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EQUILIBRIUM UNEMPLOYMENT CYCLES

by

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ABSTRACT

The global dynamics of an equilibrium model of aggregate unemployment that focuses on the job/worker matching process is studied for the case of increasing returns to scale in production and constant returns to scale in the matching process. In anticipation of a particular solution to the wage bargaining problem, unmatched expected present value maximizing workers and employers respectively invest in search and recruiting effort that collectively determines the aggregate matching rate. An equilibrium is a dynamic path for the aggregate number of matches generated by best response investment decisions under rational expectations.

There are no periodic solutions if the wage bargain is efficient and the future is discounted. However, when the difference between the elasticity of the matching rate with respect to aggregate search effort and the workers' share of match capital is small and positive, an endogenous cycle exists on an open set of small positive discount rates. The cycle can be either stable or unstable and is unstable if an only if the economy is sufficiently productive. Finally, when a cycle is consistent with initial conditions, an alternative non-periodic equilibrium also exists that yields higher permanent income now and less unemployment in the future.

[1]Useful comments and corrections suggested by Peter Howitt and other participants of the Conference on Search, Unemployment, and Sectoral Reallocation held at the University of Western Ontario, April 20-21, are gratefully acknowledged.

1. Introduction

In Mortensen [1989], I suggested that a dynamic version of Pissarides' [1986,1990] deterministic model of equilibrium unemployment could have an endogenous unemployment cycle as a solution when the production technology exhibits increasing returns attributable to external economies across sectors.[2] The validity of the conjecture is established and conditions for the local stability of the cycle are derived in this more detailed study of the model's comparative dynamics. In addition, any periodic solution is shown to be socially dominated by another non-periodic equilibrium, a fact that poses a dynamic coordination policy problem in the model.

A single good is produced and consumed by worker/employer pairs. The process by which employers and workers find one another is the focus of the model introduced in Section 2. Search and recruiting activities serve as inputs in a matching technology that produces a flow of new productive pairs as output. In equilibrium, unemployed workers and employers with vacancies make matching investment decisions so as to maximize expected discounted future consumption flows. These decisions depend on match surplus, the expected present value of the future quasi-rent flow attributable to an existing match, and on its

[2] Endogenous cycles can also be present when the production technology is standard but the matching technology is subject to increasing returns. Diamond and Fudenberg [1989] establish the existence of limit cycles in a closely related model under these conditions. Boldrin, Kiyotaki and Wright [1991] is another more recent study of a search equilibrium model with these properties.

division, determined by the wage bargain. For the model considered, the wage bargain is efficient, maximizes the present value of aggregate consumption, only if the worker's share of match capital equals the elasticity of the matching function with respect to aggregate search effort.

The model's equilibria, defined in Section 3, are bounded solutions to a planar system of differential equations involving the number of matched employer/worker pairs and the capital value of the typical match. Given a zero pure rate of time discount and an efficient wage bargain, the system's dynamics are Hamiltonian, i.e., the solution paths are level curves of a known function of the number of matches and their value, a fact established in Section 4. As a corollary, multiple steady state solutions generally exist composed of alternating saddle points and centers under the assumption that productivity per match increases with aggregate employment. Non-periodic equilibria converge to the saddle points and a family of closed orbits, representing oscillating periodic equilibria, surrounds each center. Many equilibria exist for some initial conditions when there is more than one steady state solution. The alternative equilibria correspond to different self fulfilling expectations about the future course of the economy modelled.

Because the model is structurally unstable given no discounting and efficient wage bargaining, the conclusions drawn don't extend even approximately to the small positive discount rate and inefficient bargaining cases without qualification. For example, that no periodic equilibria exist given an efficient wage bargain and a positive discount rate is a corollary of the principal result of Section 5. Still, a perturbation argument, presented in Section 6, implies that there is a limit cycle for each element in an open set of discount rates when the workers' share of match capital is less than the search effort elasticity of the matching function provided that these deviations from efficiency and no discounting are sufficiently small. The limit cycle can be either stable or unstable and is unstable if and only if the economy is productive enough in a well defined sense. Finally, an equilibrium unemployment cycle yields less income than an alternative non-periodic equilibrium that also exist for the same initial condition, a fact that justifies stabilization policy designed to coordinate on the better equilibrium.

Computed examples are reported to illustrate and extend the analytic results. The special case studied, introduced in Section 7, is characterized by constant elasticity production, matching, and cost of search functions and a constant workers' share of match surplus. In this case, there are at most three steady states. Exactly three exist if the economy is sufficiently productive and if the elasticity of the matching rate per unemployed worker with respect to the value of a match and the elasticity of match productivity with respect to the aggregate number of matches are positive and large enough. Parameter values based on explicit and implicit estimates reported in the literature suggest that this necessary condition for multiple equilibria is empirically reasonable. Finally, actual cycles in unemployment are identified for these same parameter values, a discount rate of 1% per quarter, and an appropriate division of match capital. These computed solutions to the constant elasticity version of the model are reported and illustrated in Section 8. The final section, Section 9, includes a brief description of the empirical properties of the cycle.

2. A Search Economy[3]

Agents are of two types, workers and employers who provide jobs. A single consumption good is produced by each matched worker/employer pair at a rate that depends on aggregate employment. Match formation is the output of a transactions process with aggregate search effort and recruiting activity serving as inputs. Let $f(n)$ denote the output of the typical match expressed as a function of the current number of matches, n . Every worker is either matched with a job, employed, or seeking such a match, unemployed. Letting the unit interval represent the total available worker force, the fraction seeking employment is $1-n$. Aggregate search effort is the product $s(1-n)$ where s denotes the search effort of the representative worker not employed. The cost per searching worker expressed in terms of the produced good is denoted

[3]The model presented here is a version of that developed in Pissarides [1986,1990] and studied in Mortensen [1989]. The reader is referred to these works for extensive motivation and more detail.

as $c(s)$. Employers both participate in existing job matches and seek new employees. The recruiting cost flow per vacancy is denoted as a and the aggregate recruiting effort is reflect in the number of job vacancies, represented as v . The rate at which new matches form is a function of aggregate search and recruiting effort; it is denoted as $m(v,s(1-n))$. Finally, the law of motion for employment (its time derivative) is

$$(1) \quad \dot{n} = m(v,s(1-n)) - \delta n$$

where δ is an exogenous separation or job destruction rate.

Increasing returns in production is assumed in the sense that match productivity, $f(n)$, is an increasing function of the aggregate number of matches. Hall [1988] suggests the possibility of increasing returns in production at the aggregate level in the U.S. and Caballero and Lyons [1989] provide evidence that external economies across sectors can account for it. The assumption that the job/worker matching function, $m(v,s(1-n))$, is concave and homogeneous of degree one is the second principal specification restriction. Empirical justification can be found in Blanchard and Diamond [1989] for the U.S. labor market and in Pissarides [1986] for the U.K. Finally, both the job destruction rate, δ, and the cost of filling a vacancy, a , are strictly positive by assumption throughout the paper and the cost of search function, $c(s)$, is strictly increasing and convex with the property that $c(0) = c'(0) = 0$.

In search equilibrium, workers and employers decide the extent of their individual search and recruiting investment in response to expectations about future private return. Let J and W respectively represent the gross capital value of being matched to an employer and worker respectively expressed in term of the expected present value of future consumption conditional on the specified state. Analogously, let V represent the employer's capital value of holding a job vacant and U the worker's capital value of being unemployed.

Under the assumption that existing vacancies are matched with equal likelihood, the probability that any one will be matched per period is (approximately) equal to the ratio of the aggregate meeting rate to the total number, $m(\cdot)/v$.[4] The return to holding a job vacant is the product of this probability and the net gain in capital value attributable to filling the job, denoted as $J-V$. As vacancies are created so long as this return exceeds the cost a by virtue of free entry, the equilibrium number of vacancies relative to aggregate search effort, defined as

$$(2) \quad x \equiv \frac{v}{s(1-n)},$$

equates the two. Formally,

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[4]Under the assumption that any particular vacancy is matched with equal probability, the meeting process is Poisson with arrival frequency m/v.

$$(2a) \quad a = (J-V)\mathrm{m}(v, s(1-n))/v = (J-V)\mathrm{m}(x,1)/x$$

given the assumed concavity and homogeneity of the matching technology.

Similarly, if each worker is matched at random with probability equal to the worker's own search effort relative to the aggregate, the gross total return to search effort at own intensity $\tilde{s}$ is $(W-U)\tilde{s}\mathrm{m}(v, s(1-n))/s(1-n)$ where s is the common search intensity of all the other workers. Hence, the required equality of the marginal return and cost of search effort implies that the common equilibrium search intensity solves

$$(2b) \quad c'(s) = (W-U)\mathrm{m}(v, s(1-n))/s(1-n) = (W-U)\mathrm{m}(x,1)$$

when the net capital value of becoming employed, $W-U$,is positive and is zero otherwise.

The asset value of an occupied job J, of employment W, of holding a job vacant V, and of remaining unemployed U, respectively solve the asset pricing equations

$$(3a) \quad rJ = f(n) - w - \delta(J-V) + \dot{J}$$

$$(3b) \quad rW = w - \delta(W-U) + \dot{W}$$

$$(3c) \quad rV = (J-V)\mathrm{m}(v, s(1-n))/v - a + \dot{V}$$

$$(3d) \quad rU = (W-U)m(v, s(1-n))/(1-n) - c(s) + \dot{U}$$

where w denotes the wage and r represents the pure rate of time discount. The equations of (3) reflect the fact that the return on the value of an asset is equal to the expected current income flow plus anticipated capital gain. When matched, the gross income attributable to the match is the employer's (worker's) share of match product less the expected loss per period attributable to the possibility of job separation, the product of the separation rate and the capital loss suffered by the employer (worker) when separation occurs. In the case of a vacancy (an unemployed worker), anticipated income is the product of an employer's (worker's) share of match capital value and the probability per period that a vacancy (unemployed worker) is matched less the flow of recruiting (search) cost incurred per period.[5]

The match surplus accruing to the participating worker and employer pair is given by the identity

$$(4) \quad p \equiv J + W - V - U.$$

This capital surplus is divided between the worker and employer by the terms of a bargain that is allowed to be contingent on the tightness of the market as reflected in the magnitude of the

[5]Note that (3c) and the equilibrium condition for vacancies (2b) imply V(t) = 0 for all t. This result represents the zero profit condition associated with the model's free entry assumption.

ratio of vacancies to search effort.[6] Specifically, let $\theta(x)$ represent the worker's share of the match surplus expressed as a non-decreasing function of x. Given this specification, the outcome of the bargaining can be written as follows:

(5a) $J - V = [1-\theta(x)]p.$

(5b) $W - U = \theta(x)p.$

By virtue of equations (2) and (5), the equilibrium values of the ratio of vacancies to aggregate search effort, x , and search intensity per unemployed worker, s, are the unique solutions to

(6a) $p[1-\theta(x)]m(x,1)/x = a$

(6b) $p\theta(x)m(x,1) = c'(s).$

Given that the matching technology is increasing and concave in its inputs and exhibits constant returns to scale, that the cost of search in strictly convex, and that the workers' share of match surplus is a non-decreasing function of market tightness, these equations imply that both the ratio of vacancies to aggregate search effort and the search intensity per unemployed

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[6]Pissarides [1990] assumes the relative shares, which reflect relative "bargaining power," are constants rather than functions. This restriction is neither necessary nor universally plausible. Even the assumption that the share depends only on market tightness is too restrictive in general. For example, see Mortensen [1989] for different interpretations of the share function arising from alternative institutional arrangements for wage determination including efficiency wage and insider-outsider specifications.

worker are zero when a match has no value and are strictly increasing function of the capital value of a match when positive by virtue of the implicit function theorem. As a consequence, the equilibrium rate at which worker transit from non-employment to employment is an increasing functions of the value of a match when positive and is zero otherwise, i.e., the equations of (6) imply that the rate of transition from unemployment to employment is

$$(7) \quad m(v, s(1-n))/(1-n) = sm(x,1) \equiv h(p) \quad \text{where}$$

$$h'(p) > 0 \ \forall\, p > 0, \ h(p) = 0 \ \forall\, p \le 0.$$

The equations of (6) also imply

$$(8) \quad g(p) \equiv p\theta(x)\, sm(x,1) - c(s) = \int_0^p \frac{\theta(x(q))}{\lambda(x(q))} h(q)\, dq$$

where $x(p)$ is the increasing function implicitly defined by (6.a) and where

$$(9) \quad \lambda(x) \equiv 1 - xm_1(x,1)/m(x,1)$$

is the elasticity of the matching function with respect to aggregate search effort by virtue of an argument presented in the appendix. Note that the right side of (8) is the difference between the total return and cost of search effort. Hence, $g(p)$ represents the net value flow imputable to search activity, i.e., the permanent income attributable to the search activity of an unemployed worker.[7]

By differentiating the identity (4) with respect to time and then substituting appropriately from the equations of (2), (3), (5), (6), and (8), one finds that the surplus capital value of a match satisfies the following differential equation:

$$(10a) \quad \dot{p} = (r+\delta)p + g(p) - f(n).$$

By substituting from equation (7) into (1), one completes the derivation of the dynamical system of interest:

$$(10b) \quad \dot{n} = h(p)[1-n] - \delta n.$$

The wage equation that supports the agreed upon division of match capital is derived as follows. As a consequence of (3b) and (3d), equations (5b) and (8) imply

$$w = g(p) + \theta(p)[(r+\delta)p - \dot{p}].$$

Hence,

$$(11) \quad w = \theta(p)f(n) + [1-\theta(p)]g(p)$$

is obtained by substituting from (10a). As $g(p)$ is the imputed income attributable to search activity while unemployed by virtue

[7]In spite of the risk to notational purity, we let $\theta(p) = \theta(x(p))$ and $\lambda(p) = \lambda(x(p))$ in the sequel.

of its definition in equation (8), it represents the opportunity cost of accepting employment. Hence, the wage function that supports any particular bargaining outcome represented by the workers' share of match capital is the analogous weighted average of match product and the income forgone by accepting employment.

Equilibrium matching models embody two external effects that tend to be offsetting. On the one hand, an increase in an employer's recruiting or a worker's search activity induces a congestion effect that diminishes the probability per time period that any other agent on the same side of the market will become matched. On the other hand, the same increase in activity benefits agents on the other side of the market in the sense that their match likelihoods are increased. When the matching function is homogenous of degree one as assumed here, the effects of these two externalities cancel for particular match surplus shares. Specifically, equilibrium search effort and vacancies maximize social welfare in the sense of expected discounted aggregate consumption only if the worker's share of match rent is equal to the elasticity of the matching function with respect to search effort, i.e., $\theta \equiv \lambda$.[8] The fact that net income attributable to unemployed search, $g(p)$, is equal to the integral of the transition rate from unemployment to employment

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[8]If equilibrium is unique and there are no external effects, then this condition is also sufficient. See Pissarides [1990] for discussion and proofs.

in this case by virtue of equation (8) is of considerable importance in the sequel.

As agents are risk neutral discounted consumption maximizers, aggregate permanent income, defined as

$$(12a) \quad y \equiv r(J+W)n + rU(1-n) + rVv,$$

is an appropriate measure of social welfare in the context of this model. This definition together with (4), (5), (8), the asset pricing equations of (3) and the equilibrium conditions of (2) imply that

$$(12b) \quad y = r(J+W-U)n + rU$$

$$= rpn + r\int_t^{\infty} g(p(\tau))\, e^{-r(\tau-t)}\, d\tau .$$

Specifically, the first equality follows from the definition (12a) and the fact that rents attributable to holding a job vacant are bid away in equilibrium, i.e., V = 0 by virtue of (2a) and (3c). The second equality is an implication of the definition of the capital surplus associated with a match, p, given in (4) and the integral form of the differential equation $\dot{U} = rU - g(p)$ obtained by substituting from equations (5b) and (8) into equation (3d).

In its second form, equation (12b) implies that aggregate income is equal to the interest on capital surplus associated with the existing number of job/worker matches plus interest on the present value of the imputed earnings flow associated with search, which is an exponential weighted average of that future flow. Income as defined is also related to the standard national income and the GNP measure. By substituting for rp from equation (10a), by replacing rU with its equivalent $g(p) + \dot{U}$, and then by rearranging the result appropriately, one obtains

$$(12c) \quad y = nf(n) + (1-n)g(p) - \delta pn + \dot{p}n + \dot{U}, \text{ where}$$

$$U(t) = \int_t^\infty g(\tau)e^{-r(\tau-t)}d\tau .$$

As the first term is the current consumption flow produced per period, the second is the gross value added by investment in matching activity made per period, and the third term represents an adjustment for match capital "depreciation" attributable to job separation and destruction, the first three terms together equal GNP.[9] Because the last two terms represent capital value appreciation adjustments to aggregate income, our measure of welfare is GNP plus the anticipated change in the values of wealth.

[9]Of course, GNP as actually measured includes neither the investment in search nor match depreciation adjustment.

3. Search Equilibrium

A rational expectations (perfect foresight) equilibrium is necessarily a solution to the ordinary differential equation system (10). Although the initial number of job/worker matches, n_0 , is historically predetermined, the initial capital value of a match, p_0 , can be chosen at any value consistent with the transversality condition, the requirement that the present value of match capital at any date in the indefinite future be bounded. Formally, match capital is a "jump" variable determined in a forward looking manner to be any value consistent with rational expectations about the future evolution of the economy in this sense. In the Appendix, I demonstrate that only those solutions that originate and remain in the bounded rectangle $B = [0,\overline{p}]\times[0,1]$ where $\overline{p}$ is the unique solution to $(r+\delta)\overline{p} + g(\overline{p}) = f(1)$ satisfy the transversality condition. Of course, multiple equilibria may exist. Conditional on the initial number of matches n_0 , the number of rational expectations equilibria is equal to the number of choices of the initial capital value p_0 that satisfy this criterion.

The singular curves, implicitly defined by the conditions $\dot{p} = 0$ and $\dot{n} = 0$, are both upward sloping, as illustrated in Figure 1, by virtue of the assumption that $f(n)$ is increasing and the fact that $h(p)$ is increasing in p . Because along the $\dot{n} = 0$ curve n is bounded above by 1, because p on the $\dot{p} = 0$ is bounded above by $\overline{p}$, and because $h(0) = 0$ implies that the $\dot{n} = 0$ curve passes through the origin, at least one and generally several steady state solutions exist at the intersections of the two curve.

The **phase portrait** for the case of three steady states is illustrated in Figure 1. The direction arrows illustrate local dynamics which one can derive from (10) using well known techniques. As indicated in the figure, the lowest (L) and the highest (H) steady states are saddle points (every other steady state is a saddle in general) while the steady state in the middle (M) (any in between in general) is either a source, center, or sink depending on the sign of the expression

$$(13)\qquad \frac{\partial\dot{p}}{p} + \frac{\partial\dot{n}}{n} = r - \left[1 - \frac{\theta(x(p))}{\lambda(x(p))}\right] h(p),$$

which is equal to the sum of the real parts of the eigen values of the system[10]

As both L and H are saddle points, there are unique solution converging to each along, their respective **stable manifolds**, which are equilibria. When M is a sink, all converging solutions are also equilibria. Unfortunately, this local information is not sufficient to answer most questions of interest. Do multiple equilibria exist, i.e., are there two or more solution paths

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[10]The lower equilibrium saddle point L is at the origin as illustrated if and only if match productivity at zero employment is zero, i.e. f(0) = 0. However, the topology of the phase portrait is the same if this special condition does not hold. Specifically, L always exists and consequently one can linearly transform the variables so the L is at the new origin. All the results that follow will also hold for (10) expressed in terms of the transformed variables.

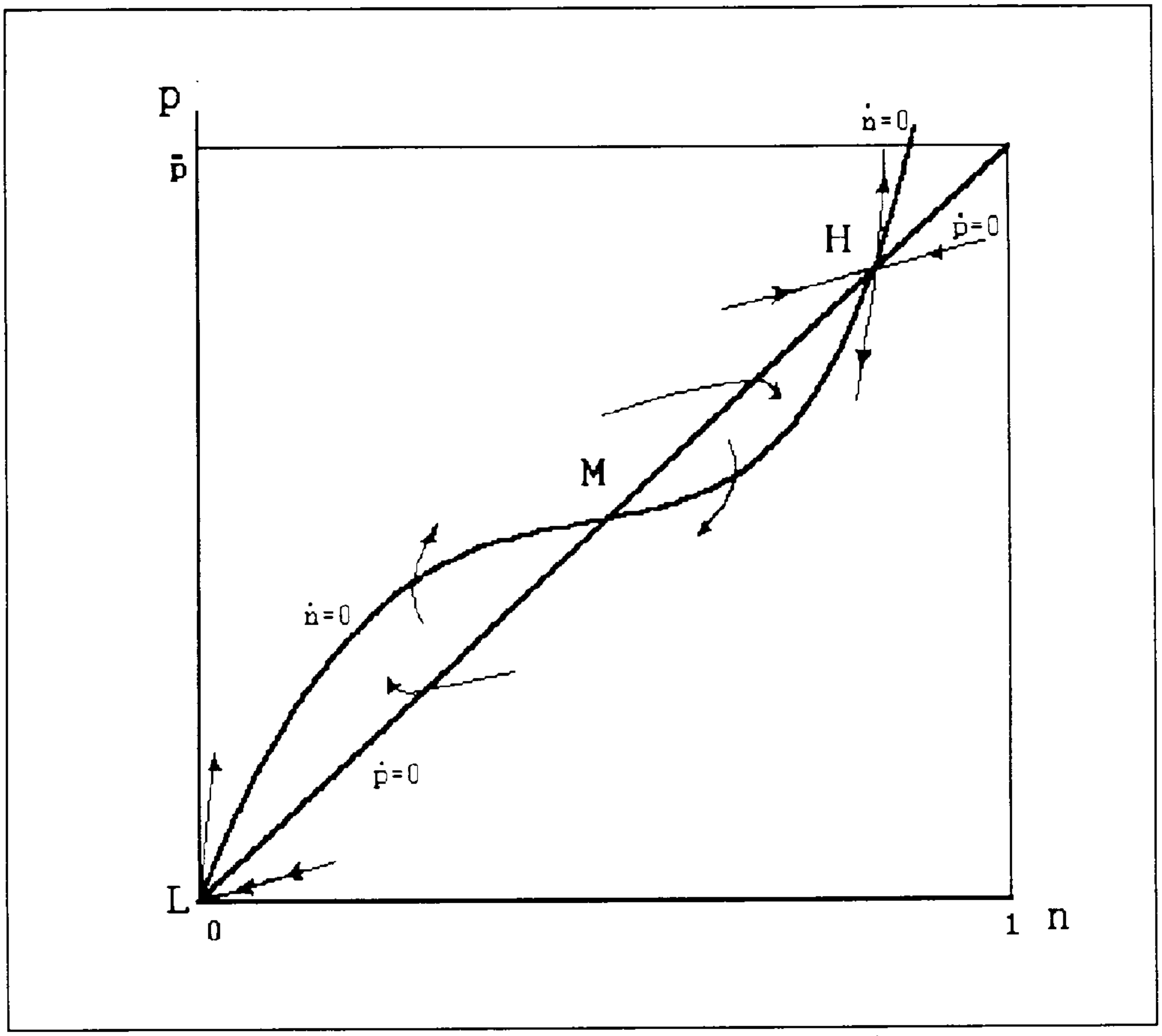

Figure 1: Local Search Equilibrium Dynamics

contained in *B* associated with at least some initial values of n? If so, is there an equilibrium path that converge to L and another that tends to H for every possible initial value of n? Finally, are there equilibria that never converge to a steady state? Answers to any one of these questions require a global analysis.

4. Hamiltonian Dynamics:[11]

Limit cycles of finite period are connected solution paths contained in the rectangle B that do not include any steady state points, trajectories known as **closed orbit.** When both the wage bargain is efficient and the pure time rate of discount is zero, the following fact implies that a family of closed orbits exists surrounding each non-saddle steady state.

Proposition 1: Suppose that $r = 0$ and $\theta(x) \equiv \lambda(x)$. Every solution to (10) is a level curve of the Hamiltonian function

$$(14) \quad \mathrm{H}(p,n) = \int_0^n \mathrm{f}(y)\,dy + (1-n)\int_0^p \mathrm{h}(q)\,dq - \delta pn$$

where

$$(15a) \quad \mathrm{h}(p) = sm(x,1)$$

given that s and x solve

$$(15b) \quad pm_1(x,1) = a$$

$$(15c) \quad pm_2(x,1) = c'(s)$$

when $p > 0$ and $s = x = 0$ otherwise.

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[11]The analysis in this section and the one that follows applies arguments developed by Matsuyama [1989] in his analysis of a similar mathematical structure.

Proof: When $\theta(x) \equiv \lambda(x)$, the equations of (15) are implied by the equations of (6) and equation (9) and the assumption that the meeting function is homogenous of degree one. By virtue of (14), $H_p(p,n) = \dot{n}$ and $H_n(p,n) = -\dot{p}$ from (10). Consequently, $\dot{H} = \dot{n}\dot{p} - \dot{p}\dot{n} = 0$, i.e., $H(p,n)$ is a constant along any solution path.

□

The Hamiltonian function, defined in (14), can be used to characterize the global dynamic as follows: Suppose that three steady state exists as illustrated in Figure 1. Because $\partial H/\partial p = \dot{n} > (<)\ 0$ in the region above (below) the $\dot{n} = 0$ singular curve, the value of the Hamiltonian function increases (decreases) with p in that region holding n fixed. Consequently, the fact that any solution path is a level curve of the function implies that the left stable (unstable) manifold converging to (diverging from) H lies above (below) the unstable (stable) manifold diverging from (converging to) L if the value of the Hamiltonian at H exceeds the value at L. Conversely, if the value at L exceeds that at H, the relative positions of the separatrixes associated with L and H are reversed. Because the Hamiltonian function takes on a relative minimum at M, M is a center surrounded by a family of closed orbits.[12] Hence, there

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[12]Simply note that $\partial H/\partial n = -\dot{p}$ is positive (negative) for all values of n to the right (left) of M in Figure 1 holding p constant and that $\partial H/\partial p = \dot{n}$ is positive (negative) for all values p above (below) M holding

are only two different topological cases, those illustrated as Figures 2a and 2b and the boundary case that separates them, illustrate as Figure 2c.

The panels of Figure 2 represent phase diagrams or portraits associated with the system (10), the family of solution trajectories (p(t),n(t)) (level curves of H) that originate within the rectangle $B = [0,\overline{p}] \times [0,1]$, in the case of a zero discount rate and an efficient wage bargain. The arrows in the figure indicate the direction of motion through time. The distinction between cases I, in Figure 2a, and case II, represented in Figure 2b, are the relative sizes of the values of the Hamiltonian at points L and H. Specifically, for values of parameters on the boundary of the two cases, illustrated inFigure 2c, the value of the Hamiltonian at H is equal to that at L by definition.

In the boundary case, the unstable solution trajectory from one saddle must connect it to the other as the second's stable trajectory forming a **heteroclinic orbit**, a closed loop connecting more than one stationary point of a system, because both M and H are on the same level curve of the Hamiltonian. As the Hamiltonian function takes on its global minimum at M, the geometry of the phase portrait and the fact that each solution trajectory traverses a level curve of $H(p,n)$ imply that the

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n constant.

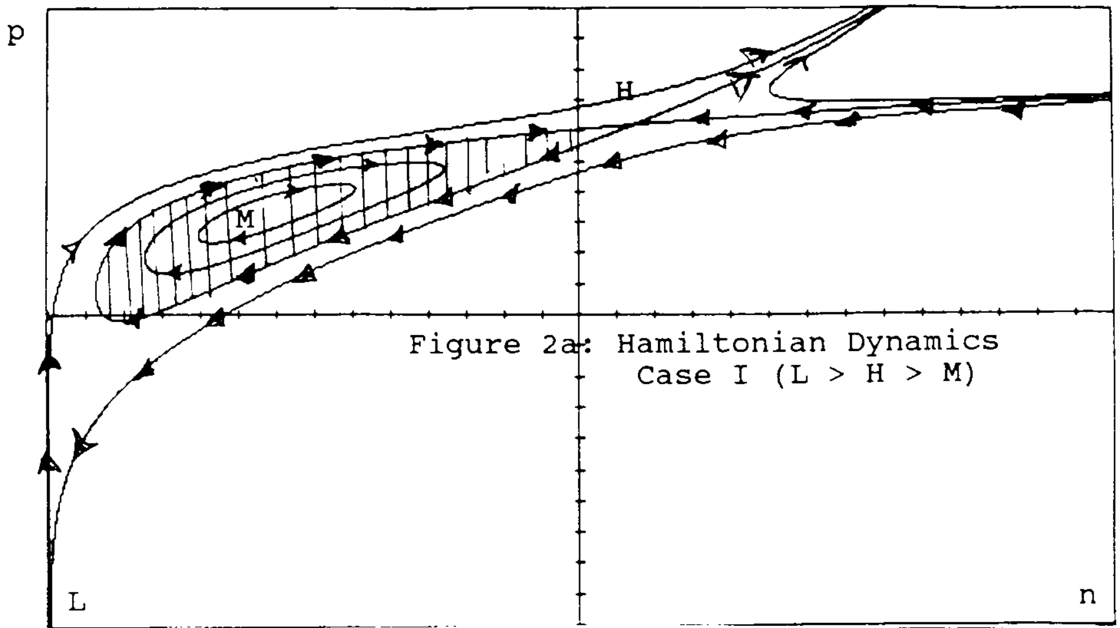

Figure 2a: Hamiltonian Dynamics
Case I (L > H > M)

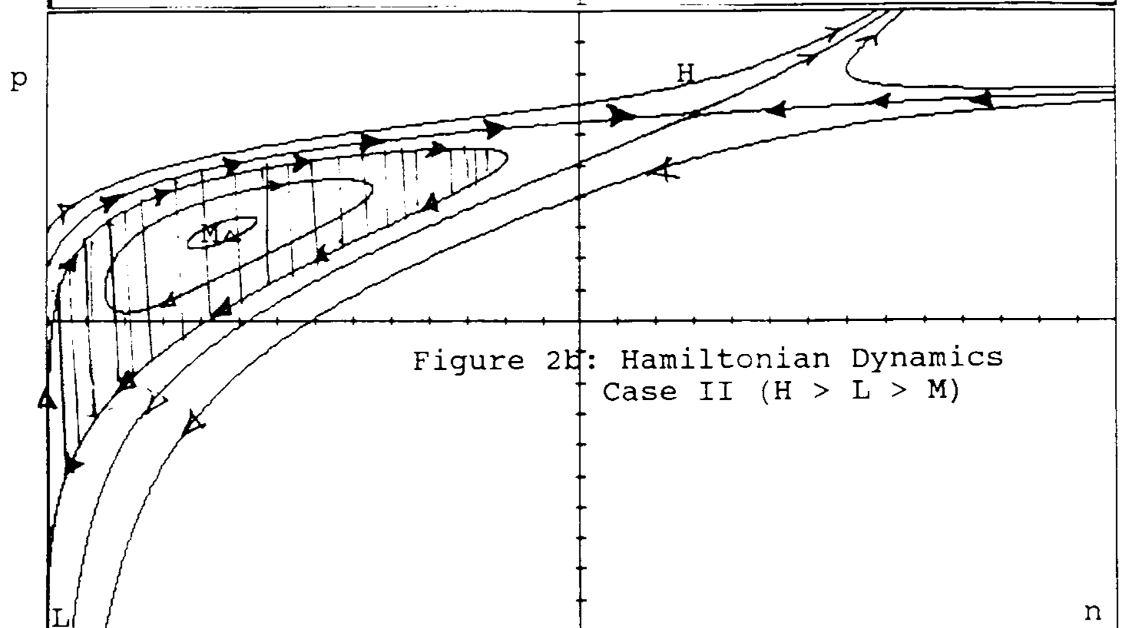

Figure 2b: Hamiltonian Dynamics
Case II (H > L > M)

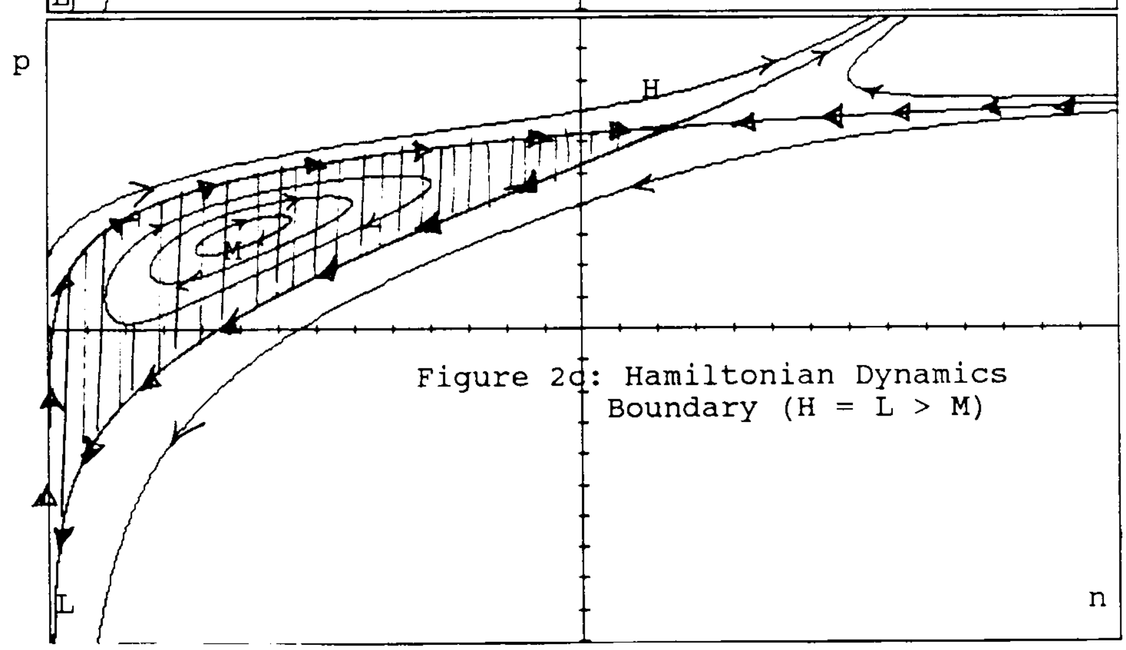

Figure 2c: Hamiltonian Dynamics
Boundary (H = L > M)

Hamiltonian is larger at L and along the stable manifold converging to L than along the **homoclinic orbit** connecting H to itself in case I and is larger in value at H and along it stablemanifold than at L an on the homoclinic orbit connecting L to itself in case II.

The set of equilibria for all possible initial values of employment, those that originate and remain in the rectangle $B = [0,1] \times [0,\bar{p}]$, include the stable manifolds associated with the two saddles, L and H, indicated in the figure by the heavily barbed curves, and the continuous family of closed orbits around M, located in the shaded areas of Figure 2. That multiple search equilibria exist for some initial values of employment when there are multiple steady states is now obvious. Indeed, for initial values of employment near the steady state level at M a continuum of periodic equilibria exist as well as two non-periodic equilibria converging to L and H respectively. The multiplicity arises because more than one future path is consistent with the expectation that it will be realized in the future. Specifically, "bullish" expectations are self fulfilling along the solution trajectory leading to H, "bearish" expectations generate a future history leading to L, and the expectation of future oscillations with a fixed period and amplitude are self fulfilling on the equilibria represented by any one of the closed orbits surrounding M in Figure 2.

5. Positive Discounting and a Large Workers' Share: No Cycle

Bendixson's criterion is a well known sufficient condition for the non-existence of closed orbits (See Guckenheimer and Holmes [1986, p.44] for a formal statement.) In our case, the condition requires that the right side of (13) not be identically zero and not change sign on the rectangle $B = [0,\bar{p}] \times [0,1]$. The following is an important consequence for our study.

Proposition 2: If $r > 0$ and $\theta(p) \geq \lambda(p)$ on $[0,\bar{p}]$, no equilibrium is periodic.

Hence, if they exist at all under positive discounting, limit cycles require a workers' share of match capital strictly less than that consistent with matching efficiency.

The global dynamics of the system under the conditions of Proposition 2 are as illustrated in Figure 3. Again there are two case analogous to those illustrated in Figure 2. As there are no closed orbits and M is unstable by virtue of (13) and the hypothesis of Proposition 2, M is necessarily the source of the stable manifold that converges to H in case I and to L in case II. The other stable manifold converging to L in case I and converging to H in case II is also an equilibrium. Although multiple equilibria continue to exist, note that there is no equilibrium paths converging to H in case I for sufficiently small initial values of employment nor to L in case II for sufficiently large initial employment level.

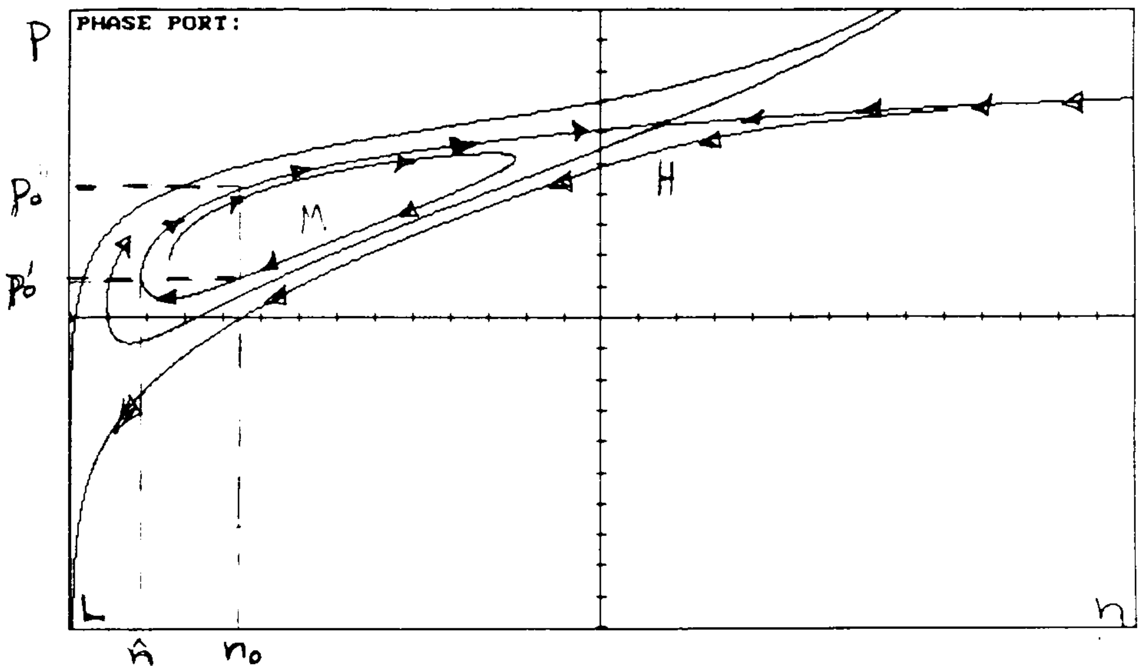

Figure 3a: Efficient Wage Bargain (Case I)

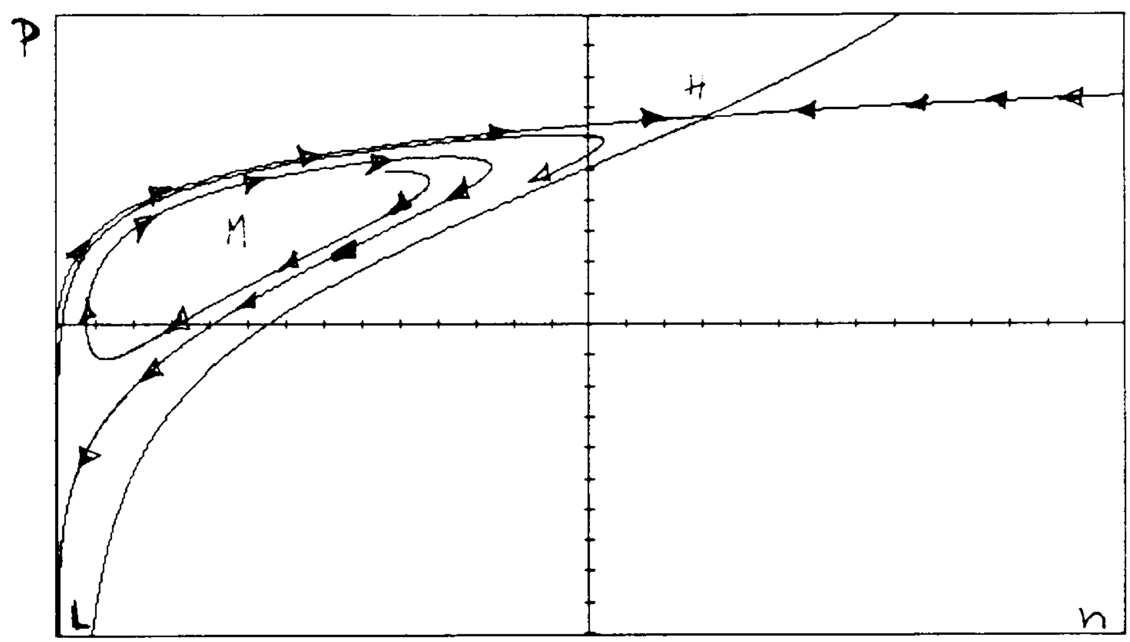

Figure 3b: Efficient Wage Bargain (Case II)

When there are two or more equilibria, equation (12b) can be used to compare them with respect to social welfare. Consider the equilibrium trajectory converging to H in case I. As that path is emitted from M andoscillates around M in any neighborhood (at least for values of r near zero), multiple equilibria generally exist corresponding to the different values of p on the path associated with the given initial value of employment. For example, given $n = n_0$, both p_0' and p_0'' in Figure 3a are alternative initial equilibrium values of match capital. Because $p_0'' > p_0'$ implies that the future time path $p(\tau)$ starting at (p_0'', n_0) dominates that starting at (p_0', n_0) and because $g(p)$ is increasing, income at the starting point with the higher capital price is larger by virtue of (12b). Formally, letting $p'(t)$ and $p''(t)$ represent these two time paths, $p''(t) > p'(t)$ implies that the difference in the two income at the initial date,

$$(16) \quad y(p_0'', n_0) - y(p_0', n_0) =$$

$$(p_0''-p_0')\,n_0 + \int_t^\infty [g(p''(\tau)) - g(p'(\tau))]\, e^{-r(\tau-t)}\, d\tau$$

is strictly positive. Similarly, the income on any path converging to the high employment steady state H dominates that converging to L in both case I and case II. In sum, the equilibrium paths are income ranked and the income rank is the same as the value of match capital rank. Hence, the income maximizing equilibrium is that defined by the highest equilibrium capital value associated with any initial level of employment.

That equilibria are socially ranked poses a coordination problem. The implementation of a policy designed to induce the economy to select the best equilibrium is obviously suggested. Furthermore, the cross sector production externality assumed also implies that equilibrium employment, even at the high steady state H, is too low, i.e., an employment subsidy is justified as well. Such a policy is of particular importance in case I if the initial value of employment is below the critical number $\hat{n}$, the minimum initial employment level consistent with the existence of an equilibrium path converging to the high employment steady state, H.[13] Because the critical value can be shown to decrease with an employment subsidy and because case I becomes case II for a sufficiently large subsidy, a subsidy of sufficient magnitude is both necessary and sufficient to achieve the high employment steady state when the initial level of employment is below the critical value.

6. Positive Discounting and A Small Workers' Share: Limit Cycle

Solutions to a differential equation system are said to be **structurally stable** if the topological properties of the phase portrait are invariant to small perturbations of the system's parameters. **Bifurcations** occurs at points in the parameter space

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[13]The same kind of policy issue arises in Matsuyama [1989] as well.

associated with structurally instability. Homoclinic orbits such at that linking H to itself in Figure 2a and that linking L to itself in Figure 2b are structurally unstable. **Saddle-loop** bifurcation is said to occur as a consequence of variation in a parameter, the discount rate in our case, through the bifurcation point, $r = 0$, given an efficient wage bargain. On either side of the bifurcation point, the homoclinic orbit spits into two separate paths, the stable and unstable manifolds associated with the saddle point on the homoclinic orbit are distinct as illustrated in Figure 3. In this section, perturbation techniques (See Guckenheimer and Holmes [1986, Chapter 4].) are applied to establish that a saddle loop bifurcation occurs as well at a positive discount rates when the difference between the workers' share of match capital and that required for matching efficiency is small and negative.

Typically saddle-loop bifurcation signals the existence of a family of closed orbit associated with values of the parameters near the bifurcation point. (See Guckenheimer and Holmes [1986, Sections 6.1].) Furthermore, under regularity conditions satisfied by our model, the limit cycle identified is stable (unstable) if the sum of the eigen values of the saddle connected by the homoclinic orbit is negative (positive) at the bifurcation point. Although this test fails in the special case of efficient matching and a zero discount rate because the eigen values sum identically to zero everywhere, the test implies that the identified limit cycle is stable in case I and is unstable in case II when a saddle loop bifurcation occurs for a positive discount rate.

The generic existence of a limit cycle is established and a formula for finding discount rates that yield such cycles is derived using Melnikov's perturbation method. For the purpose of presenting results, define the system

$$(17) \qquad \dot{x} = F(x) + \varepsilon G(x) \quad \text{where } x = \begin{pmatrix} p \\ n \end{pmatrix},$$

$$F(x) = \begin{pmatrix} F_1(x) \\ F_2(x) \end{pmatrix} = \begin{pmatrix} \delta p + \int_0^p h(q)\,dq - f(n) \\ h(p)\,[1-n] - \delta n \end{pmatrix},$$

$$G(x) = \begin{pmatrix} G_1(x) \\ G_2(x) \end{pmatrix} = \begin{pmatrix} r - [\int_0^p h(q)\,dq - g(p)\,] \\ 0 \end{pmatrix},$$

and ε is a small positive constant. Note that $F(x)$ is the **vector field** associated with the Hamiltonian function H(x), defined in (14), and that $\varepsilon G(x)$ is a (small) perturbation of that field representing the distortion (of size ε) attributable to positive discounting and inefficiency. Of course, by design the system (17) is (10) when the discount rate and the deviation of the workers' share from the search elasticity of the matching function are both small. Finally, let the saddle point of interest, denoted as S, be H (L) when the parameters are such that case I (case II) of the Hamiltonian dynamic obtains and define

$$(18) \quad \Gamma(S) \equiv \{x \in \mathbf{R}^2 |\ H(x) \le H(S)\}$$

as the area surrounded by the graph of the homoclinic orbit containing S = H (L), the shaded area in Figure 2a (2b) in case I (in case II).

For any small perturbation of the system away from the Hamiltonian case, indexed by the value of ε , the saddle point of interest, H in case I and L in case II, continues to exist but is displaced slightly (See Guckenheimer Holmes, Lemma 4.5.1 and 4.52.). The following result pertains to these perturbed saddle points:

Proposition 3: For $\varepsilon > 0$ sufficiently small, a homoclinic orbit containing H (L) exist in case I (case II) if and only if $r = \hat{r}$ where

$$(19) \quad \hat{r} = \frac{\int_{\Gamma(S)} \left(\int_0^p [1 - \frac{\theta(q)}{\lambda(q)}]\, h(q)\, dq \right) dp dn}{\int_{\Gamma(S)} dp dn}, \quad S = \mathrm{H\ (L)}.$$

Furthermore, if the workers' share is less than the elasticity of the matching function with respect to search effort, i.e., $\theta(p) < \lambda(p)\ \forall\, p \in [0, \bar{p}]$, then solutions to (17) include a stable (unstable) limit cycle with finite period for any positive value of r sufficiently close to and on one side of $\hat{r}$ in case I (case II).

Proof: As the proposition is a direct implication of Theorems 4.5.3 and Theorems 6.1.1 in Guckenheimer and Holmes [1986] taken together, the task is to verify the conditions of both.

By virtue of equation (4.5.15) in the same source, the conclusion of Theorem 4.5.3 is the first assertion of this proposition because the **Melnikov function** is

$$M = \int_{\Gamma} [\frac{\partial G_1(x)}{\partial p} + \frac{\partial G_2(x)}{\partial n}]\,dx = \int_{\Gamma} [r + \left(\int_0^p [\frac{\theta(q)}{\lambda(q)} - 1]\,h(q)\,dq\right) dp dn$$

in our case where the second equality follows from the equations of (17) and equation (8). In particular, the graphs of the stable and unstable manifolds associated with the saddle point of interest combine with the saddle point to form a homoclinic loop if the parameters are such that M is zero and the two graphs have no common points otherwise by virtue of the referenced theorem.

Given the homoclinic orbit at $r = \hat{r}$, the fact just established, the second assertion is the conclusion to Theorem 6.1.1 if the sum of the characteristic roots of (17) evaluated at H (L) in case I (case II) are negative (positive). As $F(x)$ is a Hamiltonian vector field with eigen values that sum to zero, the sum of the eigen values of (17) is given by

$$trace\frac{d\dot{x}}{dx} = \varepsilon\left(\frac{\partial G_1(x)}{\partial p} + \frac{\partial G_2(x)}{\partial n}\right) = \varepsilon\left(r + \int_0^p [\frac{\theta(q)}{\lambda(q)} - 1]\, h(q)\, dq\right).$$

Because the value of p at H (L) in case I (case II), denoted as $p_H(p_L)$, maximizes (minimizes) the value of $\int_0^p [1 - \frac{\theta(q)}{\lambda(q)}]\, h(q)\, dq$ on Γ under the hypothesis, the maximum (minimum) exceeds (is less than) $\hat{r}$. Consequently,

$$\frac{\partial G_1(x)}{\partial p} + \frac{\partial G_2(x)}{\partial n} = \hat{r} - \int_0^{p_S} [1 - \frac{\theta(q)}{\lambda(q)}]\, h(q)\, dq < (>) = 0 \text{ as } S = H\ (L).$$

□

As in the previous section, multiple equilibria exist and are socially ranked by aggregate income with rank reflected in the value of match capital given initial employment by virtue of equation (12). Specifically, in case I, the segment of the stable trajectory converging monotonically to the high employment steady state, H, from the end point $(\hat{p},\hat{n})$ dominates all others equilibria provided that $n_0 < \hat{n}$. When the existence of a limit cycle is consistent with initial employment, convergence to the H is also feasible because all closed orbits lie to the right of the critical initial employment level $\hat{n}$. Still, any solution trajectory converging to the cycle has a higher income than that associated with the monotone solution converging to L since the average of the future values of $g(\tau)$, the second term on the right of (12b), is larger on the limit cycle than on the L stable manifold for the same initial employment level. In case II, the stable manifold converging to H exists for all initial conditions and yields a higher income than any alternative. Although the limit cycle is unstable in this case, M is stable. Any trajectory converging to M dominates a trajectory converging to L from the same initial employment level and a lower initial value of capital in the sense of aggregate income. Again in this case, policies that coordinate on the best equilibrium and policies that subsidize employment are warranted.

7. The Constant Elasticity Case

The derivation of empirically verifiable conditions under which multiple steady states actually exist, conditions that are necessary for a cycle, generally requires a more concrete specification of the model. In this section, a parametric version of the model is studied, that obtained under the assumptions of a constant elasticity matching function, a constant elasticity cost of search function, a constant elasticity of match productivity with respect to employment and, finally, a constant workers' share. Formally, let

$$(20a) \quad \mathrm{m}(s(1-n), v) = b(s(1-n))^{\lambda} v^{1-\lambda}$$

$$(20b) \quad \mathrm{c}(s) = cs^{\gamma}.$$

Then the equations of (6) and (7) imply a transition to employment function of the form

$$(21a) \quad \mathrm{h}(p,\theta) = \alpha p^{k}$$

where

$$(21b) \quad \alpha = \left(\frac{1-\theta}{ab}\right)^{\frac{(1-\lambda)\gamma}{\lambda(\gamma-1)}}\left(\frac{b\theta}{c\gamma}\right)^{\frac{1}{\gamma-1}}$$

$$(21c) \quad k = \frac{\lambda+\gamma(1-\lambda)}{\lambda(\gamma-1)}.$$

Finally, let

$$(22) \quad \mathrm{f}(n) = \beta n^{\ell}.$$

For this specification, the singular curve associated with (10) can be expressed as follows:

$$(23a) \quad \dot{n} = 0 \leftrightarrow n = \frac{\alpha p^{k}}{\delta+\alpha p^{k}}.$$

$$(23b) \quad \dot{p} = 0 \leftrightarrow n = \left(\frac{(r+\delta)(1+k)p + (\theta/\lambda)\alpha p^{1+k}}{\beta(1+k)}\right)^{\frac{1}{\ell}}.$$

Although the origin is always a steady state solution, the existence of other steady states depends on the values of the constants. Caballero and Lyons [1989] estimate the aggregate returns to scale of about 1.3 taking account of external effects which suggests a value for ℓ of about .3. For any $\ell < 1$ and $k > 0$, the value of n along the $\dot{p} = 0$ singular curve is a strictly convex function of p by virtue of (23b). That the value of n along the $\dot{n} = 0$ curve is a concave function of p when $k \leq 1$ and has a logistic shape (first convex and then concave for values of p greater than some critical value) when $k > 1$ is an implication of (23a). Hence, an elastic transition rate function ($k > 1$) is necessary for three steady states and there are at most three.

Provided that the productivity parameters α and β are large enough, a sufficient condition for the existence of three steady state solutions is that the $\dot{p} = 0$ singular curve relation has a relatively greater slope at the origin than does the $\dot{n} = 0$ singular curve. This sufficient condition is equivalent to the following lower joint bound on the elasticity of the transition to employment with respect to the value of a match and the elasticity of match output with respect to aggregate employment:

Proposition 4: For the constant elasticity model defined by equation (20)-(22), three steady states exist if $(k-1)\ell > 1$ and α and β are sufficiently large.

Proof: See the Appendix.

Given the point estimate $\ell = 0.3$, the sufficient condition is satisfied for all k greater that 4.33. Blanchard and Diamond [1989] obtain a point estimate of λ equal to 0.4. I argue elsewhere (Mortensen [1990]) that this estimate combined with values of γ implicit in the estimated magnitudes of the transition rates to employment elasticities with respect to predicted wage rates reported in Burdett <u>et al</u> [1984] are consistent with values of k between 4 to 12. In sum, the sufficient condition for three steady state solutions is empirically plausible.

For the parameterized model, the wage bargain is efficient when the worker's share of match surplus, θ, equals the elasticity of matching function with respect to aggregate search effort, the parameter λ . When in addition there is no discounting, the solution trajectories to (10) are the level curves of the Hamiltonian

$$(24) \qquad H(p,n) = \frac{\beta n^{1+\ell}}{1+\ell} + \frac{\alpha(1-n)p^{1+k}}{1+k} - \delta pn$$

by virtue of equations (14) and (15). Finally, since the partial derivatives of the Hamiltonian function are zero at steady states (see the proof to Proposition 2), the difference between its value at H and L is increasing in the productivity of both the matching technology and production technology as represented by the parameters α and β respectively. Hence, Hamiltonian dynamics case II is associated with relatively high values of these parameters and case I with relatively low values.

8. Computed Examples with Cycles

Proposition 3 can guarantee the existence of a limit cycle only for small positive discount rates and small negative deviations of workers' share from the elasticity of the matching function with respect to search effort. In this section, limit cycles are computed and displayed for plausible values of the discount rate and the other parameters of the constant elasticity version of the model. These examples extend the analytic results by showing that cycles exist even when the differences between workers' share and the search elasticity of the matching function are quite large.

The values of the elasticity of the transition to employment and the elasticity of the production function used in the computations are $k = 8$ and $\ell = .3$ respectively. Letting the unit time period equal one quarter, reasonable values for the discount and job separation rates are $r = .01$ and $\delta = .15$ respectively. For the purposes of the demonstration, α is set equal to unity. To obtain an examples of cases I and II, β is set at 0.173 and at 0.18 respectively. Finally, the value of the ratio θ/λ is varied to generate sub-cases of interest. The values of p and n at the stationary points M and H and the sums of the real parts of the eigen values at each stationary point, represented as $\sum Re(M)$ and $\sum Re(H)$, for different values of the the worker share to search elasticity ratio are reported in Table I for case I and in Table II for case II. The associated phase portraits computed for each specification of the parameters are illustrated in Figure 4 for case I and Figure 5 for case II.[14]

[14]The computer program PHASER written and developed by Kocak [1989] was used to compute and to display each phase portrait.

Table I: Stationary Point Values and Eigen Value Sums

$\alpha = 1$, $\beta = .173$, $\delta = .15$, $r = .01$, $k = 8$, $\ell = .3$

$\frac{\theta}{\lambda}$	M	$\sum Re(M)$	H	$\sum Re(H)$
0.75	.685,.245	-0.002	.835,.611	-0.049
0.80	.675,.224	0.001	.829,.598	-0.035
0.90	.678,.229	0.006	.794,.510	-0.009

Hopf bifurcation is a local form of structural instability that occurs at any point in the parameter space for which the eigen values at some stationary state are purely imaginary. Since the eigen values at M form a complex pair Hopf bifurcation occurs at M for a value of the workers' share to search elasticity ratio that equates the sum of the two real parts to zero, a value close to but less than 0.8 and but larger than 0.75 given the results reported in the third column of Table I. According to the Hopf bifurcation theorem (See Guckenheimer and Holmes [1986, pp 150-1]53].), closed orbits exist revolving around M in a one sided neighborhood of the bifurcation value of the ratio. However, the limit cycles that exists in this case can be either stable or unstable and the criterion for determining which is difficult to evaluate. For this reason the test associated with saddle-loop bifurcation is more useful.

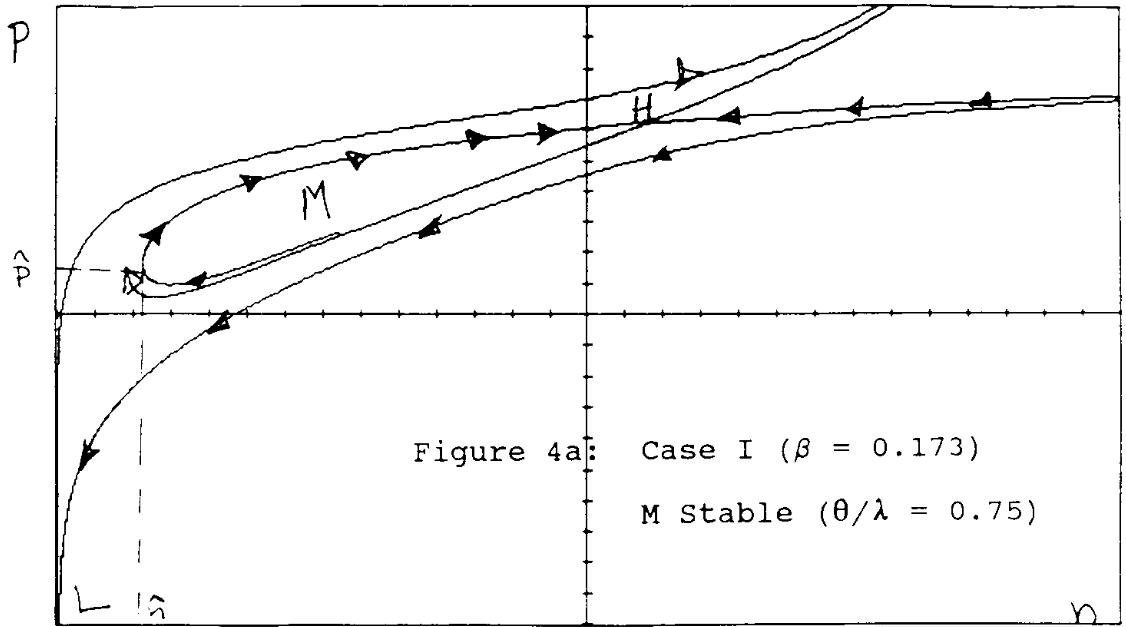

Figure 4a: Case I (β = 0.173)

M Stable (θ/λ = 0.75)

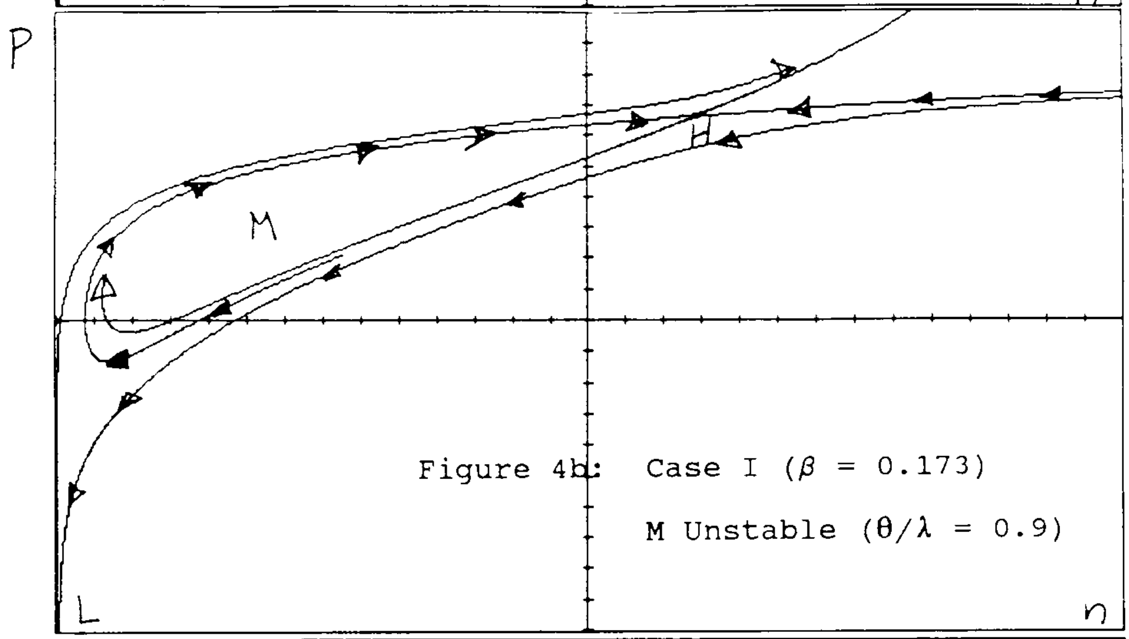

Figure 4b: Case I (β = 0.173)

M Unstable (θ/λ = 0.9)

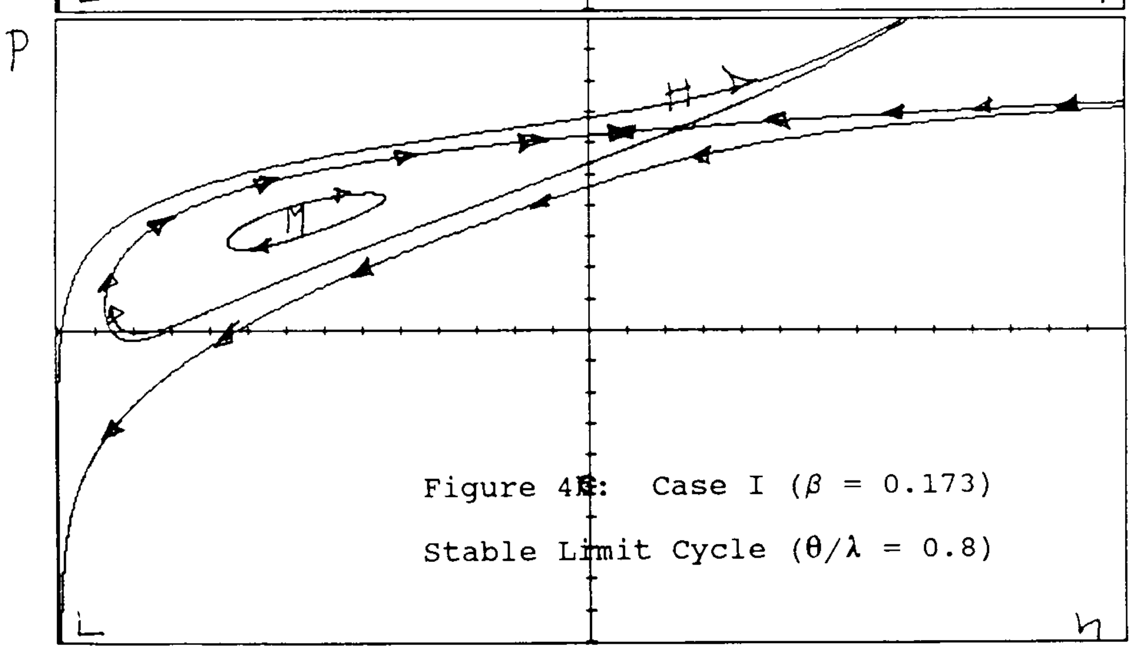

Figure 4c: Case I (β = 0.173)

Stable Limit Cycle (θ/λ = 0.8)

The computed phase portraits illustrated in Figure 4a and 4b confirm that saddle-loop bifurcation occurs at some value of the ratio θ/λ in the interval [.75,.9]. Specifically, a comparison of 4b and 4c reveals that the left-side stable and unstable manifolds associated with H reverse their relative positions as the ratio increase from 0.8 to .9. Since solution trajectories cannot cross, the two trajectories must coincide for some value of the ratio between the these two.

Finally, because the sum of H's eigen values for all θ/λ on the interval are negative given the numbers reported in the last column of Table I, the limit cycles like those illustrated in Figure 4c are stable by virtue of the saddle-loop bifurcation theorem. As an additional confirmation of the result, note that M is locally unstable when the ratio equals 0.8 (the real part of its eigen values are positive from Table I) but yet the left unstable manifold from H is converging toward M in Figure 4c. These facts are consistent only if at least one stable limit cycle surrounds M.

In case II, Hopf bifurcation occurs as well, at a value of the worker share to search elasticity a little larger than 0.65 according to the third column of Table II. The Figures 5a and 5b also imply saddle-loop bifurcation, at some value of θ/λ between 0.65 and 0.75. However, in this case L is the saddle point on the homoclinic orbit surrounding M. Since the sum of its eigen values is the discount rate by virtue of (13) and the fact that $h(0) = 0$, any period solution that arises in this case is unstable by virtue of the saddle-loop bifurcation theorem.

Table II: Stationary Point Values and Eigen Value Sums

$\alpha = 1,\ \beta = .18,\ \delta = .15,\ r = .01,\ k = 8,\ \ell = .3$

$\frac{\theta}{\lambda}$	M	$\sum Re(M)$	L	$\sum Re(L)$
0.55	.640,.159	-0.003	0,0	0.01
0.65	.642,.162	-0.0001	0,0	0.01
0.75	.671,.165	0.003	0,0	0.01

9. The Economics of the Cycle

I conclude this admittedly technical study with a more heuristic discussion of the time series properties of the employment cycle identified in the paper. Obviously, the source of the cumulative process driving fluctuations is the positive external economy that generates increasing returns. However, the existence of periodic nature of the employment movement also requires a propagation mechanism, supplied in this case by the delays implicit in the matching technology and the incentive structure that conditions investment in matching inputs. An important reason for interest in the model is the fact that the cyclic movements in unemployment and vacancies implied by this

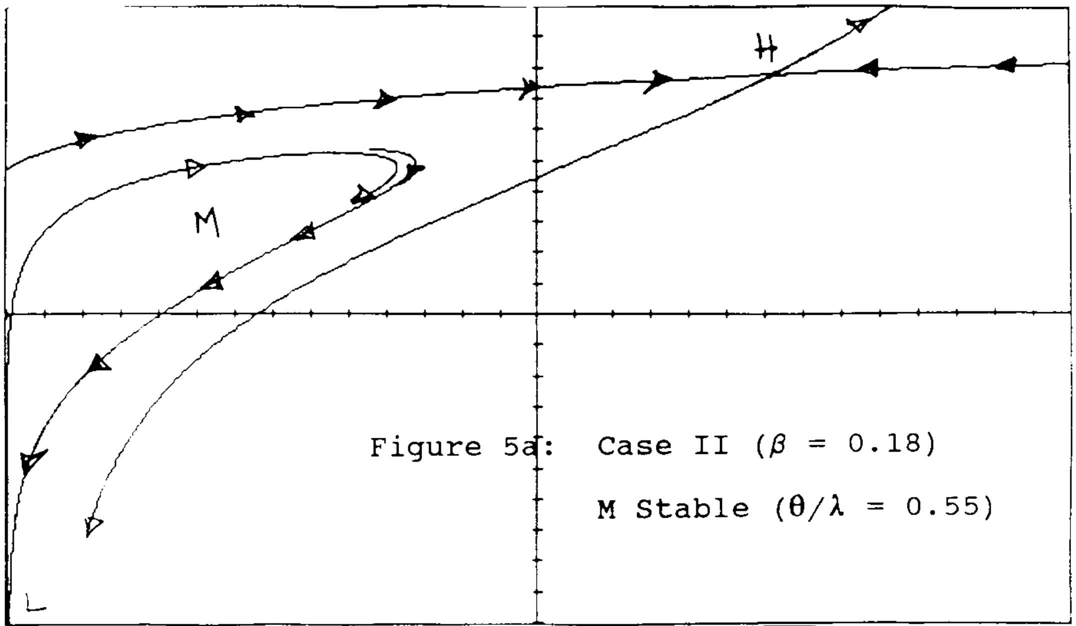

Figure 5a: Case II (β = 0.18)

M Stable (θ/λ = 0.55)

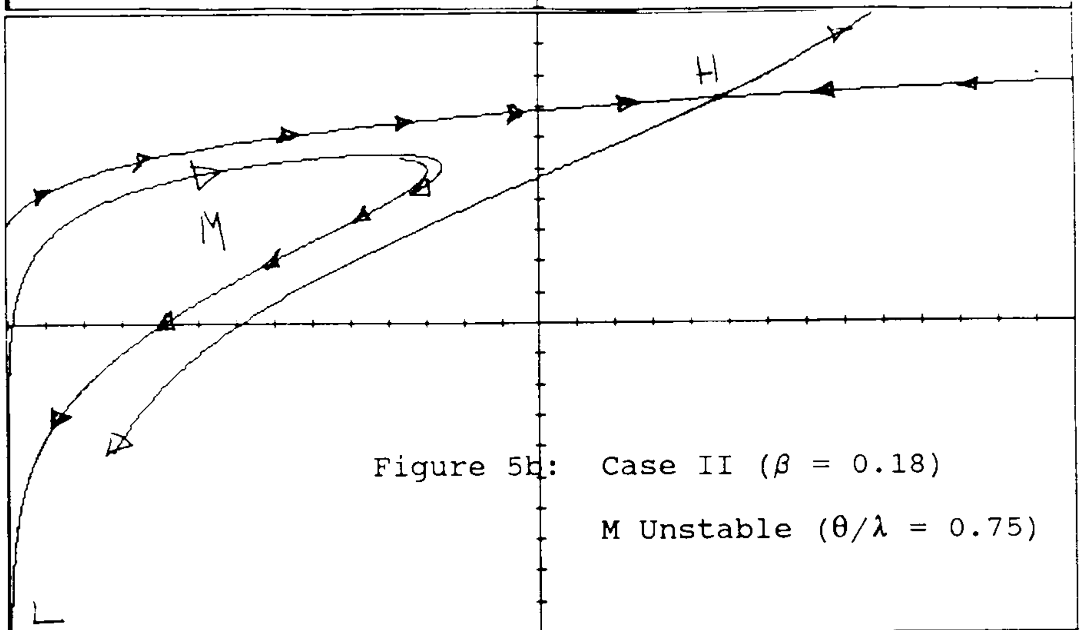

Figure 5b: Case II (β = 0.18)

M Unstable (θ/λ = 0.75)

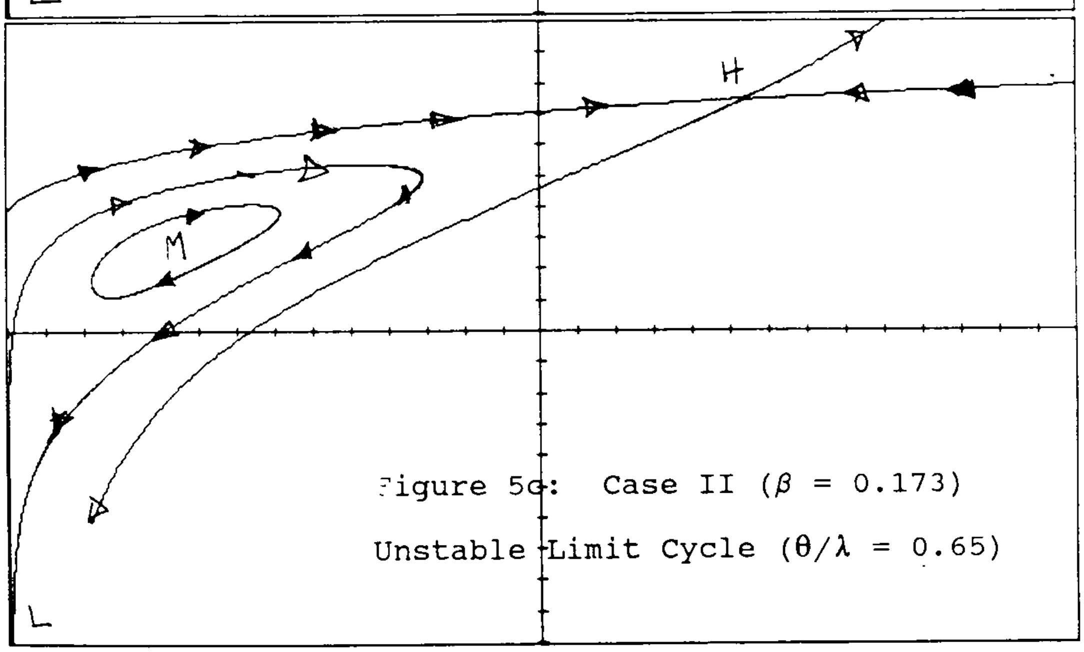

Figure 5c: Case II (β = 0.173)

Unstable Limit Cycle (θ/λ = 0.65) propagation mechanism are consistent with important stylized facts. In addition, the predicted timing of the turning points in important aggregate measures of economic activity--employment, income, and the stock price of operating firms--are also consistent with those observed.

The expectation of rising employment in the future stimulates investment in match formation now as a consequence of increasing returns. Given sufficiently optimistic (pessimistic) expectations, this cumulative process can result in smooth monotone convergence to the high (low) employment steady state. However, if agents are a little less (more) optimistic, the only perfect foresight employment time series turns down (up) before the high (low) employment steady state is reached. In the downswing, the cumulative process works in reverse; the expectation of falling employment in the near future induces less investment in matching now which ultimately confirms the expectation. Eventually the process reverses itself again to complete the cycle, even though search intensity and vacancies are low, simply because there are so many unemployed.

This cycle can be characterized in two different but theoretically equivalent ways. The first is an output/asset market description based on the fact that the stock price of an operating firm in the model is proportional to match capital, p , at least when the employers' share, $1-\theta$, is a constant. Since aggregate income, y, as defined in (12), is increasing in both p and n , income is maximum (minimum) at some point on the closed orbit representing the cycle, say that depicted in Figure 3c, between the points at which p and n attain their maximum (minimum) values. Hence, as in the real world, the model implies stock prices lead income and income leads employment at both the top and the bottom of the cycle.

With it focus on the fact that vacancies, search effort per worker, and the ratio of vacancies to aggregate search effort are all increasing function of p , the second characterization is a description of the cycle in labor market terms. Specifically, the model provides an explanation for counter-clockwise movement in vacancies and unemployment about the Beveridge Curve, the downward sloping relationship between vacancies and unemployment given "normal" effort consistent with no change in unemployment, i.e. that implicit in the steady state condition $\dot{u} = m(v,\bar{s}u) - \delta(1-u) = 0$ where $\bar{s}$ is defined as search effort averaged over the cycle. During the upswing, the ratio of vacancies to unemployment increases and search effort rises relative to its average so that the realized vacancy-unemployment pair move in a counter-clockwise direction above the Beveridge curve. Eventually, this process reverses as vacancies and search effort fall in response to falling expectations about future productivity. In the downturn, the labor market moves in the direction of higher unemployment relative to vacancies along a path below the Beveridge curve.

Although convergence to a endogenous deterministic limit cycle is a generic theoretical possibility in the model, I do not argue that observed fluctuations are actually generated by such a structure. The reason I find the model of interest arises from the fact that there are other equilibria associated with intermediate employment levels that share the cycle's qualitative time series properties even when their paths eventually converge to some steady state, e.g. the path starting at (p_0', n_0) in Figure 2. This fact suggests that these same properties would also be evident in a real business cycle version of the model in which the primary source of disturbance in economic activity is an exogenous technology shock process. Developing the model in this direction is a promising avenue for future research.

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Appendix

A.1 Derivation of (8)

Use the identity that defines $g(\cdot)$ in (8) and (6b) to show that

$$(A1) \quad g(p) = sc'(s) - c(s)$$

where the pair (s, x) is the unique solution to (6) restated here as

$$(A2) \quad p\theta(x)m(x,1) = c'(s)$$

$$(A3) \quad p[1-\theta(x)]m(x,1) = ax.$$

By adding these two equation to obtain

$$pm(x,1) - ax = c'(s)$$

and then differentiating the result, one finds

$$c''(s)\frac{\partial s}{\partial p} = m + [pm_1(x,1) - a]\frac{\partial x}{\partial p}.$$

Now, equation (A3) and the definition (9) of the search effort elasticity imply

$$1 + [1 - \lambda(x)]\,\frac{\partial x}{\partial p}\,\frac{p}{x} = \frac{\partial x}{\partial p}\,\frac{p}{x}$$

By combining these two facts one obtains

$$c''(s)\,\frac{\partial s}{\partial p} = m(x,1)\,\frac{\theta(x)}{\lambda(x)}$$

after rearranging terms. Finally, by differentiating (A1) with respect to p, and substituting appropriately from above and the second equation of (7), find

$$(A4) \qquad g'(p) = sc''(s)\,\frac{\partial s}{dp} = \frac{\theta(x(p))}{\lambda(x(p))}\mathrm{h}(p)\,.$$

Finally, (8) is equivalent given the fact that $\mathrm{h}(0) = 0$.

A.2 Equilibria are Bounded

The task is to prove the following: Every solution trajectory which leaves the rectangle $B = [0,\overline{\mathrm{p}}]\times[0,1]$ where $\overline{\mathrm{p}}$ is the unique solution to

$$(A5) \qquad (r+\delta)\overline{\mathrm{p}} + g(\overline{\mathrm{p}}) = f(1)$$

violates the transversality condition

$$(A6) \quad \lim_{t\to\infty} \{p(t)\, e^{-rt} dt\} = 0,$$

the requirement that the present value of a match in the indefinite future has no finite current present value. Because $g(p)$ is non-negative and $f(n)$ is increasing, equations (10a) and (A5) imply

$$\dot{p} > (r+\delta)p - f(1) \;\forall p > \bar{p}.$$

Consequently, for any $n_0 \in [0,1]$ and $p_0 \geq \bar{p}$, the associated solution to (10) has the property that

$$p(t) \geq q(t) \;\forall t \geq 0$$

where $q(t)$ is the unique solution to

$$\dot{q} = (r+\delta)q - f(1), \; q_0 = p_0 \; .$$

Since $q(t)$ does not satisfy (A6), $p(t)$ can't either. As $h(p) = g(p) = 0 \;\forall\, p \leq 0$ implies

$$\frac{\dot{p}}{p} = (r+\delta) - f(n)/p \geq r+\delta \;\forall\, p < 0,$$

the assertion also holds for initially negative match values.

A.3 Proof of Proposition 4

Given α and β sufficiently large, $\ell \leq 1$, and $k > 1$, a sufficient condition for multiple steady states is that the $\dot{n} = 0$ singular curve intersect the $\dot{p} = 0$ curve from below at the origin when both are represented as functions of p . In the conclusion, $(k-1)\ell > 1$ is asserted to be this sufficient condition for the parameterization studied. The complication in the proof arises because the first and some higher order derivatives of both expressions are zero at the origin. Hence, the derivation requires showing that the limit of the ratio of the slope of $\dot{n} = 0$ to the slope of $\dot{p} = 0$ is equal to zero if and only if the inequality holds.

The equations of (23) imply that the slopes of the two curves can be expressed as follows:

$$(A6a) \qquad \frac{dn}{dp}\Big|_{\dot{n}=0} = \left(\frac{\alpha\delta k p^{k-2}}{(\delta+\alpha p^k)^2}\right)\times\left(\frac{k-1}{2k} - \frac{\alpha p^k}{\delta+\alpha p^k}\right)$$

$$(A6b) \qquad \frac{dn}{dp}\Big|_{\dot{p}=0} = \frac{1}{\ell}(r+\delta+\alpha p^k)\left(\frac{(r+\delta)p + \frac{\alpha p^{k+1}}{1+k}}{\beta}\right)^{\frac{1}{\ell}-1}.$$

As expressions of the form $a + bp^k$ converge to the constant a as p tends to zero given $k > 1$, the limit of the ratio of the first expression to the second is equal to the limit of the following term:

$$\frac{\frac{\alpha\delta k(k-1)p^{k-2}}{\delta^2}}{\frac{r+\delta}{\ell}((r+\delta)p)^{\frac{1}{\ell}-1}} = \left(\frac{\alpha\ell k(k-1)}{\delta(r+\delta)^{\frac{1}{\ell}}}\right)p^{k-1-\frac{1}{\ell}}.$$

The assertion follows.

$\square$