty$. Because $\prod_{k=t}^{\tau} p_1^k p_2^k \leq 1$ for any $\tau \geq t$, and, by (2.1) $P(\beta^{\tau+1} D^0) \to 0$ and $P(\beta^{\tau} D^0) \to 0$ as $\tau \to \infty$, we have the claim of Lemma 3. //

Here, we prove that the behavior strategy combination $\hat{b}^0$ called central is a subgame perfect equilibrium of the game $\Gamma(0, D^0)$ with the time continuation property. From Lemma 2 and Lemma 3 we have that for every t

(5.5) $$\Pi_i^t(b_i^t, \hat{b}_j^t) = D_i P(\beta^t D^0) \text{ for any } b_i^t \in B_i^t.$$

By (5.5) $\Pi_i^t(\hat{b}^t) = D_i P(\beta^t D^0)$, thus

$$\Pi_i^t(b_i^t, \hat{b}_j^t) \leq \Pi_i^t(\hat{b}_i^t, \hat{b}_j^t) \text{ for every t and any } b_i^t \in B_i^t,$$

i.e., bank i has no incentives to deviate from its strategy $\hat{b}_i^t$ for every t. Thus the strategy combination $\hat{b}^0$ called central induces a Nash equilibrium on every subgame of the game $\Gamma(0, D^0)$, i.e., $\hat{b}^0$ is a subgame perfect equilibrium of the game $\Gamma(0, D^0)$.

The central equilibrium $\hat{b}^0$ satisfies the time continuation property, because for every $t \leq -1$ the strategy combination $\hat{b}^t$ is a subgame perfect equilibrium of the retrospective extension $\Gamma(t, \beta D^0)$ of the game $\Gamma(0, D^0)$ and the realization probability $\prod_{\tau=t}^{-1} \hat{p}_1^\tau \hat{p}_2^\tau$ of the game $\Gamma(0, D^0)$ is positive.

The following lemma will be used to prove that the behavior strategy combination $\tilde{b}^0$ called alternating is a subgame perfect equilibrium of the game $\Gamma(0, D^0)$.

Lemma 4. For any $b_i^t \in B_i^t$ it holds

1) for even t

$$H_i^t(b_i^t, \tilde{b}_j^t) - D_i P(\beta^t D^0) = \begin{cases} [H_i^{\tau+1}(b_i^{\tau+1}, \tilde{b}_j^{\tau+1}) - D_i P(\beta^\tau D^0)] \prod_{k=t}^{\tau} p_i^k \tilde{p}_j^k \\ \quad + \sum_{s=(t+2)/2}^{\tau/2} [D_i P(\beta^{2s-1} D^0) - D_i P(\beta^{2s-2} D^0)](1 - p_i^{2s-1}) \prod_{k=t}^{2s-2} p_i^k \tilde{p}_j^k \\ \quad \text{for } \tau = t+2, t+4, \ldots, \\ \\ [H_i^{\tau+1}(b_i^{\tau+1}, \tilde{b}_j^{\tau+1}) - D_i P(\beta^{\tau+1} D^0)] \prod_{k=t}^{\tau} p_i^k \tilde{p}_j^k \\ \quad + \sum_{s=t/2}^{(\tau-1)/2} [D_i P(\beta^{2s+1} D^0) - D_i P(\beta^{2s} D^0)](1 - p_i^{2s+1}) \prod_{k=t}^{2s} p_i^k \tilde{p}_j^k \\ \quad \text{for } \tau = t+1, t+3, \ldots; \end{cases}$$

2) for odd t

$$H_i^t(b_i^t, \tilde{b}_j^t) - D_i P(\beta^{t-1} D^0) = \begin{cases} [H_i^{\tau+1}(b_i^{\tau+1}, \tilde{b}_j^{\tau+1}) - D_i P(\beta^\tau D^0)] \prod_{k=t}^{\tau} p_i^k \tilde{p}_j^k \\ \quad + \sum_{s=(t+1)/2}^{\tau/2} [D_i P(\beta^{2s-1} D^0) - D_i P(\beta^{2s-2} D^0)](1 - p_i^{2s-1}) \prod_{k=t}^{2s-2} p_i^k \tilde{p}_j^k \\ \quad \text{for } \tau = t+1, t+3, \ldots,^6 \\ \\ [H_i^{\tau+1}(b_i^{\tau+1}, \tilde{b}_j^{\tau+1}) - D_i P(\beta^{\tau+1} D^0)] \prod_{k=t}^{\tau} p_i^k \tilde{p}_j^k \\ \quad + \sum_{s=(t-1)/2}^{(\tau-1)/2} [D_i P(\beta^{2s+1} D^0) - D_i P(\beta^{2s} D^0)](1 - p_i^{2s+1}) \prod_{k=t}^{2s} p_i^k \tilde{p}_j^k \\ \quad \text{for } \tau = t+2, t+4, \ldots. \end{cases}$$

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[6] We use the convention $\prod_{k=t}^{t-1} p_i^k \tilde{p}_j^k = 1$.

Proof. 1) Let t be even. From (5.2) and (3.4), we have

$$H_i^t(b_i^t, \tilde{b}_j^t) - D_i P(\beta^t D^0) = (1 - p_i^t) D_i P(\beta^t D^0) + p_i^t \tilde{p}_j^t H_i^{t+1}(b_i^{t+1}, \tilde{b}_j^{t+1}) - D_i P(\beta^t D^0)$$

$$= \left[H_i^{t+1}(b_i^{t+1}, \tilde{b}_j^{t+1}) - D_i P(\beta^t D^0) \right] p_i^t \tilde{p}_j^t.$$

Conducting further substitution by using (5.2) and (3.5) for t + 1, we have

$$H_i^t(b_i^t, \tilde{b}_j^t) - D_i P(\beta^t D^0) =$$

$$\left[(1 - p_i^{t+1}) D_i P(\beta^{t+1} D^0) + p_i^{t+1}(1 - \tilde{p}_j^{t+1}) D_i P(\beta^{t+2} D_i) + p_i^{t+1} \tilde{p}_j^{t+1} H_i^{t+2}(b_i^{t+2}, \tilde{b}_j^{t+2}) - D_i P(\beta^t D^0) \right] p_i^t \tilde{p}_j^t$$

$$= H_i^{t+2}(b_i^{t+2}, \tilde{b}_j^{t+2}) p_i^t \tilde{p}_j^t p_i^{t+1} \tilde{p}_j^{t+1} - D_i P(\beta^{t+2} D^0) p_i^t \tilde{p}_j^t p_i^{t+1} \tilde{p}_j^{t+1} + D_i P(\beta^{t+2} D^0) p_i^t \tilde{p}_j^t p_i^{t+1} \tilde{p}_j^{t+1}$$
$$+ D_i P(\beta^{t+1} D^0) p_i^t \tilde{p}_j^t (1 - p_i^{t+1}) + D_i P(\beta^{t+2} D_i) p_i^t \tilde{p}_j^t p_i^{t+1}(1 - \tilde{p}_j^{t+1}) - D_i P(\beta^t D^0) p_i^t \tilde{p}_j^t$$

$$= \left[H_i^{t+2}(b_i^{t+2}, \tilde{b}_j^{t+2}) - D_i P(\beta^{t+2} D^0) \right] p_i^t \tilde{p}_j^t p_i^{t+1} \tilde{p}_j^{t+1} + D_i P(\beta^{t+1} D^0) p_i^t \tilde{p}_j^t (1 - p_i^{t+1})$$
$$- \left[D_i P(\beta^{t+2} D_i) - D_i P(\beta^{t+2} D^0) \right] p_i^t \tilde{p}_j^t p_i^{t+1} \tilde{p}_j^{t+1} + D_i P(\beta^{t+2} D_i) p_i^t \tilde{p}_j^t p_i^{t+1} - D_i P(\beta^t D^0) p_i^t \tilde{p}_j^t$$

$$= \left[H_i^{t+2}(b_i^{t+2}, \tilde{b}_j^{t+2}) - D_i P(\beta^{t+2} D^0) \right] p_i^t \tilde{p}_j^t p_i^{t+1} \tilde{p}_j^{t+1} + D_i P(\beta^{t+1} D^0) p_i^t \tilde{p}_j^t (1 - p_i^{t+1})$$
$$- \left[D_i P(\beta^{t+2} D_i) - D_i P(\beta^{t+2} D^0) \right] p_i^t \tilde{p}_j^t p_i^{t+1} \frac{P(\beta^{t+2} D_i) - P(\beta^t D^0)}{P(\beta^{t+2} D_i) - P(\beta^{t+2} D^0)}$$
$$+ D_i P(\beta^{t+2} D_i) p_i^t \tilde{p}_j^t p_i^{t+1} - D_i P(\beta^t D^0) p_i^t \tilde{p}_j^t$$

$$= \left[H_i^{t+2}(b_i^{t+2}, \tilde{b}_j^{t+2}) - D_i P(\beta^{t+2} D^0) \right] p_i^t \tilde{p}_j^t p_i^{t+1} \tilde{p}_j^{t+1} + D_i P(\beta^{t+1} D^0) p_i^t \tilde{p}_j^t (1 - p_i^{t+1})$$
$$+ D_i P(\beta^t D^0) p_i^t \tilde{p}_j^t p_i^{t+1} - D_i P(\beta^t D^0) p_i^t \tilde{p}_j^t$$

$$= \left[H_i^{t+2}(b_i^{t+2}, \tilde{b}_j^{t+2}) - D_i P(\beta^{t+2} D^0) \right] p_i^t \tilde{p}_j^t p_i^{t+1} \tilde{p}_j^{t+1} + \left[D_i P(\beta^{t+1} D^0) - D_i P(\beta^t D^0) \right] p_i^t \tilde{p}_j^t (1 - p_i^{t+1}),$$

i.e., Lemma 4.1) is true for $\tau = t + 1$.

Using (5.2) and (3.4) for t + 2, we obtain

$$H_i^t(b_i^t, \tilde{b}_j^t) - D_i P(\beta^t D^0) = \left[H_i^{t+3}(b_i^{t+3}, \tilde{b}_j^{t+3}) - D_i P(\beta^{t+2} D^0) \right] p_i^t \tilde{p}_j^t p_i^{t+1} \tilde{p}_j^{t+1} p_i^{t+2} \tilde{p}_j^{t+2}$$
$$+ \left[D_i P(\beta^{t+1} D^0) - D_i P(\beta^t D^0) \right] p_i^t \tilde{p}_j^t (1 - p_i^{t+1}),$$

i.e., Lemma 4.1) is true for $\tau = t + 2$. By induction, we obtain the claim of Lemma 4.1).

2) By analogous induction, we obtain the claim of Lemma 4.2). //

Here we prove that the behavior strategy combination $\tilde{b}^0$ called alternating is a subgame perfect equilibrium of the game $\Gamma(0, D^0)$ with the time continuation property. For a complete proof it is necessary to show that banks i and j do not have any incentives to deviate from their strategies in $\tilde{b}^t = (\tilde{b}_i^t, \tilde{b}_j^t)$ for t = 0, 1, ... Since the proof for the bank j goes allong the same lines, we show only that bank i does not have any incentives to deviate from its strategy $\tilde{b}_i^t$ for t = 0, 1, ...

Since $\tilde{p}_i^{2s-1} = 1$ for s = 1, 2, ... , it follows from Lemma 3 and Lemma 4 that for even t,

(5.6) $$H_i^t(\tilde{b}^t) = D_iP(\beta^t D^0),$$

and for odd t,

(5.7) $$H_i^t(\tilde{b}^t) = D_iP(\beta^{t-1}D^0).$$

From Lemma 4.1) we have that for even t and any $b_i^t \in B_i^t$

$$H_i^t(b_i^t, \tilde{b}_j^t) - D_iP(\beta^t D^0) \le [H_i^{\tau+1}(b_i^{\tau+1}, \tilde{b}_j^{\tau+1}) - D_iP(\beta^{\tau+1}D^0)]\prod_{k=t}^{\tau} p_i^k \tilde{p}_j^k \quad \text{for } \tau = t+1, t+3, \dots .$$

Using Lemma 3.2) we obtain

$$H_i^t(b_i^t, \tilde{b}_j^t) - D_iP(\beta^t D^0) \le 0 \quad \text{for even t and any } b_i^t \in B_i^t.$$

Applying (5.6) we have

$$H_i^t(b_i^t, \tilde{b}_j^t) \le H_i^t(\tilde{b}^t) \text{ for even t and any } b_i^t \in B_i^t,$$

i.e., bank i has no incentives to deviate from its strategy $\tilde{b}_i^t$ for every even t.

In the same way, we prove that bank i has no incentives to deviate from its strategy $\tilde{b}_i^t$ for every odd t. From Lemma 4.2) we have that for odd t and any $b_i^t \in B_i^t$

$$H_i^t(b_i^t, \tilde{b}_j^t) - D_iP(\beta^{t-1}D^0) \le [H_i^{\tau+1}(b_i^{\tau+1}, \tilde{b}_j^{\tau+1}) - D_iP(\beta^{\tau}D^0)]\prod_{k=t}^{\tau} p_i^k \tilde{p}_j^k \quad \text{for } \tau = t+1, t+3, \dots .$$

Using Lemma 3.2) we obtain

$$H_i^t(b_i^t, \tilde{b}_j^t) - D_iP(\beta^{t-1}D^0) \le 0 \quad \text{for odd t and any } b_i^t \in B_i^t.$$

Applying (5.7) we have

$$H_i^t(b_i^t, \tilde{b}_j^t) \leq H_i^t(\tilde{b}^t) \quad \text{for odd t and any } b_i^t \in B_i^t,$$

i.e., bank i has no incentives to deviate from its strategy $\tilde{b}_i^t$ for every odd t. Thus the strategy combination $\tilde{b}^0$ called alternating induces a Nash equilibrium on every subgame of the game $\Gamma(0, D^0)$, i.e., $\tilde{b}^0$ is a subgame perfect equilibrium of the game $\Gamma(0, D^0)$.

The alternating equilibrium $\tilde{b}^0$ satisfies the time continuation property, because for every $t \leq -1$ the strategy combination $\tilde{b}^t$ is a subgame perfect equilibrium of the retrospective extension $\Gamma(t, \beta D^0)$ of the game $\Gamma(0, D^0)$ and the realization probability $\prod_{\tau=t}^{-1} \tilde{p}_1^\tau \tilde{p}_2^\tau$ of the game $\Gamma(0, D^0)$ is positive.

Now, we prove that a strategy combination $\bar{b}^0$ called mutating is a subgame perfect equilibrium of the game $\Gamma(0, D^0)$. By definition of $\bar{b}^0$, $\bar{b}^t = \tilde{b}^t$ for $t = \tau + 3,\ \tau + 4,\ \ldots$. Since the alternating equilibrium $\tilde{b}^0$ is subgame perfect, the mutating equilibrium $\bar{b}^0$ induces a Nash equilibrium on the subgames $\Gamma(\tau + 3, \beta^{\tau+3}D^0)$, $\Gamma(\tau + 4, \beta^{\tau+4}D^0)$, ... It remains to consider the subgames $\Gamma(0, D^0), \Gamma(1, \beta D^0),\ \ldots, \Gamma(\tau + 2, \beta^{\tau+2}D^0)$.

First, we consider the subgames $\Gamma(\tau + 1, \beta^{\tau+1}D^0)$ and $\Gamma(\tau + 2, \beta^{\tau+2}D^0)$. Since the local strategies of banks in periods $\tau + 1$ and $\tau + 2$ are asymmetric, we have to look at the incentives of each bank to deviate separately.

Consider the payoffs to bank i from the game $\Gamma(\tau + 2, \beta^{\tau+2}D^0)$. From (5.2) we have
$H_i^{\tau+2}(b_i^{\tau+2}, \bar{b}_j^{\tau+2}) = (1 - p_i^{\tau+2})D_iP(\beta^{\tau+2}D^0) + p_i^{\tau+2}(1 - \bar{p}_j^{\tau+2})D_iP(\beta^{\tau+3}D_i) + p_i^{\tau+2}\bar{p}_j^{\tau+2}H_i^{\tau+3}(b_i^{\tau+3}, \bar{b}_j^{\tau+3})$.

Because $H_i^{\tau+3}(b_i^{\tau+3}, \bar{b}_j^{\tau+3}) = H_i^{\tau+3}(b_i^{\tau+3}, \tilde{b}_j^{\tau+3})$ for all $b_i^{\tau+3} \in B_i^{\tau+3}$, thus using (3.7) we have

$$H_i^{\tau+2}(\bar{b}^{\tau+2}) = (1 - \bar{p}_j^{\tau+2})D_iP(\beta^{\tau+3}D_i) + \bar{p}_j^{\tau+2}H_i^{\tau+3}(\tilde{b}^{\tau+3}).$$

Since also $H_i^{\tau+3}(b_i^{\tau+3}, \tilde{b}_j^{\tau+3}) \leq H_i^{\tau+3}(\tilde{b}^{\tau+3})$ for all $b_i^{\tau+3} \in B_i^{\tau+3}$, we have

$$H_i^{\tau+2}(b_i^{\tau+2}, \bar{b}_j^{\tau+2}) \leq (1 - p_i^{\tau+2})D_iP(\beta^{\tau+2}D^0) + p_i^{\tau+2}(1 - \bar{p}_j^{\tau+2})D_iP(\beta^{\tau+3}D_i) + p_i^{\tau+2}\bar{p}_j^{\tau+2}H_i^{\tau+3}(\tilde{b}^{\tau+3})$$

$$= D_iP(\beta^{\tau+2}D^0) + p_i^{\tau+2}\Big[- D_iP(\beta^{\tau+2}D^0) + D_iP(\beta^{\tau+3}D_i) - \bar{p}_j^{\tau+2}(D_iP(\beta^{\tau+3}D_i) - H_i^{\tau+3}(\tilde{b}^{\tau+3}))\Big].$$

The term in the last square bracket is positive, because $H_i^{\tau+3}(\tilde{b}^{\tau+3}) = D_iP(\beta^{\tau+3}D^0)$ by (5.6) for $\tau + 3$, and $\bar{p}_j^{\tau+2}$ is given by (3.7). Thus

$$H_i^{\tau+2}(b_i^{\tau+2}, \bar{b}_j^{\tau+2}) \leq D_iP(\beta^{\tau+2}D^0) + 1 \times \Big[- D_iP(\beta^{\tau+2}D^0) + D_iP(\beta^{\tau+3}D_i) - \bar{p}_j^{\tau+2}(D_iP(\beta^{\tau+3}D_i) - H_i^{\tau+3}(\tilde{b}^{\tau+3}))\Big]$$

$$= (1 - \bar{p}_j^{\tau+2})D_iP(\beta^{\tau+3}D_i) + \bar{p}_j^{\tau+2}H_i^{\tau+3}(\tilde{b}^{\tau+3}).$$

Therefore

$$H_i^{\tau+2}(b_i^{\tau+2}, \bar{b}_j^{\tau+2}) \;\le\; H_i^{\tau+2}(\bar{b}^{\tau+2}) \text{ for all } b_i^{\tau+2} \in B_i^{\tau+2},$$

i.e., bank i has no incentives to deviate from its strategy $\bar{b}_i^{\tau+2}$ in the game $\Gamma(\tau + 2, \beta^{\tau+2}D^0)$.

Now, consider the payoff to bank j from the game $\Gamma(\tau + 2, \beta^{\tau+2}D^0)$. From (5.2) and (3.7) we have

$$H_j^{\tau+2}(\bar{b}_i^{\tau+2}, b_j^{\tau+2}) \;=\; (1 - p_j^{\tau+2})D_jP(\beta^{\tau+2}D^0) + p_j^{\tau+2}H_j^{\tau+3}(\bar{b}_i^{\tau+3}, b_j^{\tau+3}).$$

Because $H_j^{\tau+3}(\bar{b}_i^{\tau+3}, b_j^{\tau+3}) = H_j^{\tau+3}(\tilde{b}_i^{\tau+3}, b_j^{\tau+3})$ for all $b_j^{\tau+3} \in B_j^{\tau+3}$, thus using (3.7) we have

$$H_j^{\tau+2}(\bar{b}^{\tau+2}) \;=(1 - \bar{p}_j^{\tau+2})D_jP(\beta^{\tau+2}D^0) + \bar{p}_j^{\tau+2}H_j^{\tau+3}(\tilde{b}^{\tau+3}).$$

Substituting $H_j^{\tau+3}(\tilde{b}^{\tau+3}) = D_jP(\beta^{\tau+2}D^0)$ which follows from (5.7) for $\tau + 3$, we obtain

$$H_j^{\tau+2}(\bar{b}^{\tau+2}) \;=\; D_jP(\beta^{\tau+2}D^0).$$

Since also $H_j^{\tau+3}(\tilde{b}_i^{\tau+3}, b_j^{\tau+3}) \le H_j^{\tau+3}(\tilde{b}^{\tau+3})$ for all $b_j^{\tau+3} \in B_j^{\tau+3}$, we have

$$H_j^{\tau+2}(\bar{b}_i^{\tau+2}, b_j^{\tau+2}) \;\le\; (1 - p_j^{\tau+2})D_jP(\beta^{\tau+2}D^0) + p_j^{\tau+2}H_j^{\tau+3}(\tilde{b}^{\tau+3}) = D_jP(\beta^{\tau+2}D^0).$$

Therefore

$$H_j^{\tau+2}(\bar{b}_i^{\tau+2}, b_j^{\tau+2}) \le H_j^{\tau+2}(\bar{b}^{\tau+2}) \text{ for all } b_j^{\tau+2} \in B_j^{\tau+2},$$

i.e., bank j has no incentives to deviate from its strategy $\bar{b}_j^{\tau+2}$ in the game $\Gamma(\tau + 2, \beta^{\tau+2}D^0)$.

Consider the payoff to bank i from the game $\Gamma(\tau + 1, \beta^{\tau+1}D^0)$. From (5.2) we have that for any $b_i^{\tau+1} \in B_i^{\tau+1}$

$$H_i^{\tau+1}(b_i^{\tau+1}, \bar{b}_j^{\tau+1}) \;=\; (1 - p_i^{\tau+1})D_iP(\beta^{\tau+1}D^0) + p_i^{\tau+1}(1 - \bar{p}_j^{\tau+1})D_iP(\beta^{\tau+2}D_i) + p_i^{\tau+1}\bar{p}_j^{\tau+1}H_i^{\tau+2}(b_i^{\tau+2}, \bar{b}_j^{\tau+2}).$$

Because $H_i^{\tau+2}(\bar{b}^{\tau+2}) = (1 - \bar{p}_j^{\tau+2})D_iP(\beta^{\tau+3}D_i) + \bar{p}_j^{\tau+2}D_iP(\beta^{\tau+3}D^0)$, thus

$$H_i^{\tau+1}(\bar{b}^{\tau+1}) = (1 - \bar{p}_i^{\tau+1})D_iP(\beta^{\tau+1}D^0) + \bar{p}_i^{\tau+1}(1 - \bar{p}_j^{\tau+1})D_iP(\beta^{\tau+2}D_i) + \bar{p}_i^{\tau+1}\bar{p}_j^{\tau+1}H_i^{\tau+2}(\bar{b}^{\tau+2})$$

$$= (1 - \bar{p}_i^{\tau+1})D_iP(\beta^{\tau+1}D^0) + \bar{p}_i^{\tau+1}(1 - \bar{p}_j^{\tau+1})D_iP(\beta^{\tau+2}D_i)$$
$$+ \bar{p}_i^{\tau+1}\bar{p}_j^{\tau+1}[D_iP(\beta^{\tau+3}D_i) - \bar{p}_j^{\tau+2}(D_iP(\beta^{\tau+3}D_i) - D_iP(\beta^{\tau+3}D^0))]$$

$$= (1 - \bar{p}_i^{\tau+1})D_iP(\beta^{\tau+1}D^0) + \bar{p}_i^{\tau+1}D_iP(\beta^{\tau+2}D_i)$$
$$- \bar{p}_i^{\tau+1}\bar{p}_j^{\tau+1}[D_iP(\beta^{\tau+2}D_i) - D_iP(\beta^{\tau+3}D_i) + \bar{p}_j^{\tau+2}(D_iP(\beta^{\tau+3}D_i) - D_iP(\beta^{\tau+3}D^0))].$$

Substituting (3.6) we obtain

$$H_i^{\tau+1}(\bar{b}^{\tau+1}) = D_iP(\beta^{\tau+1}D^0).$$

Since also $H_i^{\tau+2}(b_i^{\tau+2}, \bar{b}_j^{\tau+2}) \le H_i^{\tau+2}(\bar{b}^{\tau+2})$ for all $b_i^{\tau+1} \in B_i^{\tau+1}$, we have

$$H_i^{\tau+2}(b_i^{\tau+2}, \bar{b}_j^{\tau+1}) \le (1 - p_i^{\tau+1})D_iP(\beta^{\tau+1}D^0) + p_i^{\tau+1}(1 - \bar{p}_j^{\tau+1})D_iP(\beta^{\tau+2}D_i) + p_i^{\tau+1}\bar{p}_j^{\tau+1}H_i^{\tau+2}(\bar{b}^{\tau+2})$$

$$= (1 - p_i^{\tau+1})D_iP(\beta^{\tau+1}D^0) + p_i^{\tau+1}(1 - \bar{p}_j^{\tau+1})D_iP(\beta^{\tau+2}D_i)$$
$$+ p_i^{\tau+1}\bar{p}_j^{\tau+1}[D_iP(\beta^{\tau+3}D_i) - \bar{p}_j^{\tau+2}(D_iP(\beta^{\tau+3}D_i) - D_iP(\beta^{\tau+3}D^0))]$$

$$= (1 - p_i^{\tau+1})D_iP(\beta^{\tau+1}D^0) + p_i^{\tau+1}D_iP(\beta^{\tau+2}D_i)$$
$$- p_i^{\tau+1}\bar{p}_j^{\tau+1}[D_iP(\beta^{\tau+2}D_i) - D_iP(\beta^{\tau+3}D_i) + \bar{p}_j^{\tau+2}(D_iP(\beta^{\tau+3}D_i) - D_iP(\beta^{\tau+3}D^0))].$$

Substituting (3.6) into the last expression we obtain

$$H_i^{\tau+1}(b_i^{\tau+1}, \bar{b}_j^{\tau+1}) \leq D_iP(\beta^{\tau+1}D^0) \text{ for all } b_i^{\tau+1} \in B_i^{\tau+1}.$$

Thus

$$H_i^{\tau+1}(b_i^{\tau+1}, \bar{b}_j^{\tau+1}) \leq H_i^{\tau+1}(\bar{b}^{\tau+1}) \quad \text{for all } b_i^{\tau+1} \in B_i^{\tau+1},$$

i.e., bank i has no incentives to deviate from its strategy $\bar{b}_i^{\tau+1}$ in the game $\Gamma(\tau + 1, \beta^{\tau+1}D^0)$.

Now, consider the payoff to bank j from the game $\Gamma(\tau + 1, \beta^{\tau+1}D^0)$. From (5.2) we have

$$H_j^{\tau+1}(\bar{b}_i^{\tau+1}, b_j^{\tau+1}) = (1 - p_j^{\tau+1})D_jP(\beta^{\tau+1}D^0) + p_j^{\tau+1}(1 - \bar{p}_i^{\tau+1})D_jP(\beta^{\tau+2}D_j)$$
$$+ \bar{p}_i^{\tau+1}p_j^{\tau+1}H_j^{\tau+2}(\bar{b}_i^{\tau+2}, b_j^{\tau+2}).$$

Because $H_j^{\tau+2}(\bar{b}^{\tau+2}) = D_jP(\beta^{\tau+2}D^0)$, thus

$$H_j^{\tau+1}(\bar{b}^{\tau+1}) = (1 - \bar{p}_j^{\tau+1})D_jP(\beta^{\tau+1}D^0) + \bar{p}_j^{\tau+1}(1 - \bar{p}_i^{\tau+1})D_jP(\beta^{\tau+2}D_j) + \bar{p}_i^{\tau+1}\bar{p}_j^{\tau+1}D_jP(\beta^{\tau+2}D^0)$$
$$= D_jP(\beta^{\tau+1}D^0) - \bar{p}_j^{\tau+1}[D_jP(\beta^{\tau+1}D^0) - D_jP(\beta^{\tau+2}D_j) + \bar{p}_i^{\tau+1}(D_jP(\beta^{\tau+2}D_j) - D_jP(\beta^{\tau+2}D^0))].$$

Substituting (3.6) we obtain

$$H_j^{\tau+1}(\bar{b}^{\tau+1}) = D_jP(\beta^{\tau+1}D^0).$$

Since also $H_j^{\tau+2}(\bar{b}_i^{\tau+2}, b_j^{\tau+2}) \leq H_j^{\tau+2}(\bar{b}^{\tau+2})$ for all $b_j^{\tau+1} \in B_j^{\tau+1}$, we have

$$H_j^{\tau+1}(\bar{b}_i^{\tau+1}, b_j^{\tau+1}) \leq (1 - p_j^{\tau+1})D_jP(\beta^{\tau+1}D^0) + p_j^{\tau+1}(1 - \bar{p}_i^{\tau+1})D_jP(\beta^{\tau+2}D_j) + \bar{p}_i^{\tau+1}p_j^{\tau+1}H_j^{\tau+2}(\bar{b}^{\tau+2})$$
$$\leq (1 - p_j^{\tau+1})D_jP(\beta^{\tau+1}D^0) + p_j^{\tau+1}(1 - \bar{p}_i^{\tau+1})D_jP(\beta^{\tau+2}D_j) + \bar{p}_i^{\tau+1}p_j^{\tau+1}D_jP(\beta^{\tau+2}D^0)$$
$$= D_jP(\beta^{\tau+1}D^0) - p_j^{\tau+1}[D_jP(\beta^{\tau+1}D^0) - D_jP(\beta^{\tau+2}D_j) + \bar{p}_i^{\tau+1}(D_jP(\beta^{\tau+2}D_j) - D_jP(\beta^{\tau+2}D^0))].$$

Substituting (3.6) into the last expression we obtain

$$H_j^{\tau+1}(\bar{b}_i^{\tau+1}, b_j^{\tau+1}) \leq D_jP(\beta^{\tau+1}D^0) \text{ for all } b_j^{\tau+1} \in B_j^{\tau+1},$$

Thus

$$H_j^{\tau+1}(\bar{b}_i^{\tau+1}, b_j^{\tau+1}) \leq H_j^{\tau+1}(\bar{b}^{\tau+1}) \qquad \text{for all } b_j^{\tau+1} \in B_j^{\tau+1},$$

i.e., bank j has no incentives to deviate from its strategy $\bar{b}_j^{\tau+1}$ in the game $\Gamma(\tau + 1, \beta^{\tau+1}D^0)$.

Consider the payoff to bank i = 1, 2 from the game $\Gamma(t, \beta^tD^0)$ for $t = 0, 1, ..., \tau$. By definition of $\bar{b}^0 = ((\bar{p}_1^0, \bar{p}_1^1, ...), (\bar{p}_2^0, \bar{p}_2^1, ...))$, $\bar{p}_i^t = \hat{p}_i^t$ for $t = 0, 1, ..., \tau$. It follows from Lemma 2 that

$$H_i^t(b_i^t, \bar{b}_j^t) - D_iP(\beta^t D^0) = [H_i^{\tau+1}(b_i^{\tau+1}, \bar{b}_j^{\tau+1}) - D_iP(\beta^{\tau+1} D^0)] \prod_{k=t}^{\tau} p_i^k \hat{p}_j^k.$$

Because $H_i^{\tau+1}(\bar{b}^{\tau+1}) - D_iP(\beta^{\tau+1}D^0) = 0$, thus

$$H_i^t(\bar{b}^t) = D_iP(\beta^t D^0).$$

Since $H_i^{\tau+1}(b_i^{\tau+1}, \bar{b}_j^{\tau+1}) \leq H_i^{\tau+1}(\bar{b}^{\tau+1})$ for all $b_i^{\tau+1} \in B_i^{\tau+1}$, we also have

$$H_i^t(b_i^t, \bar{b}_j^t) - D_iP(\beta^t D^0) \leq 0 \quad \text{for } t = 0, 1, \ldots, \tau \text{ and for all } b_i^t \in B_i^t.$$

Thus we obtain

$$H_i^t(b_i^t, \bar{b}_j^t) \leq H_i^t(\bar{b}^t) \quad \text{for } t = 0, 1, \ldots, \tau \text{ and for all } b_i^t \in B_i^t,$$

i.e., bank i has no incentives to deviate from its strategy $\bar{b}_i^t$ in the game $\Gamma(t, \beta^t D^0)$ for $t = 0, 1, \ldots, \tau$.

Therefore the strategy combination $\bar{b}^0$ called mutating induces a Nash equilibrium on every subgame of the game $\Gamma(0, D^0)$, i.e., $\bar{b}^0$ is a subgame perfect equilibrium of the game $\Gamma(0, D^0)$.

A mutating equilibrium $\bar{b}^0$ satisfies the time continuation property, because for every $t \leq -1$ the strategy combination $\bar{b}^t$ is a subgame perfect equilibrium of the retrospective extension $\Gamma(t, \beta D^0)$ of the game $\Gamma(0, D^0)$ and the realization probability $\prod_{\tau=t}^{-1} \bar{p}_1^\tau \bar{p}_2^\tau$ of the game $\Gamma(0, D^0)$ is positive. //

6. Conclusions

In the present paper, we consider the problem of debt overhang, formulating the problem as an infinite horizon game with two banks as players. We find that there exist three types of subgame perfect equilibria with the time continuation property which are called central, alternating and mutating, respectively (Theorem 1). However, the strategies of banks in every equilibrium do not differ much from the strategies of the unique central equilibrium. Therefore the average attainable payoffs are almost uniquely determined.

Our game constitutes a dynamic version of the one-shot game of lender banks given in Kaneko and Prokop (1991). Kaneko and Prokop (1991) focused on the behavior of a large number of banks in a short period. Since the underlying story is dynamic, an important extension of the previous analysis is a direct dynamic approach to the banks' behavior. Sacrificing the insight into the behavior of a large number of banks obtained in Kaneko and Prokop (1991), we investigate the long-run behavior of banks in the presence of the secondary market for debts.

The central equilibrium gives the local equilibrium strategies in each period which coincide with the equilibrium of the one-period game of Kaneko and Prokop (1991). This link between the two formulations allows us to use the one-period approach without losing the dynamic nature of the problem, i.e., we obtain a decomposition property of our dynamic game into the one-period games. Both approaches are complementary in that the one-period model is static but enables us to discuss the effects of a large number of banks, and the dynamic model helps us to understand the long-run behavior of banks but it is too complex to consider the behavior of many banks.

The Folk theorem does not hold for our repeated game, since a deviation of one bank causes its exit from the game and takes away a possibility of punishment by the other bank. In each equilibrium of the dynamic game, the average attainable payoff is almost uniquely determined on the contrary to the Folk Theorem.

In every equilibrium each bank waits in every period with a relatively high probability, and when the interest factor is close to 1, the probability of waiting is close to 1, as well (Theorem 3). We also show that when the price function is approximated by some homogenous function for the large values of debt, the probability of a bank waiting in period t becomes constant as t increases (Theorem 4). The constant is close to 1 as long as the interest factor is relatively low. It suggests that the situation of debt overhang may remain unchanged over time.

In addition, we show that along the central equilibrium path, in every period a bank with a higher loan exposure has a higher probability of waiting than the other one (Theorem 2).

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