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LINEAR MEASURES, THE GINI INDEX AND THE INCOME-EQUALITY TRADEOFF*

by

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Abstract

The paper provides an axiomatization of linear inequality measures as a representation of a binary relation on the subspace of income profiles having the same total income. Interpreting the binary relation as a preference (of, say, a policymaker), we extend the axioms to the whole space of income profiles, and find that they characterize linear social evaluation functions.

The axiomatiziation seems to suggest that a policymaker who has a linear measure of inequality on a subspace should have a linear evaluation on the whole space.

In particular, we find that an extension of the preferences reflected in the Gini index to the whole space is represented by a linear combination of total income and the Gini index.

1. Introduction

The paper provides an axiomatization of linear evaluation functions[1] as a representation of a binary relation on profiles of incomes. In particular, the Gini index is obtained by a natural strengthening of the main axiom. The paper is closely related to the work on the measurement of inequality but the interpretation of the binary relation on the income profiles is different. Here the binary relation is interpreted as a preference order of society or a policymaker. Thus, if f and g are income profiles, $f \succ g$ means that if the policymaker had to choose between f and g she would choose f. While the literature on inequality measurement (notably Atkinson (1970)) clearly recognizes the relationship between social welfare and inequality, its ultimate goal is to measure inequality per se. Thus, it attemps to find a mathematical representation for a binary relation $\succ^*$ with the following interpretation: $f \succ^* g$ means that f is more egalitarian than g. Clearly we would not expect that $\succ$ and $\succ^*$ coincide on the whole space of income profiles. For example, a typical assumption on $\succ^*$ is relative invariance which means that the level of inequality does not change when all incomes are multiplied by the same factor. When one considers the relation $\succ$, such a property is obviously unreasonable. However, if we restrict our attention to some subspace of income profiles where the total income is fixed, then it seems reasonable (or at least of interest) to assume that on such a subspace $\succ$ and

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[1]The term linear in the literature on income distribution means linear after arranging incomes in an increasing order. Formally, let $f \in R^n$, $f = (f_1,\ldots,f_n)$ be an income profile and let $\tilde{f}$ be the profile that is obtained from f by arranging the incomes in an increasing order. We say that a social evaluation function J(f) is linear if there exists numbers $a_1,\ldots,a_n$ such that $J(f) = \sum_{i=1}^{n} a_i\tilde{f}_i$.

$\succ^*$ do coincide. In other words, when the total income is fixed the policymaker determines her preferences according to her judgment about the level of inequality.[2] If this assumption is made our results can be interpreted as suggesting that if a policymaker has a linear measure of inequality on some subspace then she should have a linear evaluation on the whole space. In particular, the evaluation on the whole space can be represented as the sum of total income and an appropriate inequality index (i.e., a function that represents the preference on the subspace).

Specifically, we consider two domains:

1. the subspace of income distributions with fixed total income;
2. the whole space.

We first provide an axiomatization of a linear evaluation function on the subspace (the Gini index is obtained by a natural strengthening of the main axiom). Then we show that an "innocent" modification of one of the axioms implies a linear evaluation on the whole space. We do not see how this modification can be rejected while the original axiom is accepted. However, this, of course, is an intuitive claim and the reader will judge it for himself. The Gini index has acquired a special status and it is therefore of interest to derive the representation of a Gini-index on the whole space. Specifically, let $\succeq$ be a preference order on the whole space with the property that its restriction to some subspace is a Gini preference order (i.e., the Gini index represents the preference). Let $f = (f_1, \ldots, f_n)$ denote an income profile. If $\succeq$ still satisfies our axioms then it can be represented by a

[2]For example, Ebert (1987) (and in a different way Sheshinsky (1972)) studies preferences on the whole space that are derived from a preference order on pairs of total income and an inequality index, where the inequality index represents the preference on the subspace.

function J(f) where $J(f) = \sum_{i=1}^{n} f_i - \delta\sum_{1\le i<j\le n} |f_j - f_i|$ for some $0 < \delta < 1/(n-1)$. Thus, our result is that a Gini preference on the whole space can be represented by a linear combination of total income and the Gini index. Let us emphasize that, on the whole space, J(f) is different from the Gini index which is equivalent[3] to

$$\frac{1}{\sum_{i=1}^{n} f_i} \sum_{1\le i<j<n} |f_j - f_i|.$$

This of course is not surprising since the interpretation of J(f) and the Gini index on the whole space is different. The Gini index is an inequality index; it can represent the preference of a decision maker only on the subspace where the total income is fixed. When two profiles that do not belong to the same subspace are compared, the total income should be taken into account as well.

Linear evaluation functions have been studied by Donaldson and Weymark (1980), Meheran (1976), Weymark (1981), and Yaari (1988), among others. Yaari's axiomatization is actually similar to the one provided here when the whole space is considered. (The main difference is that in Yaari's model there is a continuum of individuals. This makes the proof different.) However, all this work refers to the whole space of income distributions. Our focus is on the subspace and on the relationship between a linear measure of inequality (and in particular the Gini index) and a linear evaluation function on the whole space.

The paper is organized as follows. Section 2 contains a description of the axioms and the results. The proofs are given in Section 3.

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[3]When the size of the population is fixed.

2. Notations and Results

Let $N = \{1,\ldots,n\}$ be the set of individuals. Let the income profiles (or simply "profiles") be

$$F = \{f\colon N \to \mathbb{R} \mid f(i) \geq 0 \ \forall \ i \in N\},$$

to be identified with $\mathbb{R}^n_+$. In the sequel we will not distinguish between $f(i)$ and f_i.

Since we will be interested in subspaces across which total income is constant, it will prove useful to define, for $C \geq 0$

$$F^C = \{f \in F \mid \textstyle\sum_{i \in N} f_i = C\}$$

Some of our axioms will involve an assumption of order preservation, i.e., that two profiles do not change the income-ordering of individuals. In general, this condition (on pairs of profiles) is called comonotonicity (see Schmeidler (1989)). However, since we will anyway impose a symmetry axiom, it will facilitate notations to simply focus on monotone profiles. Define, then,

$$F_M = \{f \in F \mid f_i \leq f_{i+1} \text{ for } 1 \leq i \leq n - 1\}$$

and

$$F_M^C = F^C \cap F_M.$$

For a permutation $\pi\colon N \to N$, and $f \in F$, define $\pi f \in F$ by $(\pi f)_i = f_{\pi(i)}$. Obviously, $\pi(F^C) = F^C$ for all $C \geq 0$. For every $f \in F$ define $f^{(\bullet)} = (f^{(1)}, f^{(2)}, \ldots, f^{(n)})$ to be the element of F_M for which there exists a permutation π: $N \to N$ satisfying $f^{(0)} = \pi f$. Note that $f^{(\bullet)}$ is uniquely defined even if π is not.

For $f \in F$ and $i,j \in N$, i is said to <u>f-precede</u> j iff $f_i \le f_j$ and there is no $k \in N$ for which $f_i < f_k < f_j$.

Let $\succeq \subseteq F \times F$ be a binary relation, to be interpreted as a <u>preference</u> relation. We will now formulate some axioms on $\succeq$. These axioms are parametrized by a set of profiles $H \subsetneq F$, where the theorems will be stated using axioms $H = F$, F^C, F_M or F_M^C.

<u>A1(H). Weak Order</u>: $\succeq$ is complete, i.e., for every $f,g \in H$, $f \succeq g$ or $g \succeq f$, and transitive: for every $f,g,h \in H$, $f \succeq g$ and $g \succeq h$ imply $f \succeq h$.

We use the regular notations of $\preceq = \succeq^{-1} = \{(f,g) | (g,f) \in \succeq\}$; $\sim = \succeq \cap \preceq$; $\succ = \succeq \setminus \sim$; $\prec = \preceq \setminus \sim$.

<u>A2(H). Continuity</u>: For every $f \in H$ the sets $\{g \in H | g \succ f\}$, $\{g \in H | g \prec f\}$ are open in H. (That is, in the topology on H induced by the natural topology on $\mathbb{R}^n$.)

<u>A3(H). Symmetry</u>: For every $f,g \in H$, if there is a permutation π: $N \to N$ such that $g = \pi f$, then $f \sim g$.

<u>A4(H). Monotonicity</u>: For every $f,g \in H$, if $f_i \ge g_i$ for all $i \in N$ and $f_j > g_j$ for some $j \in N$, then $f \succ g$.

The next two axioms all require some consistency between choices. They have a flavor of Savage's sure-thing principle (Savage (1954)), but they also presuppose that utility is linear in income.

<u>A5(H). Order-preserving Gift</u>: For every $f, g, f', g' \in H \cap F_M$ and $i \in N$, if $f_j = f'_j$ and $g_j = g'_j$ for all $j \neq i$, and $f'_i = f_i + t$, $g'_i = g_i + t$ for some $t \in \mathbb{R}$, then $f \geq g$ iff $f' \geq g'$.

A5 says that the preference between two profiles of f and g, which agree on the social income-ordering, should not change if the same individual i receives a "gift" t in both f and g, provided that the resulting profiles f' and g' respect the same ordering. The logic behind it, which one may accept or reject, can best be seen if one first considers the cases it excludes: if, for instance, f and g do not agree on the social income-ordering, individual i may be the poorest in f and the richest in g. Increasing his income would therefore have a different effect on the inequality in f and in g, and the preference between them may well change. Similarly, even if both f and g are monotone, a gift of t to individual i may make him richer than (i + 1) in f, but leave him poorer than (i + 1) in g. Again, this asymmetric impact on inequality may give rise to preference reversal.

It is only in the cases where the above do not happen that A5 can be invoked to deduce that preference reversal should not occur.

Note that for $H \subseteq F^C$ A5 is vacuously satisfied, since for $t \neq 0$, f' and g' do not belong to F^C if f and g do. Hence, we will need a total-income preserving version of it, which will deal with transfers (from one individual to another) rather than gifts. However, we formulate it in a potentially stronger form, as explained below.

A6(H) Order-Preserving Transfer: For all $f,g,f',g' \in H$ and $i,j \in N$, if the following hold:

(i) i f-, g-, f'- and g'- precedes j;

(ii) $f'_i = f_i + t$ $\quad g'_i = g_i + t$

$f'_j = f_j - t$ $\quad g'_j = g_j - t$ for some $t > 0$; abd

(iii) $f'_k = f_k$ $\quad g'_k = g_k$ for $k \notin \{i,j\}$,

then $f \succeq g$ iff $f' \succeq g'$.

To better undersatnd A6, let us first consider the case $H = F^C_M$ (as will be done in Theorem A below). For this H, A6 is the "natural" reformulation of A5 when one is restricted to a constant-total-income hyperplane. Indeed, $A5(F_M)$ implies $A6(F^C_M)$, as is easily verified.

However, we will also use A6 for $H = F^C$ (in theorems B and D), in which case it makes a stronger claim: starting out with some $f,g \in F^C$, f' preserves the order of f, as does g' with respect to g. But A6 requires that there will be no preference reversal even if f and g do not agree on the social ordering. In particular, the pair (i,j) may be the poorest in f and the richest in g, yet a transfer from j to i should not, according to $A6(F^C)$, change the preference between f and g.

Thus the distinction between general linear welfare functions on F^C and the more specific Gini index will be whether A6 is required to hold on F^C_M or on all of F^C.

A7(H). Inequality Aversion: For all $f, f' \in F_M \cap H$ and $1 \le i < n$, if

$$f'_j = f_j, \qquad \text{for all } j \notin \{i, i+1\}$$

$$f_i' = f_i + t, \quad f_{i+1}' = f_{i+1} - t, \quad \text{for some } t > 0,$$

then $f' \succ f$.

A7 simply states that a transfer of money from an individual to the next-richest one, in such a way that the social income-ordering is preserved, will result in a strictly preferred social profile. Thus, A7 is a weak version of the famous Dalton-Pigou principle which states that a transfer of money from a rich person to a poor person, which leaves the rich person richer, will reduce inequality.

We will say that "$\succsim$ satisfies A_n on H" if $\succsim$ satisfies $A_n(H)$.

We can finally formulate our main results:

<u>Theorem A</u>: For every $C > 0$ and every $\succsim \subseteq F^C \times F^C$ the following are equivalent:

(i) $\succsim$ satisfies A1, A2, A3, and A7 on F^C, and A6 on F^C_M.

(ii) there is a vector $p = (p_1, \ldots, p_n)$ with $p_1 > p_2 > \ldots > p_n$ such that for all $f, g \in F^C$,

$$f \succsim g \iff \sum_{i \in N} p_i f^{(i)} \geq \sum_{i \in N} p_i g^{(i)}.$$

Furthermore, in this case the vector p (in (ii)) is unique up to a positive linear transformation (p.l.t.).

<u>Theorem B</u>: For every $C > 0$ and every $\succsim \subseteq F^C \times F^C$ the following are equivalent:

(i) $\succsim$ satisfies A1, A2, A3, A6 and A7 on F^C;

(ii) for all $f, g \in F^C$,

$$f \succeq g \iff \sum_{1\le i<j\le n} |f_i - f_j| \le \sum_{1\le i<j\le n} |g_i - g_j|.$$

<u>Theorem C</u>: For every $\succeq \subseteq F^C \times F^C$ the following are equivalent:

(i) $\succeq$ satisfies A1, A2, A3, A5, and A7 on F;

(ii) there is a vector $p = (p_1,\ldots,p_n)$ with $p_1 > p_2 > \ldots > p_n > 0$ such that for all $f,g \in F$.

$$f \succeq g \iff \sum_{i\in N} p_i f^{(i)} \ge \sum_{i\in N} p_i g^{(i)}.$$

Furthermore, in this case the vector p (in which (ii)) is unique up to multiplication by a positive scalar.

<u>Theorem D</u>: For every $\succeq \subseteq F \times F$, the following are equivalent:

(i) $\succeq$ satisfies A1, A2, A3, A4, A5, A6 and A7 on F;

(ii) There is a number δ, $0 < \delta < 1/(n - 1)$, such that for all $f,g \in F$,

$$f \succeq g \iff \sum_{i=1}^{n} f_i - \delta \sum_{1\le i<j\le n} |f_i - f_j| \ge \sum_{i=1}^{n} g_i - \delta \sum_{1\le i<j\le n} |g_i - g_j|.$$

Furthermore, in this case the coefficient δ (in (ii)) is unique.

Theorem A provides a characterization of linear evaluation functions on the subspace. Theorem B states that the Gini index is obtained if, in addition to the assumptions in Theorem A, we require that the order-preserving transfer axiom will apply to every pair of profiles f and g (and not only to pairs of profiles that agree on the social ordering). Theorems C and D are the counterparts of Theorems A and B, respectively, when the whole space of income profiles is considered and when we add A5(F). Thus, Theorem C characterizes a linear evaluation function on the whole space, while Theorem D provides the representation of an extension of the preferences, reflected in the Gini index, to the whole space. Note that the preference on the whole space is represented by a linear combination of total income and the Gini index.

We now want to suggest that a decision maker that satisfies A6(F^C_M) on the subspace should satisfy A5(F) on the whole space. As we noted, A5(F) implies A6(F^C_M) because a transfer from i to j can be obtained by a gift to j and a "negative" gift to i. So, A6(F^C_M) is obtained by putting a certain restriction on the application of A5(F), namely, that a gift to one person should be offset by a negative gift to another person. We do not see why a decision maker will accept A5(F) with the restriction but not without it. One could, for example, object to A5(F) on the grounds that the effect of a given gift to individual i on social welfare should depend on total income (and not only on the rank of the individual involved). However, if the decision maker satisfies A6(F^C_M) on the subspace, it means that the effect of a given transfer between two individuals on the evaluation function depends only on the rank of the individuals involved but not on their absolute level of income. Hence, our response to the above objection is that it seems inconsistent for a decision maker to evaluate a change in individual i's income according to total income, but not to take into account i's own income. We hope that our results are of interest whether the above view is accepted or not. However, if this view is accepted then the implication is that a decision maker who has a linear evaluation function on the subspace should have a linear evaluation on the whole space as well. In particular, if the preference of the decision maker on the subspace corresponds to her evaluation of inequality then the results can be interpreted as follows: if a decision maker has a linear inequality measure on some subspace than she should have a linear evaluation function on the whole space. In particular, if the decision maker evaluates inequality according to the Gini index, then her evaluation on the whole space is a linear combination of total income and the Gini index.

3. Proofs and Related Analysis

This section is organized as follows. We first introduce some preliminaries, mostly regarding the Choquet integral, to be used in the proof. We then continue to prove the theorems according to their stated order. Theorems A and C are basically Choquet-integral representations of preferences on F^C and on F, respectively (with the additional restriction of inequality aversion). While Theorem C is a rather trivial corollary of a result in Wakker (1989), Theorem A does not rely on any known characterization, and thus involves some work. Theorems B and D are quite simple given A and C, respectively.

Before we begin, let us note that in all theorems the necessity of the axioms (i.e., (ii) => (i)) is straightforward, and we therefore focus on proof of sufficiency. Finally, the uniqueness results (in Theorems A, C and D) are easy to verify, and will become even more transparent given the sufficiency proofs.

3.1 Preliminaries

In Schmeidler's seminal papers (see Schmeidler (1982, 1986, 1989)) nonadditive measures were introduced into decision theory. By a nonadditive measure we refer (in our framework) to a set function $v: 2^N \to \mathbf{R}$ with $v(\emptyset) = 0$, $v(N) = 1$, satisfying $v(A) \geq v(B)$ whenever $A \supseteq B$. A nonadditive measure v is symmetric if $v(A) = v(\pi A)$ for every $A \subsetneq N$ and every permutation $\pi: N \to N$. Obviously, v is symmetric iff there exists and nondecreasing $\phi: [0,1] \to [0,1]$ with $\phi(0) = 0$, $\phi(1) = 1$, such that $v(A) = \phi(|A|/n)$.

For $f \in F$, the Choquet integral of f w.r.t. (with respect to) v (defined in Choquet (1953-4)) is

$$\int fdv = \int_N fdv = \int_0^\infty v(\{i \mid f_i \geq t\})dt.$$

It is easily verified that, for $f \in F_M$, setting $f_0 = 0$,

$$\int fdv = \sum_{i=1}^n [f_i - f_{i-1}]v(S_i) = \sum_{i=1}^n f_i \; [v(S_i) - v(S_{i+1})]$$

where $S_i = \{i, \ldots, n\}$ (and $S_{n+1} = \emptyset$).

Define $f, g, \in F$ to be comonotonic if there are no $i,j \in N$ with $f_i > f_j$, $g_i < g_j$. Equivalently, f and g are comonotonic if there is a permutation $\pi: N \to N$ such that $\pi f, \pi g \in F_M$. It is well known that the Choquet integral is linear on cones of pairwise comonotonic profiles. In particular,

$$\int (\lambda f)dv = \lambda \int fdv \text{ for } \lambda > 0, \; f \in F.$$

and

$$\int (f + c1_N)dv = \int fdv + c$$

where 1_A is the indicator function of $A \subsetneq N$, and whenever $f, f + c1_N \in F$.

There is, by now, a wide collection of axiomatizations of preferences which are representable by a Choquet integral of a certain utility function w.r.t. some nonadditive measure. The first one is due to Schmeidler (1982,

1989) who used the framework of Anscombe-Aumann (1963). Gilboa (1987) provided an axiomatization in the framework of Savage (1954), which requires, in our interpretation, infinitely many individuals. Wakker (1989) axiomatized these preferences on Γ^n where Γ is some connected separable topological space. Sarin and Wakker (1990) provided an axiomatization for a domain which may be thought of as a hybrid between Savage's and Anscombe-Aumann's.

All of these axiomatizations are greatly simplified if one presupposes that the range of the functions involved is $\mathbb{R}$ (or $\mathbb{R}_+$), and that utility is linear, as is implicitly assumed by the Gini index. It seems that, at least given that simplification, the most convenient result to use is the "Main Theorem" of Wakker (1989, Vi.5, p. 117).

Quoting Wakker's theorem would require too many additional definitions, which are unlikely to enlighten the reader. We will therefore use it when necessary, referring the reader to Wakker (1989).

3.2 Proof of Theorem A

In this section we state and prove the following result, which is merely a restatement of Theorem A:

Proposition 3.2.1: Let $\succeq$ satisfy A1, A2, A3, and A7 on F^C and A6 on F^C_M. Then there exists a nonadditive measure v such that for all $f,g \in F^C$

$$f \succeq g \iff \int f dv \geq \int g dv.$$

This proposition can hardly come as a shocking surprise to anyone familiar with related literature. Yet it did not seem to fit nicely into any of the known theorems in the literature. We therefore provide an independent proof.

First, note that, in view of A3, it suffices to provide a vector $p = (p_1, \ldots, p_n)$ such that for all $f, g \in F_M^C$,

$$f \succeq g \iff \sum_{i \in N} p_i f_i \geq \sum_{i \in N} p_i g_i .$$

(Since $p + c1_N$ would provide the same representation for all $c \in \mathbf{R}$, the p_i's can later be assumed positive.)

We first need some auxiliary results, which will strengthen the main axiom, i.e., A6(F_M^C). It will prove useful to focus on the interior of F_M^C,

$$(F_M^C)^0 = \{f \in F^C | 0 < f_1 < f_2 < \ldots < f_n\}.$$

All the following lemmata and claims in this subsection are steps in the proof of the proposition, and presuppose its provisions.

We will now show that a transfer from individual j to i, which respects comonotonicity, does not induce preference reversal.

<u>Lemma 3.2.2</u>: For all $f, g, f', g' \in (F_M^C)^0$, and all $i, j \in N$, $i \neq j$, if

$$f'_k = f_k, \qquad g'_k = g_k \text{ for all } k \notin \{i,j\}$$

and for some $t \in \mathbf{R}$

$$f'_i = f_i + t \qquad g'_i = g_i + t$$

$$f'_j = f_j - t \qquad g'_j = g_j - t,$$

then

$$f \geq g \text{ iff } f' \geq g'.$$

<u>Proof</u>: W.l.o.g. (without loss of generality) assume $i < j$. Furthermore, w.l.o.g. we can also assume that $t > 0$, for if $t < 0$ one can switch the roles of f and f', g and g'.

For $h \in (F_M^C)^0$, let

$$d(h) = \min_{1 \leq i < n} (h_{i+1} - h_i).$$

Choose $\epsilon > 0$ such that $\epsilon < (1/2)\min\{d(f), d(f'), d(g), d(g')\}$. For $i \leq k \leq j$ and $0 \leq r \leq \lfloor t/\epsilon \rfloor = M$, define

$$f_{r,k} = f - r\epsilon e^j + (r - 1)\epsilon e^i + \epsilon e^k$$

$$g_{r,k} = g - r\epsilon e^j + (r - 1)\epsilon e^i + \epsilon e^k$$

where

$$e^\ell \in \mathbb{R}^n \text{ satisfies } (e^\ell)_\ell = 1, \ (e^\ell)_s = 0 \text{ for } s \neq \ell.$$

Further, let $\delta = t - M\epsilon \geq 0$ and define

$$f_{M+1,k} = f - (M\epsilon + \delta)e^j + M\epsilon e^i + \delta e^k$$

$$g_{M+1,k} = g - (M\epsilon + \delta)e^j + M\epsilon e^i + \delta e^k.$$

Note that for all $i \leq k \leq j$ and $0 \leq r \leq M + 1$, $f_{r,k}, g_{r,k} \in (F_M^C)^0$ and that

$$f_{o,j} = f; \qquad g_{o,j} = g;$$

$$f_{M+1,i} = f'; \qquad g_{M+1,i} = g';$$

and

$$f_{r,i} = f_{r+1,j}; \qquad g_{r,i} = g_{r+1,j} \text{ for } r \le M.$$

Finally, for $k > i$, A6(F_M^C) implies that

$$f_{r,k} \succeq g_{r,k} \iff f_{r,k-1} \succeq g_{r,k-1}$$

for all $0 \le r \le M + 1$, whence the result follows. ■

Remark 3.2.3: Note that for F_M^C the same result cannot be similarly proven. Consider, for instance, $f = (0,1,2)$ and $f' = (1,1,1)$. Although one can obtain f' from f by a single order-preserving transfer from 3 to 1, no (finite) sequence of order-preserving transfers between "adjacent" individuals would yield f' from f.

Obviously, an "infinite" sequence will do the trick, i.e., f' can be obtained as the limit of f_n, where each f_n can be obtained from f by a finite sequence of adjacent transfers. However, starting from $f \succ g$, continuity of $\succeq$ only guarantees $f' \succeq g'$, which is not sufficient for our purposes.

This is the main reason to focus on $(F_M^C)^0$ (rather than F_M^C) first. Only when enough structure is proven to exist in the preference over $(F_M^C)^0$ will we use continuity to derive representation on its boundary as well.

We will also need the corresponding extension of A7:

Lemma 3.2.4: Let $f, f' \in (F_M^C)^0$ where for some $1 \le i < j \le n$ and some $t > 0$,

$$f_i' = f_i + t \qquad f_j' = f_j - t$$

and

$$f_k' = f_k \qquad \text{for all } k \notin \{i,j\},$$

then $f' \succ f$.

Proof: As in Lemma 3.2.2, by successive applications of A7 and transitivity. ∎

A further extension of A5b is the following:

Lemma 3.2.5: Let $f,g,f',g' \in (F_M^c)^0$ where $f' = f + t$, $g' = g + t$ for some $t \in \mathbb{R}^n$. Then $f \succeq g$ iff $f' \succeq g'$.

Proof: We will prove the following claim for all $0 \le k \le n$. For all $f,g,f',g' \in (F_M^c)^0$ such that $f' = f + t$, $g' = g + t$ for some $t \in \mathbb{R}^n$ and $t_i = 0$ for all $1 \le i \le k$, $f \succeq g$ iff $f' \succeq g'$. Note that for $k = 0$ this is the desired result.

The proof is by induction on $(n - k)$. For $k = n - 1$ or $k = n$ the claim is trivial since $f' = f$ and $g' = g$. Assume, then, the claim was proven for $k = r + 1$ and consider the case $k = r$.

Let there be given, then, f,f',g,g' and t as in the claim. Note that $f_i' = f_i$ and $g_i' = g_i$ for $1 \le i \le r$. Assume w.l.o.g. that $t_{r+1} < 0$. (If $t_{r+1} > 0$, reverse the roles of f and f', g and g'.)

Define $\hat{f},\hat{g}$ as follows:

$$\hat{f}_i = f_i = f_i' \qquad \hat{g}_i = g_i = g_i' \text{ for } i \le r$$

$$\hat{f}_{r+1} = f_{r+1}' \qquad \hat{g}_{r+1} = g_{r+1}'$$

$$\hat{f}_i = f_i \qquad \hat{g}_i = g_i \text{ for } r + 1 < i < n$$

$$\hat{f}_n = f_n - t_{r+1} \qquad \hat{g}_n = g_n - t_{r+1}.$$

It is easily verifiable that $\hat{f},\hat{g} \in (F_M^C)^0$. Furthermore, $f \succeq g$ iff $\hat{f} \succeq \hat{g}$ by Lemma 3.2.2. However, by the claim for $k = r + 1$, $\hat{f} \succeq \hat{g}$ iff $f' \succeq g'$. ∎

Equipped with these tools, we now turn to the proof. The general strategy is as follows: for every triple (i,j,t) where $1 \le i < j \le n$ and $t \in \mathbb{R}$, consider the "improvement" obtained if, in a given profile, a transfer of t from j to i takes place. We will show that these triples can be ranked (in terms of the "size" of improvement) regardless of the base profile. We will further show that this binary relation is homogeneous, i.e., that the "improvement" in (i,j,t) is "greater" than that implied by the transfer (k,ℓ,s) iff the same holds for $(i,j,\alpha t)$ and $(k,\ell,\alpha s)$ for $\alpha > 0$.

This homogeneity will give rise to coefficients $\sigma_{ijk\ell}$, which will provide the "substitution rate" between transfers from j to i and transfers from ℓ to k. We will show that a given profile is equivalent to a profile generated from it by offsetting transfers (according to these substitution rates).

Next we will use these coefficients $\sigma_{ijk\ell}$ to define the "weights" p_i, and will show that the weighted average $J(f) = \sum_i p_i f_i$ is also unaltered when offsetting transfers are made. Finally, each profile f will be "normalized" (in some appropriate sense) by a sequence of offsetting transfers and it will only remain to show that $J(\bullet)$ represents $\succeq$ on the "normalized" profiles.

Let us begin by using the following notation: for $f \in F^C$, $i,j \in N$ and

$t \in \mathbb{R}$, let $f_{(i,j,t)} \in \mathbb{R}^n$ be given by $f_{(i,j,t)} = f - te^j + te^i$.

Denote $T = \{(i,j,t) | 1 \le i < j \le n,\ t \in \mathbb{R}\}$.

<u>Lemma 3.2.6</u>: Let $f, g \in F^C$, and $(i,j,t),(k,\ell,s) \in T$ be such that

$$f_{(i,j,t)}, f_{(k,\ell,s)}, g_{(i,j,t)}, g_{(k,\ell,s)} \in (F_M^C)^0.$$

Then, $f_{(i,j,t)} \succsim f_{(k,\ell,s)}$ iff $g_{(i,j,t)} \succsim g_{(k,\ell,s)}$.

<u>Proof</u>: Let $t = g - f$, and use Lemma 3.2.5 (with $\tilde{f} = f_{(i,j,t)}$, $\tilde{g} = f_{(k,\ell,s)}$; $\tilde{f}' = g_{(i,j,t)}$, $\tilde{g}' = f_{(k,\ell,s)}$). ∎

In view of this lemma, we will write $(i,j,t) \succsim (\succ, \sim)\ (k,\ell,s)$ iff there exists $f \in F^C$ such that $f_{(i,j,t)} \succsim (\succ, \sim)\ f_{(k,\ell,s)}$ and both are members of $(F_M^C)^0$.

<u>Lemma 3.2.7</u>: Let $(i,j,t),(k,\ell,s) \in T$ with $t,s > 0$, and assume that for some $f, g \in F^C$, $f_{(i,j,t)}, g_{(i,j,-t)}, f_{(k,\ell,s)}, g_{(k,\ell,-s)} \in (F_M^C)^0$. Then $(i,j,t) \succsim (k,\ell,s)$ iff $(i,j,-t) \precsim (k,\ell,-s)$.

<u>Proof</u>: Define $h = (g_{(i,j,-t)})_{(k,\ell,-s)} \in F^C$ and note that

$$g_{(i,j,-t)} = h_{(k,\ell,s)} \text{ and } g_{(k,\ell,-s)} = h_{(i,j,t)}.$$

Considering h and f and applying Lemma 3.2.6 one obtains the desired conclusion. ∎

<u>Lemma 3.2.8</u>: Suppose that for $(i,j,t),(k,\ell,s) \in T$ and $\alpha > 0$ there exist $f,g \in (F_M^C)^0$ such that $f_{(i,j,\alpha t)}$, $f_{(k,\ell,\alpha s)}$, $g_{(i,j,t)}$, $g_{(k,\ell,s)} \in (F_M^C)^0$. Then $(i,j,t) \succeq (k,\ell,s)$ iff $(i,j,\alpha t) \succeq (k,\ell,\alpha s)$.

(Notice that in the statement above the existence of f and g is only required to guarantee that the triples involved are comparable.)

<u>Proof</u>: First consider the case $t,s > 0$. Let us begin with $\alpha = r \in \mathbf{N}$, and prove by induction on r.

Assume $f \in (F_M^C)^0$ and $(i,j,t),(k,\ell,s) \in T$ are such that

$$f_{(i,j,t)}, f_{(k,\ell,s)}, f_{(i,j,rt)}, f_{(k,\ell,rt)} \in (F_M^C)^0$$

(Note that if f and g are given as in the lemma, for $\alpha > 1$, $t,s > 0$ we also have $f_{(i,j,t)}, f_{(k,\ell,s)} \in (F_M^C)^0$.)

Define, for $0 \leq \nu,\mu \geq r$ with $\nu + \mu \leq r$,

$$h_{\nu,\mu} = (f_{(i,j,\nu t)})_{(k,\ell,\mu s)}.$$

Note that

$$h_{\nu,\mu} \in (F_M^C)^0 \text{ for all } \nu,\mu \geq 0 \text{ with } \nu + \mu \leq r$$

$$h_{0,0} = f$$

$$h_{r,0} = f_{(i,j,rt)},\ h_{0,r} = f_{(k,\ell,rs)}.$$

Further observe that, for $0 \leq \nu \leq r - 1$ and $\mu = r - \nu \geq 1$,

$$h_{\nu,\mu} = (h_{\nu,\mu-1})_{(k,\ell,s)}; \; h_{\nu+1,\mu-1} = (h_{\nu,\mu-1})_{(i,j,t)},$$

whence, by Lemma 3.2.6,

$$h_{\nu+1,\mu-1} \succeq h_{\nu,\mu} \text{ iff } (i,j,t) \succeq (k,\ell,s),$$

and by transitivity

$$h_{r,0} \succeq h_{0,r} \text{ iff } (i,j,t) \succeq (k,\ell,s).$$

We therefore conclude that for a rational α, $(i,j,\alpha t) \succeq (k,\ell,\alpha s)$ iff $(i,j,t) \succeq (k,\ell,s)$, whenever there are f and g as in the provisions of the lemma.

Next consider irrational $\alpha > 0$. It will here be useful to distinguish indifference from strict preference. If $(i,j,t) \sim (k,\ell,s)$, then for every rational α (for which the involved triples are comparable) $(i,j,\alpha t) \sim (k,\ell,\alpha s)$ and the conclusion follows by continuity of $\succeq$. If, however, $(i,j,t) \succ (k,\ell,s)$ we invoke Lemma 3.2.4 to deduce $(k,\ell,s) \succ (i,j,0)$ whence, again by continuity, there is $\hat{t} \in (0,t)$ such that $(i,j,\hat{t}) \sim (k,\ell,s)$. Therefore, $(i,j,\alpha\hat{t}) \sim (k,\ell,\alpha s)$. However, Lemma 3.2.4 also implies $(i,j,\alpha t) \succ (i,j,\alpha\hat{t})$.

We can now turn to the cases in which s, t or both are not strictly positive. Obviously, by 3.2.4 again, if $t > 0 \geq s$ or $t \geq 0 > s$, for all $i < j$ and $k < \ell$ $(i,j,t) \succ (k,\ell,s)$ and the same holds for αt, αs where $\alpha > 0$. Finally, the case $s,t < 0$ follows from $s,t > 0$ in view of Lemma 3.2.7. ∎

As in the above proof we conclude that for every $i < j$, $k < \ell$ and for every small enough $t > 0$ there is an $\bar{s} > 0$ such that $(i,j,t) \sim (k,\ell,\bar{s})$. Furthermore, for $s < \bar{s}$ we have $(i,j,t) \succ (k,\ell,s)$ and for $s > \bar{s}$ $(k,\ell,s) \succ (i,j,t)$. Since, moreover, for $\alpha > 0$, $\alpha\bar{s}$ would correspond to αt, we define

$$\sigma_{ijk\ell} = t/\bar{s} > 0.$$

<u>Conclusion 3.2.9</u>: For all $(i,j,t),(k,\ell,s) \in T$, with $t,s > 0$, if there is $f \in (F_M^C)^0$ such that $f_{(i,j,t)}, f_{(k,\ell,s)} \in (F_M^C)^0$, then $(i,j,t) \succeq (\succ,\sim)(k,\ell,s)$ iff $t \geq (>,=)\ \sigma_{ijk\ell}s$.

To facilitate notations, we extend $\succeq$ to all of T, using 3.2.9 (together with 3.2.4 and 3.2.7) as the definition of $\succeq$ if such an f does not exist. (For instance, if $t > C$.)

We will need two properties of the "substitution rates" $\sigma_{ijk\ell}$:

<u>Lemma 3.2.10</u>: For all $i,j,k,\ell,r,q \in N$ with $i < j$; $k < \ell$; $r < q$,

$$\sigma_{ijk\ell}\sigma_{k\ell rq} = \sigma_{ijrq}.$$

<u>Proof</u>: Take $t,s,u > 0$ such that $(i,j,t) \sim (k,\ell,s) \sim (r,q,u)$. Then,

$$\sigma_{ijk\ell} = t/s$$

$$\sigma_{k\ell rq} = s/u$$

and, since $(i,j,t) \sim (r,q,u)$

$$\sigma_{ijrq} = t/u = (t/s)(s/u) = \sigma_{ijk\ell}\sigma_{k\ell rq}. \qquad \blacksquare$$

Lemma 3.2.11: For all i, j, k, r, ℓ with $i < j$, $r < k < \ell$,

$$\sigma_{ijrk} + \sigma_{ijk\ell} = \sigma_{ijr\ell}.$$

Proof: Let $t,u,s,v > 0$ satisfy

$$(i,j,t) \sim (r,k,s)$$

$$(i,j,u) \sim (k,\ell,s)$$

$$(i,j,v) \sim (r,\ell,s)$$

whence

$$\sigma_{ijrk} = t/s$$

$$\sigma_{ijk\ell} = u/s$$

$$\sigma_{ijr\ell} = v/s.$$

Further assume w.l.o.g. that all of t,u,s,v are small enough so that there is an $f \in F^C$ for which $f_{(i,j,t)}, f_{(i,j,u)}, f_{(i,j,v)}, f_{(i,j,t+u)}, f_{(r,k,s)}$, $f_{(k,\ell,s)}, f_{(r,\ell,s)} \in (F_M^C)^0$.

Then we have

$$f_{(i,j,t)} \sim f_{(r,k,s)}$$

and, by Lemma 3.2.2,

$$(f_{(i,j,t)})_{(i,j,u)} \sim (f_{(r,k,s)})_{(i,j,u)}.$$

However, since $(i,j,u) \sim (k,\ell,s)$,

$$(f_{(r,k,s)})_{(i,j,u)} \sim (f_{(r,k,s)})_{(k,\ell,s)}$$

which implies

$$(f_{(i,j,t)})_{(i,j,u)} \sim (f_{(r,k,s)})_{(k,\ell,s)}$$

or

$$f_{(i,j,t+u)} \sim f_{(r,\ell,s)}$$

whence $t + u = v$.

Finally, $t/s + u/s = v/s$ and

$$\sigma_{ijrk} + \sigma_{ijk\ell} = \sigma_{ijr\ell}. \quad \blacksquare$$

We can finally define the "weights" p_i for $i \in N$: Let $p_1 = 0$, $p_2 = -1$ and for $2 < k \leq n$, $p_k = -\sigma_{121k}$. (So that

$$\sigma_{121k} = (p_1 - p_k)/(p_1 - p_2).)$$

Next define, for all $f \in F^C$,

$$J(f) = \sum_{k=1}^{n} p_k f_k,$$

We now wish to show that equivalent transfers have identical effect on J:

Lemma 3.2.12: Let $(i,j,t),(k,\ell,s) \in T$ satisfy $(i,j,t) \sim (k,\ell,s)$. Then for all $f \in F^C$ such that $f_{(i,j,t)}, f_{(k,\ell,s)} \in (F_M^C)^0$,

$$J(f_{(i,j,t)}) = J(f_{(k,\ell,s)}).$$

Proof: Notice that

$$J(f_{(i,j,t)}) = J(f) + t(p_i - p_j) = J(f) + t(\sigma_{121j} - \sigma_{121i})$$

(with $\sigma_{1211} = 0$).

Similarly,

$$J(f_{(k,\ell,s)}) = J(f) + s(\sigma_{121\ell} - \sigma_{121k}).$$

However, by Lemma 3.2.11,

$$\sigma_{121j} - \sigma_{121i} = \sigma_{12ij}$$

$$\sigma_{121\ell} - \sigma_{121k} = \sigma_{12k\ell}.$$

Hence,

$$J(f_{(i,j,t)}) = J(f) + t\sigma_{12ij}$$

$$J(f_{(k,\ell,s)}) = J(f) + s\sigma_{12k\ell}.$$

However, $t = s\sigma_{ijk\ell}$ and, by Lemma 3.2.10,

$$t\sigma_{12ij} = s\sigma_{ijk\ell}\sigma_{12ij} = s\sigma_{12k\ell} \qquad \blacksquare$$

As for the converse:

Lemma 3.2.13: Suppose that for some $f \in (F_M^C)^0$ and $(i,j,t),(k,\ell,s) \in T$ such that $f_{(i,j,t)}, f_{(k,\ell,s)} \in (F_M^C)^0$, $J(f_{(i,j,t)}) = J(f_{(k,\ell,s)})$. Then $(i,j,t) \sim (k,\ell,s)$.

Proof: By the computations of the previous proof one obtains

$$t\sigma_{12ij} = s\sigma_{12k\ell}$$

i.e.,

$$t/s = \sigma_{12k\ell}/\sigma_{12ij} = \sigma_{ijk\ell},$$

which suffices by 3.2.9. $\blacksquare$

We are approaching the final steps of the proof. It will be useful, however, to have explicit mention of the following:

Lemma 3.2.14: Suppose that $(i,j,t),(k,\ell,s)$ satisfy $(i,j,t) \sim (k,\ell,s)$, and assume that $f \in F$ satisfies

$$f, f_{(i,j,t)}, f_{(k,\ell,s)}, (f_{(i,j,t)})_{(k,\ell,-s)} \in (F_M^C)^0.$$

Then $f \sim (f_{(i,j,t)})_{(k,\ell,-s)}$.

Proof: Since $f_{(i,j,t)} \sim f_{(k,\ell,s)}$, we may use Lemma 3.2.2 to obtain

$$(f_{(i,j,t)})_{(k,\ell,-s)} \sim (f_{(k,\ell,s)})_{(k,\ell,-s)} = f. \qquad \blacksquare$$

The next step is to show that for every $f \in (F_M^C)^0$ there is an $\hat{f} \in (F_M^C)^0$ such that $f \sim \hat{f}$ and $J(f) = J(\hat{f})$, where $\hat{f}$ is "normalized" in some sense. If we could choose $\hat{f}$ from F_M^C, we would like it to be of the form $(\alpha,\alpha,\ldots,\alpha,\beta)$ where $\beta > \alpha$, and then to show, for the unique α and β determined by $\hat{f} \in F^C$ and $J(\hat{f}) = J(f)$, that $\hat{f}$ can be obtained from f by a sequence of pairs of offsetting transfers $(i,j,t),(k,\ell,-s)$ where $t = \sigma_{ijk\ell}s$.

However, given that our results are guaranteed to hold for $(F_M^C)^0$, we have to choose $\hat{f}$ which is strictly monotone. This does not make a fundamental difference, though it complicates both the statement and the proof.

<u>Lemma 3.2.15</u>: Given $f \in (F_M^C)^0$ there is an $\bar{\epsilon} > 0$ such that for all $\epsilon \in (0,\bar{\epsilon})$ there are $\alpha = \alpha(\epsilon)$ and $\beta = \beta(\epsilon)$ such that $\hat{f}_\epsilon \sim f$ and $J(\hat{f}_\epsilon) = J(f)$ where $\hat{f}_\epsilon \in (F_M^C)^0$ is defined by

$$(\hat{f}_\epsilon)_i = \alpha + (i - 1)\epsilon \text{ for } i < n$$
$$(\hat{f}_\epsilon)_n = \beta.$$

Furthermore, $\alpha(\epsilon)$ and $\beta(\epsilon)$ are given by

$$\alpha = [\sum_{i=1}^{n-1} p_i - (n-1)p_n]^{-1}[J(f) - p_nC - \epsilon\sum_{i=2}^{n-2} ip_i + \epsilon p_n(n-2)]$$

$$\beta = [\sum_{i=1}^{n-1} p_i - (n-1)p_n]^{-1}$$
$$[C\sum_{i=1}^{n-1} p_i - (n-1)J(f) - \epsilon(n-2)\sum_{i=1}^{n-1} p_i + \epsilon(n-1)\sum_{i=2}^{n-1} ip_i]$$

<u>Proof</u>: It is straightforward to check that should $\hat{f}_\epsilon$ belong to F^C and satisfy $J(\hat{f}_\epsilon) = J(f)$, α and β can be computed from these two equations and should equal the expressions above.

These horrendous expressions are given here explicitly for two reasons. First, we must convince the reader that this system has a unique solution (which is obvious since $p_i > p_j$ for $i < j$), and that for small enough ϵ it will be in $(F_M^C)^0$. To this end, note that--again, since p_i are monotonically decreasing--$J(f) = \sum_{i=1}^n f_i p_i > p_n C$. Hence, $\alpha > 0$ for small ϵ. Further,

$$\beta - \alpha = [\sum_{i=1}^{n-1} p_i - (n-1)p_n]^{-1}[C\sum_{i=1}^{n} p_i - nJ(f)] + O(\epsilon),$$

where the first expression is positive.

The second reason will become clear later on, when we shrink ϵ to zero and claim convergence of $\hat{f}_\epsilon$.

Let, then, $\bar{\epsilon}$ be small enough to guarantee $\hat{f}_\epsilon \in (F_M^C)^0$ and to satisfy $\bar{\epsilon} < (f_i - f_{i-1})$ for all $1 < i \le n$.

Considering $\epsilon \in (0, \bar{\epsilon})$, we will show that for every $1 \le k \le n - 1$ there exists f_ϵ^k such that: (i) $(f_\epsilon^k)_i - (f_\epsilon^k)_{i-1} = \epsilon$ for $2 \le i \le k$; (ii) $(f_\epsilon^k)_i - (f_\epsilon^k)_{i-1} \ge \epsilon$ for $k < i \le n$; (iii) $f_\epsilon^k \sim f$; and (iv) $J(f_\epsilon^k) = J(f)$. The proof is by induction on k, and the existence of $f_\epsilon^{n-1} = \hat{f}_\epsilon$ will complete the proof of the lemma.

For $k = 1$, condition (i) is vacuous and we may take $f_\epsilon^1 = f$. Assume, then, that f_ϵ^k was found and consider f_ϵ^{k+1} (for $k \le n - 2$).

If $(f_\epsilon^k)_{k+1} - (f_\epsilon^k)_k = \epsilon$, we may choose $f_\epsilon^{k+1} = f_\epsilon^k$. Assume then,

$$(f_\epsilon^k)_{k+1} - (f_\epsilon^k)_k = \epsilon + t, \text{ for } t > 0.$$

One may obtain f_ϵ^{k+1} from f_ϵ^k by the following: we will make an identical transfer s > 0 from k + 1 to each of 1,...,k, and a transfer of r > 0 from k + 1 to n, such that these transfers will offset each other. However, in order to use Lemmata 3.2.6 and 3.2.12 (which will guarantee the preservation of properties (iii) and (iv)) we need to split this transfer into a sequence of pairs of offsetting transfers.

First, we set

$$r = t[\sum_{i=1}^{k} (p_1 - p_{k+1}) + (k+1)(p_{k+1} - p_n)]^{-1} \sum_{i=1}^{k} (p_i - p_{k+1})$$

$$s = t[\sum_{i=1}^{k} (p_1 - p_{k+1}) + (k+1)(p_{k+1} - p_n)]^{-1}(p_{k+1} - p_n)$$

Notice that r,s > 0, and that

$$r + (k+1)s = t$$

$$r(p_{k+1} - p_n) = s \sum_{i=1}^{k} (p_i - p_{k+1}).$$

The first equation guarantees that the difference $(f_\epsilon^k)_{k+1} - (f_\epsilon^k)_k$ will be decreased by t exactly. The second one--that the overall transfers will preserve the value $J(f_\epsilon^k)$.

We now split these transfers as follows: for $1 \le i \le k$, define

$$r_i = (p_i - p_{k+1})/(p_{k+1} - p_n)s.$$

Note that $r_i > 0$ and $\sum_{i=1}^{k} r_i = r$. Next define $f_\epsilon^{k,0} = f_\epsilon^k$ and for $1 \le i \le k$,

$$(f_\epsilon^{k,i}) = ((f_\epsilon^{k,i-1})_{(k+1-i,k+1,s)})_{(k+1,n,-r_{k+1-i})}.$$

That is, we first transfer r_k from k + 1 to n and s from k + 1 to k. Only then do we transfer r_{k-1} from k + 1 to n, and s from k + 1 to k - 1, and so forth.

Thus, $f_\epsilon^{k,i} \in (F_M^C)^\circ$ for $0 \le i \le k$. Furthermore,

$$(k + 1 - i, k + 1, s) \sim (k + 1, n, r_{k+1-i})$$

by 3.2.13. Hence, by 3.2.14, $f_\epsilon^{k,i} \sim f_\epsilon^{k,i-1}$ and $J(f_\epsilon^{k,i}) = J(f_\epsilon^{k,i-1})$ for all $i \le k$. In particular, for i = k we obtain f_ϵ^{k+1} satisfying (i)-(iv), and this completes the proof of the lemma. ∎

We can finally return to F_M^C, including its boundary:

<u>Lemma 3.2.16</u>: For every $f \in F_M^C$ there is $\hat{f} \in F_M^C$ with the following properties:

(i) $\hat{f}_i = \alpha$ for $i < n$

$\hat{f}_n = \beta$

for some $\beta \ge \alpha$;

(ii) $\hat{f} \sim f$; and

(iii) $J(\hat{f}) = J(f)$.

<u>Proof</u>: Starting with $f \in (F_M^C)^\circ$, we obtain $\hat{f}_\epsilon$ for every $\epsilon \in (0, \bar{\epsilon})$ by 3.2.15. Letting ϵ tend to zero, the explicit formulae of 3.2.15 show that $\hat{f}_\epsilon$ converge to some $\hat{f} \in F_M^C$, satisfying (i). Condition (ii) would follow from the continuity of $\succeq$, while (iii) follows from the continuity of the (linear) functional J.

As for $f \in F_M^C \backslash (F_M^C)^\circ$, let $f_n \in (F_M^C)^\circ$ satisfy $f_n \to f$, and let $\hat{f}_n$ be the corresponding profile for f_n. Given the explicit formulae of 3.2.15 we know that $\hat{f}_n \to \hat{f}$ where $\hat{f}$ satisfies (i). By continuity of J, (iii) also holds. Finally, since $\succeq$ is continuous, it is also closed (as a subset of $\mathbb{R}^{2n}$), and since $f_n \sim \hat{f}_n$, $f \sim \hat{f}$. ∎

It therefore suffices to show that J represents $\succeq$ on the one-dimensional half-space $\{(\alpha,\alpha,\ldots,\alpha\beta) | (n - 1)\alpha + \beta = C,\ \beta \geq \alpha \geq 0\}$. Indeed, we have

Lemma 3.2.17: Let $\alpha,\beta,\gamma,\delta$ satisfy $\beta \geq \alpha \geq 0$, $\delta \geq \gamma \geq 0$,

$$(n - 1)\alpha + \beta = (n - 1)\gamma + \delta = C.$$

Then the following are equivalent:

(i) $\beta - \alpha < \delta - \gamma$;

(ii) $J((\alpha,\alpha,\ldots,\alpha,\beta)) > J((\gamma,\gamma,\ldots,\gamma,\delta))$;

(iii) $g \equiv (\alpha,\alpha,\ldots,\alpha,\beta) \succ (\gamma,\gamma,\ldots,\gamma,\delta) \equiv f$.

Proof: the equivalence of (i) and (ii) follows from the definition of J, combined with the observation $p_i > p_j$ for $i < j$. To see that (i) implies (ii), consider $f_\epsilon \in F^C$ defined by

$$(f_\epsilon)_i = \gamma + (i - 1)\epsilon \qquad 1 \leq i < n$$

$$(f_\epsilon)_n = \delta - (1/2)i(i - 1)\epsilon.$$

If ϵ is small enough for $f_\epsilon \in F_M^C$, f_ϵ can be obtained from f by successive adjacent transfers of size ϵ. Thus, by A7, $f_\epsilon \succ f$. Choose $\epsilon > 0$ such that

$(f_\epsilon)_n > \beta$ and $(f_\epsilon)_{n-1} < \alpha$.

Similarly, for $\delta > 0$ consider g_δ defined by

$$(g_\delta)_i = \alpha + (i - 1)\delta \qquad 1 \le i < n$$

$$(g_\delta)_n = \beta - (1/2)i(i - 1)\delta.$$

Consider small enough δ such that $g_\delta \in F^C_M$. Since $f_\epsilon, g_\delta \in (F^C_M)^0$, g_δ may be obtained from f_ϵ by successive adjacent transfers. Hence, $g_\delta \succ f_\epsilon$. Letting δ tend to zero, one concludes that $g \succeq f_\epsilon \succ f$, and (iii) is proven. Finally, since $\beta - \alpha = \delta - \gamma$ immediately implies $f \sim g$, (iii) => (i) follows. ∎

Normalizing the numbers $\{p_i\}$ such that they are all positive (by adding a constant throughout) and sum up to 1 (by multiplying by a positive constant), and employing symmetry of $\succeq$, we conclude the proof of Proposition 3.2.1. ∎

3.3 Proof of Theorem B

Given Theorem A, we know that for every $f, g \in F^C$

$$f \succeq g \text{ iff } \textstyle\sum_{i\in N} p_i f^{(i)} \ge \sum_{i\in N} p_i g^{(i)}$$

for some $p_1 > p_2 > \ldots > p_n$. We now further assume that A6 holds on all of F^C.

3.3.1 Lemma: For every $i \le \dot{i} \le n - 1$,

$$p_i - p_{i+1} = p_1 - p_2.$$

Proof: Assume w.l.o.g. $i > 1$. Choose $f \in (F^C_M)^0$. Let $\epsilon > 0$ satisfy $\epsilon < (1/2)(f_{i+1} - f_i)$ for all $i \le i \le n - 1$, and define $f' = f + \epsilon e^1 - \epsilon e^2 \in (F^C_M)^0$.

Define a pertumutation π: $N \to N$ as follows:

(i) if $i > 2$, $\pi(1) = i$ $\pi(2) = i + 1$

$\pi(i) = 1$ $\pi(i + 1) = 2$

and $\pi(k) = k$ for $k \notin \{1, 2, i, i + 1\}$.

(ii) if $i = 2$, $\pi(1) = 2$ $\pi(2) = 3$ $\pi(3) = 1$

and $\pi(k) = k$ for $k \notin \{1, 2, 3\}$.

Define $g = \pi f$, and notice that in g, individuals 1 and 2 are ranked as the i- and (i + 1)-poorest, respectively. Finally, define $g' = g + \epsilon e^1 - \epsilon e^2$.

By symmetry, $f \sim g$. However, individual 1 f-, g'-, g- and g'- precedes 2, and $A6(F^C)$ implies that $f' \sim g'$. Hence

$$J(f') - J(f) = J(g') - J(g)$$

where J is defined as in Section 3.2. This implies

$$\epsilon(p_1 - p_2) = \epsilon(p_i - p_{i+1})$$

which completes the proof of the lemma.

To complete the proof of Theorem B, recall that for every $a > 0$ and $b \in \mathbb{R}$ the vector $q = (q_1, \ldots, q_n)$ defined by $q_i = ap_i + b$ also satisfies

$$f \succsim g \text{ iff} \Leftrightarrow \sum_{i \in N} q_i f^{(i)} \ge \sum_{i \in N} q_i g^{(i)}$$

for all $f, g \in F^c$. Setting

$$a = 2(n - 1)/(p_1 - p_n)$$

and

$$b = (n - 1)[1 - 2p_1/(p_1 - p_n)]$$

yields

$$q_i = n + 1 - 2i$$

for $i \leq i \leq n$.

On the other hand,

$$-\sum_{1 \leq i < j \leq n} |f_i - f_j| = \sum_{i \in N} (n + 1 - 2i) f^{(i)},$$

and the theorem is proven. ■

3.4 Proof of Theorem C

We first state and prove the following:

Proposition 3.4.1: Let $\succsim$ satisfy A1, A2, A3, A4 and A5 on F. Then there exists a nonadditive measure v such that for all $f, g \in F$,

$$f \succsim g \iff \int f dv \geq \int g dv.$$

Proof: Consider $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that $\beta - \alpha = \delta - \gamma$. Take $f, g, f', g' \in F_M$ and $i \in N$ such that $f_j = f_j'$ and $g_j = g_j'$ for all $j \neq i$, and $f_i = \alpha$; $f_i' = \beta$; $g_i = \gamma$; $g_i' = \delta$. By A5a, $f \succsim g$ iff $f' \succsim g'$. By monotonicity, if in the above structure $\beta - \alpha > \delta - \gamma$ and $f \succsim g$, then $f' \succ g'$. Hence, the relation $\succsim$ does not reveal "comonotonic contradictory tradeoffs as consequences" (see the definition in Wakker (1989, p. 113)) and, together with A1 and A2, one can use Wakker's theorem VI.5.1 (p. 117) to conclude that there are u: $\mathbb{R}_+ \to \mathbb{R}$ and a nonadditive measure v such that

$$f \succeq g \Longleftrightarrow \int u(f)dv \geq \int u(g)dv.$$

(Notice that the axiom of symmetry is actually redundant. However, if it is to be dropped, A5a should be strengthened to hold for <u>any</u> four pairwise comonotonic f,f',g,g'.)

Furthermore, one obtains

$$\beta - \alpha \geq \delta - \gamma \Longleftrightarrow u(\beta) - u(\alpha) \geq u(\delta) - u(\gamma)$$

whence u must be linear and the result follows. ■

Given this result, imposing A7 proves Theorem C.

3.5 <u>Proof of Theorem D</u>

Given Theorem C, we know that for all f,g ∈ F

$$f \succeq g \text{ iff} \Longleftrightarrow \sum_{i \in N} p_i f^{(i)} \geq \sum_{i \in N} p_i g^{(i)}$$

with $p_1 > p_2 > \ldots > p_n > 0$. Furthermore, A6 is known to hold on F, and in particular also on F^C for all C > 0. Thus, by Lemma 3.3.1, $p_i - p_{i+1} = p_1 - p_2$ for all $1 \leq i \leq n - 1$.

Setting

$$\alpha = 2/(p_1 + p_n) > 0$$

and

$$\delta = \frac{1}{(n - 1)} \frac{p_1 - p_n}{p_1 + p_n}$$

it is easy to verify that for all $1 \leq i \leq n$

$$\alpha p_i = 1 + (n - 2i + 1)\delta .$$

It only remains to note that, for all $f \in F$,

$$\textstyle\sum_{i\in N} [1 + (n - 2i + 1)\delta] f^{(i)} = \sum_{i\in N} f_i - \delta \sum_{1\leq i<j\leq n} |f_i - f_j|$$

and that $\delta < 1/(n - 1)$ follows from $p_n > 0$. ∎

4. Concluding Remarks

1. In both Theorems And C, one may use Schmeidler's comonotonic independence axiom rather than A5 or A6. (See Schmeidler (1989).) As a matter of fact, this axiom would have greatly simplified the proof of Theorem A, since it provides a von-Neumann-Morgenstern (1947) representation on F_M^C.

However, we found axioms A5 and A6 much more natural for our context. It is somewhat difficult to justify the mixture operation in this framework without resorting to uncertainty, which would have cluttered the issue. We therefore used A6, at the cost of a long subsection 3.2. Note, however, that with comonotonic independence one can use precisely the same axiom for F^C and for F (as in the case of A6(F^C)). Therefore, accepting the Gini index for inequality measurement on F^C while rejecting linear tradeoff between inequality and total income requires one to justify why comonotonic independence makes sense on F^c but does not on F.

2. It is sometimes convenient to model a population as a continuum of agents, say [0,1], endowed with a σ-algebra, say, the Borel sets. All our axioms will have natural counterparts if one assumes a nonatomic σ-additive measure on these Borel sets to be given in the model. Symmetry is then required to hold with respect to the group of measurable and measure-preserving permutations, and the continuity of $\succeq$ should be stipulated with respect to convergence in the measure. In this topology one may approximate every profile by a simple profile, which is constant on every element of an equi-measure finite partitions. With the representations obtained above, the derivation of similar ones for this setup is then straightforward.

3. It is easy to verify that, in each of the theorems, the axioms are independent. We omit the simple examples (some of which appeared above).

4. Finally, note that Theorem A provides a Choquet-representation on F^c only for inequality-averse preferences. This condition, however, is not crucial. We used it since it eliminates some non-insightful complications into the proof, while being natural in our context.

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