Discussion Paper No. 935

ABSORBENT STABLE SETS*

by

Itzhak Gilboa**

and

Dov Samet***

April 1991

*We wish to thank John Geanakoplos, Ehud Kalai, Michael Maschler, David Schmeidler and Yair Tauman for comments and references.

**Department of Managerial Economics and Decision Sciences, J. L. Kellogg Graduate School of Management, Northwestern University, Evanston, IL 60208.

***Recanati School of Business, Tel Aviv University, Tel Aviv 69978, Israel.

Abstract

This paper suggests a general framework to deal with learning, dynamics and evolution in games and economic environments. In this general set-up we define a (set-valued) solution concept and prove some properties, including existence.

We then discuss more specific dynamical processes and show that the general solution concept provides quite intuitive results for various contexts.

1. Introduction

Recent years have witnessed a growing interest in dynamic processes, emphasizing learning and/or evolution, in game and economic theory. Various models describe the interaction of agents, who range on the rationality gamut from genes to Bayesian-rational decision makers, which can be as few as two or as numerous as the continuum. (A very partial list of references include Boylan (1990), Canning (1989), Crawford (1990), Fudenberg-Kreps (1988), Fudenberg-Levine (1990), Gilboa-Matsui (1989), Gilboa-Schmeidler (1989), Kalai-Lehrer (1990, 1991), Kandori-Mailath-Rob (1991), Matsui (1989, 1990), and Swinkels (1990).)

In many of these studies, the specification of the dynamics per se is not clearly separated from the "solution concept" applied to it. The latter seems to be tailored for the specific dynamical model, and thus tends to be ad hoc.

With some diffidence, we would like to propose the opinion (to which we do not fully subscribe) that one cannot hope to find a universally applicable dynamical process for learning or evolution. Various set-ups call for different, sometimes intrinsically incompatible, dynamics. The quest for universal learning/evolutionary laws may be as futile as the attempt to specify a single utility function for a "typical" decision maker. Our goal in this paper is, therefore, primarily to provide a general framework which can accommodate a wide range of phenomena of interest, and define a "solution concept" that will be applicable to all these phenomena. When analyzing a specific process of learning and/or evolution process one should use the particular dynamics as a parameter to this solution concept, very much like utility functions and the game structure are used to apply Nash equilibrium.

We start by presenting the general framework, which models a system as a pair (X,R) where X is interpreted as a state space, and R is a binary relation to it, to be thought of as the "may evolve into" relation. We will assume that X is a compact metric space, and define a set-valued solution concept--called "asborbent stable set" (ASS)--which is, roughly, a minimal compact set, small neighborhoods of which do not lead (according to R) away from it.

At first sight, our framework may seem extremely restrictive: we only consider dynamic systems for which the possible evolutions depend solely on the current state. The classical reply is, of course, that one should define one's "states" to be elaborate enough in order to contain all relevant information. Thus, for instance, an economic system's "state" may specify not only prices and quantities, but also beliefs and expectations to the extent that these are of relevance. However, some nontrival restriction may be hidden in the assumption that X is compact in some interesting metrizable topology.

In this general framework, we show that every system has at least one ASS, that ASS's are closed and disjoint, and that their definition presupposes some transitive and topological closure of the relation R. We also compare ASS's of different relations between which inclusion holds and prove some other results.

Thus we provide a uniform framework and general results for dynamic models. Furthermore, treating the relation R as a parameter of the solution concept allows us to formally compare different dynamics in terms of the

ASS's they give rise to.

We continue by studying some more specific models, such as best-response dynamics in games, assuming myopic (but otherwise quite rational) economic agents, or better-response dynamics, which is more appropriate for the analysis of gene behavior, and so forth.

The rest of the paper is organized as follows. Section 2 presents the general model, defines ASS and proves some general results. It also includes a generalized definition of CSS (Gilboa-Matsui (1989)) and a comparison of the two set-valued "solution concepts." In Section 3 we apply the concept of ASS to games played with large, randomly matched populations and myopic best response. We compare our results to Gilboa-Matsui's (1989) cyclically stable sets. Section 4 analyzes games with similar assumptions, but replacing their "best response" by a "better response." Thus we can compare the behavior of (more rational) human agents to that of (less rational) genes by applying the same solution concept--ASS--to slightly different dynamics. In Section 5 we briefly discuss some learning and/or evolutionary models in game and economic theory, and test the scope of our general model in terms of those that can be embedded in it. Finally, Section 6 contains some concluding remarks.

2. The General Model

A (dynamical) **system** is a pair (X,R) where X is a compact metric space of states and $R \subseteq X^2$ is a binary relation on X. We write xRy when $(x,y) \in R$, and this should be interpreted as "x may evolve into y" or "y is accessible from x."

We define $\tilde{R}$: $X \twoheadrightarrow X$, the associated correspondence, by $\tilde{R}(x) =$

$\{y \in X | xRy\}$. A correspondence $\Phi: X \twoheadrightarrow X$ is extended to 2^X by $\Phi(A) = \cup_{x \in A} \Phi(x)$ for $A \subseteq X$.

Topological properties relating to R should be read as referring to the product topology on X^2. Thus, for instance, R is closed iff $\tilde{R}$ is upper hemi-continuous. On the other hand, topological set-properties ascribed to $\tilde{R}$ should be understood pointwise. Thus, $\tilde{R}$ is closed if and only if $\tilde{R}(x)$ is a closed set (in X) for all $x \in X$.

A set $A \subseteq X$ is <u>R-self-absorbent</u>, or simply <u>absorbent</u> if $x \in A$ and $y \notin A$ imply not (xRy)--that is, if $\tilde{R}(A) \subseteq A$.

We note the following.

<u>Observation 2.1</u>: The class of absorbent sets is closed under arbitrary unions and intersections.

Notice that this class is also nonempty as $\emptyset$ and X are trivially absorbent.

A set $A \subseteq X$ is (R-)<u>stable</u> if: (i) A is nonempty; and (ii) there are $\{F_n\}_{n\geq 1}$, with $F_n \subseteq X$ closed and absorbent, such that $A \subseteq F_n^0$ for all $n \geq 1$ and $A = \cap_{n\geq 1} F_n$ (where B^0 denotes the interior of B).

<u>Observation 2.2</u>: In the definition of stability above, one may restrict one's attention to decreasing sequences $\{F_n\}$ (i.e., $F_{n+1} \subseteq F_n$ for all $n \geq 1$) without loss of generality.

<u>Proof</u>: For an arbitrary $\{F_n\}_{n\geq 1}$, define $\{F'_n\}_{n\geq 1}$ by $F'_n = \cap_{k=1}^{k} F_n$. These sets are obviously closed and absorbent by (2.1). Moreover, $A \subseteq F_k^0$ for $1 \leq k \leq n$. Hence, $A \subseteq \cap_{k=1}^{n} (F_k^0) = (F'_n)^0$. Finally, $\cap F'_n = \cap F_n = A$. //

Note that every stable set is compact and, by (2.1), also absorbent.

This definition of stability basically says that small perturbations outside of a stable set A cannot lead to "remote" points. However, stable sets may be very large; indeed, X itself is always stable. We therefore define our "solution concept" as follows: a set $A \subseteq X$ is (R-)Absorbent Stable Set (R-ASS) if it is a minimal stable set (with respect to set inclusion). English permitting, we will use the term "ASS" as an adjective and as a noun interchangeably.

Theorem 2.3:[1] For every system (X,R) there is at least one R-ASS.

Proof: Let $\mathcal{B}$ be the set of all stable sets. As noted above, $X \in \mathcal{B}$ so that $\mathcal{B}$ is nonempty. $\mathcal{B}$ is partially ordered by set inclusion, and in order to apply Zorn's lemma all we have to do is convince ourselves (and possibly also the reader) that for every decreasing chain there is a minimal element.

Let Δ be a $\geq$-linearly ordered set of indices, and let $\{A_\alpha\}_{\alpha\in\Delta}$ be such a chain, i.e., $A_\alpha \in \mathcal{B}$ and $\alpha \geq \beta$ implies $A_\alpha \subseteq A_\beta$.

Define $A = \cap_{\alpha\in\Delta} A_\alpha$. Note that each A_α is compact; hence, A is compact and nonempty. We now wish to show that there are indices $\{\alpha_n\}_{n\geq 1}$, $\alpha_{n+1} \geq \alpha_n$, such that $A = \cap_n A_{\alpha_n}$.

Let $\{B_n\}_{n\geq 1}$ be a decreasing sequence of open sets in X such that $A = \cap_{n\geq 1} B_n$. (For instance, let $B_n = \cup_{x\in A} N_{1/n}(x)$ where $N_\varepsilon(x)$ =

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[1] Both the definition and existence proof for ASS's were inspired, to some extent, by Kalai-Samet (1984) and Gilboa-Matsui (1991). We later found out that our general framework is similar to those of Maschler-Peleg (1976) and Kalai-Schmeidler (1977).

$\{y \in X | d(x,y) < \varepsilon\}$ and $d(\bullet,\bullet)$ is a metric for X.) Fix $n \geq 1$. For all $\alpha \in \Delta$, $A_\alpha \cap (B_n)^c$ is compact. If it is also nonempty for all $\alpha \in \Delta$, then $\cap_\alpha [A_\alpha \cap (B_n)^c] = A \cap (B_n)^c \neq \emptyset$, which is a contradiction. Hence, for some $\alpha_n \in \Delta$, $A_{\alpha_n} \subseteq B_n$. Picking such α_n for each n we obtain $A \subseteq \cap_n A_{\alpha_n} \subseteq \cap_n B_n = A$.

A_{α_n} being stable, there is a sequence $\{F_{n,k}\}_{k\geq 1}$ such that $F_{n,k}$ are closed, absorbent and satisfy $A_{\alpha_n} \subseteq F^0_{n,k}$ and $A_{\alpha_n} = \cap_k F_{n,k}$. Obviously, $A \subseteq A_{\alpha_n} \subseteq F^0_{n,k}$ and $A = \cap_n A_{\alpha_n} = \cap_{n,k} F_{n,k}$. Hence, A is stable, and by Zorn's lemma, there exists an ASS. //

Before we proceed to analyze and apply the concept of ASS, we would like to compare it to cyclically stable sets (CSS) defined in Gilboa-Matsui (1989) as a solution concept for finite games. Gilboa-Matsui defined a specific accessibility relation, a version of which we will introduce in the sequel. At this point, however, we may suggest the following generalization of their definition of CSS: given a system (X,R), a set $A \subseteq X$ is an <u>(R-)CSS</u> if: (i) A is nonempty; (ii) for $x,y \in A$, xRy; and (iii) for $x \in A$, $y \in A^c$, not (xRy).

In other words, A is a CSS iff A is nonempty and for all $x \in A$, $\tilde{R}(x) = A$.

Recalling that $\tilde{R}$ is said to be closed (nonempty) iff $\tilde{R}(x)$ is closed (nonempty) for all $x \in X$, we have:

<u>Theorem 2.4 (Gilboa-Matsui)</u>:[2] Let (X,R) be a system and assume that R is

[2] It turns out that both the theorem and proof we provide are very similar to Theorem 3 in Kalai-Schmeidler (1977).

transitive and $\tilde{R}$ is closed and nonempty. Then (X,R) has at least one CSS.

Since Gilboa-Matsui formulated the theorem in a much more specific framework, we provide here the general proof. It is, however, a straightforward adaption of the existence result in their paper.

Proof: Let $\mathcal{B}$ be $\{\tilde{R}(x) | x \in X\}$. We first show, using Zorn's lemma, that $\mathcal{B}$ has a set-inclusion minimal element. Let $\{x_\alpha\}_{\alpha\in\Delta} \subseteq X$ be such that $\{\tilde{R}(x_\alpha)\}_{\alpha\in\Delta}$ is a decreasing sequence, i.e., $R(x_\alpha) \subseteq R(x_\beta)$ for $\alpha \geq \beta$.

Since $\tilde{R}$ is closed and nonempty, $\tilde{R}(x_\alpha)$ is nonempty and compact for all $\alpha \in \Delta$. Hence, $A \equiv \cap_\alpha \tilde{R}(x_\alpha)$ is nonempty and compact. Choose $x \in A$. If xRy, then, by transitivity, $x_\alpha Ry$ for all $\alpha \in \Delta$. Hence, $\tilde{R}(x) \subseteq A$, which means that $\{\tilde{R}(x_\alpha)\}_\alpha$ has an lower $\supseteq$-bound in $\mathcal{B}$. This implies the existence of $A_0 = \tilde{R}(x_0)$ which is minimal with respect to inclusion.

We wish to show that for all $y \in A_0$, $\tilde{R}(y) = A_0$. The inclusion $\tilde{R}(y) \subseteq A_0$ follows from transitivity, while the converse follows from minimality. This completes the proof. //

Comparing the two notions of "stable sets" we note the following:

Remark 2.5: In general, CSS's may fail to exist even if R satisfies any two of the conditions of Theorem 2.4 (nonemptiness and closedness of $\tilde{R}$, and transitivity of R). The example $R = \emptyset$ clearly shows that nonemptiness cannot be dropped from the theorem's provisions. Next consider $X = [0,1]$ and $R = \{(x,y) | 0 \leq x \leq y < 1\} \cup \{1\} \times [0,1]$. R is transitive and $\tilde{R}$ is nonempty, but it is not closed and there is no CSS. Finally, if $\tilde{R}$ is nonempty and closed but R fails to be transitive, it is not difficult to see that CSS's may fail to exist. (For instance, take $X = \{0,1,2\}$ and $R = \{(0,1),(1,2),(2,0)\}$.)

By contrast, ASS's will always exist.

Given a relation R, R-ASS's, as opposed to R-CSS's, can be thought of as taking the transitive and topological closure of R.

Formally, given a relation R, one may consider the class of relations:

$$\mathcal{E} = \{R' \subseteq X^2 \mid R \subseteq R',\ \tilde{R}' \text{ is closed, } R' \text{ is transitive}\}.$$

Since $X^2 \in \mathcal{E}$, $\mathcal{E}$ is nonempty. Thus, we may define

$$\bar{R} = \cap_{R' \in \mathcal{E}} R'.$$

It is easy to check that $\bar{R} \in \mathcal{E}$. Hence, it is its (unique) minimal element and may be dubbed the <u>semi-closure</u> of R. We note that:

<u>Theorem 2.6</u>: Let (X,R) be a system. A set $A \subseteq X$ is R-ASS iff it is $\bar{R}$-ASS.

<u>Proof</u>: We first show that a closed set $F \subseteq X$ is R-self-absorbent iff it is $\bar{R}$-self-absorbent. The "if" part is trivial since $R \subseteq \bar{R}$. As for the "only if" part, assume F is R-self-absorbent, and suppose, contrary to our claim, that $x\bar{R}y$ for $x \in F$ and $y \in F^c$. Define

$$R' = \bar{R} \setminus (F \times F^c).$$

We claim that $R' \in \mathcal{E}$ where $\mathcal{E}$ is defined as above. Indeed, $R \subseteq R'$ since F is R-self-absorbent, that is, $R \cap (F \times F^c) = \emptyset$. To see that $\tilde{R}'$ is closed, notice that for $z \in F^c$, $\tilde{R}'(z) = \tilde{\bar{R}}(z)$ and for $z \in F$, $\tilde{R}'(z) = \tilde{\bar{R}}(z) \cap F$. Hence, $\tilde{R}'(z)$ is closed for all $z \in X$. Finally, to show that R' is transitive, let $zR'w$ and $wR't$. If $z \in F$, $w \in F$ and $t \in F$ follow. Hence, $z\bar{R}w$ and $w\bar{R}t$, whence $z\bar{R}t$ and $zR't$. If, however, $z \in F^c$, $zR'w$ and $wR't$ imply $z\bar{R}w$ and $w\bar{R}t$ (by $R' \subseteq \bar{R}$). Transitivity of $\bar{R}$ implies $z\bar{R}t$, which yields $zR't$ since $z \in F^c$. Thus, we obtain $R' \in \mathcal{E}$, but $(x,y) \in \bar{R} \setminus R'$, contrary to the minimality of $\bar{R}$.

Since every closed set is R-self-absorbent iff it is $\bar{R}$-self-absorbent, every set is R-stable iff it is $\bar{R}$-stable and, perforce, every set is R-ASS iff it is $\bar{R}$-ASS. //

Remark 2.7: This result shows that the notion of ASS "presupposes" the semi-transitive closure of a given relation. Given a system (X,R), with $\tilde{\bar{R}}$ being nonempty, one may conjecture that the R-ASS's ($\bar{R}$-ASS's) would coincide with the $\bar{R}$-CSS's (guaranteed to exist by (2.4)).

However, it should be pointed out that ASS's also have some "robustness" inherent in their definition. For instance, consider $X = [0,1]$ (with the natural topology) and

$$R = \{(0,0)\} \cup \{(x,y) | 0 < x \leq y\}.$$

R satisfies all provisions of (2.4), and X indeed has two CSS's: $\{0\}$ and $\{1\}$. However, only the latter is an ASS.

We find that, at least for the applications we have in mind, the analysis is greatly simplified by defining the stable set in a way that would guarantee not only transitivity and closedness, but also robustness with respect to small perturbations. For instance, one may define a relation R to capture a dynamical process only where it is intuitively well defined, and let the ASS take care of the rest.

Remark 2.8: We have seen that even if R satisfies the conditions of (2.4), some CSS's may fail to be ASS's. It is only natural to ask whether all ASS's are CSS's, and it is quite simple to see that the answer is negative: let $X = [0,1]$ and $R = \{(0,0),(1,1)\} \cup \{(x,y)\} | x \in (0,1),\ y \in [0,1]\}$.

$\tilde{R}$ is nonempty and closed and R is transitive. The CSS's are, as in the previous example, $\{0\}$ and $\{1\}$. However, the unique ASS is $[0,1] = X$.

In both these examples, the ASS contained a CSS. Indeed, we have:

Proposition 2.9: Let (X,R) be a system where $\tilde{R}$ is nonempty and closed and R is transitive. Then every ASS contains a CSS.

This will be a corollary of:

Proposition 2.10: Let (X,R) be a system where $\tilde{R}$ is nonempty and closed and R is transitive. Let $A \subseteq X$ be nonempty, closed and absorbent. Then A contains a CSS.

Proof: Given the set A, define the subsystem (A,R) by restricting R to A and endowing A with the relative topology. A is compact; hence, it will have a CSS. However, since A is also absorbent, this CSS will also be a CSS of X. //

Another conclusion is

Corollary 2.11: Let (X,R) be a system with $\tilde{R}$ nonempty and closed and R transitive. Then for every $x \in X$ there is a CSS $C \subseteq X$ such that $C \subseteq \tilde{R}(x)$.

That is, every point x and X is "attracted" by at least one CSS.

In the examples we have seen above, the CSS's were strictly included in the respective ASS's since the notion of CSS has no "robustness" built into its definition. One may be tempted to conjecture that this fact is due to some lack of continuity of R. Indeed, R was not closed (alternatively, $\tilde{R}$ was not upper hemi-continuous) in these examples. It is therefore natural to consider the "topological closure" of R in a stronger sense, i.e., to require that R itself be closed, rather than merely $\tilde{R}$.

Formally, given an arbitrary $R \subseteq X^2$, define the full closure of R, denoted $\bar{\bar{R}}$, to be the minimal transitive and closed relation containing R. (As above, let $\bar{\bar{R}} = \cap_{R' \in \mathcal{E}} R'$ where $\mathcal{E} = \{R' \subseteq X^2 ; R \subseteq R',\ R' \text{ is closed and transitive}\}$. $\mathcal{E}$ is nonempty since $X^2 \in \mathcal{E}$ and $\bar{\bar{R}}$ is easily verified to belong to $\mathcal{E}$.)

Given R with $\tilde{R}$ nonempty, it seems natural to compare $\bar{\bar{R}}$-CSS's to R-ASS's. However, we note that:

Remark 2.12: Given R which is closed and transitive with $\tilde{R}$ nonempty, an R-ASS may still strictly include an R-CSS.

Proof: Consider $X = [0,1]$ and

$$R = \{(x,x) | x \in [0,1]\} \cup \{(1/(2k+2),\ 1/(2k-1) \mid k \geq 1\}.$$

$\tilde{R}$ is nonempty, and R is closed and transitive. Yet {0} is a CSS which is not an ASS. //

If one's research goal is to obtain equivalence of ASS and CSS, one path to follow is to slightly modify the definition of a "stable set" and redefine ASS accordingly. The interested reader is referred to the Appendix. It is far from being clear which definition is, in general, more appropriate. Fortunately, for our applications the distinction is immaterial, and we therefore stick to the original, simpler definition.

One more point, however, has to be settled before we relinquish the concept of CSS in favor of ASS: CSS's have the intuitive property that they do not intersect. Indeed, if $x \in C_1 \cap C_2$ where both C_1 and C_2 are CSS's, then $C_1 = C_2 = \tilde{R}(x)$. It is comforting to know that the same is true of ASS's. We first prove:

Proposition 2.13: If A^1 and A^2 are stable and $A^1 \cap A^2$ is nonempty, it is also stable.

Proof: Given sequences $\{F_n^1\}_{n\geq1}$ and $\{F_n^2\}_{n\geq1}$ for A^1 and A^2, respectively, define $F_n = F_n^1 \cap F_n^2$. F_n is closed and absorbent. Moreover, $A_1 \cap A_2 \subseteq (F_n^1)^0 \cap (F_n^2)^0 = F_n^0$. Finally, $A_1 \cap A_2 = \cap_{n\geq1} F_n$. //

Now we derive:

<u>Corollary 2.14</u>: If A^1 and A^2 are distinct ASS's, they are disjoint.

<u>Proof</u>: Otherwise, at least one of them is not minimal. //

It is interesting to note that although the definition of ASS has some robustness flavor, it does not "presuppose" the full closure of R. More formally, a version of Theorem 2.6 with $\bar{\bar{R}}$ instead of $\tilde{R}$ cannot be proven:

<u>Remark 2.15</u>: For a system (X,R), the set of R-ASS's and that of $\bar{\bar{R}}$-ASS's do not always coincide.

<u>Proof</u>: Consider the following example:

$$X = [0,1],\ R = \{(0,0)\} \cup \{(x,y) | 1/(n+1) < x \leq y \leq 1/n \text{ for some } n \geq 1\}.$$

R is transitive, $\tilde{R}$ is nonempty and closed, but R is not closed. Taking $F_n = [0, 1/n]$, which are closed and absorbent, we can prove that {0} is a R-ASS. However, $\bar{\bar{R}}$ is simply ≤ and the unique $\bar{\bar{R}}$-ASS is {1}. //

It will prove useful to note that:

<u>Proposition 2.16</u>: Let (X,R) be a system and let $A \subseteq X$ be R-stable. Then A contains at least one R-ASS.

<u>Proof</u>: Almost identical to that of Theorem 2.3, with the obvious

modification that only subsets of A, which are R-stable, are considered. //

One of the advantatages of a general solution concept, parameterized by the relation R, is the ability to compare formally ASS's of different relations. In particular, it is interesting to ask what happens if one relation allows for more points to be accessed from every given point.

Theorem 2.17: Let X be a compact metric space and let $R_1 \subseteq R_2 \subseteq X^2$. Then every R_2-ASS contains an R_1-ASS.

Proof: It is immediate that every R_2-self absorbent set is also R_1-self-absorbent. This implies that every R_2-stable set is also R_1-stable.

Considering an R_2-ASS, it is R_2-stable--hence, R_1-stable--and by 2.16 it contains an R_1-ASS. //

Notice that the converse if false, i.e., under the conditions of the theorem it does not hold that every R_1-ASS is also contained in an R_2-ASS. (Consider, for instance, $X = \{0,1\}$ with $R_1 = \emptyset$ and $R_2 = \{(0,1)\}$, where $\{0\}$ is an R_1-ASS which is not contained in the unique R_2-ASS $\{1\}$.) Yet, Theorem 2.17 shows that the cardinality of the set of R_1-ASS's is at least as large as that of the R_2-ASS's, and that each R_2-ASS is at least as large as (at least) one R_1-ASS. Roughly, if $R_1 \subseteq R_2$, the R_2-ASS's are larger and fewer than the R_1-ASS's.

We conclude this section with a comment regarding connectedness of ASS's. As we have seen, the definition and existence of ASS's does not depend on any property of R. Indeed, R need not even be transitive.

However, with the interpretation we bear in mind, i.e., understanding R as representing the "may follow" relation, transitivity seems quite natural. Furthermore, for applications to dynamical systems in continuous time, it seems natural that $\tilde{R}$ be connected, as the "way" from a point x to a point y "should" be along some continuous path.

Since many applications will involve connected relations, the following is of interest.

Theorem 2.18: Let (X,R) be a system in which $\tilde{R}$ is connected. Then every ASS is connected.

Proof: Let A be an ASS and assume it is not connected. This implies $A = A^1 \cup A^2$ where $A^1 \cap A^2 = \emptyset$ and A^i (i = 1,2) is nonempty and closed.

A being stable, there is a sequence $\{F_n\}$ of closed absorbent sets such that $A \subsetneq F_n^0$ and $A = \bigcap_{n\geq 1} F_n$. By (2.2) we also assume without loss of generality that $F_{n+1} \subseteq F_n$, for all $n \geq 1$.

Choose two disjoint open sets O^1 and O^2 such that $A^i \subsetneq O^i$ (i = 1,2), and define $F_n^i = F_n \cap O^i$ for i = 1,2. For large enough n, $F_n = F_n^1 \cup F_n^2$. (Otherwise, by compactness, there exists $x \in (\bigcap_n F_n) \cap (O^1 \cup O^2)^c$, which is a contradiction.) This implies that F_n^i (i = 1,2) is closed. Furthermore, since F_n is absorbent and $\tilde{R}$ is connected, F_n^i has to be absorbent, which means that A^i (i = 1,2) is stable, contrary to the minimality of A. //

3. Best Response Dynamics in Games

In this section we discuss finite games in strategic (normal) form. The interpretation we bear in mind is a "large population" one: to each player in the game there is a corresponding large population (a type). The game is played repeatedly where the role of each player is taken by an individual drawn at random from the corresponding type. A mixed strategy of a player is interpreted as the distribution of the corresponding type among the pure strategies. (For a more explicit model of random matching with large populations, see Gilboa-Matsui (1990).)

Formally, a game G is a triple:

$$G = \langle I, (S_i)_{i\in I}, (\pi_i)_{i\in I}\rangle$$

where $I = \{1,2,\ldots,n\}$ is the set of types of individuals, $S_i (i \in I)$ is the finite and nonempty set of strategies for each individual of type i and $\pi_i: \times_{j\neq i} S_j \times S_i \to \mathbb{R}$ is a payoff function for each individual of type i, where a typical value $\pi_i(s_1,\ldots,s_n;s_i)$ is the payoff for an individual of type i when he/she takes s_i, while others take $(s_1,\ldots,s_n)$. This somewhat awkward definition of the domain will simplify notations in the sequel. Let $F_i = \Delta(S_i)$ be the set of probability distributions over S_i, i.e.,

$$F_i = \Delta(S_i) = \{f_i: S_i \to \mathbb{R} \mid \Sigma_{s_i\in S_i} f_i(s_i) = 1, \text{ and } f_i(s_i) \geq 0 \text{ for all } s_i \in S_i\}.$$

We may call $\mathbf{F} = \times_{i\in I} \Delta(S_i)$ the class of strategy profiles and $f = (f_1,\ldots,f_n) \in \mathbf{F}$ a strategy profile. F is considered as a $(\Sigma_{i\in I}|S_i| - n)$-dimensional space on which Euclidean norm, $\|\bullet\|$, and linear operations are defined. Given a strategy profile $f \in \mathbf{F}$, the expected payoff for an individual of type i $(i \in I)$ if he/she takes a strategy $r_i \in S_i$ is:

$$\Pi_i(f;r_i) = \Sigma_{s\in\times_{j\in I}S_j} \Pi_{j\neq i} f_j(s_j)\pi_i(s;r_i).$$

Let $BR_i(f)$ be the set of pure strategies for individuals of type $i \in I$ that are best responses to f, i.e.,

$$BR_i(f) = \text{argmax}_{r_i\in S_i} \Pi_i(f;r_i).$$

Given $G \subseteq F$, we denote $BR_i(G) = \cup_{g\in G}BR_i(g)$.

We denote by $[s_i]$ the mixed strategy that ascribes probability 1 to $s_i \in S_i$. When no confusion is likely to arise, we will identify s_i with $[s_i]$.

Thus far we have only specified the set X = F of states. We have already (implicitly) assumed that the strategy distribution profile suffices for the determination of the profiles which may succeed it. In particular, the history of the system is assumed to be irrelevant in this model.

While this assumption cannot be claimed universally plausible and, indeed, in many situations people would predict future developments based on "trends" and so forth--it is still rather reasonable in a variety of applications. For instance, whenever "most people" happen to believe that history is irrelevant, history indeed becomes irrelevant. (Recall that no external uncertainty is assumed here.) In other words, if we embed our system in a meta-system that would also describe belief formation, a stationary system as we discuss here is likely to be "stable."

Coming to define the dynamics of this system, many ways may be followed. Indeed, so many ways that every choice may seem somewhat arbitrary. However, we do not purport to have found the "right" one. We will study one such relation, which we find reasonable for certain set-ups. Undoubtedly, it will fail to capture the essence of others. We therefore present this section more as an example of an application of ASS rather than as our proposed theory. The general procedure one should follow in analyzing a certain interaction is to model it as a game, inducing $X = F$, model the appropriate dynamics as a relation R on it, and apply ASS to (X,R).

The relation we deal with in this section is best response accessibility, denoted B, and defined as follows: for $f,g \in F$, fBg if $\exists\, \alpha \in [0,1]$ and $h \in F$ such that:

(i) $g = (1 - \alpha)f + \alpha h$;

(ii) for all $t \in [0,\alpha)$, $h_i \in BR_i\,((1 - t)f + th)$ for all $i \in I$; and

(iii) For all $i \in I$, if $f_i \in BR_i(f)$ then $h_i = f_i$.

In other words, fBg if g is on a line segment from f to some best-response point h. (This definition is very similar to those of Gilboa-Matsui (1989), Matsui (1989), and, especially, Matsui (1990).) It is required that all types would move toward h along the way from f to g, and that only those types that have an incentive to change their behavior ($f_i \notin BR_i(f)$) actually do so (for the others $h_i = f_i$).

Before we analyze the system (F,B) for some interesting games, we would like to comment on the underlying intuition behind the relation B.

One scenario which would give rise to this accessibility relation is the following: suppose that individuals get the chance to change their selected (pure) strategy only every so often, but not instantly. (One may assume this opportunity occurs in random time intervals, or possibly only once in each individual's life.) Further, suppose that the individuals are boundedly rational, and since they cannot fully analyze the dynamics (or are unaware of it), they respond to the current state f in a best-response direction h.

Alternatively, one may not regard B as specifying the complete dynamics, rather it may be interpreted merely as a class of possible small perturbations. Thus, the fact that the society's strategy profile moves from f (slightly) in a direction to a best-response h (with a small α) is simply a first approximation to a perturbation that is likely to occur if f is not a best response to itself.

Notice that B satisfies $f \in \tilde{B}(f)$ for all $f \in F$ and that $f = \tilde{B}(f)$ iff f is a Nash equilibrium (Nash (1951)) (i.e., $f_i \in BR_i(f)$ for all $i \in I$).

However, not every Nash equilibrium will constitute or even be included in a B-ASS of F. For instance, consider "the battle of the sexes":

	L	R
T	(2,1)	(0,0)
B	(0,0)	(1,2)

It is not difficult to see that the only ASS's are {(T,L)} and {(B,R)}, and that the mixed-strategy Nash equilibrium is not in any ASS.

Another classical example is "matching pennies":

	L	R
T	(1,-1)	(-1,1)
B	(-1,1)	(1,-1)

In this example, we find that the unique B-ASS is the Nash equilibrium $((1/2)[T] + (1/2)[B],\ (1/2)[L] + (1/2)[R])$.

These two examples have exactly the same CSS's as B-ASS's. (Here we refer to CSS as defined in Gilboa-Matsui (1989), which is a special case of the "R-CSS" notion defined in Section 2 above.) However, consider the game

	L	R
T	(1,1)	(1,1)
B	(1,1)	(0,0)

The (unique) CSS in this game (according to Gilboa-Matsui (1989) or Matsui's (1990) modified definition) is

$$\{([T],\alpha[L] + (1 - \alpha)[R]) | 0 \leq \alpha \leq 1\} \cup$$
$$\{(\alpha[T] + (1 - \alpha)[B],\ [L] | 0 \leq \alpha \leq 1\}.$$

However, the unique B-ASS is {(T,L)}, which may be somewhat more intuitive. This example shows the strength of the general theory developed in Section 2: the properties of ASS's and, most importantly, its existence, which guarantees some prediction of the theory, allow us to define an intuitive relation R without worrying about existence, compactness of ASS's, and so forth.

The last example may tempt one to conjecture that B-ASS's would not include strategy profiles that ascribe positive probability to weakly dominated strategies. However, this is false hope, as is shown by:

<u>Example 3.1</u>: Consider the game

		type II		
		L	C	R
	T	(1,-1)	(-1,1)	(3,-3)
type I	M	(-1,1)	(1,-1)	(3,-3)
	B	(-1,1)	(1,-1)	(2,-2)

in which B is weakly dominated by M. Yet, if we ignore R (which is <u>strictly</u> dominated by both C and L), B is indistinguishable from M. In fact, the game restricted to {T,M,B} × {L,C} is "matching pennies" with one of type I's strategies duplicated. Every Nash equilibrium ((1/2)[T] + α[M] + ((1/2 - α)[B], (1/2)[L] + (1/2)[C]) for $\alpha \in [0, 1/2]$ is a B-ASS (as a singleton) of the restricted game. It is also easy to see that every such B-ASS will also be a B-ASS of the system (F,B). //

It is not surprising, however, that strictly dominated strategies will not be played in a B-ASS:

<u>Proposition 3.2</u>: Let G be a game and let (F,B) be its associated system. Assume that for some $i \in I$, $s_i \in S_i$ is strictly dominated by $f_i \in \Delta(S_i)$. Then, if A is a B-ASS of F, $g_i(s_i) = 0$ for all $g \in A$.

<u>Proof</u>: It suffices to note that s_i is never a best response (not even a weak best response). Hence, if gBh and $g_i \neq h_i$ (for some $g,h \in F$), then $g_i(s_i) > h_i(s_i)$. Now, if C is a closed, B-self-absorbent subset of F, so will be

$$C_n = C \cap \{g \in F | g_i(s_i) \leq 1/n\}$$

for all $n \geq 1$, and C_n will be nonempty. Hence, a minimal stable set A cannot have $g_i(s_i) > 0$ for any $g \in A$. //

For a game G, it is interesting to ask what happens to B-ASS's when the structure of the game is fixed, but the payoffs are perturbed. Formally, fixing $\langle I,(S_i)_{i\in I}\rangle$ we define a correspondence $\mathcal{A}$: $(\pi_i)_{i\in I} \twoheadrightarrow F$ by $\mathcal{A}((\pi_i)_{i\in I}) = \cup \{A | A \text{ is B-ASS of } F\}$.

The following point is unlikely to surprise the reader:

Remark 3.3: $\mathcal{A}$ is neither upper- nor lower-hemi-continuous.

Proof: As for lower-hemi-continuity, consider the game

	L	R
T	$(\varepsilon,\varepsilon)$	(0,0)
B	(0,0)	(0,0)

in which for all $\varepsilon > 0$, {(T,L)} is the unique ASS, but for $\varepsilon = 0$ every $f \in F$ defines a singleton ASS.

To see that $\mathcal{A}$ fails to satisfy upper hemi-continuity, consider the game

	L	R
T	(1,1)	(0,0)
B	(0,0)	$(\varepsilon,\varepsilon)$

in which, for $\varepsilon > 0$, (B,R) is an ASS (as well as (T,L)), but for $\varepsilon = 0$ it is not. //

We conclude this section with the following.

Remark 3.4: The definition of the relation B implicitly assumes that the "rate of change" (or "birth") in each type is identical and f "moves towards h" along a straight line segment in F. Alternatively, one may want to consider the following definition: $gB'f$ if for all $i \in I$ these are $\alpha_i \in [0,1]$ and $h_i \in S_i$ such that:

(i) $g_i = \alpha_i f_i + (1 - \alpha_i)h_i$;

(ii) for all $i \in I$ and all $t \in [0,1)$, $h_i \in BR_i(((1 - t\alpha_i)f_i + t\alpha_i h_i)_{i \in I})$; and

(iii) $h_i = f_i$ or $f_i \notin BR_i(f)$.

(A version of this definition was also suggested by Gilboa-Matsui.) It seems that, while this definition is equivalent to the previous one in the neighborhood of a strict Nash equilibrium, it is quite different when the stability of a mixed-strategy Nash equilibrium is considered. In "matching pennies," for instance, the unique B'-ASS is F itself.

The choice between B and B' as "best-response dynamics" relations is not always easy to make. There are situations in which a constant change rate (across types) makes sense, there are cases in which it does not. Especially in view of the variety of reasonable definitions, the general approach of Section 2 seems to be appropriate.

4. Better-Response Dynamics

In this section we will borrow Section 3's definition of a game, but will propose a slightly different relation on F.

When we consider the model as describing evolution, with genes as implicit "decision makers," the dynamics would naturally change. While myopic behavior is not as controversial, the assumption of "best response" seems too demanding. In gene competition the best of the existing genes should prevail, but not necessarily the best of all. Until some perturbation (mutation) introduces a new gene, that gene will not appear with positive probability, even if it is a better response to the environment. Furthermore, even after the gene has been introduced, it will not automatically dominate all new-borns (as should be expected of rational economic agents once a new strategy has become a best response)--rather, it will reproduce alongside, albeit faster than, the other genes. (See also Gilboa-Matsui (1989) for a discussion of this point.)

With this intuition in mind we would like to define a relation T on F as follows: for $f, g \in F$, gTf if there are $\alpha \in [0,1)$, $h \in F$ such that:

(i) $g = (1 - \alpha)f + \alpha h$;

(ii) $h_i(s_i) > 0$ only if $f_i(s_i) > 0$, $\forall\, i \in I$, $\forall\, s_i \in S_i$;

(iii) for $i \in I$, if $s_i, r_i \in S_i$ satisfy $f_i(s_i), f_i(r_i) > 0$, $h_i(s_i) > h_i(r_i)$ implies that for all $t \in [0,\alpha)$, $\Pi_i((1 - t)f + th, s_i) \geq \Pi_i((1 - t)f + th, r_i)$;

(iv) $h_i = f_i$ or $f_i \notin BR_i(f)$ for all $i \in I$.

(where $\Pi_i(f, s_i)$ is type i's payoff from s_i when the others are playing according to f, and so forth.)

Consider the example of the "rocks-scissors-paper" game (also discussed in Gilboa-Matsui (1989)):

	R	S	P
R	(0,0)	(1,-1)	(-1,1)
S	(-1,1)	(0,0)	(1,-1)
P	(1,-1)	(-1,1)	(0,0)

where both players are of the same type, and F may be identified with $\{(p_R, p_S, p_P) | p_i \geq 0, \Sigma_{i \in \{R,S,P\}} p_i = 1\}$.

Consider, for example, f = (ε, 1 - ε, 0) for some small $\varepsilon > 0$. Then h = (1,0,0) satisfies (ii) and (iii) for t $\in$ [0,1]. Hence, hTf. Similarly, (0,0,1) T (1 - ε, 0, ε) and (0,1,0) T (0, ε, 1- ε). Hence, any T-stable set containing one of the pure strategies will also contain the other ones.

To analyze the interior of F, consider points at which none of $\{p_R, p_S, p_P\}$ is 1/3. Take such f = (p_R, p_S, p_P). Defining h as

(i) (1/2 - ε, 0, 1/2 + ε) when $p_P, p_S < 1/3$;

(ii) (0, ε, 1 - ε) when $p_R, p_P > 1/3$;

(iii) (0, 1/2 + ε, 1/2 - ε) when $p_S, p_R < 1/3$;

(iv) (ε, 1 - ε, 0) when $p_P, p_S > 1/3$;

(v) (1/2 + ε, 1/2 - ε, 0) when $p_R, p_P < 1/3$;

(vi) (1 - ε, 0, ε) when $p_S, p_R > 1/3$;

for appropriately small ε (depending on the chosen f $\in F^0$), we conclude that the unique T-ASS is F. By comparison, the unique B-ASS of this game is the Nash equilibrium {(1/3,1/3,1/3)}.

Indeed, more generally, we have:

<u>Proposition 4.1</u>: For every game G, every T-ASS contains a B-ASS.

<u>Proof</u>: Define a relation T' by dropping requirement (ii) in the definition of T above, i.e., by allowing strategies to be chosen (as "h") without already existing in the population (as "f"). It is not hard to see that, since $\tilde{T}$ and $\tilde{T}'$ coincide on R^0, a set is T-ASS iff it is T'-ASS. However, T' includes B and Theorem 2.17 completes the proof. //

In other words, the economic-rationality dynamics captured by B is "more stable" than the evolutionary one, represented by the relation T. With a non-trivial leap of imagination this simple result may be reinterpreted as saying that rationality is stabilizing: the fact that reasoning individuals should obtain the same conclusions from the same premises may make the corresponding ASS's smaller.

This example is supposed to convince the (skeptical) reader that the concept of ASS is useful. Not only is it convenient to define the dynamics in an arbitrary way and be guaranteed that some nice properties hold, the generality of ASS's may help us to better understand different dynamics by comparing the ASS's that correspond to various "accessibility" relations.

5. Additional Examples

In this section we will briefly discuss other models, and will attempt to exhibit a wider range of applications of ASS's than hitherto discussed. For obvious reasons (of time and space) we will not attempt to actually compute ASS's for each model, or to prove/disprove equivalence to the specific solution concept in each. We will only try to point out that our general framework can accommodate them.

The first class of models that comes to mind is probably systems of differential equations or differential inclusions. (In these models, singleton ASS's would typically correspond to stable steady states, and ASS's to attractors.) Providing here even a very partial list of references is rather hopeless.

Another class of models are repeated games where some statistics of the history are retained. The first such example is probably Shapley's (1964) "fictitious play," in which a player's move is best response to an averaged distribution. Since only the distribution matters for optimization at each stage, the model fits into Section 2's framework almost perfectly. (Updating this distribution, however, calls for knowledge of the number of stages. Thus, a certain compactification may be required.)

More recently, Gilboa-Schmeidler (1989) and Fudenberg-Levine (1990) have studied repeated games with infinite histories, in which each player has a bounded memory and reacts optimally to his/her own experience. Focusing on the information available to the players, one may reduce the system to a finite-dimensional one.

The models of Fudenberg-Kreps (1988) and Canning (1989) study more general belief formation and update, which may guarantee convergence of a repeated play to a one-shot Nash equilibrium. Canning's model, for instance, is formulated in terms of finite spaces which automatically lend themselves to be embedded in Section 2's framework.

Kalai and Lehrer (1990, 1991) study convergence of a repeated game play to a repeated game Nash equilibrium play. Although their results are quite general, at least a restricted version (of beliefs with finite support) may fit into our model, where the dynamic process consists of Bayes update and optimal behavior.

Taylor and Jonker (1978) and, more recently, Boylan (1990), Swinkels (1990), and Kandori, Mailath and Rob (1991) study evolutionary stability in games with explicit dynamics. In these models, as in Gilboa-Matsui (1989) and Matsui (1989, 1990), the state space is simply the product of mixed strategy spaces.

The studies mentioned above are by no means all the relevant or important ones. An exhaustive survey, though much needed in face of the growing interest in these subjects, is far beyond the scope of this paper. We only hope the few examples given here suffice to exhibit the potential scope of applications of our model.

6. Concluding Remarks

Some of the applications mentioned in Section 5 seem to be constrained by the assumption that the state space X is a compact metric space. It is natural to ask to what extent the results can be generalized.

While compactness seems essential to the analysis, the assumption of metrizability can be relaxed without devoiding the theory of all content. Some modifications are, however, needed: in the definition of a stable set, instead of a sequence of closed and absorbent sets $\{F_n\}$, one should consider arbitrary sets $\{F_\alpha\}_\alpha$ (where each F_α is closed and absorbent). The existence theorem for ASS will hold with this definition and, as a matter of fact, its proof will be simplified. (Given a nonincreasing chain of stable sets $\{A^\alpha\}_\alpha$, with $\{F^\alpha_\beta\}_\beta$, a set for A^α, one should only take their union to obtain a set for $A \equiv \bigcap_\alpha A^\alpha$.)

The existence of CSS's in every closed and absorbent set will also hold whenever R is transitive with $\tilde{R}$ closed and nonempty.

However, for many applications the topologies in which X is compact may be too weak, and in more useful topologies compactness will be lost. We therefore chose to assume metrizability and use a somewhat more intuitive definition of stable sets.

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Appendix

In this Appendix we suggest an alternative definition of "stable sets" that will clarify the relationship between ASS's and CSS's. This definition is very similar to (and probably partly inspired by) that of "stable sets" in Maschler-Peleg (1976).

Define a set $A \subseteq X$ to be stable' if:

(i) A is nonempty;

(ii) there are $\{F_n\}_{n\geq 1}, \{G_n\}$ such that $A \subseteq G_n \subseteq F_n \subseteq X$, G_n is open, F_n is closed, $\tilde{R}(G_n) \subseteq F_n$, and $A = \bigcap_{n\geq 1} F_n$.

Obviously, taking $G_n = F_n^0$ would yield our original definition. Thus, every stable set is also stable'. Remark 2.12 has shown that the converse is false.

As with the original definition, no loss of generality is involved in assuming that $G_{n+1} \subseteq G_n$ and $F_{n+1} \subseteq F_n$ for all $n \geq 1$.

If A is stable', then $A = \bigcap_{n\geq 1} F_n$ implies that A is compact. However, $A = \bigcap_{n\geq 1} G_n$ also implies that A is absorbent.

Define a set to be ASS' if it is a minimal stable' set. We note that:

Theorem A.1: For a system (X,R) there is at least one ASS'.

(The proof is identical to that of Theorem 2.3 with slight editorial modifications.)

We now have an equivalence theorem for "nice" relations R.

Theorem A.2: Let (X,R) be a system in which R is closed and transitive, and

$\tilde{R}$ is nonempty. Then a set $A \subseteq X$ is ASS' iff it is a CSS.

Proof: We first show that every CSS is stable'. Let $C \subseteq X$ be a CSS, and define F_n to be the closure of $\cup_{x \in C} N_{1/n}(x)$. ($N_\varepsilon(x) = \{y \in X | d(x,y) < \varepsilon\}$ where d is a metric for X.) R is closed; hence, $\tilde{R}$ is closed and C is compact. Hence, $C = \cap_{n \geq 1} F_n$.

To show stability of C we need to find for each F_n an open set G_n such that $C \subseteq G_n \subseteq F_n$ and $\tilde{R}(G_n) \subseteq F_n$. Fix $n \geq 1$. Consider $G_m = \cup_{x \in C} N_{1/m}(x)$. If for all $m \geq 1$, $\tilde{R}(G_m) \cap F_n^c \neq \emptyset$, choose x_m, y_m such that $x_m \in G_m$, $y_m \in F_n^c$ and $x_m R y_m$. By (sequential) compactness, there is a subsequence $\{(x_{m_k}, y_{m_k})\}_{k \geq 1}$ such that $(x_{m_k}, y_{m_k}) \to_{k \to \infty} (x,y)$. This implies $x \in C$ and $y \in F_n^c$. However, closedness of R yields xRy, in contradiction to C being a CSS. Thus, for every n there is $m_n \geq 1$ such that $\tilde{R}(G_{m_n}) \subseteq F_n$, and we have established that every CSS is stable'.

We now turn to the theorem's statement. If A is a CSS, it is stable'. However, it has to be minimal since every stable' set is absorbent and a CSS cannot contain a proper absorbent subset.

Conversely, let A be an ASS'. As in Proposition 2.9, A contains a CSS C. Since C is stable' and A is minimal, A = C follows. //

Since theorem A.2 requires that R be closed and transitive, one may be tempted to conjecture that given R with $\tilde{R}$ nonempty, R-ASS' would coincide with $\bar{\bar{R}}$-CSS. Indeed, this would follow from A.2 if R-ASS' were to coincide with $\bar{\bar{R}}$-ASS'. However, this is not the case in general, as Remark 2.11 has shown. Worse still, under the new definition of stability, Theorem 2.6

fails to hold.

Remark A.3: For a system (X,R), the set of R-ASS' is not always identical to that of $\bar{R}$-ASS'.

Proof: Consider X = [0,1] with

$$R = \{(x,y) | 0 \le x \le 1,\ x \le y \le \min(2x,1)\}$$

$\tilde{R}$ is nonempty and closed but R is not transitive. It is easy to see that {0} is R-ASS' (though not R-ASS). However, the semi-closure of R is

$$\bar{R} = \{(0,0)\} \cup \{(x,y) | 0 < x \le y \le 1\}$$

and the unique $\bar{R}$-ASS' (and unique $\bar{R}$-ASS) is {1}.

To sum, we have seen that the new definition of stability has an advantage of coinciding with CSS for transitive and closed relations, but the disadvantage of failing to "presuppose" the semi-closure of a relation.