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EQUILIBRIUM IN NON-PARTITIONING STRATEGIES

by

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Equilibrium in Non-Partitioning Strategies

Abstract

Herein we present a single example with three purposes: (1) to show the existence of equilibria in a game which violates the assumptions of currently-available general existence theorems, (2) to illustrate the importance of the "affiliation" assumption in economic games of incomplete information, by showing how even a slight relaxation can lead to the nonexistence of equilibria in monotone strategies, and, most importantly, (3) to exhibit an equilibrium point in strategies which partially reveal information without inducing posterior partitionings of the players' type spaces.

Introduction

Our example is a symmetric two-player, first-price, private-values auction game. Each of two risk-neutral players privately learns the (nonnegative) value to himself of the object being auctioned. Then, each submits a (nonnegative) sealed bid. Finally, the player submitting the higher bid receives the object, and pays the amount of his bid. (Ties, which we will find to occur with probability zero at equilibrium, are broken by a coin toss.)

Private-value auctions have been the object of substantial prior study. The novelty here is in the joint distribution of player valuations: With probability α, the valuations are independent draws from a fixed atomless distribution F; with probability $1-\alpha$, the valuations are both equal to a single draw from F. The extreme cases in which $\alpha = 0$ or $\alpha = 1$ have been treated thoroughly in the received literature. We will see that results for the intermediate cases differ substantially from previous results.

There are numerous equilibrium existence theorems for games with incomplete information (see, for example, Milgrom and Weber [1981a], which discusses several), all of which require some form of assumption concerning the continuity of the payoff structure of the game, together with the assumption that the joint distribution of player types is absolutely continuous with respect to the product of the marginal distributions. While many games with discontinuities in the payoff structure (such as auction games) have been shown to have equilibria, and specific examples of nonexistence are also available, we are unaware of any previous investigation of games which violate the absolute-continuity-of-information assumption. Here, we find that equilibria do, in fact, exist. (However, note that existence, while of interest, is not particularly striking. Every game based on discrete approximations of the players' type and action spaces is known to possess equilibria (Milgrom and Weber [1985]), so one might expect a limit of equilibria of approximating games to be at least "equilibrium-like" (Simon and Zame [1990]).)

There is a well-developed theory of auctions, in which most results are derived from the assumption that the players' private signals are "affiliated" (Milgrom and Weber [1981b]). In our two-player setting, this simply means that the players' valuations are nonnegatively correlated, given that they lie in any product set. In auctions where the private signals come from an atomless distribution, the affiliation assumption typically leads to equilibria in pure strategies which are increasing in the players' signals. Here, for all $0 < \alpha < 1$, the affiliation property does not hold: For example, if the distribution F is uniform on $[0,1]$, then, given that one player's valuation is in $\{0.25, 0.50\}$ and the other's is in $\{0.50, 0.75\}$, it is overwhelmingly likely that the first valuation is high and the second low. While the valuations are not affiliated, they still possess a strong monotonic relationship: Each player's posterior beliefs about his opponent's valuation increase stochastically with his own valuation. However, pure-strategy equilibria will be seen not to exist. This result is not an artifact of the presence of atoms in the posterior distributions: If the positive probability that the players' valuations are identical is replaced by a positive probability that the valuations are distributed on some narrow band around the diagonal, the qualitative nature of our results remains unchanged.

In many published studies of economic games with incomplete information, the players' type and action spaces are real intervals, and strategies at equilibrium (in the studied examples) turn out to be monotone (i.e., either monotone increasing — for all types $t_1 > t_2$ and actions a_1 and a_2 taken by the respective types, $a_1 \geq a_2$ — or monotone decreasing). In a monotone strategy, each type has associated with it an interval of actions, and the intervals associated with distinct types intersect only at their boundaries. Since at most countably many of the intervals can have nonempty interiors, monotone strategies are pure for all but at most a discrete set of types.

There are classes of games in which all best responses are monotone. Witness, for example, Myerson's treatment [1981] of private-value auctions in which the players' valuations are independent. Once strategies are fixed for all but one player, the last player is left to choose among actions which yield different pairs (p, e), where p is the probability that a considered action will obtain the object being sold, and e is the player's (unconditional) expected payment resulting from the considered action. With valuation v, the player will select an action — equivalently, a (p, e)-pair — that maximizes $v \cdot p - e$. Larger values of v yield steeper indifference curves (lines) in (p, e)-space, and hence a best response to the strategies of the other players must have the choice of p (and hence, of a bid) increase monotonically with v. "Separating" strategies (in which all types take distinct actions) arise when the convex hull of all available (p, e)-pairs has unique tangents; "pooling" of types (in which several types take the same action) results from nonunique tangency at some points. Similar arguments can be applied in a variety of settings.

Generally, if a player employs a monotone strategy, then the ability of an observer to learn about the player's type by observing the player's action is quite restricted: Observed actions simply partition the player's type space. The observer can rule out some types on the basis of his observation, but all types not ruled out (i.e., types which pool together on the observed action) retain their original relative likelihoods. Never can the observer state, "This type seems *somewhat* more likely, and that type *somewhat* less likely, given my observation."

It would be truly remarkable if all learning (in a rational world, in contexts where players begin with a continuum of possible types) took this restrictive, disjointed form. Our example shows that learning through observation can indeed have a more continuous nature: Observers can, in some settings, make the statement in the previous paragraph.

Nonexistence of equilibria in monotone strategies

For $\alpha \in \{0, 1\}$, the auction game studied here is known to have a unique equilibrium point in pure (monotone increasing) strategies. The equilibria for these two extreme cases are described later in this paper. But first, we show that no equilibria in monotone strategies exist for intermediate values of α.

Let V be the support of the distribution F. We represent a strategy for a player in behavioral form, as a family $G = \{G(\cdot|v)\}_{v \in V}$ of cumulative distributions over the space of bids. Associated with every such strategy is a marginal distribution H of bids defined by $H(b) = \int G(b|v)\, dF(v)$. Against any strategy G of the opposing player, a player with valuation v who bids b has expected payoff

$$(v-b)\cdot[\alpha H(b) + (1-\alpha)G(b|v)] ,$$

as long as neither H nor $G(\cdot|v)$ has an atom at b. (Ties are a notational nuisance which we will be able to avoid in most of our analysis.)

It is easy to show that no symmetric equilibria in monotone strategies exist. The argument which follows covers asymmetric possibilities, as well. Assume $0 < \alpha < 1$, and assume, for purposes of eventual contradiction, that an equilibrium point in monotone strategies G_1 and G_2 exists. Let H_1 and H_2 be the associated marginal distributions of bids.

Take any valuation v such that $F(v) > \alpha$, $G_1(\cdot|v)$ is concentrated at b_1, and $G_2(\cdot|v)$ is concentrated at b_2. (Because the strategies are monotone, G_1 and G_2 are each concentrated at a single point for all but countably many values of v. Since F is assumed to be nonatomic, valuations v which satisfy the conditions are abundant.)

At equilibrium, bidders can never bid more than their valuations. Therefore, since $\alpha > 0$, a bidder with any valuation v above the lowest possible valuation $v_\ell = \min_{v \in V} v$ must have a positive expected payoff at equilibrium (bidding any amount between v and v_ℓ will yield a positive payoff with positive probability), and hence must bid strictly less than his valuation.

It follows that, at equilibrium, a player's marginal bid distribution cannot have atoms. (Were there an atom, either the other player would never bid at or just below the atom, and bids at the atom could be profitably dropped, or the other player would have valuations above the atom with which he bids at or just below, and some of those bids could be profitably raised to just above the atom.) Consequently, we can avoid below the cumbersome notation required for dealing with ties, since we must have $H_1(b_1) = H_2(b_2) = F(v)$.

If $b_1 = b_2$, either player can gain from a small increase in bid, when his valuation is v. So, without loss of generality, assume $b_1 > b_2$.

Let $\epsilon = 1 - F(v)$, $\delta = 1 - H_2(b_1)$, and $\gamma = H_1(b_1) - H_1(b_2)$. For player 1 not to gain (when his valuation is v) from dropping his bid to just above b_2, it must be that

$$(v-b_1)\, [\alpha\cdot(1-\delta) + (1-\alpha)\cdot 1] \geq (v-b_2)\, [\alpha\cdot(1-\epsilon) + (1-\alpha)\cdot 1] .$$

For player 2 not to gain from raising his bid to just above b_1, it must be that

$$(v-b_2)\,[\alpha\cdot(1-\epsilon-\gamma) + (1-\alpha)\cdot 0] \geq (v-b_1)\,[\alpha\cdot(1-\epsilon) + (1-\alpha)\cdot 1]\ .$$

Solving both inequalities for $(v-b_1)/(v-b_2)$ and combining the results, we must have

$$\frac{\alpha(1-\epsilon-\gamma)}{1-\alpha\epsilon} \geq \frac{1-\alpha\epsilon}{1-\alpha\delta}\ .$$

Since $1-\epsilon-\gamma < 1-\alpha\epsilon$ and $1-\alpha\delta < 1$, this in turn implies that $\alpha > 1-\alpha\epsilon > 1-\alpha(1-\alpha)$, an impossibility. Hence our original assumption, that there exists an equilibrium point in monotone strategies, must be incorrect.

Equilibrium characterization

Here we shall seek a symmetric equilibrium point in behavioral strategies. G will denote the sought-for symmetric equilibrium strategy, and H will be the marginal distribution of bids induced by G.

Let $\underline{b}(v)$ represent the lowest bid in the support of $G(\cdot\,|\,v)$, and $\overline{b}(v)$ the highest. For G to be a best response to itself, a player with valuation v must have the same expected payoff $K(v)$ for all bids b in the support of $G(\cdot\,|\,v)$. Hence, we must have

$$(v-b)\cdot[\alpha H(b) + (1-\alpha)G(b\,|\,v)] = K(v) = (v-\underline{b}(v))\cdot\alpha H(\underline{b}(v))\ ,$$

and therefore

(1) $$G(b\,|\,v) = \left[\frac{\alpha}{1-\alpha}\right]\cdot\left[\frac{v-\underline{b}(v)}{v-b}\cdot H(\underline{b}(v)) - H(b)\right]\ ,$$

for all $\underline{b}(v) \leq b \leq \overline{b}(v)$.

For any $b < \underline{b}(v)$, the expected payoff is simply $(v-b)\cdot\alpha H(b)$, which can be at most $K(v)$; for any $b > \underline{b}(v)$, this expression must be less than $K(v)$ (since $G(b\,|\,v) > 0$). Therefore, the expression is maximal at $\underline{b}(v)$ and, assuming that H is differentiable, $(v-\underline{b}(v))\cdot H'(\underline{b}(v)) = H(\underline{b}(v))$, or equivalently,

(2) $$\underline{b}^{-1}(b) = b + H(b)/H'(b)\ .$$

Finally,

(3) $$H(b) = \int G(b\,|\,v)\,dF(v)\ .$$

(Note that this is the only point in our analysis where the distribution F appears.)

Now, H determines $\underline{b}$ through (2), together they determine G through (1), and G redetermines H through (3). This suggests an iterative calculation to numerically solve for the symmetric equilibrium strategy. Results of the calculation are given in the next section.

Numerical results

For purposes of illustration, we take the distribution F to be uniform on $[0, 1]$, i.e., $F(v) = v$. The graphs in Figures 1 through 3 cover the cases $\alpha = 0.95$, 0.50, and 0.05. The graphs serve the dual roles of illustrating the nature of the equilibrium strategies in the three cases, and confirming the results of the calculations at the two extremes, where equilibrium behavior is close to that known to hold at the limits when $\alpha = 1$ and $\alpha = 0$.

(For those interested in computational details: H was initially taken to be uniform on $[0,1]$. In each subsequent iteration, H was computed at 1000 equally-spaced points, and interpolated using cubic splines; all intermediate functions were numerically computed from the splined function. For smaller values of α, the values of H converged to four decimal places within six iterations. For $\alpha = 0.95$, the result of each even-numbered iteration was averaged with the previous iterate to increase numerical stability.)

When $\alpha = 1$ (i.e., when the players' valuations are independent, and both distributed the same as the random variable V), the unique equilibrium point arises when both players follow the strategy of bidding $b(v) = E[V \mid V \leq v]$. When V is uniformly distributed, this simplifies to $b(v) = v/2$. In Figure 1, where $\alpha = 0.95$ and the players' valuations are nearly independent, the symmetric equilibrium strategy is quite similar to this; as α approaches 1, both $\underline{b}(v)$ and $\overline{b}(v)$ converge to $b(v)$. Note that the expected-payoff graph is rotated 90 degrees to more clearly show that, for any valuation v, randomization over bids between $\underline{b}(v)$ and $\overline{b}(v)$ is indeed a best response to G.

Figure 2 shows an intermediate case.

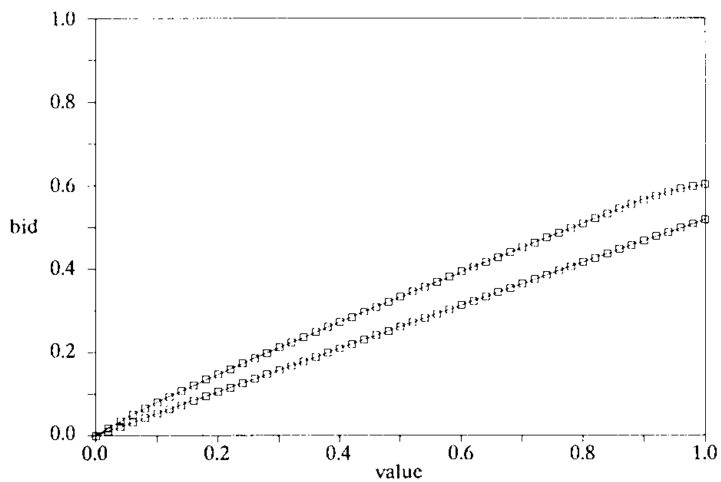
When $\alpha = 0$ (i.e., when the players' valuations are identical), the unique equilibrium point arises when both bidders bid their valuations, i.e., when $b(v) = v$. Figure 3 illustrates the case in which $\alpha = 0.05$, i.e., the valuations are very likely to be equal. The striking feature of equilibrium behavior in this case is that, as α approaches 0, the support of the joint distribution of bids and types is *not* converging to the diagonal, although the strategy G, reinterpreted in distributional form (Milgrom and Weber [1985]) is weakly converging to $b(v)$. The support of G converges (in the Hausdorff metric) to the triangle spanned by the lines $b = v$ and $b = v/2$.

Prospects

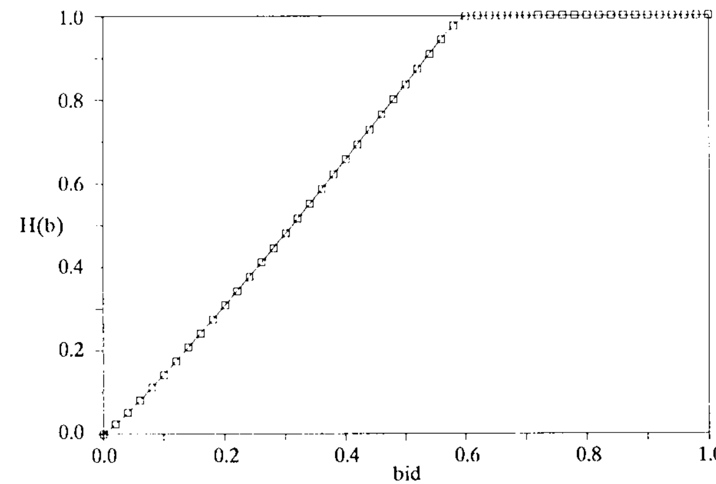
Under what circumstances might we expect equilibria in non-partitioning strategies to arise? In our example here, a player's need is to keep a competitor who might know his type from being able to predict perfectly his action. An alternative motivation would be to keep a competitor who can observe his action from being able to infer perfectly his type. Such an inference can only be costly in a multi-stage game, wherein the competitor has a subsequent move.

A number of multi-stage games have been found to have equilibria employing monotone strategies at each stage. In most cases, those games have had the property that player types at one extreme wish to accurately signal their existence in early stages. (For example, in bargaining games, types in an

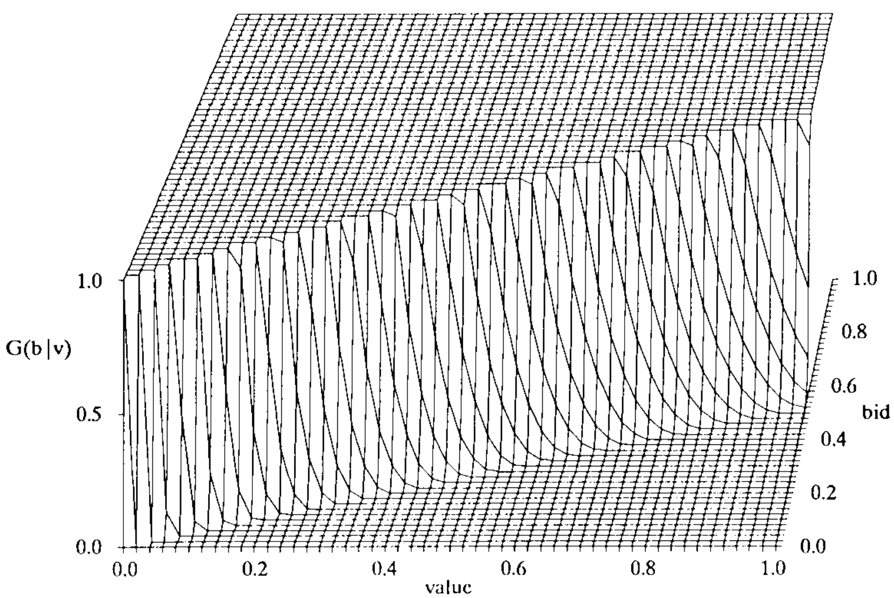
Figure 1: Pr $(V_1 = V_2) = 0.05$



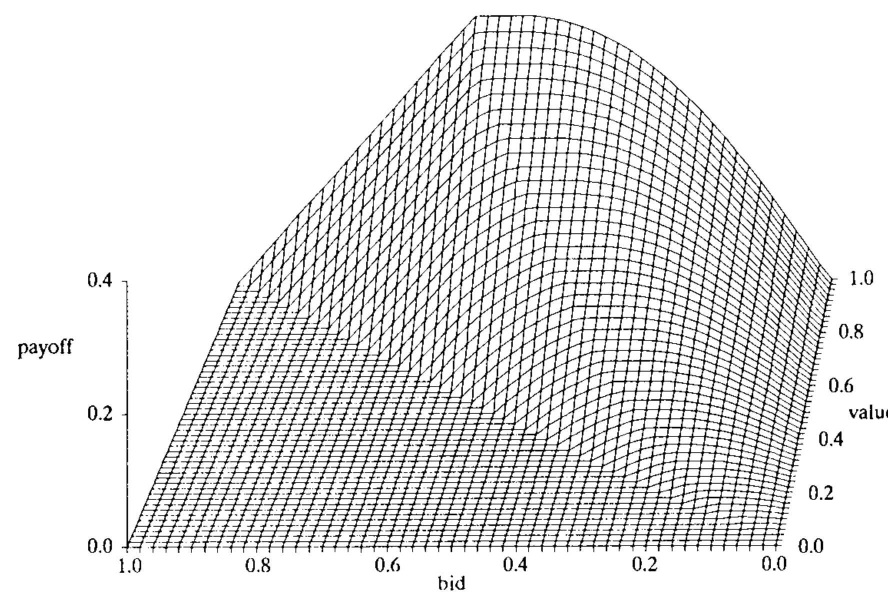
Support of Equilibrium Bid Distribution



Marginal Bid Distribution

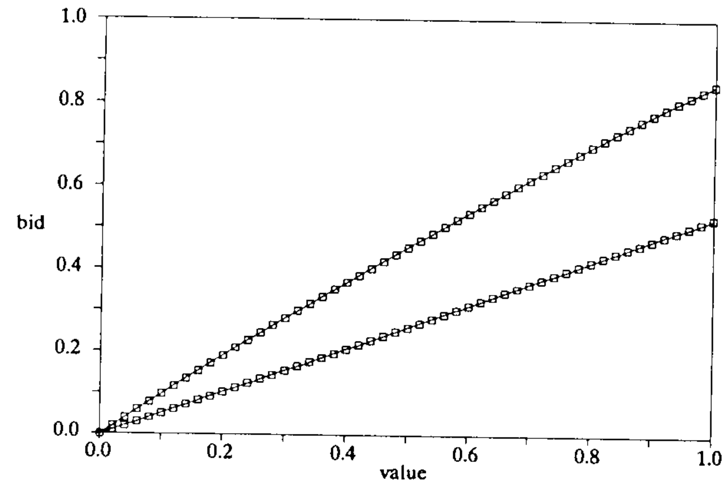


Equilibrium Behavior Strategy

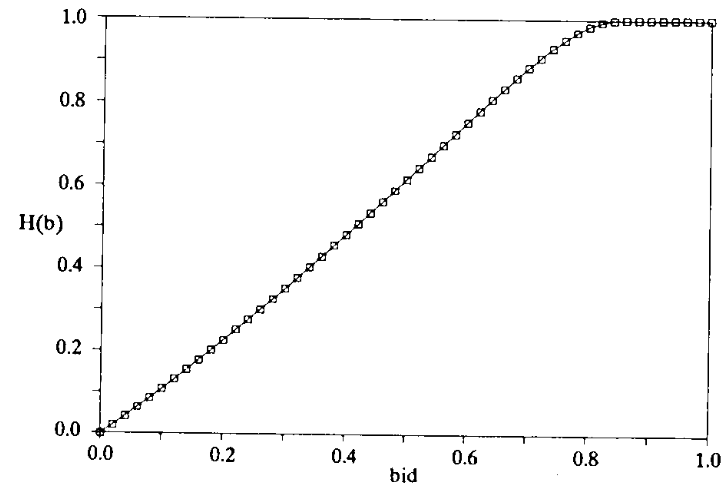


Expected Payoff against Equilibrium Strategy

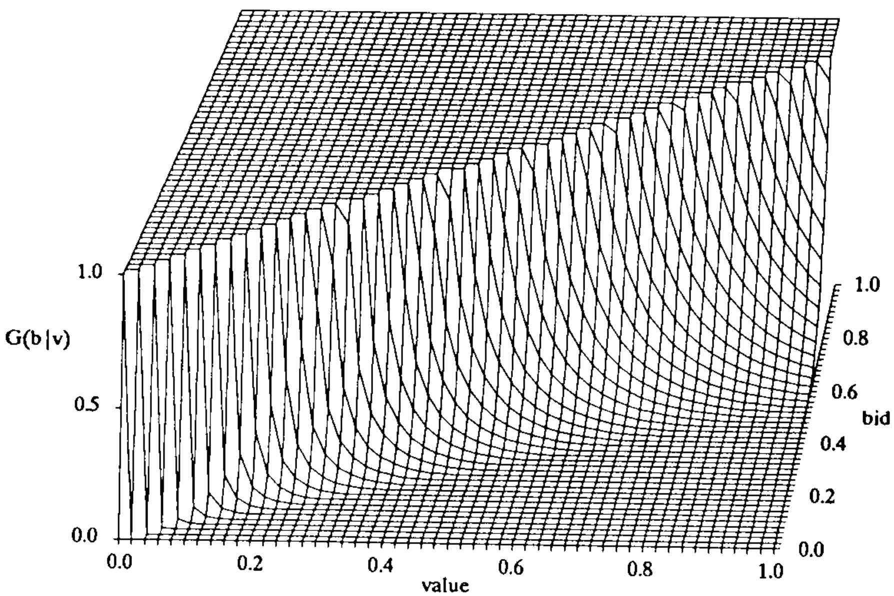
Figure 2: $\Pr(V_1 = V_2) = 0.5$



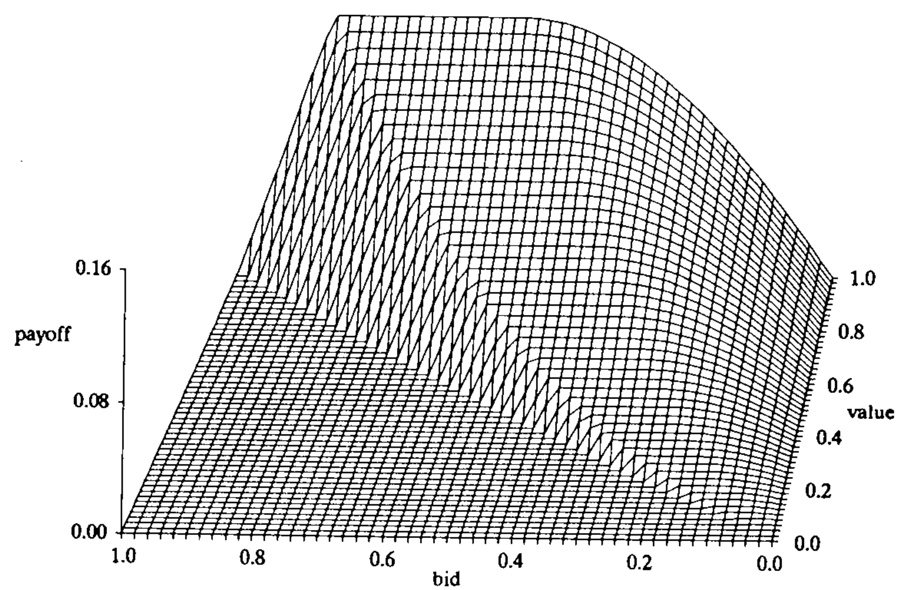
Support of Equilibrium Bid Distribution



Marginal Bid Distribution

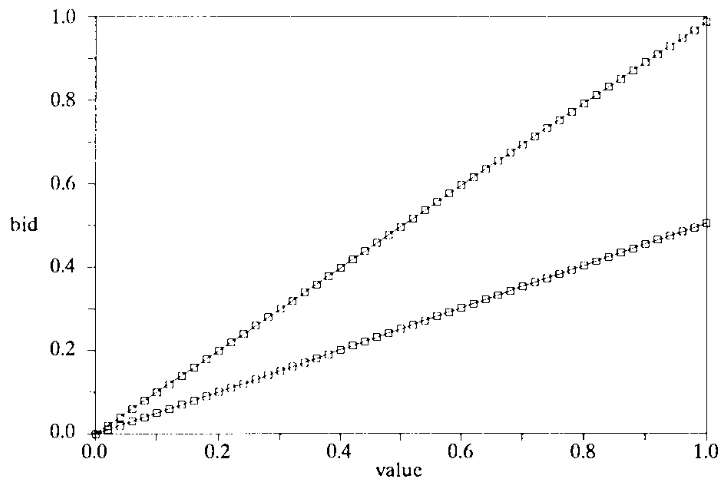


Equilibrium Behavior Strategy

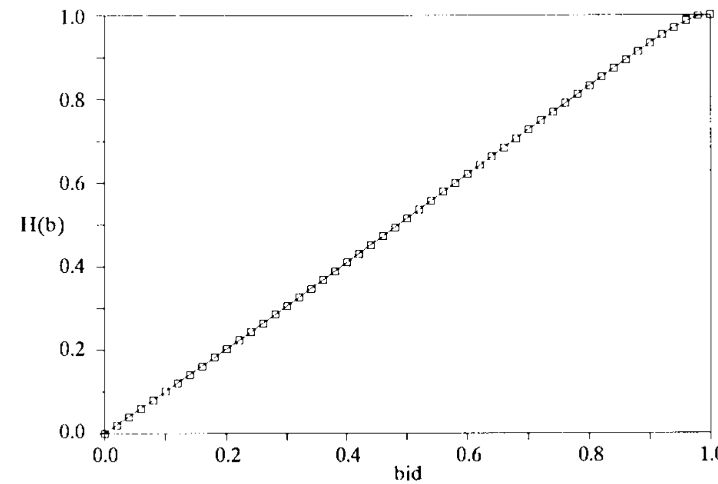


Expected Payoff against Equilibrium Strategy

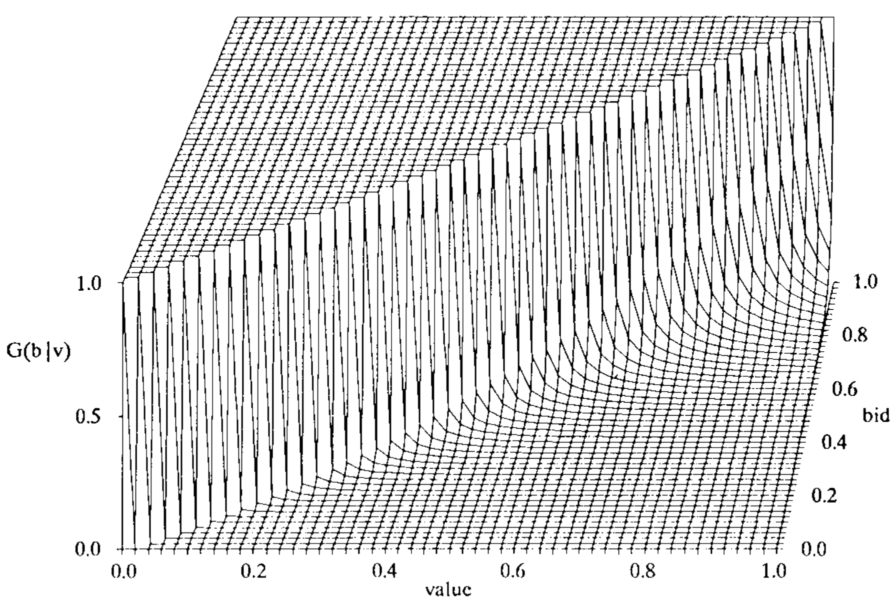
Figure 3: Pr $(V_1 = V_2) = 0.95$



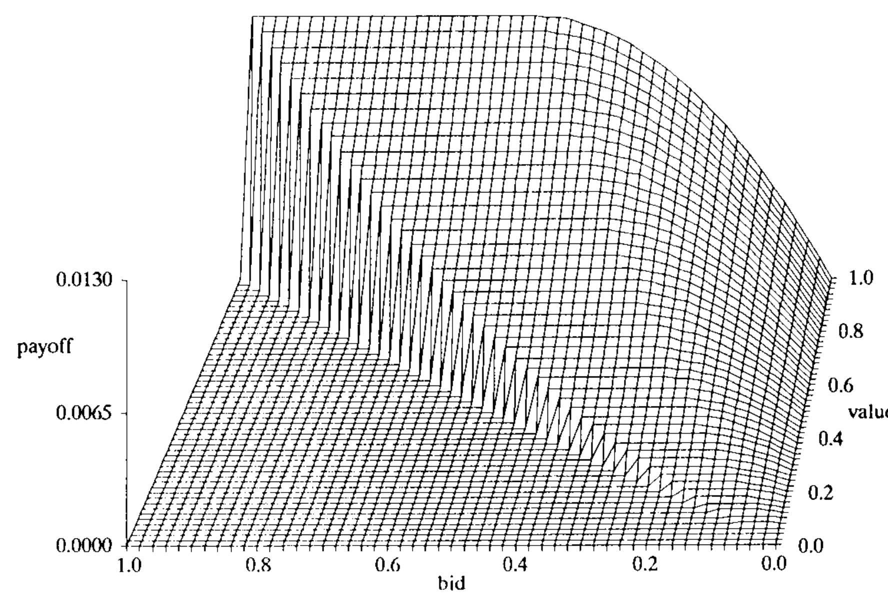
Support of Equilibrium Bid Distribution



Marginal Bid Distribution



Equilibrium Behavior Strategy



Expected Payoff against Equilibrium Strategy

extremely "strong" position want that position to be known. In repeated Cournot duopoly games where players have private knowledge of their production costs, types with extremely low costs wish to signal.) In a companion paper, we will provide an example of a two-stage game in which, at equilibrium, a player must employ a non-partitioning strategy in the first stage. The distinguishing feature of that example is that, in the second stage, every one of the player's types gains from the maintenance of residual uncertainty in the competitor's posterior beliefs, i.e., no type has a pure incentive to signal.

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