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A DYNAMIC EQUILIBRIUM MODEL OF SEARCH, PRODUCTION AND EXCHANGE

by

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Abstract

We study a general equilibrium model where agents search for production and trading opportunities, that generalizes the existing literature by considering a large number of differentiated commodities and agents with idiosyncratic tastes. Thus, agents must choose nontrivial exchange as well as production strategies. We consider decreasing, constant, and increasing returns to scale in the matching technology, and characterize the circumstances under which there exist multiple steady state equilibria, or multiple dynamic equilibria for given initial conditions. We also characterize the existence of dynamic equilibria that are limit cycles. Equilibria are not generally optimal, and when multiple equilibria coexist they may be ranked. Pareto optimal allocations are also described and contrasted to those that obtain in equilibrium. We analyze comparative statics and find that certain intuitive results do not necessarily hold without restrictions on the stochastic structure.

I. Introduction

This paper analyzes a general equilibrium model of production and exchange. We extend the standard search equilibrium framework, as described, for example, in the work of Diamond,[1] by assuming a large number of differentiated commodities and agents with idiosyncratic tastes. In the standard framework, agents search for production opportunities, or projects, which always yield one unit of homogeneous output at some project-specific production cost. In our framework, each project yields one unit of a particular type of commodity. As in the standard model, there is an exchange sector where agents search for trading partners, and when they meet, if mutually agreeable, they exchange and consume. Because of the large number of differentiated commodities, in our model agents face a non-trivial decision over trading strategies, in addition to the decision over production strategies analyzed in the standard model.

Elsewhere, Kiyotaki and Wright (1989, 1990) use specialized versions of this framework to study the role of fiat currency as a medium of exchange. Having many differentiated commodities is essential for modeling the role of money endogenously, since this is exactly what makes pure barter difficult. The extension to monetary economics provides one motivation for the multi-good generalization of the standard approach; but the non-monetary version seems interesting enough in its own right to warrant an extended analysis. Without the complication of fiat money, we are able to consider a very general specification, and to completely characterize not only steady

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[1] See, for example, Diamond (1982, 1984a, 1984b), or Diamond and Fudenberg (1989). See also Mortensen (1989) and the references contained therein.

states, but also dynamic equilibria. We study constant (CRS), decreasing (DRS), and increasing (IRS) returns to scale in the matching technology. The latter two cases can lead to multiple steady state equilibria, and also a continuum of dynamic equilibrium trajectories for given initial conditions, while the constant return case always entails a unique nondegenerate steady state and a unique equilibrium path leading to it. These results contrast with the case of homogenous commodities, where multiplicity is possible only under IRS

A qualitative implication of the type of multiplicity obtained here is that economies having the same initial conditions may asympotically converge toward very different attractors. In other words, given initial conditions, the dynamics of the system are such that there can exist some equilibria converging to one attractor and other equilibria converging to a second, completely different, attractor. These different trajectories can be thought of as being indexed by expectations. If expectations are initially optimistic, the economy can set off along one path that justifies the initial optimism, while if agents are initially pessimistic, the economy can set off along a completely different path. All of these paths are consistent with rational expectations.[2]

We also analyze the welfare implications of the model, some of which differ from the standard one-good model. For instance, in the simplest case of constant returns, the equilibrium allocation is inefficient here,

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[2] One can indeed show that the standard search model, as described in Diamond and Fudenberg (1989), would display similar properties under certain conditions. Formally, analogous results have been derived by Boldrin (1990) and Matsuyama (1991) in the context of two very different models of economic growth and industrialization.

although it would be efficient in the standard model. We also analyze comparative statics, and find that some intuitively reasonable results cannot generally be guaranteed. For example, an increase in the arrival rate of production opportunities does not necessarily reduce the number of agents in the production sector, nor does it even increase the rate at which agents exit from that sector.[3] However, a straightforward restriction on the stochastic structure, log-concavity, can be used to rule out the counterintuitive cases.

The paper is organized as follows. Section 2 describes the model and defines equilibrium. Section 3 introduces the dynamic analysis by studying the CRS and DRS cases. Section 4 discusses in detail the dynamic behavior of the IRS case. Section 5 derives some welfare implications of the different equilibrium paths. Section 6 contains the comparative statics. Section 7 concludes. Most of the proofs are relegated to an appendix.

2. The Basic Model

Consider an economy with a continuum of infinite lived agents, with total measure normalized to one, indexed by points around a circle of circumference two. There is also a continuum of commodities indexed by points on the same circle. Goods are indivisible and come in units of size one. They are also perfectly storable, but only a single unit at a time. Individuals have idiosyncratic tastes for these goods: the agent indexed by point i most prefers the good indexed by i, and receives utility u(z) from

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[3] Agents searching for production opportunities are often identified as <u>unemployed</u> in the search literature; on this interpretation, reducing the frictions involved in the job finding process does not necessarily reduce the unemployment rate or even increase the exit rate from unemployment.

consuming a unit of good j, where z is a the distance between i and j along the circle, and u: [0,1] → ℝ is twice continuously differentiable with $u'(z) < 0$. Thus, one can think of position on the circle as representing a characteristic such as color, and utility as decreasing in the difference between a good's actual color and a consumer's favorite color. We do not require u(•) to be either concave or convex, in general, but for some results below we do need to assume that $u'(z) + zu''(z) \leq 0$ (which is automatically satisfied if u is concave).

We assume that $u(0) > 0 > u(1)$, where 0 is also the utility from consuming nothing. We note here for future reference that the distance along the circle between an agent's ideal commodity and a commodity type drawn at random from the circle is uniformly distributed in [0,1].

To acquire commodities, agents search in a production sector at a cost in terms of disutility of w_0 per unit of continuous time. They locate potential production projects stochastically according to a Poisson process with constant arrival rate $\alpha > 0$. Each project yields a unit of good i at a cost in terms of instantaneous disutility c, where i is drawn randomly from the circle and c is drawn independently from the cumulative distribution function (CDF), F(c). Both commodity type i and cost c are observed upon location of a project, before a production decision is made. Let the greatest lower bound of the support of F be $\underline{c}$; for some results, it is important whether $\underline{c} > 0$ or $\underline{c} = 0$. We will also typically assume F(c) is differentiable and strictly increasing in order to simplify the presentation.[4]

Individuals do not consume their own output; rather, once production takes place, agents proceed to an exchange sector where they look to trade their output for something else that they can consume. In the exchange sector, there is a search (or storage) cost w_1 per unit of time, and potential trading partners are located according to a Poisson process with arrival rate $B \geq 0$. The number of meetings in the exchange sector depends on the number of agents searching, N, and we write $m = m(N)$. This means the arrival rate for a representative agent is $B = B(N) = m(N)/N$. We always assume $m(0) = 0$. In the CRS case, $m(N) = \beta N$, and the arrival rate is constant, $B(N) = \beta$. By analogy, $B' > 0$ is referred to as increasing returns (m convex) and $B' < 0$ as decreasing returns (m concave) in the matching technology.

In any case, locating a partner does not necessarily imply an opportunity to trade, since the partner may not want what you have. If both agents want what the other has, then they swap inventories one-for-one; otherwise, they part company. For now, we simply denote the probability that a randomly located trader is willing to accept good i at date t by $\theta_t(i)$. When a good is accepted in trade, there is a disutility cost $\varepsilon > 0$ that must be paid by the receiver. Let $u_\varepsilon(z) = u(z) - \varepsilon$; then we assume $u_\varepsilon(0) > 0$, and define $z_\varepsilon < 1$ by $u_\varepsilon(z_\varepsilon) = 0$. Thus, z_ε is the greatest

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[4] It is worth noting that all of our results would still obtain in the special case in which $\underline{c} = 0$ and $F(c) \equiv 1$ for all $c \geq 0$, i.e., when there is no search problem on the production side. This means that the existence of search among differentiated commodities in the exchange sector is enough to bring about the complicated structure of the set of equilibria described below.

distance from one's ideal good that generates nonnegative utility.

It may seem natural to assume that after trading an agent can consume and return to production at any time, or stay and try to make an additional trade. However, we assume that upon accepting a good in trade, the receiver must decide immediately if he is going to consume it or try to trade it again for something else. Obviously, this assumption is not restrictive in a steady state, although it could potentially be restrictive along a dynamic path.

Further, in this paper we consider only symmetric equilibria, in which no agent or commodity type plays any special role, and in particular the decision to accept good j by agent i will depend only on the distance between i and j. It turns out that under these conditions, there is no indirect trade, and agents accept a good in exchange if and only if they are going to consume it.

To understand this, observe that our notion of a symmetric equilibrium implies that the probability of a randomly located individual accepting good i is in fact independent of i: $\theta_t(i) = \theta_t$ for all i at every date t. Now consider an individual holding good i deciding whether to accept a trade for good j. If he is not going to consume good j, by accepting it he pays transaction cost ϵ and gains no benefit, since his trading position is not enhanced as long as $\theta_t(j) = \theta_t(i) = \theta_t$. Hence, he will not accept unless he is going to consume j. This implies that all trades are made for direct consumption, and there is no indirect exchange. Furthermore, once a trade is made, consumption and the return to production will take place immediately, simply because agents discount the future.[5]

We proceed using dynamic programming. Let the state variable j indicate sector, j = 0 for production, and j = 1 for exchange, and let V_{jt} be the optimal value function for a representative agent in sector j at date t. Note that V_{it} does not depend on the type of good in storage for the same reason that θ_t does not: the probability of a random trader accepting good i is independent of i. Then the continuous time version of Bellman's equations that must be satisfied by the V_{jt} are

$$(2.1) \qquad rV_{0t} = -w_0 + \alpha\int_0^\infty \max[0, V_{1t} - V_{0t} - c]dF(c) + \dot{V}_{0t}$$

$$(2.2) \qquad rV_{1t} = -w_1 + B(N_t)\theta_t \int_0^1 \max[0, V_{0t} - V_{1t} + u_\varepsilon(z)]dz + \dot{V}_{1t}$$

where r is the constant discount rate.[6]

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[5]We are not claiming there could not exist asymmetric equilibria where certain goods become focal points for indirect trade--we simply choose to concentrate on symmetric outcomes. A similar model with a finite number of agent and commodity types is used in Kiyotaki and Wright (1989) to discuss asymmetric equilibria, in which some commodities are used in indirect trades as media of exchange, or commodity money (see also Kehoe, Kiyotaki and Wright (1990), Aiyagari and Wallace (1990a,b), and Marimon, McGrattan and Sargent (1990)).

[6]Intuitively, these equations set the return to searching in sector j, rV_j, equal to the flow yield w_j plus two "capital gain" terms, an expected option value plus a pure time change in V_j. In (2.1), the expected option value is the rate at which projects arrive times the value of rejecting or accepting the opportunity, whereas in (2.2), it is the rate at which partners arrive times the probability that they are willing to trade times the value of rejecting or accepting their offer. A simple way of deriving

The maximization problems in (2.1) and (2.2) are solved by <u>reservation strategies</u>: (i) accept a production project at date t if and only if the cost c is less than k_t, where

$$(2.3) \qquad k_t \equiv V_{1t} - V_{0t};$$

(ii) accept a trade offer at date t if and only if the distance z between the good being offered and your ideal good is less than x_t, where

$$(2.4) \qquad u_\varepsilon(x_t) \equiv V_{1t} - V_{0t}.$$

If we insert (2.3) and (2.4) into (2.1) and (2.2), we have

$$(2.5) \qquad rV_{0t} = -w_o + \alpha s_0(k_t) + \dot{V}_{0t}$$

$$(2.6) \qquad rV_{1t} = -w_1 + B(N_t)\theta_t s_1(x_t) + \dot{V}_{1t}$$

where, to reduce notation, we introduce the functions

$$s_0(k) \equiv \int_0^k (k - c)dF(c) = \int_0^k F(c)dc$$

$$s_1(x) \equiv \int_0^x [u(z) - u(x)]dz = \int_0^x zdu(z).$$

Because s_0 and s_1 are important, we catalog some of their basic properties in the following lemma. The proofs are easy and left to the

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(2.1) and (2.2) formally is described in the Appendix.

render. Note, however, that the convexity of $s_1(\bullet)$ requires the assumption $u'(z) - zu''(z) \leq 0$, and then the inequality $\Sigma(x) \geq 0$ asserted in the lemma follows directly from convexity.

Lemma 1: For all k, $s_0(k)$ is positive, increasing and convex. For all x, $s_1(x)$ is positive, increasing, convex, and satisfies

$$\Sigma(x) \equiv xs_1'(x) - x_1(x) \geq 0.$$

For arbitrary time paths of N_t and θ_t a solution to the representative agent's decision problem is given by a set of nonnegative functions $\{x_t, k_t, V_{0t}, V_{1t}\}$ satisfying equations (2.3)-(2.6), and the condition $x_t \leq z_\varepsilon$ for all t. This condition rules out the consumption of goods yielding negative utility, and also implies that $V_{1t} - V_{0t} = u_\varepsilon(x_t) \geq 0$ for all t, or, that the value to being in the exchange sector is never less than the value to being in the production sector (which would conflict with an obvious free disposal condition). As discussed earlier, we concentrate on symmetric outcomes, which means all agents use the same reservation strategies. This implies the probability that an agent chosen at random will accept a trade for a given good at date t equals the probability that the distance between it and his ideal good is less than his reservation distance: $\theta_t = pr(z \leq x_t) = x_t$ (given the uniform distribution of agent and commodity types).

Since the probability that a representative agent accepts a trade is x_t, the probability of two agents trading in any particular meeting is given by the probability that each has something that the other finds acceptable,

x_t^2. This further implies that the measure of agents in the exchange sector evolves according to the law of motion

(2.7) $$\dot{N}_t = \alpha F(k_t)(1 - N_t) - m(N_t)x_t^2.$$

Putting these results together, we have the following definition of a (symmetric) equilibrium for the model.

<u>Definition</u>: An <u>equilibrium</u> with initial condition N_0 at $t = 0$ consists of nonnegative time paths for $\{k_t, x_t, V_{0t}, V_{1t}, \theta_t, N_t\}$ defined for all $t \in [0,\infty)$ and satisfying:

a. $N(0) = N_0$;

b. Conditions (2.3)-(2.6) and $x_t \le z_\varepsilon$ for all t, implying agents are following maximizing strategies given N_t and θ_t;

c. Condition (2.7) and $\theta_t = x_t$ for all t, implying expectations are rational.

3. <u>Equilibria with Constant or Decreasing Returns to Scale</u>

Without any loss in generality, we set $w_0 = w_1$ here to ease the presentation. Then, subtracting (2.5) from (2.6), we obtain

$$r(V_1 - V_0) = B(N)\theta s_1(x) - \alpha s_0(k) + \dot{V}_1 - \dot{V}_0$$

(time subscripts will be omitted from now on). The equilibrium conditions $\theta = x$, $k = V_1 - V_0 = u_\varepsilon(x)$, and the conditions $\dot{V}_1 - \dot{V}_0 = u'(x)\dot{x}$, can be used to reduce this expression to

$$ru_\varepsilon(x) = B(N)xs_1(x) - \alpha s_0[u_\varepsilon(x)] + u'(x)\dot{x},$$

which implies

(3.1) $$\dot{x} = T(N,x) \equiv (1/u'(x))\{ru_\varepsilon(x) + \alpha s_0[u_\varepsilon(x)] - B(N)xs_1(x)\}$$

Also, $k = u_\varepsilon(x)$ can be used to reduce (2.7) to

(3.2) $$\dot{N} = S(N,x) = \alpha F[u_\varepsilon(x)](1 - N) - m(N)x^2.$$

Equations (3.1) and (3.2) define a dynamical system over the set $\bar{U} = \{(N,x): 0 \leq N \leq 1,\ 0 \leq x \leq z_\varepsilon\}$. Given an initial value of $N_0 \in [0,1]$ and an initial value of $x_0 \in [0,z_\varepsilon]$, any solution $[N(t),x(t)]$ to this system going through (N_0,x_0) at $t = 0$ and remaining in $\bar{U}$ for all $t \geq 0$ constitutes an equilibrium for the model.[7] The boundaries of $\bar{U}$ are: (i) $U_1 = \{(N,x): x = 0,\ 0 \leq N \leq 1\}$; (ii) $U_2 = \{(N,x): x = z_\varepsilon,\ 0 \leq N \leq 1\}$; (iii) $U_3 = \{(N,x): 0 < x < z_\varepsilon,\ N = 0\}$; (iv) $U_4 = \{(N,x): 0 < x < z_\varepsilon,\ N = 1\}$. Any trajectory satisfying (3.1) and (3.2) and starting in U_1 or U_2 will leave $\bar{U}$ and never return, while any trajectory starting in U_3 or U_4 will enter U, and remain in U at least for small $t > 0$.

We now characterize the behavior of the system on the set $U = \text{int}(\bar{U})$. We analyze separately the cases where $B' = 0$, $B' < 0$, and $B' > 0$, as they

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[7]The standard theory of ordinary differential equations (see, e.g., Lefschetz (1957)) assures that, for any (N_0,x_0) in $U = \text{int}(\bar{U})$, there exists a unique local solution to (3.1)-(3.2). Such a trajectory can be uniquely extended for all $t \geq 0$ if it remains bounded.

have rather different properties. We also sometimes make a distinction between the cases in which $\underline{c} = 0$ and $\underline{c} > 0$, where we recall that $\underline{c}$ is the greatest lower bound of the production cost distribution $F(c)$. The case where $B' > 0$ for small N and $B' < 0$ for large N (which may be the most realistic) can be understood by combining the results obtained in the two cases separately. In what follows, we call the set of points in $\bar{U}$ satisfying $S(N,x) = 0$ and $T(N,x) = 0$ the S-locus and T-locus, respectively, and a critical point or steady state of the system is a point (N,x) solving $S(N,x) = T(N,x) = 0$.

3.1 Constant Returns to Scale

We begin with the simplest situation where $m(N) = \beta N$, which implies $B(N) = \beta$, for all $N > 0$. The horizontal line $\{(N,x)\colon 0 \le N \le 1 \text{ and } x = z_\beta\}$, where z_β is the unique solution in $(0,z_\varepsilon)$ to

$$(3.4) \qquad ru_\varepsilon(x) + \alpha s_0[u_\varepsilon(x)] - \beta x s_1(x) = 0$$

describes the T-locus. With regard to the S-locus, it is easy to see that if $\underline{c} = 0$ and $F'(0) > 0$, it begins at the point $(0,z_\varepsilon)$ and slopes downward to the point $(0,1)$. On the other hand, if $\underline{c} > 0$ and we define z_c by $u_\varepsilon(z_c) = \underline{c}$, then the S-locus coincides with the vertical axis over the nondegenerate interval $[z_c,z_\varepsilon]$ and slopes downward from the point $(0,z_c)$ to the point $(0,1)$.

In either case, if we assume that $\underline{c}$ is small enough so that $z_c > z_\beta$, there exists exactly one nondegenerate steady state, with $x = z_\beta$ and $N = N_\beta$ given by

(3.5) $$N_{\beta} = \alpha F[u_{\varepsilon}(z_{\beta})] \quad (\alpha F[u_{\varepsilon}(z_{\beta})] - \beta z_{\beta}^{2}).$$

A straightforward local analysis of the system around (N_{β}, z_{β}), together with consideration of its global behavior yields the following result, depicted in Figure 1.

<Insert Figure 1 here>

Proposition 1: Assume CRS and $z_c > z_{\beta}$; then there exists a unique equilibrium path for any initial condition $N_0 \in [0,1]$. This equilibrium satisfies $x_t = z_{\beta}$ for all t and $N_t \rightarrow N_{\beta}$ as $t \rightarrow +\infty$.

Proof: See Appendix.

3.2 Decreasing Returns to Scale

The assumptions on $B(\cdot)$ in this case are: $B(0) = \beta$, $B(N) > 0$ and continuous for all $N > 0$, and $B'(N) < 0$ for all $N > 0$. This kind of functional form may seem unrealistic although a negative relation between arrival rates and number of individuals searching may well arise (due to "congestion") at high levels of N. The extreme case we are considering is nevertheless worthy of study because it reveals that IRS in the matching technology is not necessary to obtain a multiplicity of interior steady states nor to originate a continuum of competitive equilibria in search models.[8]

The T-locus slopes upward, while the S-locus slopes downward when the elasticity of B is less than 1, but may well slope upward, if this elasticity is greater than 1. As long as $z_\beta < z_c$ at least one nondegenerate steady state exists. The existence of others depends on the elasticity of B(N), but it is clear that if a second intersection of the S-locus and T-locus exists then a third must also exist, since the T-locus must eventually reach the point (0,1). In other words, there is generically an odd number of steady states.

Proposition 2 describes the global dynamics for the case where the elasticity of B(N) is everywhere less than 1, which guarantees the existence of exactly one interior steady state. Notice that it is in all relevant respects the same as the CRS case (see Figure 1). If the elasticity condition is violated there can be multiple interior steady states. We will not pursue this here because it turns out to be formally equivalent to one of the IRS cases analyzed in the next section.

<u>Proposition 2</u>: Assume DRS and $|NB'/B| \leq 1$ for all N; then there exists a unique equilibrium path for any initial condition $N_0 \in [0,1]$. The equilibrium path coincides with the global stable manifold of the unique interior critical point of (3.1) and (3.2).

4. Equilibria with Increasing Returns to Scale

Consider now the case in which B is increasing with $B(0) = 0$, $B'(N) > 0$ $\forall N > 0$. In this case, the T-locus begins at $(0, z_\varepsilon)$ and slopes downward in

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[8] We will show in Section 4 that IRS is also not sufficient for this purpose.

the (N,x) plane to the point $(1,z_1)$, where z_1 is the unique value of $x \in (0,z_\varepsilon)$ that solves

$$ru_\varepsilon(x)+ \alpha s_0[u_\varepsilon(x)] - B(1)xs_1(x) = 0.$$

The qualitative behavior of the S-locus is the same as above, it slopes downward from $(0,z_\varepsilon)$ to $(1,0)$, and is coincident with the vertical axis between z_ε and z_c when $\underline{c} > 0$. It lies below the T-locus in a neighborhood of $N = 1$ because $z_1 > 0$, and as long as $\underline{c} > 0$, it will be below the T-locus at $N = 0$ also. Hence, there exists an even number of interior steady states, $E^j = (N^j,x^j)$, $j = 0,1,\ldots,2n$. Assuming $n > 0$, we will concentrate most of our analysis on the quite general situation shown in Figure 2, in which there are three critical points, E^0, E^1, and E^2, with $0 = N^0 < N^1 < N^2$ and $z_\varepsilon = x^0 > x^1 > x^2$.

4.1 Local Analysis

Around some critical point E^j the linear part of (3.1)-(3.2) is described by:

$$\begin{bmatrix} \dot{x} \\ \dot{N} \end{bmatrix} = \begin{bmatrix} T_x & T_N \\ S_x & S_N \end{bmatrix} \begin{bmatrix} x - x_j \\ N - N_j \end{bmatrix} + \sigma(\|x\|^2, \|N\|^2) \tag{4.1}$$

where $\sigma(\bullet,\bullet)$ is of order smaller or at most equal to $\|x - x_j\|^2$ and $\|N - N_j\|^2$. The determinant of the Jacobian matrix is

$$\det = T_xS_N - S_xT_N = S_xT_x\{dx/dN_{|T=0} - dx/dN_{|S=0}\}. \tag{4.2}$$

Hence, det is positive or negative depending on whether the T-locus is steeper or flatter than the S-locus at their intersection.

At the steady state with the greatest value of N (E^2 in the figure), the T-locus is flatter than the S-locus, and so det < 0. At the steady state with the next highest value of N (E^1 in the figure), the T-locus is steeper than the S-locus, and therefore det > 0. Finally, at the steady state E^0 with N = 0 and $x = z_\varepsilon$ the local structure of the vector field depends on the nature of our assumptions on $\underline{c}$ and F. We need to distinguish two cases:

(i) $\underline{c} > 0$, and therefore $F(0) = F'(0) = 0$;

(ii) $\underline{c} = 0$, $F(0) = 0$, and $F'(0) > 0$.

In case (i) both S_N and S_x are zero at E^0, which implies det = 0; the steady state is a degenerate critical point with one eigenvalue being zero. In case (ii) S_N is zero but S_x is positive, which implies the determinant is positive at E^0. By the implicit function theorem, the S-locus is flat at E^0 while the T-locus is downward sloping.

When $\underline{c} = 0$, one cannot exclude the situation in which the T- and S-loci intersect only once in U. It is therefore possible to have a unique interior steady state even in the presence of IRS when $\underline{c} = 0$.

In summary, under the assumption of IRS in the matching technology the following two situations are possible: (i) when $\underline{c} > 0$ either there are no interior stationary states or there is an even number of them; (ii) when $\underline{c} = 0$ and $F'(0) > 0$ there always exists at least one interior steady state and when there are more than one there are an odd number.

We begin with the case where $\underline{c} = 0$ and there are only two steady states: $E^0 = (0, z_\varepsilon)$ and $E^1 = (N^1, x^1)$, as in Figure 3.

<Insert Figure 3 here>

The global characterization of the dynamic equilibria for this case is:

<u>Proposition 3</u>: Assume IRS, $\underline{c} = 0$ and a unique interior steady state. Then for every N_0 in [0,1] there is a unique equilibrium. When $N_0 = 0$ such an equilibrium coincides with the stationary point $(0, z_\varepsilon)$ for all $t \geq 0$. When $N > 0$, $N_0 \neq N^1$ such an equilibrium starts at the unique value $x(0)$ such that $(N_0, x(0))$ belongs to $W^s(E^1)$ (the global stable manifold of E^1) and converges to E^1.[9]

<u>Proof</u>: See Appendix.

We now proceed to the case where $\underline{c} > 0$. In this case there is an even number of interior steady states. We will concentrate on the case in which there are two of them (Figure 2): $E^0 = (0, z_\varepsilon)$, $E^1 = (N^1, x^1)$ and $E^2 = (N^2, x^2)$ in order of increasing values of N.

The local characterization is simple:

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[9]To be complete we have to add that, in exceptional and structurally unstable cases, the stable manifold of the saddle point E^1 may go all the way up to E^0 and therefore connect the two critical points. The fact that the S-locus is horizontal at E^0, prevents $W^u(E^1)$ (the global unstable manifold of E^1) from getting into E^0.

Proposition 4: Assume IRS and $\underline{c} > 0$. Then the boundary steady state E^0 is a degenerate saddle-node (or saddle-focus) at which a transcritical bifurcation occurs. The first interior steady state, E^1, is (generically) either a sink or a source depending on parameter values, whereas the second one, E^2, is a regular saddle.

Proof: See the Appendix.

<Insert Figure 4 here>

Figure 4 reports the local behavior of the flow around the boundary steady state for the case in which the slope of the T-locus at E^0 is, in absolute value, less than r/T_N, which is the slope of the (local) center manifold. This occurs when $T_N > r$, whereas the opposite is true in the other case.

4.2 Global Dynamics

The presence of a center manifold at E^0 can complicate our analysis purposelessly. We will therefore restrict attention to the case in which the center manifold $(W^c(E^0))$ is unique with the motion on it converging to E^0. Even with this simplification the structure of the equilibrium set remains quite complicated, as a number of different configurations are generated by different combinations of parameter values.

One has to distinguish, first of all, between the case in which E^1 is a sink and the case in which it is a source. Second, the stable $(W^s(E^2))$ and unstable $(W^u(E^2))$ global manifolds of the saddle point E^2 may or may not be representable as graphs of functions in (N,x) space. When the latter situation occurs for either of them, dynamic equilibria that are limit cycles may originate and, under the circumstances detailed below, they may become the attractors of other equilibria beginning nearby.

From an economic point of view, the relevant features of the set of equilibria can be summarized as follows:

(i) to a given initial condition N_0, one can often associate a multiplicity (indeed, a continuum) of dynamic equilibria;

(ii) the latter need not converge to the same attractor: depending on essentially arbitrary expectations, equilibria beginning at a common value N_0 may converge to very different asymptotic positions;

(iii) on the other hand, as the initial condition N_0 moves in $[0,1]$, different sets of attractors emerge for the equilibria beginning at N_0: initial conditions matter in this economy;

(iv) expectations may nevertheless matter more than initial conditions: under certain configurations, even when N_0 is very close to one, an equilibrium path exists that asymptotically converges to E^0 if expectations are pessimistic enough. Conversely, very optimistic expectations will produce equilibria converging to E^2 from initial conditions near $N_0 = 0$;

(v) cyclic equilibria exist and, under certain circumstances, a cycle can be the global attractor of almost all the equilibria departing from any given initial condition $0 \leq N_0 \leq 1$.

Propositions 5 and 6 are dedicated to a precise mathematical description of the various cases. They are inevitably tedious and the impatient reader should make use of Figures 5a-5e and 6a-6e to facilitate geometrical understanding. Shaded areas describe "basins of attraction" for the attractor contained within. They correspond to a continuum of equilibria having a common asymptotic behavior.

We consider first the case in which E^1 is a sink, in Proposition 5. Then we consider the case in which E^1 is a source in Proposition 6.

Proposition 5: Assume IRS, $\underline{c} > 0$, and that E^1 is a sink. The global dynamics of the system can be as in either one of the five figures 5a-e. The latter originate three distinct configurations of the equilibrium set.

1. (Figure 5a.) There exists a repelling limit cycle γ around E^1 and $W^u(E^2)$ and $W^s(E^2)$ are the graphs of two functions from N to x. Denote by $\underline{N} < N^1$ and $\bar{N} > N^1$ the westmost and eastmost points of γ, respectively. Then:
   a. for $N_0 < \underline{N}$ there are only two equilibria given by $W^c(E^0)$ and $W^s(E^2)$;
   b. for $\underline{N} \le N_0 \le \bar{N}$ there are four types of equilibria: $W^c(W^0)$, $W^s(E^2)$, γ, and the continuum of paths converging to E^1;
   c. for $\bar{N} < N_0 \le \tilde{N}$, where $\tilde{N}$ denotes the eastmost point on $W^c(E^0)$, there are again two equilibria given by $W^c(E^0)$ and $W^s(E^2)$;
   d. for $N_0 > \tilde{N}$ there is a unique equilibrium given by $W^s(E^2)$.
2. (Figures 5b and 5c.) There exists a repelling limit cycle γ around E^1 and either $W^u(E^2)$ or $W^s(E^2)$ or both are not the graph of a function

from N into x. Let $\underline{N}$, $\bar{N}$, and $\tilde{N}$ be as in 1. and denote with $\underline{\underline{N}} < \underline{N}$ the westmost point on $W^s(E^2)$. Then:

a. for $N_0 < \underline{\underline{N}}$ there exists only one equilibrium: $W^c(E^0)$;

b. for $\underline{\underline{N}} \leq N_0 < \underline{N}$ and $\bar{N} < N_0 \leq \tilde{N}$ there exist two equilibria: $W^c(E^0)$ and $W^s(E^2)$;

c. for $\underline{N} \leq N_0 \leq \bar{N}$ there are four types of equilibria: $W^c(E^0)$, $W^s(E^2)$, γ, and the continuum of paths converging to E^1;

d. for $N_0 > \tilde{N}$ there exists only one equilibrium: $W^s(E^2)$.

3. (Figures 5d and 5e.) There exists no limit cycle around E^1 and $W^c(E^0)$ is the graph of a function from N into x. Let $\underline{\underline{N}}$ be as defined in 2. Then:

a. if $0 \leq N_0 < \underline{\underline{N}}$ there exists a unique equilibrium: $W^c(E^0)$;

b. for $N_0 \geq \underline{\underline{N}}$ there exist three types of equilibria: $W^c(E^0)$, $W^s(E^2)$, and the continuum of equilibria converging to E^1.

Proof: See Appendix.

<Insert Figures 5a-e here>

Proposition 6: Let the hypotheses of Proposition 5 be true but assume that E^1 is now a source. The global dynamics of the system in this case can be as in either one of the Figures 6a-e. The latter are associated with three distinct configurations of the equilibrium set:

1. (Figure 6a.) There are no limit cycles around E^1 and both $W^u(E^2)$ and $W^s(E^2)$ are graphs of functions from N into x. Then:

a. for all $0 \leq N_0 \leq \tilde{N}$, where $\tilde{N}$ is as in Proposition 5, there are two equilibria: $W^c(E^0)$ and $W^s(E^2)$;

b. if $N_0 > \tilde{N}$ there is a unique equilibrium $W^s(E^2)$.

2. (Figures 6b and 6c.) There are no limit cycles around E^1 but either $W^u(E^2)$ or $W^s(E^2)$ or both are not graphs of a function. Then:

a. for $0 \le N_0 < \underline{\underline{N}}$ there is only an equilibrium: $W^c(E^0)$;

b. for $\underline{\underline{N}} \le N_0 \le \tilde{N}$ there are two equilibria: $W^c(E^0)$ and $W^s(E^2)$;

c. for $N_0 > \tilde{N}$ there is a unique equilibrium: $W^c(E^0)$.

3. (Figures 6d and 6e.) there is an attracting limit cycle γ around E^1 and $W^c(E^0)$ is the graph of a function. Then:

a. for all $0 \le N_0 < \underline{\underline{N}}$ there is a unique equilibrium: $W^c(E^0)$;

b. for all $\underline{\underline{N}} \le N_0 < \underline{N}$ and $\bar{N} < N_0$ there are three types of equilibria: $W^c(E^0)$, $W^s(E^2)$, and the continuum of equilibria converging to γ;

c. for all $\underline{N} \le N_0 \le \bar{N}$ there are four types of equilibria: the three in (b) and the limit cycle γ.

<Insert Figures 6a-e here>

Proof: See Appendix.

From a practical point of view one is obviously interested in finding computable conditions that can be used to discriminate between the various cases. Checking the local stability-instability of E^1 is only a matter of linearizing and computing the associated eigenvalues. The interesting thing is to have a simple set of conditions that could detect the existence of the (attractive or repulsive) limit cycles. The Andronov-Hopf bifurcation theorem turns out to be quite useful for this purpose.

Proposition 7: Assume that the trace of (4.1) equals zero, i.e., that $B'(N_1)x_1^2N_1 + B(N_1)s_1(x_1)/u'(x_1) = \bar{r} \geq 0$. Then there exists a $\mu > 0$, such that for all r in either $(\bar{r} + \mu)$ or $(\bar{r} - \mu)$ the system (3.1), (3.2) has a limit cycle around the critical point E^1. If such a cycle is unique it will be locally asymptotically stable when it exists for $r \in (\bar{r} + \mu)$ and unstable when it exists for $r \in (\bar{r} - \mu)$.

Proof: See Appendix.

5. Welfare

We now turn to a discussion of welfare. In the IRS case with multiple steady states, it is easy to see that those with greater values of N are Pareto superior.[10] Thus, for the IRS case, consider two steady states (N_1,x_1) and (N_2,x_2), with $N_2 > N_1$ and, therefore, $x_2 < x_1$. Since (2.5) implies $rV_0 = -w_0 + \alpha s_0[u_\varepsilon(x)]$ in steady state, V_0 is decreasing in x. Since (2.4) implies $V_1 = V_0 + u_\varepsilon(x)$, V_1 is also decreasing in x. Hence, all agents are better off in the high N-low x outcome. We now argue that the steady state with the greatest N is not generally Pareto optimal in our model. To make the point in a simple way, we assume that c = 0 with probability 1 (production is free, although agents still have to search for projects). We begin with the case of CRS, and consider IRS below. We also

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[10]The same is not true in the case of DRS: in this instance higher values of N go together with higher values of x (the T-locus is upward sloping) and the simple argument we give below may not go through or may even be reversed.

set $w_0 = w_1 = 0$ to reduce notation.[11]

When $c \equiv 0$, equation (2.1) immediately implies $rV_0 = \alpha(V_1 - V_0)$ in a steady state, and thus $V_0 - V_1 = -rV_1/(\alpha + r)$. Further, if we let <u>any</u> value of x determine the reservation trade, and not necessarily the individual utility maximizing value of x, then by equation (2.2), in steady state we have

$$\text{(5.1)} \qquad rV_1 = \beta\theta x(V_0 - V_1) + \beta\theta \int_0^x u_\varepsilon(z)dz$$
$$= -(\beta\theta x/(\alpha + r))rV_1 + \beta\theta \int_0^x u_\varepsilon(z)dz.$$

Differentiating and setting $\partial V_1/\partial x = 0$ yields the first order condition for the value of x that maximizes V_1. If we substitute this into (5.1) and use the equilibrium condition $\theta = x$, we arrive at

$$\text{(5.2)} \qquad u_\varepsilon(x) = (\beta/(\alpha + r))xs_1(x).$$

The x that satisfies this equation is the unique nondegenerate steady state equilibrium for this case. We claim it is not Pareto optimal.

To see why, consider a social planner's problem of choosing x to maximize V_1 and V_0.[12] If we insert $\theta = x$ into (5.1) and <u>then</u> differentiate, we find that the first order condition for the planner's choice of x

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[11] With CRS and c = 0, the standard model predicts the unique nondegenerate steady state is efficient, which illustrates one way that having differentiated commodities makes an important difference.

[12] There is no ambiguity as to the correct welfare criterion for a social planner in the special case under consideration (CRS and free production), because V_1 and V_0 are proportional.

satisfies

$$2xrV_1 = (\alpha + r)xu_\varepsilon(x) + (\alpha + r) \int_0^x u_\varepsilon(z)dz.$$

Substituting this into (5.1) and simplifying yields

$$(5.3) \qquad u_\varepsilon(x) = (\beta/(\alpha + r))xs_1(x) - (1/x)\int_0^x u_\varepsilon(z)dz.$$

The solution to this equation is the socially optimal reservation good. Comparing (5.2) and (5.3), the right side of the latter has an additional term, which is negative, and thus the planner chooses a greater x than the equilibrium value.

In equilibrium, therefore, there are too many people in exchange and too few in production. Agents searching for production opportunities are often identified as unemployed in this literature, and so we could say that there is too little unemployment in laissez-faire. The intuition is somewhat different from existing theories. In this model, individuals are only willing to trade when they meet a partner who has something they want, which means a good within the distance x of their ideal. In order for trade to occur, both partners have to have something the other finds acceptable, and therefore individuals often get denied things they want because their partners do not want what they have. All agents would be better off if they would all lower their standards. This would reduce the amount of time spent in exchange and increase the time spent in the production process.[13]

We now consider the IRS case. An argument similar to the one used above implies the decentralized steady state is characterized by (5.2), except that now β is replaced by $B(N(x))$, where $N(x)$ solves $\dot{N} = \alpha(1 - N) - B(N)Nx^2 = 0$. Now the steady state equilibrium satisfies

$$(5.4) \qquad u_\varepsilon(x) = (B(N(x))/(\alpha + r))xs_1(x).$$

There can, of course, be more than one solution to (5.4); for the sake of argument, consider the best steady state (with the lowest x and greatest N). As above, it can be shown that the optimal reservation level satisfies

$$(5.5) \qquad u_\varepsilon(x) = (B(N(x))/(\alpha + r))xs_1(x) - [1/x + B'N'/B] \int_0^x u_\varepsilon(z)dz$$

(notice that when $B' = 0$, this reduces to equation (5.3)). By comparing (5.4) with (5.5), we see that whether the optimal x is above or below the equilibrium x depends on the sign of the expression is square brackets in (5.5).

The first term in the bracketed expression represents the effect discussed above for the case of CRS--individuals neglect the external effect of x on the frequency of trade, which leads to too little trade and too little unemployment. However, the second term inside the square brackets in (5.5) is negative, since $N' < 0$ along the $\dot{N} = 0$ locus. This effect tends to

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[13]This gives some insight into the welfare improving role of fiat currency in the versions of this model studied in Kiyotaki and Wright (1989, 1990). A generally accepted fiat medium of exchange increases the frequency of trade=, and there is too little trade in laissez-faire.

make x too big, which reduces N and therefore the arrival rate B. As in the standard one-good model, this effect leads to too much unemployment in equilibrium. The net result obviously depends on which effect dominates. In any case, our generalization does point out an important qualification to the prediction of the standard model. Such a qualification becomes more relevant in the DRS case, even when only one stationary state exists. In this instance the sign of B' is negative and the whole expression within square brackets becomes positive. Therefore, the equilibrium level of x is unequivocally too low and N is too high: efficiency requires an increase in "unemployment" and a decrease in the "quality" of the goods accepted in exchange.

6. Comparative Statics

In this section we study the effects of changes in the exogenous variables, concentrating on the case of CRS. In this case, the unique nondegenerate equilibrium is characterized by

$$(6.1) \qquad \phi(x) \equiv ru_\varepsilon(x) + \alpha s_0[u_\varepsilon(x)] - \beta x s_1(x) + w_1 - w_0 = 0.$$

If we let $\Delta = -1/\phi' > 0$, then (6.1) and $k = u_\varepsilon(x)$ yield:

$$\begin{aligned}
\partial x/\partial w_1 &= \Delta > 0 & \qquad \partial k/\partial w_1 &= u'\Delta < 0 \\
\partial x/\partial w_0 &= -\Delta < 0 & \qquad \partial k/\partial w_0 &= -u'\Delta > 0 \\
\partial x/\partial r &= \Delta u_\varepsilon > 0 & \qquad \partial k/\partial r &= u'\Delta u_\varepsilon < 0 \\
\partial x/\partial \alpha &= \Delta s_0 > 0 & \qquad \partial k/\partial \alpha &= u'\Delta s_0 < 0 \\
\partial x/\partial \beta &= -\Delta s_1 < 0 & \qquad \partial k/\partial \beta &= -u'\Delta s_1 > 0
\end{aligned}$$

$\partial x/\partial \varepsilon = \Delta(r + \alpha\Gamma) < 0 \qquad \partial k/\partial \varepsilon = \Delta\beta(x^2 u' \quad s_1) < 0$

All of these are fairly intuitive, and interpretation is left to the reader.

We would also like to consider changes in the production cost distribution, $F(c)$. An efficient way to parameterize things is to suppose that F belongs to a family of CDFs indexed by σ, $\{F(c,\sigma)\}$, with the property that $\sigma_2 > \sigma_1$ implies $F(c,\sigma_2)$ second order stochastically dominates $F(c,\sigma_1)$; that is, $\sigma_2 > \sigma_1$ implies

$$\int_0^K [F(c,\sigma_2) - F(c,\sigma_1)]dc \le 0 \text{ for all } K \ge 0.$$

Dividing by $\sigma_2 - \sigma_1$ and taking the limit as $\sigma_2 \to \sigma_1$ reduces this to

$$\int_0^K F_2(c,\sigma)dc \le 0 \text{ for all } K \ge 0,$$

where $F_2(c,\sigma) = \partial F/\partial\sigma$. Differentiating (6.1), we now immediately obtain the result

$$\text{(6.3)} \qquad \partial x/\partial\sigma = \Delta \int_0^K F_2(c,\sigma)dc \le 0.$$

Also, since $k = u_\varepsilon(x)$, we have $\partial k/\partial\sigma = u'\partial x/\partial\sigma \ge 0$.

We conclude that changing $F(c)$ so that it second order stochastically increases makes agents more demanding in exchange and less demanding in production. Several interesting effects are special cases of this general result. For example, suppose that the mean of $F(c,\sigma)$ is independent of σ; then reducing σ implies an increase in risk in standard mean preserving spread sense. Hence, more risk raises x and lowers k, so that agents become less demanding in trade but more demanding in production. For other cases of interest, recall that second order is implied by first order stochastic dominance. In particular, reducing all costs by a fixed amount (a translation of F), or proportionally (a scale transformation of F), first order and therefore second order stochastically decreases F, which reduces σ and thereby raises x and lowers k.

Another way to show this last result is to define <u>net cost</u> as $c_n = (1 - \gamma)c - \tau$, where we could interpret γ and τ as proportional and lump sum production subsidies. An increases in τ is equivalent to a translation of the CDF, while an increase in γ is equivalent to a scale transformation. Generalizing the arguments leading to (6.1), the steady state condition becomes

$$\Phi(x) = w_1 - w_0 + ru_\varepsilon(x) + \alpha(1 - \gamma)s_0(k) - \beta x s_1(x), \tag{6.4}$$

where now $k = (u_\varepsilon(x) + \tau)/(1 - \gamma)$. Differentiation yields $\partial x/\partial\tau > 0$ and $\partial x/\partial\gamma > 0$, as argued above using stochastic dominance. However, we also find after simplification that

$$\begin{aligned} \partial k/\partial\tau &= (\Delta/(1 - \gamma))[\beta s_1 - (r + \beta x^2)u'] > 0 \\ \partial k/\partial\gamma &= (\Delta/(1 - \gamma))[k\beta s_1 - (kr + k\beta + \alpha s_0)u'] > 0. \end{aligned} \tag{6.5}$$

The results in (6.5) may appear to contradict our earlier conclusion, that reducing c by a fixed amount or proportionally should lower k.

Fortunately, the contradiction is only apparent. Let $k_n = (1 - \gamma) - \tau$ define the net reservation cost. Then $k_n = u_\varepsilon(x)$, and hence we immediately have $\partial k_n/\partial\tau = u'\partial x/\partial\tau < 0$ and $\partial k_n/\partial\gamma = u'\partial x/\partial\gamma < 0$. In other words, the net reservation cost definitely falls when all costs are reduced by a fixed amount or proportionally, consistent with our earlier results, and it is the "after-subsidy" value of k that is relevant. As shown in Figure 7, a lump sum cost subsidy increases k but by less than the amount of the subsidy, so that k_n falls. This makes agents more willing to take on expensive projects, in gross terms, but not in net terms. Figure 7 also shows the effect of a translation of $F(\bullet)$, which results in exactly the same net effect.

We now investigate how the steady state value of N depends on the parameters. Let $H_0 = \alpha F(k)$ be the hazard rate in production (the rate at which projects arrive times the probability they are accepted), and let $H_1 = \beta x^2$ be the hazard rate in the exchange sector (the rate at which partners meet times the probability they agree to trade). As discussed above, agents looking for an opportunity in the production sector are sometimes referred to as unemployed. On this interpretation, $1/H_0$ is the average duration of an unemployment spell, $1/H_1$ is the average time between such spells, and $1 - N = H_1/(H_0 + H_1)$ is the aggregate unemployment rate. Notice that an increase in x or a fall in k, holding α and β constant, will raise H_1, lower H_0, and lower N. Thus, for example, our earlier results imply that an increase in $w_1 - w_0$ or r reduces N and leads to more unemployment.[14]

The impact of changes in the arrival rates, α and β, are complicated, because they affect N directly as well as indirectly through x and k. To develop some intuition, we begin with the case where $F(k) = 1$, which means that all projects are acceptable (this would certainly be true, e.g., if $c = 0$ with probability 1). We also assume $w_0 = w_1$ from now on to reduce the notation. Then $N = \alpha/(\alpha + \beta x^2)$, and after simplification we find

$$\partial N/\partial\alpha = \Phi\beta[2rk - (r + \alpha)xu' + \beta\Sigma] > 0$$

(6.6)

$$\partial N/\partial\beta = \Phi\alpha X[(r + \alpha)u' - \beta\Sigma] < 0,$$

where $\Phi = \Delta X/(\alpha + \beta x^2)^2 > 0$, and $\Sigma = xs_1' - x_2 \geq 0$ by Lemma 1. At least when $F(k) = 1$, then we have the reasonable result that an increase in α or decrease in β increases the number of agents in the exchange sector.

Returning to the general case, one can easily verify $\partial H_1/\partial\alpha > 0$, $\partial H_1/\partial\beta > 0$ and $\partial H_0/\partial\beta > 0$, but perhaps surprisingly, the effect of α on $H_0 = \alpha F[u_\varepsilon(x)]$ gives us some trouble. Differentiating, we have

$$(6.7)\qquad \partial H_0/\partial\alpha = F + \alpha F'u'\partial x/\partial\alpha = F + \alpha F'u'\Delta \int_0^k F(c)dc.$$

The sign of (6.7) cannot be determined in general. This is analogous to a result in the partial equilibrium job search model, where an increase in the

[14]An increase in ε is more complicated since it lowers both H_1 and H_0, and the net effect apparently cannot be determined in general. The same is true for changes in the production cost distribution.

offer arrival rate need not increase the probability of leaving unemployment (because the reservation wage increases). Burdett (1981) was the first to show the counter intuitive case, where more frequent offers increases the duration of unemployment, can be ruled out by assuming that the wage offer distribution is log-concave, and our goal is to derive some similar results here.[15]

The important fact about log-concavity for our purposes is that it implies the left and right truncated mean functions.

$$\mu_1(K) = \int_0^K cdF(c)/(1 - F(K)) \text{ and } \mu_r(K) = \int_0^K cdF(c)/F(K).$$

have slopes that are less than one (see, e.g., Goldberger (1983)). In particular, $\mu_r' = F(c)dcF'/F^2 \leq 1$. Inserting μ_r' into (6.7) and simplifying,

$$\partial H_0/\partial\alpha = \Delta F[\beta s_1 - (r + \beta x^2)u' + \alpha Fu'(\mu_r' - 1)].$$

Since log-concavity implies $\mu_r' \leq 1$, it guarantees $\partial H_0/\partial\alpha > 0$, and therefore increasing the arrival rate α will necessarily raise the hazard rate and lower the average length of a spell in the production sector.[16]

Even if an increase in α raises H_0, this does not mean that it will necessarily lower the steady state proportion of agents in the production

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[15]Log-concavity means that log[f(c)] is a concave function, where $f = F'$ is the density. This restriction has been used frequently in search theory since Burdett's work (see Wright and Loberg (1987), e.g., for extended discussion and references).

[16]Without log-concavity, we could have $\mu_r' > 1$, and hence $\partial H_0/\partial\alpha < 0$, for small values of β and r.

sector, since an increase in α also raises H_1. After simplification, the net effect is given by

$$\partial N/\partial \alpha = \bar{\Phi}[\beta s_1 - (r + \beta x^2)u' - \alpha F u'(1 = \mu_r') - 2\alpha s_0/x]$$

where $\bar{\Phi} > 0$. The sign of this expression cannot be determined in general. But if we assume log-concavity then $\mu_r' \leq 1$, and after some algebra, we have

$$\partial N/\partial \alpha \geq \Delta \beta x F[\beta s_1 - (r + \beta x^2)u' - 2\alpha s_0/x]$$
$$= \Delta \beta x F[\beta \Sigma - ru' + 2rk/x] > 0,$$

where, again, $\Sigma \geq 0$ by Lemma 1. We conclude that log-concavity not only guarantees that an increase in α raises H_0, but by more than enough to compensate for the increase in H_1, so that the net effect is to increase N.

7. Conclusion

This paper has analyzed the dynamic and steady state behavior of an equilibrium search model with heterogeneous commodities. Some of the results are similar to the standard one-good search model, while others are rather different (e.g., welfare properties). The multiplicity of dynamic equilibrium paths displayed by the model are very robust: as described by Propositions 5 and 6, two economies beginning at identical initial conditions can converge to remarkably different attractors. Which of these trajectories we follow depends on the way expectations are formed, and on the emergence of focal points upon which agents can coordinate their beliefs. The model is essentially silent on these issues and, given the initial distribution of agents across sectors, self-fulfilling "optimism" or "pessimism" can drive $N(t)$, and $x(t)$ along a wide variety of different paths.

Very simple versions of this model have already been applied to monetary economics. Future work may attempt to apply the model to other substantive areas in economics, or may attempt to incorporate the complications introduced in this paper into the versions of the model with fiat money. There are also other types of dynamic phenomena that we have not investigated, such as the emergence of "sunspot equilibria" in which the economy fluctuates stochastically even though the fundamental structure is deterministic and time invariant.

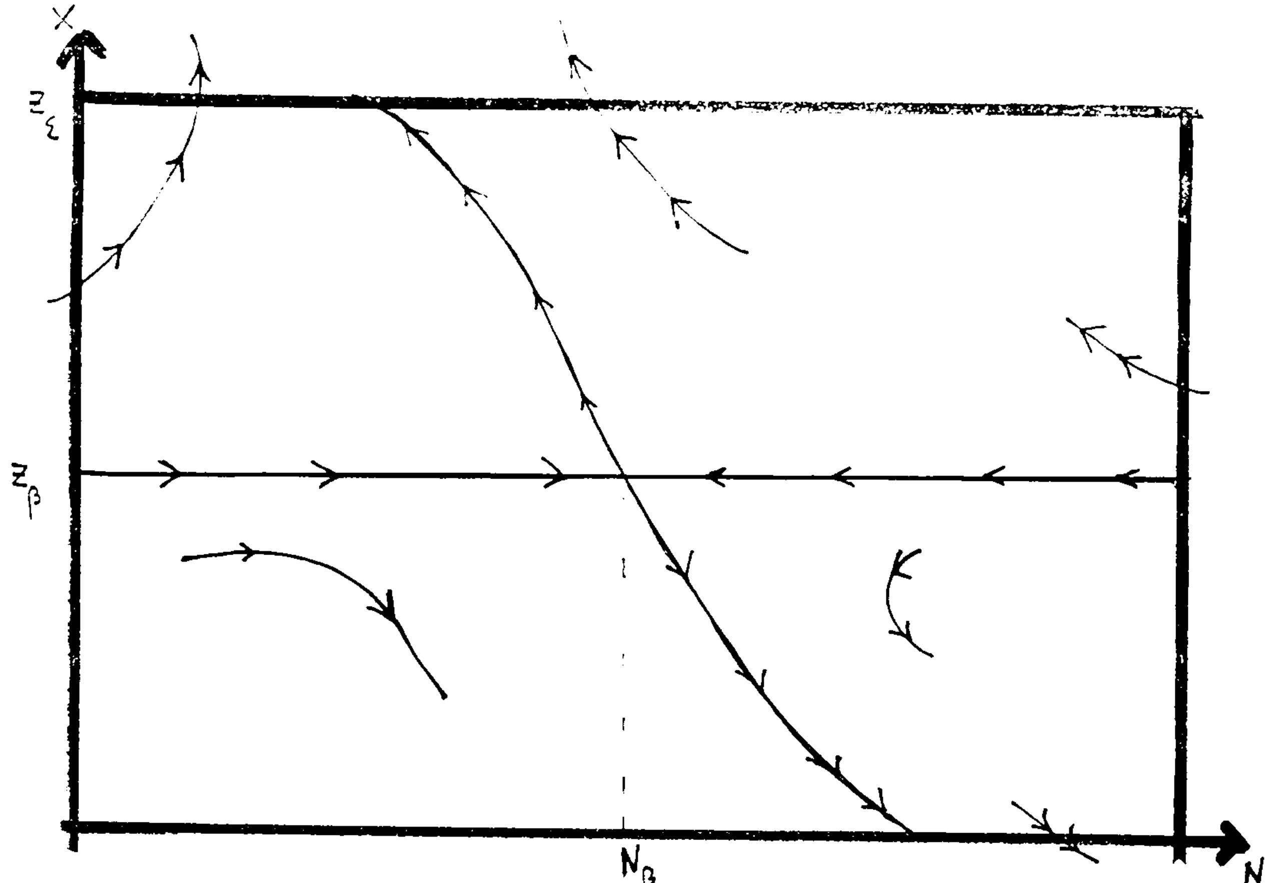


FIGURE 1

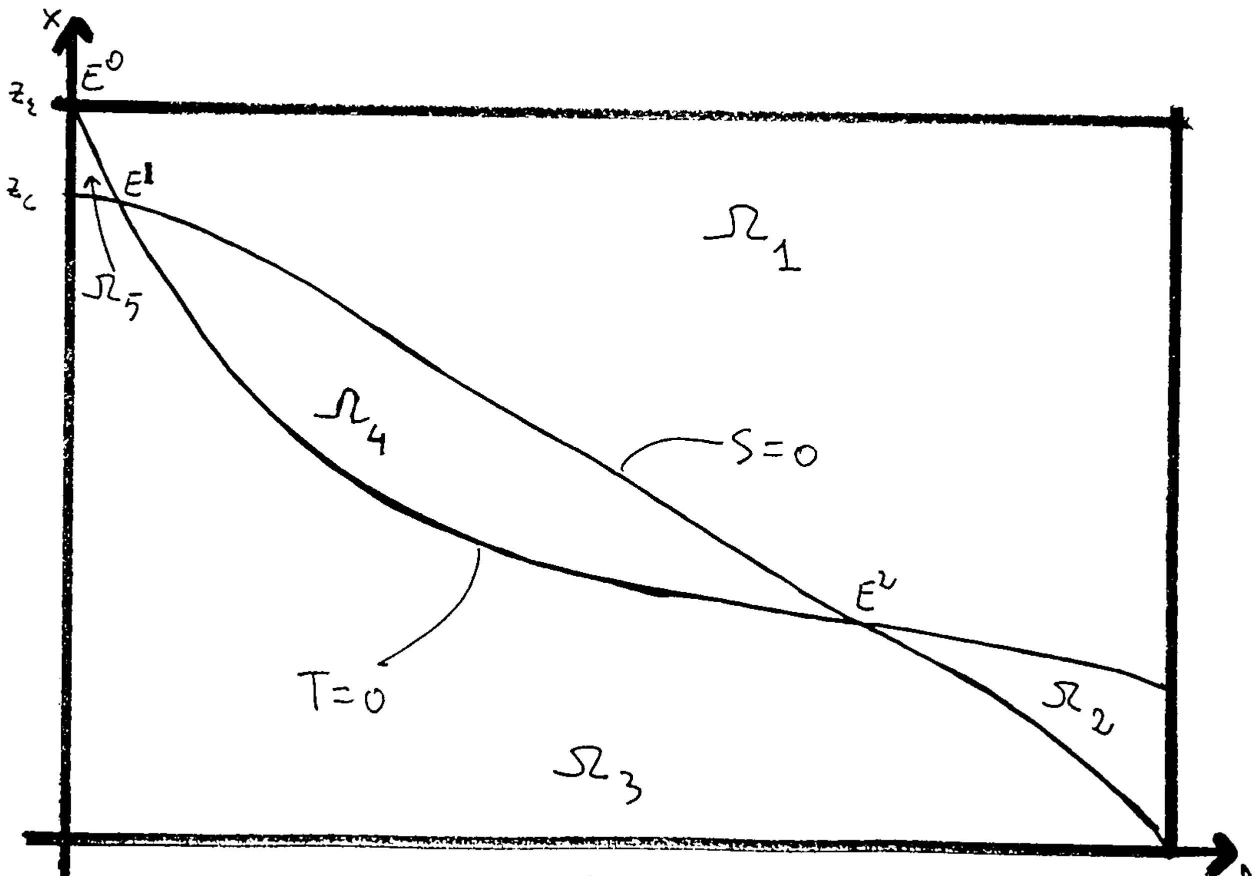


FIGURE 2

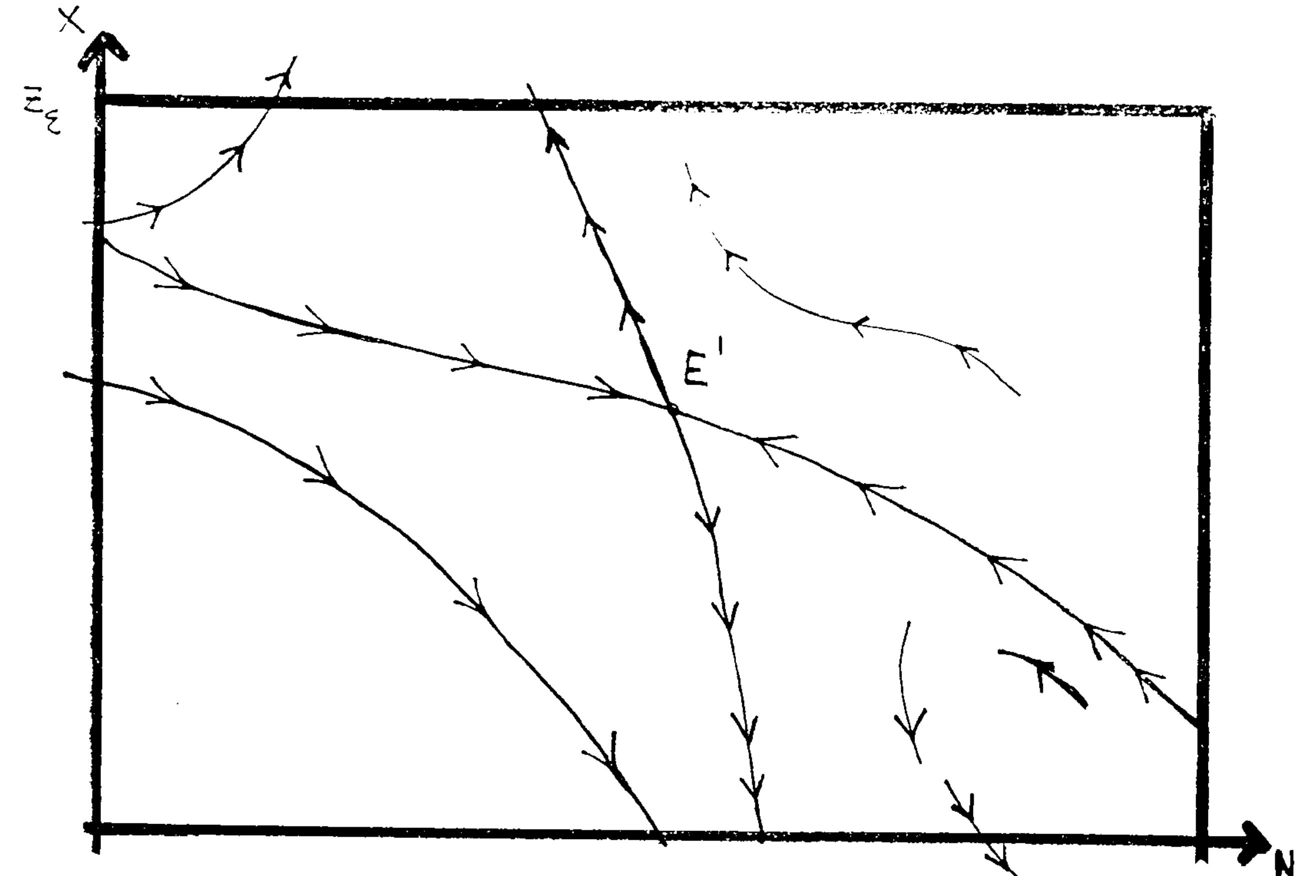


FIGURE 3

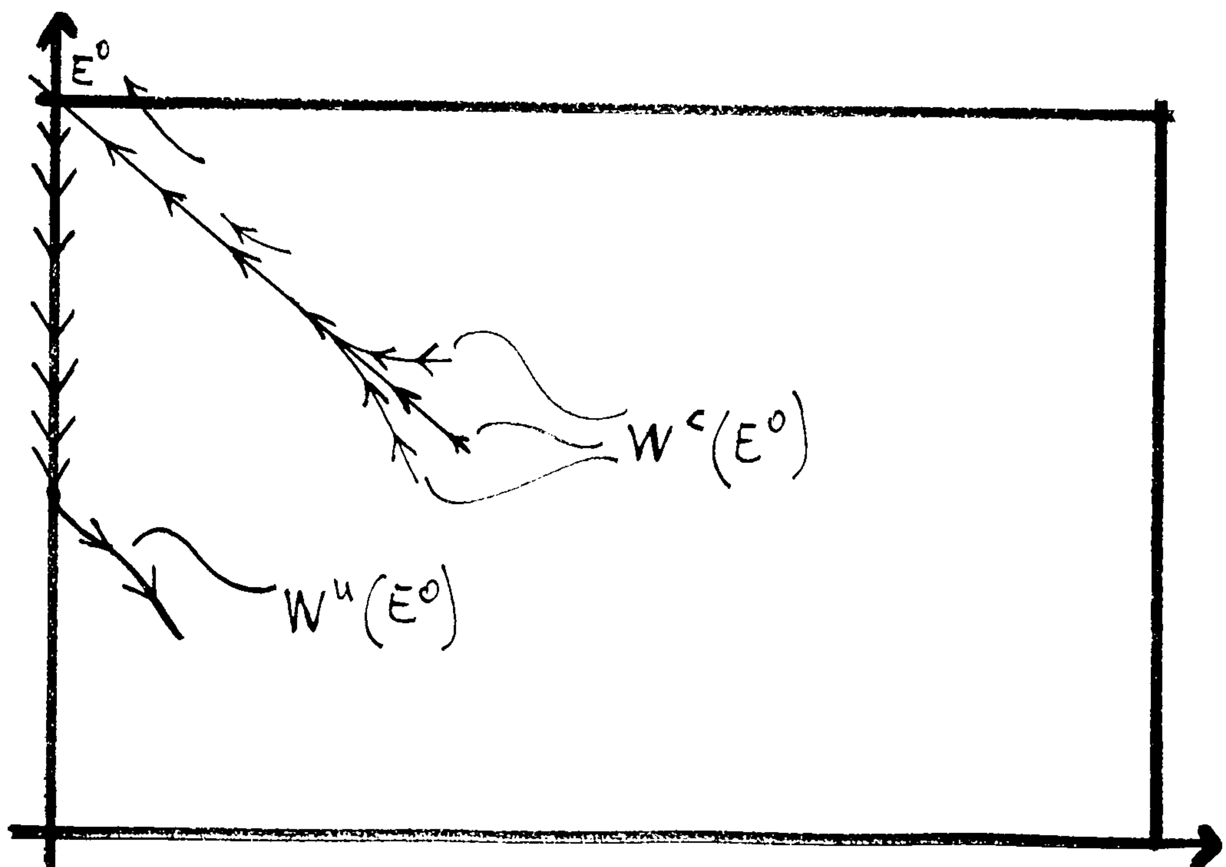


FIGURE 4

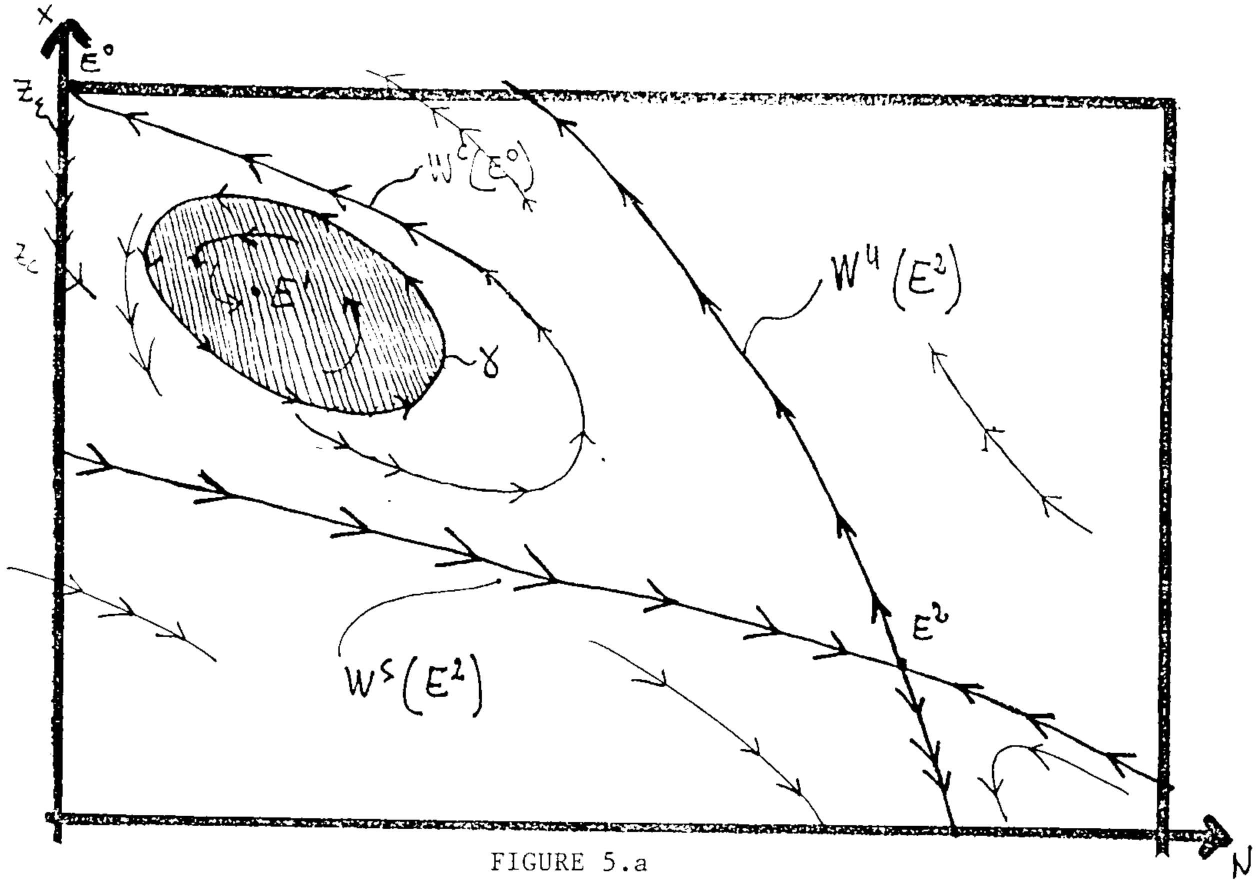


FIGURE 5.a

Shaded areas denote continua of equilibria.

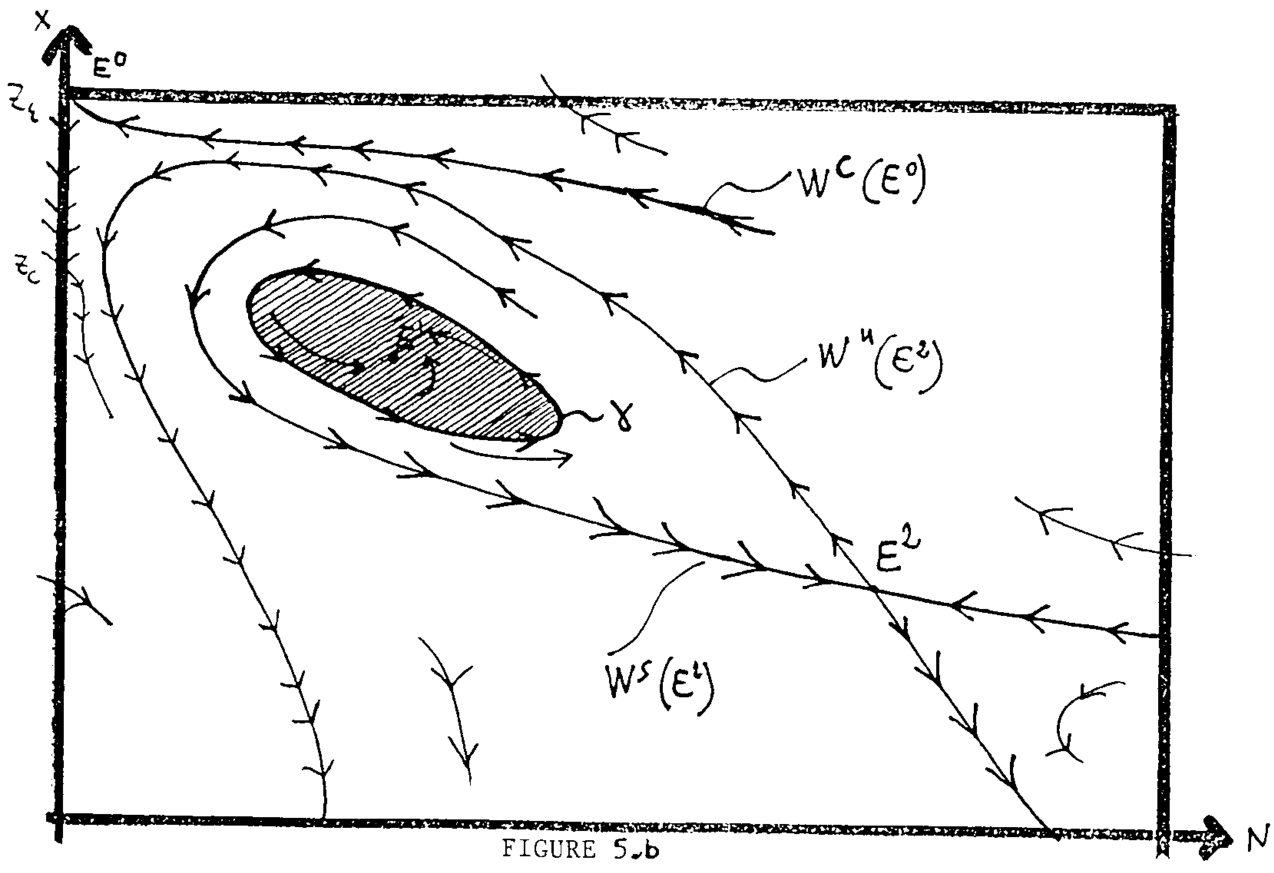


FIGURE 5.b

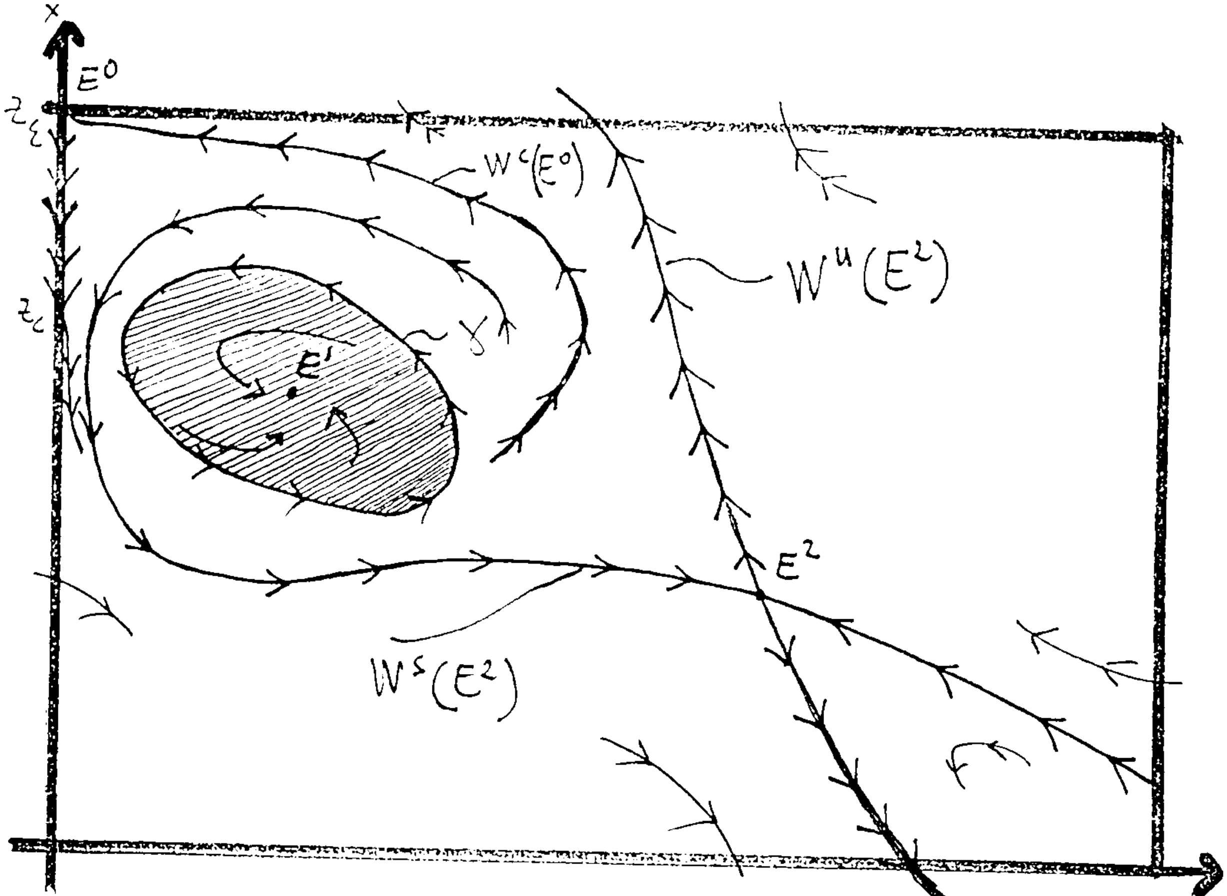


FIGURE 5.c

Shaded areas denote continua of equilibria.

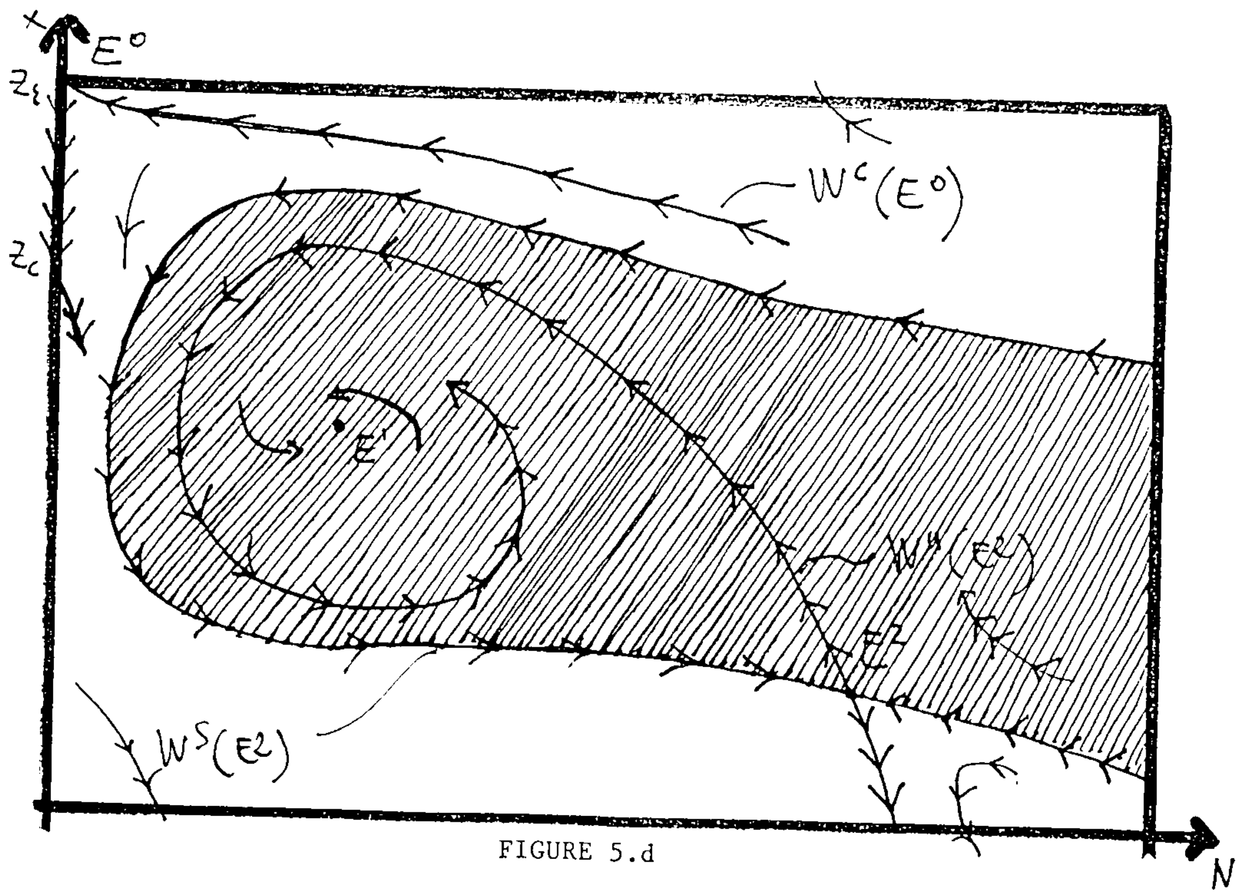


FIGURE 5.d

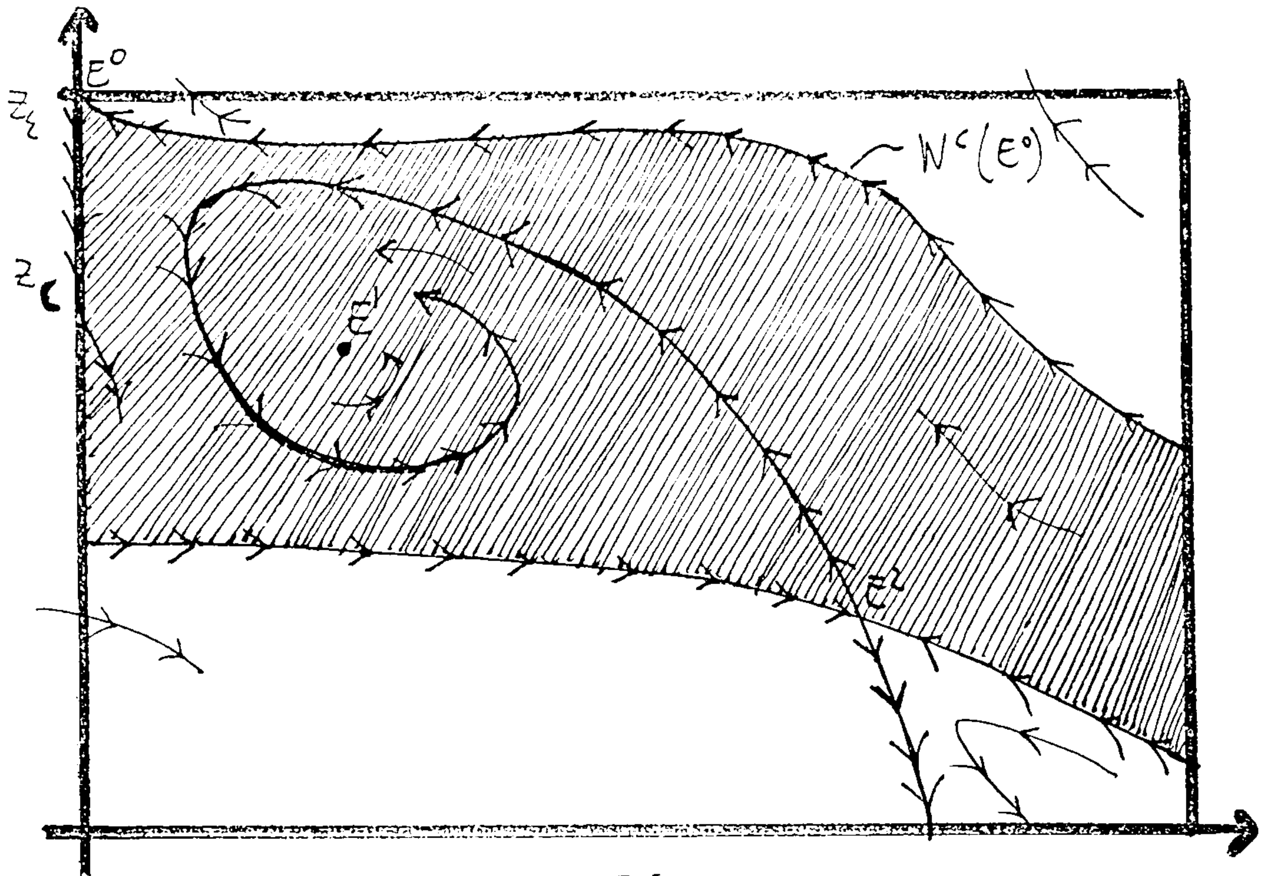


FIGURE 5.e

Shaded areas denote continua of equilibria.

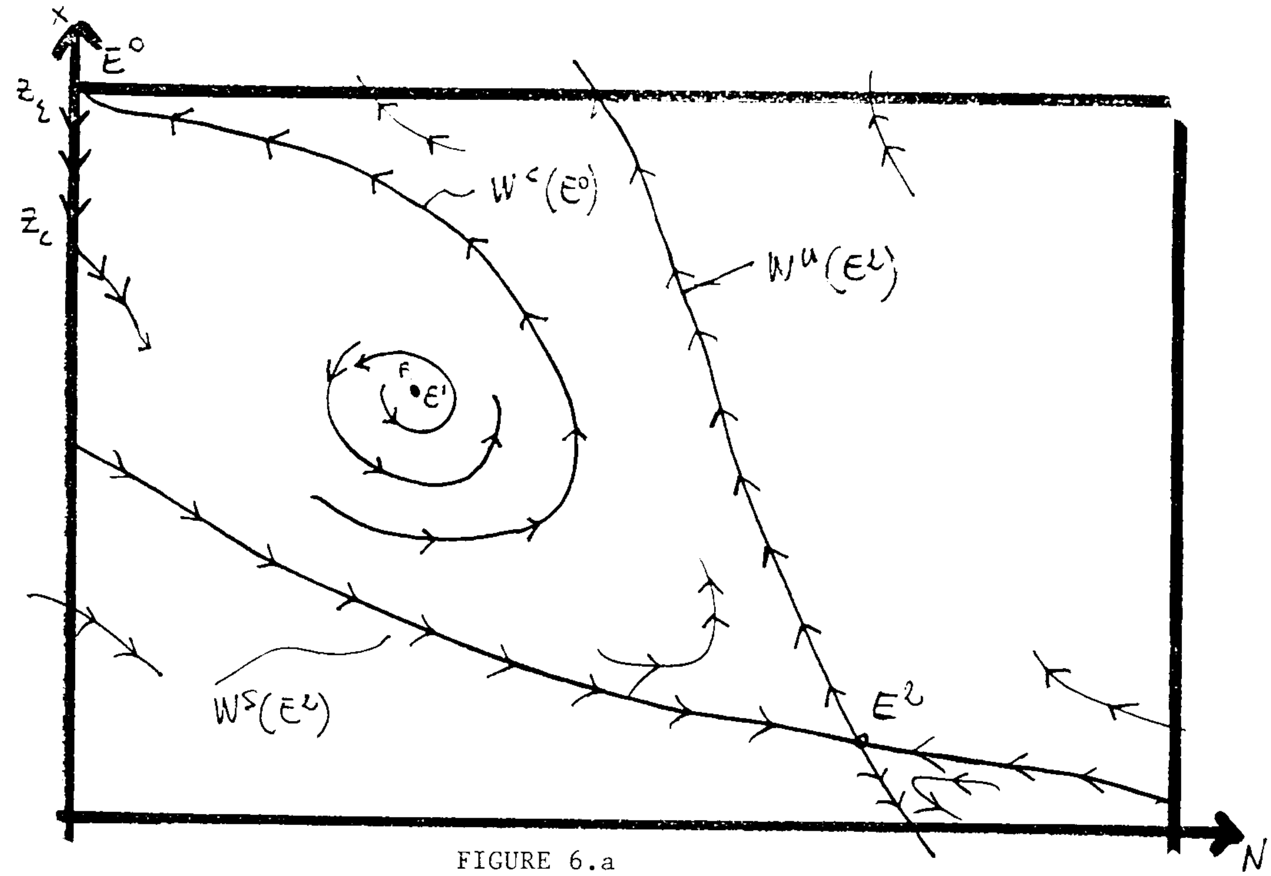


FIGURE 6.a

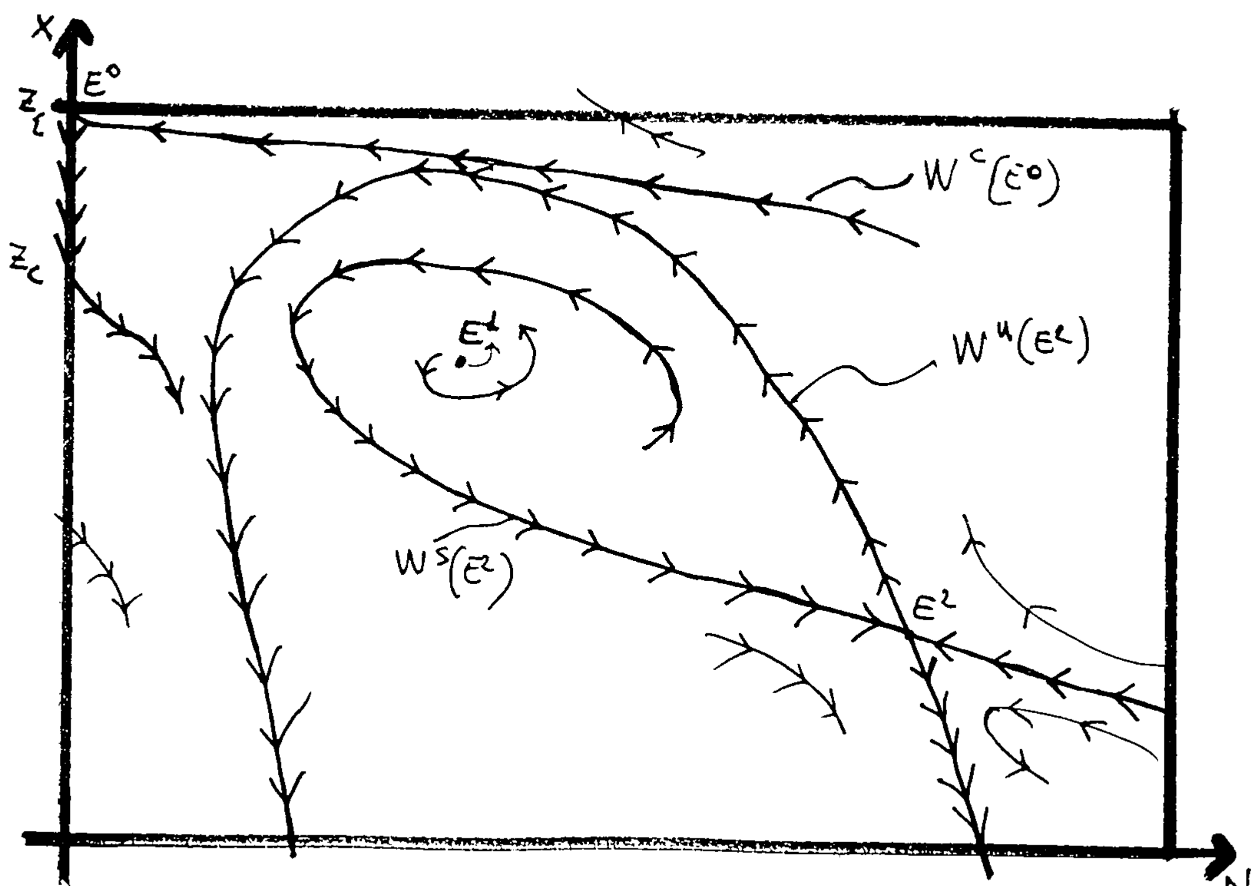


FIGURE 6.b

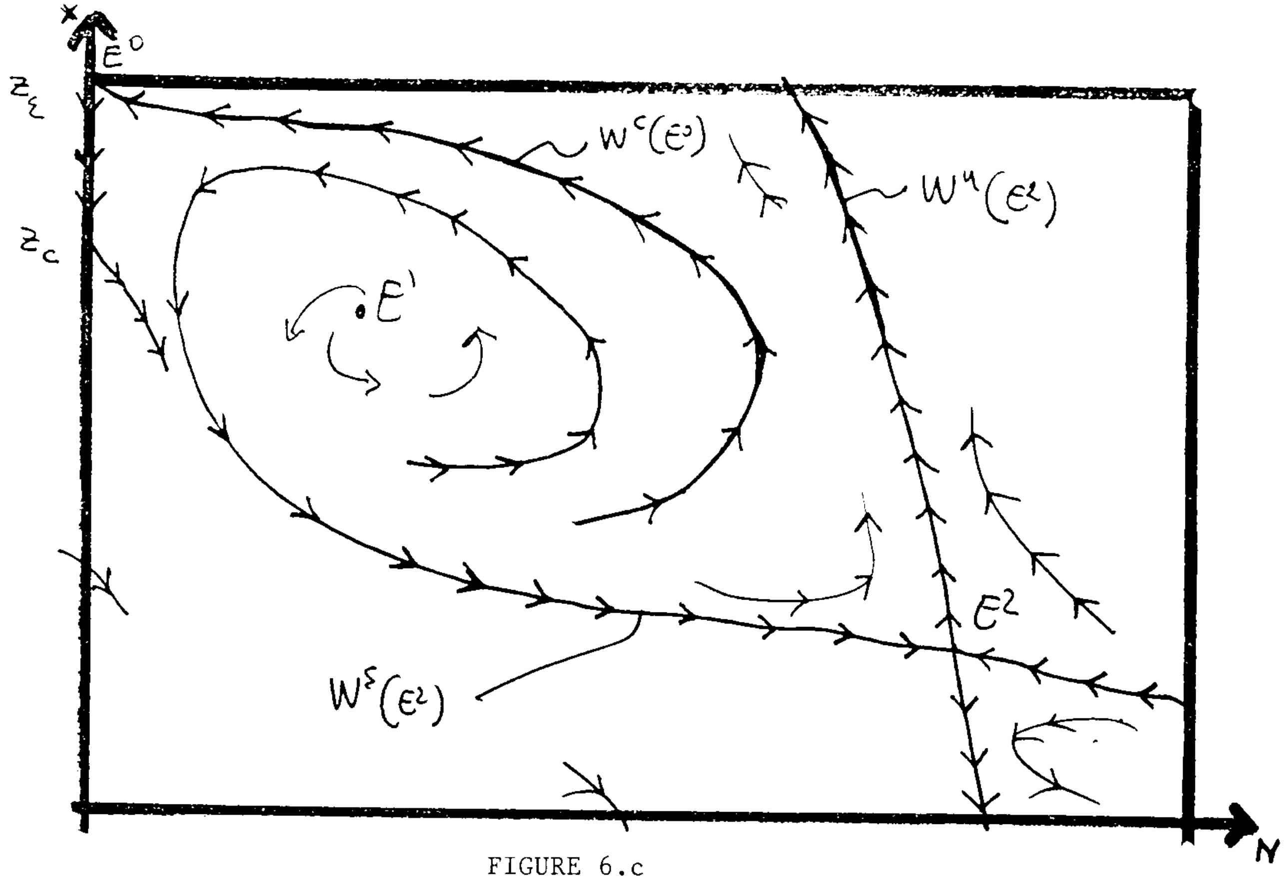


FIGURE 6.c

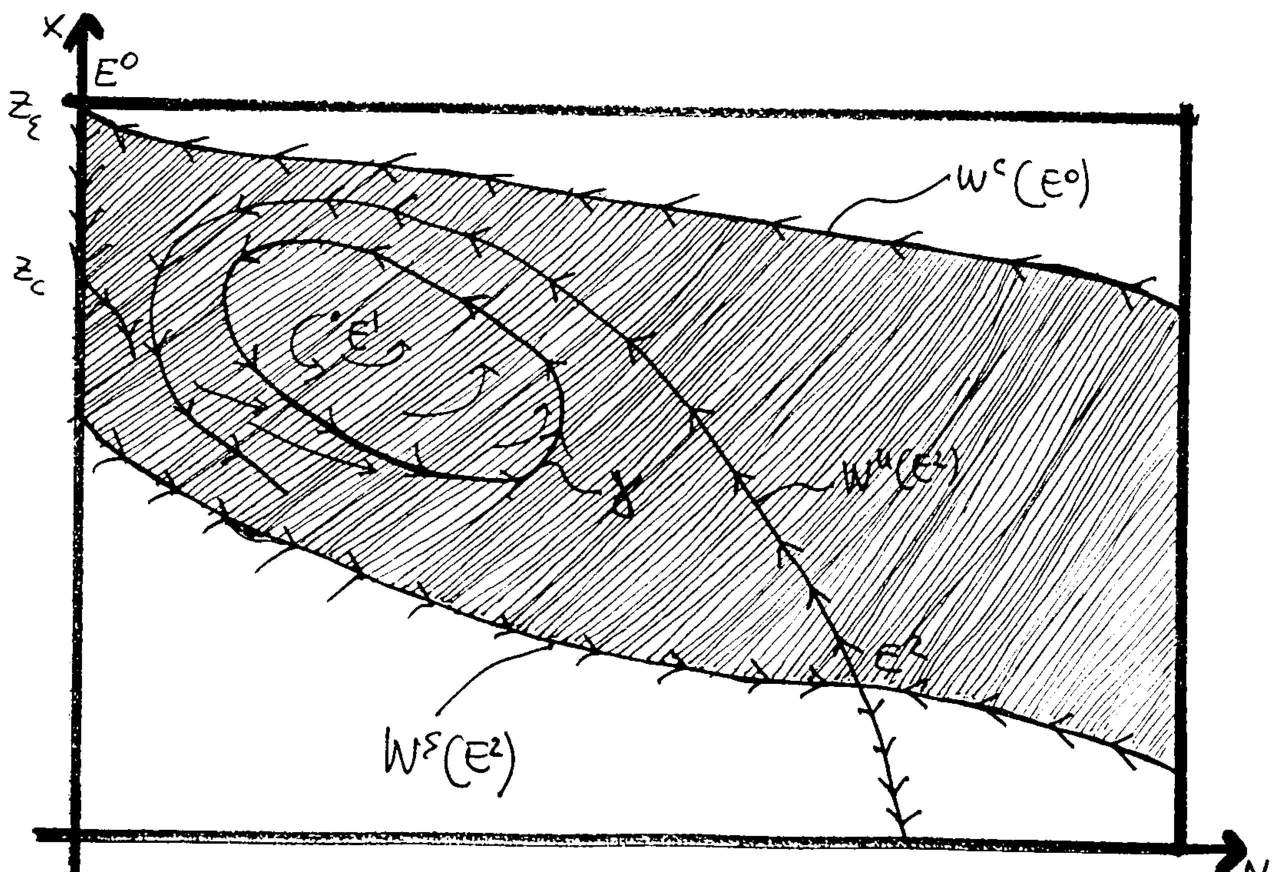


FIGURE 6.d

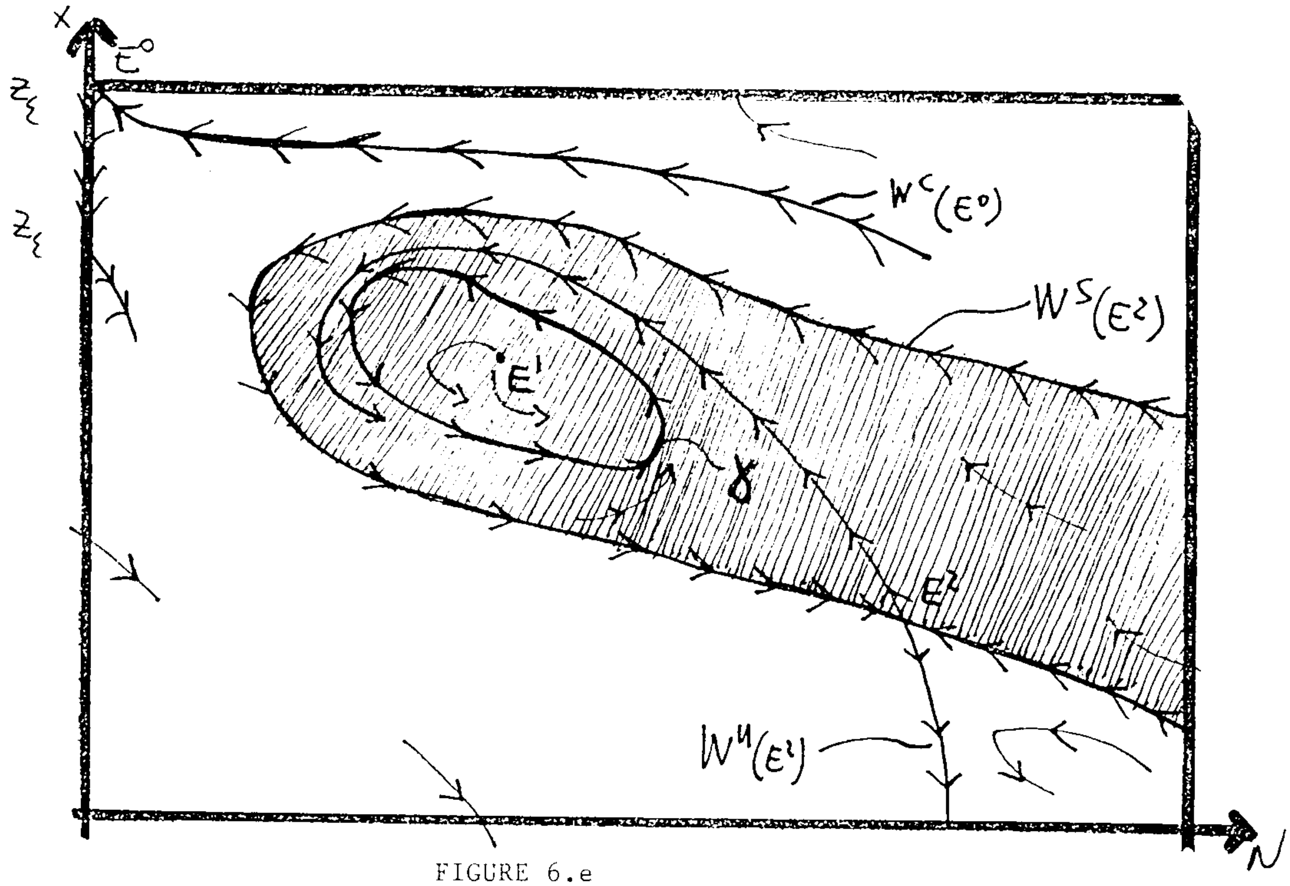
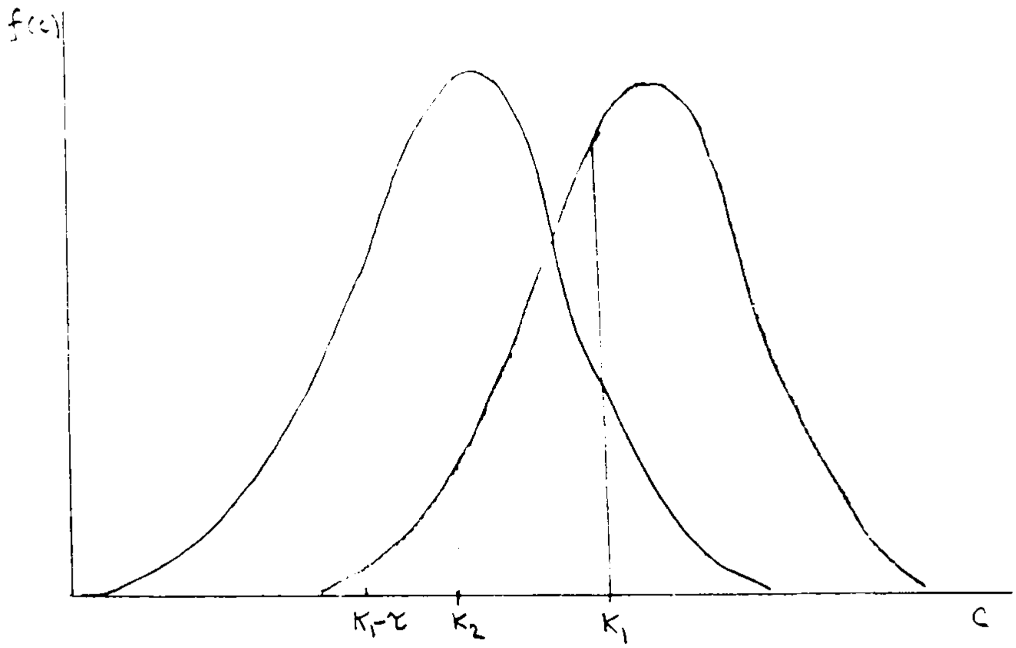


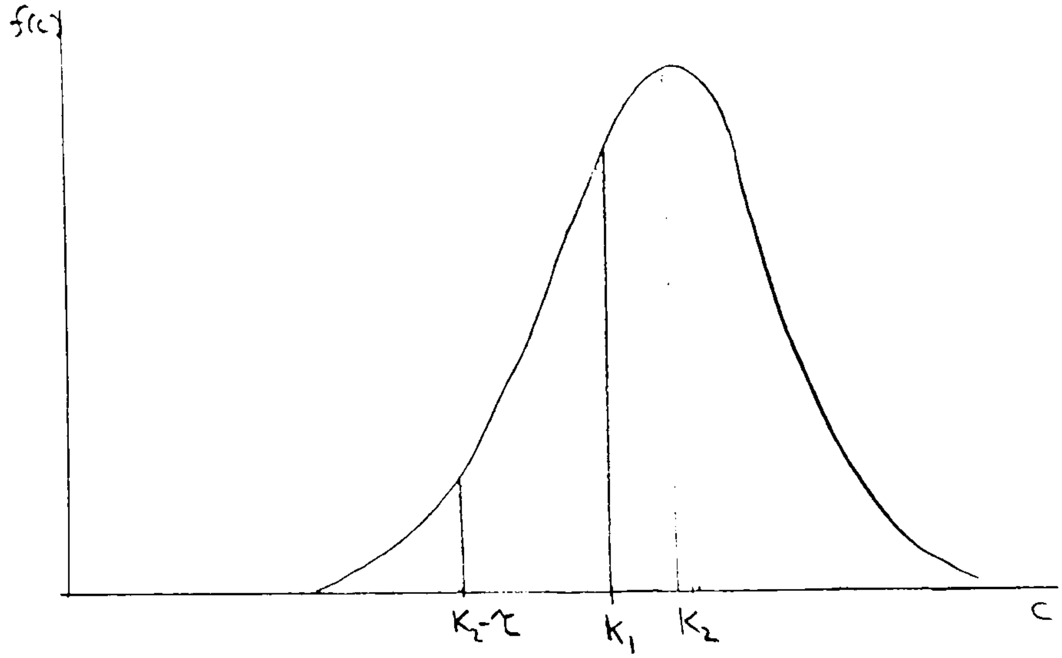
FIGURE 6.e

Shaded areas denote continua of equilibria.

Figure 7



1. translating f to the left distance τ reduces K, but by less than τ.



2. Subsidizing costs by lump sum τ increases K, but by less than τ.

Appendix

Derivation of (2.1) and (2.2)

Here we discuss a simple way to derive the continuous time dynamic programming equations (2.1) and (2.2). Consider a discrete time model with periods of length Δt. Then the value of being in state 0, for example, is the instantaneous search cost plus the discounted value of next period, which is the probability of finding a project times the expected maximum value of rejecting it or accepting it and switching to exchange, plus the probability of not finding one times the value to remaining in production

$$V_{0t} = -w_0\Delta t - o(\Delta t) + e^{-r\Delta t}\{\alpha\Delta t \int_0^\infty \max[V_{0,t+\Delta t}, V_{t,t+\Delta t} - c]dF(c) + (1 - \alpha\Delta t)V_{0,t+\Delta t}\}$$

where $o(\Delta t)$ represents the probability of two or more Poisson arrivals in a period, and hence satisfies $o(\Delta t)/\Delta t \to 0$ as $\Delta t \to 0$. Rearranging the previous expression,

$$((1 - e^{-r\Delta t})/\Delta t)V_{0t} = -w_0 + o(\Delta t)/\Delta t + e^{-r\Delta t}\{\alpha\int_0^\infty \max[0, V_{1,t+\Delta t} - V_{0,t+\Delta t} - c]dF(c) + (V_{0,t+\Delta t} - V_{0t})/\Delta t\}$$

Letting $\Delta t \to 0$ we arrive at (2.1). The derivation of (2.2) is similar.

Proof of Proposition 1: We consider in detail only the case of $\underline{c} = 0$. The same proof goes through when $\underline{c} > 0$ with a minor adjustment for the behavior of the flow on the upper portion of the boundary segment U_3. The only critical point is $E^{\beta} = (N_{\beta}, z_{\beta})$. Standard linearization techniques show that E^{β} is a regular saddle with one negative eigenvalue $\lambda_1 = S_N < 0$ and one positive eigenvalue $\lambda_2 = T_x > 0$. The associate (local) stable and unstable manifolds are tangent to the eigenspaces spanned by $\{[0,1]\}$ and $\{1, (T_x - S_N)/S_x\}$, respectively. Global stable $W^s(E^{\beta})$ and unstable $W^u(E^{\beta})$ manifolds also exist (see Irwin (1980)). The first is easily seen to coincide with the straight line $x = z_{\beta}$, as the latter is an invariant set of the flow and $W^s(E^{\beta})$ is tangent to it around (N_{β}, z_{β}). As for $W^u(E^{\beta})$ the fact that, around E^{β}, it is steeper than the S-locus (because $T_x - S_N$ is larger than $-S_N$) implies that it will never cross the S-locus again and it will therefore point outward as indicated in Figure 1. It is representable as a function, as points on the stable or unstable manifolds have to satisfy $dx/dN = T(N,x)/S(N,x)$ and $W^u(E^{\beta})$ cannot cross the S-locus. The remaining part of the global behavior of the system can be filled in using our knowledge of the boundary dynamics. Q.E.D.

<u>Proof of Proposition 3</u>: Linearization around E^0 gives the following two eigenvalues: $\lambda_{1,2} = (1/2)\{r \pm (r^2 + 4T_N S_x)^{1/2}\}$. Therefore, $Re(\lambda_i) > 0$ for $i = 1,2$, and E^0 is a source. The position of the two local, unstable manifolds is given by span $\{\lambda_1/S_x, 1\}$ and span $\{\lambda_2/S_x, 1\}$. This, together with the fact that the S-locus is flat in a small neighborhood of E^0 fully characterize the local behavior of the solution to (3.1)-(3.2). To check that the other stationary state is a saddle is trivial. The same type of argument used in the Proof of Proposition 1 applies here to $W^u(E^1)$ and $W^s(E^1)$. Q.E.D.

<u>Proof of Proposition 4</u>: That the flow around E^2 has a saddle point structure is immediate from linearization. Denote by λ_1 the positive and by λ_2 the negative root. A little algebra shows that $W^s(E^2)$ and $W^u(E^2)$ are negatively sloped, with the unstable one being steeper than the stable one. Moreover, the unstable manifold is steeper than the S-locus, which is steeper than the T-locus, which is steeper than the stable manifold at E^2.

As for E^1, the Jacobian of the linearized system has a positive determinant there, so both roots will have real parts of the same sign. The trace of the Jacobian is:

$$S_N + T_x = r - B(N)s_1(x)/u'(x) - x^2NB'(N)$$

and therefore the two roots may have either a positive (source) or a negative (sink) real part. They are complex when $(S_N - T_x)^2 < -4S_xT_N$.

Now let us consider the boundary point E^0. Linearization shows that there exists one positive root (λ_1) and one zero root (λ_2). The associated eigenspaces are, respectively:

$$E^u(E^0) = \text{span } \{1,0\} \text{ and}$$

$$E^c(E^0) = \text{span } \{1,-r/T_N\}.$$

The (local) unstable manifold will therefore coincide initially with the vertical axis. In fact, we know already that the whole segment $\{(N_0 = 0, z_c \le x \le z_\varepsilon\}$ is invariant for our flow, so it will coincide with the initial portion of the global unstable manifold $W^u(E^0)$, which then extends from $(0,z_c)$ in a southeast direction.

To understand the effect of the zero root we have to appeal to the center manifold theorem (see, e.g., Guckenheimer and Holmes (1983, p. 127)).

This guarantees the existence of a (differentiable) center manifold $W^c(E^0)$ tangent to $E^c(E^0)$ at $(0,z_\varepsilon)$ which will be invariant for the flow generated by (3.1) and (3.2). Such a manifold, nevertheless, needs not be unique (contrary to W^s and W^u, which are). Moreover, the direction of the motion over $W^c(E^0)$ cannot be inferred from the first order approximation, but requires higher order derivatives of S and T that cannot be signed under our assumptions. Q.E.D.

<u>Proof of Proposition 5</u>: Two simple facts are key to understanding our argument:

1. As the center manifold, $W^c(E^0)$, of E^0 exists and we have assumed it to be unique with the flow on it pointing toward E^0 it has to extend backward all the way from E^0 toward the boundary line U_4. It may or may not reach such a line depending on the behavior of the unstable manifold of E^2, $W^u(E^2)$: when the latter is the graph of a function it will exit $\tilde{U}$ through U_2 and therefore bound the extent to which $W^c(E^0)$ may be prolonged in direction southeast.

2. The stable and unstable manifolds of E^2 exist but need not be the graphs of two distinct functions from N into x. Recall that both $W^s(E^2)$ and $W^u(E^2)$ are solutions to our system of differential equations. Recall also that they are differentiable manifolds: points on $W^s(E^2)$ and $W^u(E^2)$ have to satisfy the relation $dx/dN = T(N,x)/S(N,x)$. Note also that, in a neighborhood of E^2, they are tangent to the eigenspaces associated with the Jacobian of the linearization of (3.1) and (3.2) at E^2.

Consider the partition of the domain given in Figure 2. $W^s(E^2)$ belongs to Ω_1 on the right of E^2 and Ω_3 on the left of E^2, whereas $W^u(E^2)$ belongs to

Ω_1 on the left side of E^2 and to Ω_3 on the right of E^2. Consider first the backward extension of the right portion of $W^s(E^2)$. Could it come from Ω_2? Clearly not, as the direction of motion in Ω_2 and on its upper boundary (the T-locus) contradicts the direction of motion on $W^s(E^2)$. It has therefore to come from outside the phase plane and enter the latter through U_4. The southeast portion of $W^s(E^2)$ is therefore the graph of a function from N into x lying completely in Ω_1. Similarly, it is easy to see that the southeast portion of $W^u(E^2)$ is also the graph of a function from N into x lying completely in Ω_3.

Now consider the northwest portion of $W^s(E^2)$; it is the graph of a function as long as it lies in Ω_3. This is the case represented in Figs. 5a and 5e. On the other hand, its backward extension cannot enter Ω_3 through the T-locus, coming from Ω_4. Once again, this would conflict with the direction of motion on $W^s(E^2)$ and on the T-locus. Nonetheless, it is clear that the directions of motion on $W^s(E^2)$ and the S-locus are consistent with $W^s(E^2)$ entering Ω_3 from Ω_5 through the S-locus, which is the lower boundary of the latter region. This amounts to saying that $W^s(E^2)$ need not enter Ω_3 from the boundary line U_3. When this occurs, $W^s(E^2)$ enter Ω_3 from Ω_5. Proceeding backward, $W^s(E^2)$ has, in turn, to enter Ω_5 from Ω_1. At this point, with $W^s(E^2)$ in Ω_1, two new possibilities arise: it may have entered Ω_1 from Ω_4 and Ω_4 from Ω_3. This case we have represented in Figures 5a, 5b and 5c. Alternatively, it may have entered Ω_1 from outside of the phase plane, cutting through the boundary line U_4. This is the case we have represented in Figure 5d. It should be now reasonably simple for the reader to carry on the same exercise for the northwest portion of the backward extension of $W^u(E^2)$. Three cases, perfectly analogous to those for $W^s(E^2)$

will arise. Obviously, not all the possible combinations of these two groups of individual configurations are admissible as solutions to our system of ODE. For example, it is clear that uniqueness of the solution to (3.1) and (3.2) for any given initial condition will rule out the case in which $W^s(E^2)$ is the graph of a function "coming from" U_3 and at the same time the backward extension of the northwest portion of $W^u(E^2)$ comes from the boundary line U_1.

These considerations prove that only the five qualitatively distinct types of phase planes given in Figures 5a-e are admissible. Notice that various non-generic and structurally unstable cases have not been considered here, as, for example, the heteroclynic case in which $W^s(E^2)$ and $W^u(E^2)$ coincide on the left side of E^2.

In order to complete the proof we need only to justify the asserted existence of (at least one) unstable limit cycle in the cases of Figures 5b and 5c. This can be easily done by a straightforward application of the following:

<u>Poincare-Bendixson Theorem</u>: A nonempty compact α- or ω-limit set of a two-dimensional flow, which contains no fixed points of the flow will contain a closed orbit.

For a proof the reader may consult Lefschetz (1957). Consider first the case of Figure 5b. Let $\varepsilon > 0$ be small enough to guarantee that all orbits with initial conditions in the open ball of radius ε around E^1 will remain in such a ball and converge to E^1. Such an ε has to exist because E^1 is a sink. Define as $\bar{\bar{N}}$ the largest value of N for which there exist at least two values of x, call them $\bar{\bar{x}}_1 < \bar{\bar{x}}_2$ such that $(\bar{\bar{N}}, \bar{\bar{x}}_i) \in W^s(E^2)$, $i = 1,2$.

Such an $\bar{\bar{N}} < N_2$ exists by construction. Now consider the closed curve γ_2 beginning at $(\bar{\bar{N}}, \bar{x}_1)$ continuing from there along $W^s(E^2)$ in direction opposite to the flow until the point $(\bar{\bar{N}}, \bar{\bar{x}}_2)$ and then coinciding with the vertical line connecting $(\bar{\bar{N}}, \bar{x}_2)$ to $(\bar{\bar{N}}, \bar{\bar{x}}_1)$. Denote with γ_1 the closed curve which is the boundary of the ϵ-ball around E^1 we constructed before. The anular region Σ contained between γ_1 and γ_2, boundaries included, has the following properties: (a) it is compact; (b) it contains no critical point of the flow; (c) it contains the α-limit set of all the orbits beginning in it. Therefore it has to contain at least one closed orbit which belongs to the flow. In fact, it may contain more than one closed orbit. Nevertheless, the outermost and the innermost among these cycles have to be repulsive and equilibria will leave them when time goes forward. If only one such limit cycle exists, as in the figures, it has to be unstable.

A brief inspection of Figures 5a and 5c will reveal that a completely similar construction can be carried out also in those cases. Our proof is therefore complete. Q.E.D.

<u>Proof of Proposition 6</u>: There is very little to prove. The five types of different configurations for the three manifolds $W^c(E^0)$, $W^s(E^2)$ and $W^u(E^2)$ have already been justified in the proof of Proposition 5. The existence of the limit cycles in the case of Figures 6d and 6e also follows from the Poincare-Bendixson theorem. The inner boundary γ_1 of the compact region Σ can be constructed here also by choosing an ϵ-ball around E^1 such that all the orbits beginning on γ_1 point toward Σ and remain there for large values of t. Such a ball exists because E^1 is a source. The outer boundary γ_2 will now be given by an appropriately chosen portion of the unstable

manifold $W^u(E^2)$ together with a vertical segment at $\bar{\bar{X}} < X_2$ connecting the lower branch of $W^u(E^2)$ to its upper branch. In this way a region Σ is created that satisfies the theorem. Notice that it is the ω-set, as opposed to the α-set, of the trajectories starting inside Σ that is now contained in Σ. For this reason the outermost and the innermost among the cycles will be attractive. Hence, the asymptotic stability statement for the case in which there is only one such cycle. Q.E.D.

<u>Proof of Proposition 7</u>: We only need to show that the condition given in the text is sufficient to guarantee that all the hypotheses of the following theorem are satisfied:

<u>Andronov-Hopf Bifurcation Theorem</u>: Suppose that the system $\dot{x} = f_\mu(x,y)$, $\dot{y} = g_\mu(x,y)$, for x, y and μ in R has a critical point at $(\bar{x},\bar{y})$ for $\mu = \bar{\mu}$. Assume that:

(H1) $D(f_\mu, g_\mu)(\bar{x},\bar{y})$ has a simple pair of pure imaginary eigenvalues, i.e., assume one can write the Taylor approximation (in deviation form) of degree three around $(\bar{x},\bar{y})$ as:

$$\dot{x} = -\omega y + a(x^2 + y^2)x - b(x^2 + y^2)y$$
$$\dot{y} = \omega x + b(x^2 + y^2)x + a(x^2 + y^2)y.$$

(H2) The real part of the two eigenvalues varies with μ around $\mu = \bar{\mu}$, i.e.,

$$d(\mathrm{Re}\lambda(\mu))/d\mu\big|_{\mu=\bar{\mu}} \neq 0.$$

(H3) The element a of the Taylor approximation in (H1) is different from zero when evaluated at $(\bar{x}, \bar{y}, \bar{\mu})$.

Then there is a unique three-dimensional center manifold of periodic solutions to the original system of ODE. If $a < 0$ these periodic solutions are stable limit cycles, while if $a > 0$ the periodic solutions are repelling.

The reader may consult Marsden and McCracken (1976) for a proof.

Denote with J the Jacobian matrix associated with the linear approximation of (3.1), (3.2) around E^1. The two eigenvalues of J are: $2\lambda_{1,2} = \text{Trace}(J) \pm (\Delta)^{1/2}$, with $\Delta = (\text{Trace}(J))^2 - 4\text{Det}(J)$. When $B'(N)x^2N + B(N)s_1(x)/u'(x)$ is larger than zero at E^1 the value $\bar{r}$, defined in the proposition, can be chosen for the interest rate. It implies $\text{Trace}(J) = 0$ and $\Delta < 0$. This yields the two purely imaginary eigenvalues at the bifurcation value $r = \bar{r}$. In fact, continuity implies that the two eigenvalues will remain complex, i.e., Δ will remain negative, for values of r in a small neighborhood of $\bar{r}$. The $\mu > 0$ in the statement of the proposition will have to be chosen appropriately to guarantee that $(\bar{r}\ \varepsilon, \bar{\mu} + \mu)$ contains the neighborhood in which Δ is negative. (H1) is then satisfied and (H2) is also satisfied as $\text{Re}(\lambda) = \text{Trace}(J)$ at the bifurcation point and Trace(J) varies smoothly with the parameter r.

Finally, condition (H3) is known to be generically true for smooth two-dimensional vector fields like ours (see Arnold (1980, Ch. 6)).

We cannot verify the sign of a. Given our assumptions, the stability-instability of the closed orbit follows from our discussion of the global behavior of the flow around E^1 (see Propositions 5 and 6). Q.E.D.

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