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GLOBAL GAMES*

by

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Abstract

Global games are real-valued functions defined on partitions (rather than subsets) of the set of players. They capture "public good" aspects of cooperation, i.e., situations where the payoff is naturally defined for all players ("the globe") together, as is the case with issues of environmental clean-up, medical research, and so forth.

We analyze the more general concept of lattice functions and apply it to partition functions, set functions and the interrelation between the two. We then use this analysis to define and characterize the Shapley value and the core of global games.

1. Introduction

Traditional cooperative game theory models a game by a set function which is interpreted as the payoffs that a coalition may guarantee itself. It is implicitly assumed that payoffs are defined for each player separately and the main question is what are the attainable payoff profiles for various possible coalitions. In a transferable utility game only one number is attached to each coalition, but it is typically interpreted as the maximum total payoff of a coalition, which is meaningful if the players may redistribute the "utilities" among themselves.

However, in many situations it seems more reasonable to say that the game's payoff--or utility--is simultaneously defined for all players. Consider, for instance, environmental problems such as air and water pollution, diminishing ozone layers, and other catastrophes. Although undoubtedly not perfectly precise, it seems safe to argue that the natural model for these problems is one in which the payoff is defined for all players together. (Or, if you will, that the utilities of the players coincide.)

Questions of art and historical treasures preservation, a cure for cancer and AIDS, indeed, the progress of science and art in general, and many other issues--though not unrelated to nations' political interests--seem to be "global," at least as a first approximation.[1]

This paper models such games and tries to cope with the question of their "solution." Mathematically speaking, a global game is a real-valued partition function: for each partition, P, of the players, i.e., for every

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[1]An additional example of "global" payoffs is the performance of a certain organization that depends on its internal structure but is not defined for separate coalitions.

profile of cooperation, there is a value f(P) describing the utility of all players ("the globe") should the structure of cooperation follow P.

It is this model of "games" for which we would like to propose solution concepts, which will be analogous to those applied to "ordinary" (transferable utility cooperative) games. Such concepts--as the Shapley value and the core--prescribe an allocation (or a set of allocations) describing how the surplus of cooperation is to be shared. The question which naturally arises at this point is: What is there to share? If the utility is identical to all players anyway, how does one share it?

The answer is that the global game does not describe the complete range of activities of the players. It is a "reduced form" model which captures a certain aspect of these activities--say, environmental clean-up or AIDS research--but does not deal with other aspects of interaction, in which each player's utility is well defined. Thus, we interpret f(P) as a transferable-utility (or "monetary") value of P-cooperation, and the question is how to divide the surplus of full cooperation f({N}) (where N is the set of players). All nations will benefit from a joint effort to clean up the atmosphere, but they still have to decide how to share the (positive) cost which exists even when they do cooperate. All nations will be better off once AIDS is cured, yet each of them also prefers to support the AIDS research to the minimal possible extent.

In a way, then, a global game may be considered as capturing the "public good" aspect of interaction, assuming there are no private goods (apart from the "monetary" transfers).[2]

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[2]Our concept differs from the cooperative games in which a coalition's payoff depends on the partition to which it belongs (Thrall and Lucas (1963)) since we focus on the "public good" aspect, i.e., on pure

The study of partition functions has shown some common features with that of set functions. Indeed, some of the results may be applied to a more general framework than both, namely, to real-valued functions as lattices. In Section 2 we deal with lattice functions in general and prove some results, most of which are known for the case of set functions. We find lattice functions to be of particular importance to cooperative game theory as they appear in a variety of models:

(1) ordinary games (on sets);

(2) global games (on partitions);

(3) k-stage games (on chains of k sets) (see Beja-Gilboa (1990));

(4) games defined on pairs (S,P) where S is a set and P is a partition containing it (see Thrall and Lucas (1963)).

In Section 3 we focus attention on partition functions and study their relation to set functions defined by them. In particular we focus on properties such as additivity, monotonicity, convexity, and total positivity (defined in Section 2); these properties also shed some light on convex ("ordinary") games and study some properties which are stronger than convexity.

Finally, Section 4 deals with solution concepts for global games. We define and characterize the (unique) Shapley value which, somewhat surprisingly, turns out to depend only on all-or-none partitions, i.e., partitions in which some subset of players fully cooperate, while the rest do not cooperate at all. We also define the core and show that convex global games have a nonempty core which includes the Shapley value.

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externality, where the "power" of the coalition is a less obvious concept.

2. Lattice Functions

A poset (partially ordered set) is a pair $(X,\geq)$ where X is a set and $\geq$ is a binary relation on it satisfying:

(i) reflexivity: $x \geq x$ for all $x \in X$.

(ii) anti-symmetry: $x \geq y$ and $y \geq x$ implies $x = y$ for all $x,y \in X$.

(iii) transitivity: $x \geq y$ and $y \geq z$ implies $x \geq z$ for all $x,y,z \in X$.

An element $x \in X$ is said to be a supremum (an infimum) of a set $A \subseteq X$ if the following hold:

(i) $x \geq y$ $(x \leq y)$ for all $y \in A$;

(ii) if $z \geq y$ $(z \leq y)$ for all $y \in A$, then $z \geq x$ $(z \leq x)$.

A supremum of a set A is denoted by $\vee A$, an infimum by $\wedge A$. In view of anti-symmetry, suprema and infima are unique. If $A = \{x,y\}$ its supremum and infimum will be denoted by $x \vee y$ and $x \wedge y$, respectively.

A poset $(X,\geq)$ is a <u>lattice</u> if for every $x,y \in X$ there exist $x \vee y$ and $x \wedge y$. We will refer to $\wedge$ and $\vee$ as binary operations. Unless otherwise stated, all lattices considered are assumed to be finite.

Obviously a lattice has a (unique) maximum and minimum, denoted by x^* and x_*, respectively. A <u>lattice function</u> is a real-valued function on X. The linear space of lattice functions on X will be identified with $\mathbb{R}^{|X|}$, but also denoted by $F(X)$ when this notation will be more suggestive.

A lattice function $f \in F(X)$ is <u>monotone</u> if $x \geq y$ implies $f(x) \geq f(y)$. It is 0-normalized if $f(x_*) = 0$, and the subspace of 0-normalized lattice functions will be denoted by $F_0(X)$. A function $f \in F(X)$ is <u>convex</u> if

$$f(x \vee y) + f(x \wedge y) \geq f(x) + f(y)$$

for all $x, y \in X$. It is <u>concave</u> if the converse inequality holds. It is <u>additive</u> if it is both convex and concave.

A lattice function f is <u>totally positive</u> if for every $x_1, x_2, \ldots, x_n \in X$

$$f(x_1 \vee x_2 \vee \ldots \vee x_n) \geq \Sigma_{\{I \mid \emptyset \neq I \subseteq \{1, \ldots, n\}\}} (-1)^{|I|+1} f(\wedge_{i \in I} x_i)$$

(This definition coincides with the Dempster-Shafer definition of belief functions for the lattice of subsets of a given set. See Dempster (1967), Shafer (1976).) It is <u>totally negative</u> if the converse inequality holds.

For $x, y \in X$ we write $x > y$ if $x \geq y$ and $x \neq y$. We also use an additional binary relation, denoted $>^*$, and defined by $x >^* y$ if $x > y$ but for no $z \in X$ $x > z > y$. (Put differently, $>^*$ is the minimal relation whose transitive closure is $>$.)

For $x \in X$ we define $g_x \in F(x)$ by $g_x(y) = 1$ if $y \geq x$ and $g_x(y) = 0$ otherwise.

<u>Proposition 2.1</u>: $\{g_x\}_{x \in X}$ is a linear basis for $F(X)$, as is $\{g_x\}_{x \in X, x \neq x_*}$ for $F_0(X)$.

<u>Proof</u>: First we show that $\{g_x\}_{x \in X}$ are linearly independent. Assume

$$\Sigma_{x \in X} \alpha_x g_x = 0.$$

Considering x_*, we obtain $\Sigma_{x \in X} \alpha_x g_x(x_*) = \alpha_{x_*} = 0$. Next consider y such that $y >^* x_*$. Obviously, $\alpha_y = 0$ follows, and the proof continues by induction.

Since there are $|X|$ functions in $\{g_x\}_{x\in X}$, they have to constitute a basis of $F(X)$. Similarly, $\{g_x\}_{x\neq x_*} \subseteq F_0(X)$ are independent and of the appropriate dimension to be a basis for $F_0(X)$. //

Given $f \in F(X)$ let $\{\alpha_x(f)\}_{x\in X}$ be the unique set of coefficients such that

$$f = \Sigma_x \alpha_x(f) g_x.$$

Note that $(f - f(x_*)g_{x_*}) \in F_0(X)$. Hence for $x \neq x_*$, $\alpha_x(f - f(x_*)g_{x_*}) = \alpha_x(f)$.

<u>Theorem 2.2</u>: Let $f \in F(X)$ be given. Then f is totally positive and monotone iff for all $x \neq x_*$, $\alpha_x(f) \geq 0$.

<u>Proof</u>: Since f is totally positive and monotone iff $(f - f(x_*)g_{x_*})$ is, we assume w.l.o.g. (without loss of generality) that $f \in F_0(X)$.

First consider the "if" part. We will first show that g_x is totally positive for every $x \in X$, and the conclusion will follow as the set of totally positive and monotone functions is a cone.

Let $x \in X$ and $x_1, \ldots, x_n \in X$ be given. We wish to show that

$$g_x(\vee_{i\leq n} x_i) \geq \Sigma_{\emptyset\neq I\subseteq\{1,\ldots,n\}} (-1)^{|I|+1} g_x(\wedge_{i\in I} x_i).$$

Let $J = \{1 \le i \le n | x \le x_j\}$. If $J = \emptyset$ the right side vanishes and the inequality holds. Otherwise, the inequality may be reduced to

$$1 \ge \Sigma_{\emptyset \ne I \subseteq J} (-1)^{|I|+1}.$$

However,

$$1 + \Sigma_{\emptyset \ne I \subseteq J} (-1)^{|I|} = \Sigma_{I \subseteq J} (-1)^{|I|} = (1 - 1)^{|J|} = 0$$

Hence, g_x is totally positive for all x, and the "if" part is proved. For the "only if" part we first prove the following.

<u>Claim</u>: Let $f \in F_o(X)$ and $x \in X$. Assume that $\{x_1, \dots, x_n\} = \{y \in X | x >^* y\}$. Then

$$f(x) - \Sigma_{\emptyset \ne I \subseteq \{1, \dots, n\}} (-1)^{|I|+1} f(\wedge_{i \in I} x_i) = \alpha_x(f).$$

<u>Proof</u>: The case $x = x_*$ is trivial. Assume, then, that $x > x_*$ and $n \ge 1$. Recall that for all $y \in X$,

$$f(y) = \Sigma_{z \in X} \alpha_z(f) g_z(y) = \Sigma_{z \le y} \alpha_z(f).$$

Hence, we have to show that

$$\Sigma_{z \le x} \alpha_z(f) - \Sigma_{\emptyset \ne I \subseteq \{1, \dots, n\}} (-1)^{|I|+1} \Sigma_{z \le \wedge_{i \in I} x_i} \alpha_z(f) = \alpha_x(f)$$

or

$$\Sigma_{z<x}\ \alpha_z(f) - \Sigma_{\emptyset\neq I\subseteq\{1,\ldots,n\}}\ (-1)^{|I|+1}\ \Sigma_{z\leq\wedge_{i\in I}x_i}\ \alpha_z(f) = 0$$

The expression on the left side equals

$$\Sigma_{z<x}\ \alpha_z(f)[1 - \Sigma_{\emptyset\neq I\subseteq\{1,\ldots,n\};z\leq\wedge_{i\in I}x_i}\ (-1)^{|I|+1}]$$

It is sufficient (and necessary) to show that the expression in brackets vanishes for all $z < x$. However, this is proven almost identically to the "if" part of the theorem. We therefore consider the claim proved. //

To complete the proof of the theorem, let $f \in F_0(X)$ be totally positive and let $x > x_*$. Let $\{x_1,\ldots,x_n\} = \{y \in X | x >^* y\}$. If $n = 1$, then

$$\alpha_x(f) = f(x) - f(x_1)$$

and nonnegativity follows from monotonicity of f. Otherwise, i.e., $n \geq 1$, we have

$$\vee_{1\leq i\leq n}\ x_i = x$$

and nonnegativity follows from the claim and the fact that f is totally positive. //

Remark 2.3: One can easily formulate the counterpart of this theorem for totally negative functions which are decreasing in $\geq$.

Observation 2.4: A totally positive (negative) function is convex (concave), though the converse is false.

Proof: Taking $\{x,y\} = \{x_1,x_2\}$ in the definition of total positivity, one gets

$$f(x \vee y) \geq f(x) + f(y) - f(x \wedge y),$$

i.e., that f is convex. To see that the converse does not hold, let $X = 2^{\{1,2,3\}}$ and $\geq\, = \,\supseteq$. Define a function (game) f by

$$f = g_{\{1,2\}} + g_{\{2,3\}} + g_{\{1,3\}} - (1/2)g_{\{1,2,3\}}.$$

It is easy to check that f is convex, but it is not totally positive.

(Switching signs all over the proof will complete it for the case of totally negative and concave function.)

Remark 2.5: Proposition 2.1 is well-known for the case of subsets (of a given set) ordered by inclusion. For this case a version of Theorem 2.2 was also proved by Dempster (1967), Shafer (1976). In their theorem, nonnegativity of $f \in F_0(X)$ replaces monotonicity. Obviously, monotonicity is a stronger requirement in general, the two coincide for totally positive functions on subset-lattices, but they do not coincide in general. In fact, nonnegativity and total positivity do not suffice for nonnegativity of all coefficients, as is shown in the following example:

$$X = \{x, y, z\}, \quad x > y > z$$

$$f(z) = 0, \ f(y) = 2, \ f(x) = 1.$$

Remark 2.6: It is also important to note that our theorem also applies to non-distributive lattices. (A lattice is distributive if

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), \ \forall \ x, y, z \in X$$

or, equivalently,

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z), \ \forall \ x, y, z \in X.$$

(See, e.g., Graetzer (1971).)

Indeed, while subset-lattices (with inclusion) are distributive, partition-lattices (with "coarser than" relation) are not. Consider the following example:

$$N = \{1, 2, 3, 4\}$$

$$P = \{\{1, 2\}, \{3, 4\}\}$$

$$Q = \{\{1, 3\}, \{2\}, \{4\}\}$$

$$R = \{\{2, 4\}, \{1\}, \{3\}\}$$

Then $Q \wedge R = \{\{1\}, \{2\}, \{3\}, \{4\}\}$ and $P \vee (Q \wedge R) = P$. But

$$P \vee Q = P \vee R = \{N\}$$

so that

$$(P \vee Q) \wedge (P \vee R) = \{N\} \neq P = P \vee (Q \wedge R).$$

3. Partition Functions

For a finite nonempty set N, a partition P is a set of nonempty pairwise disjoint subsets of N whose union is N. The set of all partitions is denoted $\mathcal{P}(N) = \mathcal{P}$. For $P \in \mathcal{P}$ and $i \in N$ let $P(i)$ be the member of P including i.

The set of partitions is partially ordered by the "coarser" than relation defined as follows: P is coarser than Q, denoted $P \geq Q$ if for every $A \in Q$ there is $B \in P$ such that $A \subseteq B$. Note that $(\mathcal{P},\geq)$ is a lattice where $P \vee Q$ is the finest partition coarser than both (the meet) and $P \wedge Q$ is the coarsest partition finer than both (the join). Note that these notations are not entirely conventional. However, with the interpretation of global games, it is more intuitive to define monotonicity with respect to the "coarser than" relation, whence the rest of the notations follow.

We will also use the terms "P is finer than Q," "$P \leq Q$," "P is a refinement of Q," and "Q is a coarsening of P."

We extend the definitions above (and below) to subsets of N. Thus, if P^A is a partition of A, P^B is a partition of B and $A \cap B = \emptyset$, $P^A \cup P^B$ is a well-defined partition of $A \cup B$.

For a partition P (of a subset A) we denote by $\mathcal{B}(P)$ the algebra (of subsets of A) generated by it. If P is a partition of A, and $B \in \mathcal{B}(P)$, P^B will denote the induced partition of B. Let P_f^A and P_c^A denote the finest and coarsest partitions of A, respectively.

In Gilboa-Lehrer (1989) we introduced the following reflexive and symmetric binary relation on partitions: $P, Q \in \mathcal{P}$ are non-intersecting if for every $A \in P$ either (i) there is $B \in Q$ such that $A \subseteq B$, or (ii) there are $\{B_i\}_{i=1}^n \subseteq Q$ such that $A = \cup_{i=1}^n B_i$.

Observation 3.1: For $P, Q \in \mathcal{P}$ the following are equivalent:

(i) P and Q are non-intersecting;

(ii) there exists $A \in \mathcal{B}(P) \cap \mathcal{B}(Q)$ such that $P^A \le Q^A$ and $P^{A^c} \ge Q^{A^c}$;

(iii) $P \cup Q = (P \wedge Q) \cup (P \vee Q)$.

A partition function $f: \mathcal{P} \to \mathbb{R}$ is partially additive if

$$f(P) + f(Q) = f(P \wedge Q) + f(P \vee Q)$$

for every non-intersecting $P, Q \in \mathcal{P}$.

Let $PA \subseteq F(\mathcal{P})$ denote the subspace of partially additive partition functions.

A game is a set function $v: 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$. Considering the lattice $(X, \ge) = (2^N, \supseteq)$, the space of games is $F_0(2^N)$ in Section 2's notations. Note that a game is additive iff

$$v(A) + v(B) = v(A \cup B)$$

for all $A, B \subseteq N$ with $A \cap B = \emptyset$.

We quote the following from Gilboa-Lehrer (1989):

Fact 3.2: A partition function is partially additive iff there is a game v such that

$$(*) \qquad f(P) = \Sigma_{A \in P} v(A), \ \forall P \in \mathcal{P}.$$

Furthermore, if both v and w satisfy (*), then (v - w) is an additive game with (v - w)(N) = 0.

For simplicity we will normalize all partition functions by subtracting $f(P_f)$, and consider the space $F_0(\mathcal{P})$. These functions are called global games. Should such a partition function f be partially additive, it has a unique game v satisfying (*) and $v(\{i\}) = 0$ for all $i \in N$. This v will be called the game associated with f, denoted v_f.

In general, we define for $f \in F_0(\mathcal{P})$ an associated game v_f by

$$v_f(A) = f(P_c^A \cup P_f^{A^c}) \text{ for } A \neq \emptyset$$

(and $v_f(\emptyset) = 0$.)

Obviously this definition coincides with the previous one for partially additive partition functions.

Proposition 3.3: If $f \in F_0(\mathcal{P})$ is monotone and convex, so is $v_f \in F_0(2^N)$.

Proof: To see that v_f is monotone let $A \subseteq B$ be given. Then $v_f(A) = f(P_c^A \cup P_f^{A^c}) \leq f(P_c^B \cup P_f^{B^c}) = v_f(B)$ follows from monotonicity of f w.r.t.

(with respect to) coarsening.

Next consider convexity. For arbitrary $A, B \subseteq N$ we have to show that

$$v(A \cup B) + v(A \cap B) \geq v(A) + v(B).$$

First assume that $A \cap B \neq \emptyset$. Then

$$(P_c^A \cup P_f^{A^c}) \wedge (P_c^B \cup P_f^{B^c}) = (P_c^{A \cap B} \cup P_f^{(A \cap B)^c})$$

and

$$(P_c^A \cup P_f^{A^c}) \vee (P_c^B \cup P_f^{B^c}) = (P_c^{A \cup B}, P_f^{(A \cup B)^c})$$

whence the desired inequality follows immediately. If $A \cap B = \emptyset$, however, we obtain

$$(P_c^A \cup P_f^{A^c}) \wedge (P_c^B \cup P_f^{B^c}) = P_f$$

and

$$(P_c^A \cup P_f^{A^c}) \vee (P_c^B \cup P_f^{B^c}) = (P_c^A \cup P_c^B \cup P_f^{(A \cup B)^c})$$
$$\leq (P_c^{A \cup B}, P_f^{(A \cup B)^c}),$$

which completes the proof. //

Similarly, one obtains the following result (the proof of which is omitted).

Proposition 3.4: If $f \in F_0(\mathcal{P})$ is monotone and totally positive, so is v_f.

For $PA_0 = PA \cap F_0(\mathcal{P})$ one may wonder whether the converse of these results also holds, i.e., whether convexity or total positivity (combined with monotonicity) of a game v is inherited by the global game $f_v \in PA_0$ defined by it. The answers are given in:

Remark 3.5: If a game v is monotone and convex, the global game f_v has to be monotone but need not be convex.

Proof: Convexity of v implies its super-additivity, that is, that

$$v(A \cup B) \geq v(A) + v(B)$$

for all $A, B \subseteq N$ with $A \cap B = \emptyset$, which implies monotonicity of f_v. (Note that monotonicity of v is not required here.)

To show that f_v need not be convex, let $N = \{1,2,3,4\}$, define

$$v(A) = \begin{cases} |A| - 1 & A \neq \emptyset \\ 0 & A = \emptyset \end{cases}$$

Thus, v is monotone and convex. However, for $P = \{\{1,2\},\{3,4\}\}$ and $Q = \{\{1,3\},\{2,4\}\}$ we get

$$f_v(P \wedge Q) + f_v(P \vee Q) = f_v(P_f) + f_v(P_c) = 3$$

and

$$f_v(P) + f_v(Q) = v(\{1,2\}) + v(\{3,4\}) + v(\{1,3\}) + v(\{2,4\}) = 4. \quad //$$

Proposition 3.6: If a game v is monotone and totally positive, so is the global game f_v.

Proof: Let $v \in F_0(2^N)$ be given. For $A \subseteq N$ let u_A denote the unanimity game on A (g_A in Section 2's notation). Let $\alpha_A(v)$ be the unique coefficients such that

$$v = \Sigma_{\emptyset \neq A \subseteq N} \alpha_A(v) u_A$$

Claim:

$$f_v = \Sigma_{\emptyset \neq A \subseteq N} \alpha_A(v) g_{(P_c^A \cup P_f^{A^c})}$$

Proof of Claim: Given $P \in \mathcal{P}$.

$$f_v(P) = \Sigma_{B \in P} v(P) = \Sigma_{B \in P} \Sigma_{A \subseteq B} \alpha_A(v)$$

$$= \Sigma_{\{A \mid P \geq (P_c^A \cup P_f^{A^c})\}} \alpha_A(v)$$

$$= \Sigma_{\{A \mid \emptyset \neq A \subseteq N\}} \alpha_A(v) g_{(P_c^A \cup P_f^{A^c})}$$

Hence, the (unique) coefficients $\{\alpha_P(f)\}_{P \in \mathcal{P}}$ such that $f = \Sigma_{P \in \mathcal{P}} \alpha_P(f) g_P$ are given by

$$\alpha_P(f) = \begin{cases} \alpha_A(v) & \text{if } P = (P_c^A \cup P_f^{A^c}) \\ 0 & \text{otherwise} \end{cases}$$

Theorem 2.2 may be now invoked twice--once for the subset lattice to deduce $\alpha_A(v) \geq 0$ for all $A \subseteq N$, and then for the partition lattice to deduce that f_v is totally positive and monotone. //

We summarize these results as follows:

$$\begin{aligned}
&\{v \mid f_v \text{ is totally positive and monotone}\} = \\
&\{v \mid v \text{ is totally positive and monotone}\} \subset \\
&\{v \mid f_v \text{ is convex and monotone}\} \subset \\
&\{v \mid v \text{ is convex and monotone}\}.
\end{aligned}$$

where the inclusions are strict. (An example showing the second inclusion was given in Remark 3.5 above: as for the first inclusion, note that the game given in Observation 2.4 above also defines a convex and monotone f_v.)

It is not too surprising that concavity of $f \in PA_0$ will not always be inherited by v_f. Indeed, the inequality

$$v(A \cup B) + v(A \cap B) \leq v(A) + v(B)$$

has to hold whenever $A \cap B \neq \emptyset$, but may be violated otherwise. More specifically:

<u>Remark 3.7</u>: $f \in PA_0$ may be monotone and concave (additive) without v_f being concave (additive).

<u>Proof</u>: Let $N = \{1,2,3\}$, and define

$$v(A) = \begin{cases} |A| - 1 & \text{if } A \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

Define $f = f_v$ (so that $v = f_v$ also holds). It is easily verified that f is additive, hence, concave. However,

$$v(\{1,2,3\}) > v(\{1,2\}) + v(\{3\})$$

so that v is not concave--hence, not additive. //

However, for a large enough set of players, additivity of f is inherited by v_f:

<u>Proposition 3.8</u>: Let $|N| \geq 4$ and assume that $f \in F_0(\mathcal{P})$ is additive. Then $f \equiv 0$.

<u>Proof</u>: Let f be given and let v_f be its associated game. We first prove that $v(A) = v(B)$ whenever $|A| = |B|$. The proof is by induction on $k = |A| = |B|$. For $k = 1$ there is nothing to prove as $v_f(\{i\}) = 0$ for all $i \in N$. We therefore assume the claim for k and prove for $(k + 1)$.

If $(k + 1) = |N|$ there is nothing to prove. Hence, assume $(k + 1) < |N|$ and let C be a subset of cardinality $k + 2$. Consider any $A, B \subseteq C$ with $A \neq B$ and $|A| = |B| = k + 1$. Using additivity of f for $P = (P_c^A \cup P_f^{A^c})$ and $Q = (P_c^B \cup P_f^{B^c})$ we obtain

$$v(C) + v(A \cap B) = v(A) + v(B)$$

Note that $|A \cap B| = k \geq 1$. Furthermore, $|C| = k + 2 > 2$. Hence, we can choose a set $D \subseteq C$ with $D \neq A$, $D \neq B$ and $|D| = k + 1$. Then

$$v(C) + v(A \cap D) = v(A) + v(D)$$

also holds. However, $v(A \cap B) = v(A \cap D)$ as $|A \cap D| = k = |A \cap B|$, so we obtain $v(B) = v(D)$. Similarly, we get $v(A) = v(B) = v(D)$. Since this holds for subsets (of size $k + 1$) of <u>any</u> set C (of size $k + 2$), it also holds for any A,B with $|A| = |B| = k + 1$.

(Note that so far we have not used the fact $|N| \geq 4$.)

Hence, there exists a function $h\colon \mathbb{N} \to \mathbb{R}$ such that $v(A) = h(|A|)$ (which implies $h(1) = 0$). Denote $\alpha = h(2)$. Assume w.l.o.g. $N = \{1,2,3,4\} \cup M$ (with $M \cap \{1,2,3,4\} = \emptyset$).

We know that whenever $A \cap B \neq \emptyset$, the following holds

$$v(A \cup B) + v(A \cap B) = v(A) + v(B)$$

For $A = \{1,2\}$ and $B = \{2,3\}$ we obtain $v(\{1,2,3\}) = h(3) = 2\alpha$.

Next consider $P_1 = \{\{1,2\},\{3,4\}\} \cup \{\{i\}\}_{i \in M}$ and $Q_1 = \{\{1,2,3\},\{4\}\} \cup \{\{i\}\}_{i \in M}$. The equality

$$f(P_1 \wedge Q_1) + f(P_1 \vee Q_1) = f(P_1) + f(Q_1)$$

implies

$$h(4) + h(2) = 2h(2) + h(3)$$

or

$$h(4) = h(3) + h(2) = 3\alpha.$$

However, for $Q_2 = \{\{1,4\},\{2,3\}\} \cup \{\{i\}\}_{i\in M}$ the equality

$$f(P_1 \wedge Q_2) + f(P_1 \vee Q_2) = f(P_1) + f(Q_2)$$

yields

$$h(4) = 2h(2) + 2h(2) = 4\alpha.$$

Together we obtain $\alpha = 0$, whence it follows by induction that $f \equiv 0$.

4. Solution Concepts

A solution concept for a global game--as for any game--should "solve" it, i.e., prescribe a certain possible outcome of it. In the context of global games, however, it is not entirely clear what do "outcome" or "solution" mean. More precisely, one has to decide whether global games should be "solved" directly, or should they first be reduced to "ordinary" games? The indirect strategy would rely on the assumption that we know how to solve games (whatever "solve" may mean), and the only problem with global games is that we do not know the "value" of each coalition. Hence, all we need to ask of a solution concept for global games is to translate the global game to an ordinary one, to which standard solution concepts may be applied.

In this paper we follow the direct solution strategy. It seems to us that this strategy eliminates arbitrary choices which will be required (using the indirect strategy) to specify an ordinary game (or set of such) for a global one.

We confine ourselves to the Shapley value and the core. A subsection is devoted to each.

4.1 The Shapley Value

An operator $\Psi\colon F_0(\mathcal{P}) \to \mathbb{R}^N$ is a Shapley value (for global games) if it satisfies the following axioms.

1. Linearity;
2. Dummy: for all $f \in F_0(\mathcal{P})$ and $i \in N$, if

$$f(P) = f(P \wedge (\{\{i\}\} \cup P_c^{N\setminus\{i\}}))$$

for all $P \in \mathcal{P}$, then $(\Psi f)(i) = 0$.

3. Interchangeable players: for all $f \in F_0(\mathcal{P})$, and $i,j \in N$ if

$$f(P \wedge (\{\{i\}\} \cup P_c^{N\setminus\{i\}})) = f(P \wedge (\{\{j\}\} \cup P_c^{N\setminus\{j\}}))$$

for all $P \in \mathcal{P}$, then $(\Psi f)(i) = (\Psi f)(j)$.

4. Efficiency:

$$\Sigma_{i\in N} (\Psi f)(i) = f(\{N\}).$$

Let us comment briefly on the interpretation of these axioms. Linearity has its usual meaning: suppose that the players in the global game f (say, environmental clean-up) are also involved in a different global game g (e.g., art treasures preservation). It is desirable that one will be able to solve each game separately and obtain the same outcome that would result from considering the two global issues together $(f + g)$. Similarly, homogeneity (that is, $\psi(\alpha f) = \alpha(\psi f)$) simply means scale invariance.

Next consider the dummy axiom. A player i is a "dummy" in a global game f if the payoff is independent of i's cooperative behavior. As formulated, we only require that for every partition P, f(P) will equal the payoff of the partition obtained from P by player i's desertion. Obviously, this also means that player i may decide to join another set in P but will still not affect the payoff. It seems reasonable that such a player will have no share in the surplus of cooperation $f(\{N\})$.

As for Axiom 3, two players i and j are "interchangeable" if for every partition P the desertion of i from his/her current coalition to form a separate coalition $\{i\}$ has the same impact on f(P) as j would have. (Notice that in the formulation given above the term f(P) was cancelled on both sides of the equality.) The requirement that i and j will get the same payoff according to ψf has a flavor of "symmetry" or "fairness" (though, as we shall see later, it is stronger than the traditional meaning of "symmetry").

Finally, the efficiency axiom simply requires that the overall surplus of cooperation, $f(\{N\})$, will be shared among the players.

We recall that the counterparts of these axioms for the context of ordinary games characterize the Shapley value for set functions (introduced in Shapley (1953)). Let it be denoted by Φ.

Theorem 4.1.1: There is a unique Shapley value ψ for the space of global games and it equals the Shapley value of the induced (ordinary) game, i.e.,

$\Psi f = \Phi(v_f)$ for all $f \in F_0(\mathcal{P})$.

Proof: First we show that $\Phi(v_f)$ is a Shapley value. Linearity and efficiency are immediate. It is also easily verified that if i is dummy in f (i,j are interchangeable in f), then i is also dummy in v_f (i,j are interchangeable in v_f) in the usual sense. I.e., $v(S) = v(S\setminus\{i\})$, $\forall\ S \subseteq N$ ($v(S\setminus\{i\}) = v(S\setminus\{j\})$, $\forall\ S \subseteq N$, $i,j \in S$), which implies that $\Phi(v_f)(i) = 0$ ($\Phi(v_f)(i) = \Phi(v_f)(j)$.) Hence, $\Phi(v_f)$ is a Shapley value.

Next we show uniqueness. Let Ψ be a Shapley value on $F_0(\mathcal{P})$. It suffices to show that Ψ is uniquely determined on $\{g_P\}_{P\in\mathcal{P},P\neq P_f}$ since this set is a basis for $F_0(\mathcal{P})$. Consider, then, g_P for some $P \neq P_f$. Assume $P = \{A_1,A_2,\ldots,A_k\} \cup P_f^{A_{k+1}}$ where $|A_\ell| > 1$ for $1 \le \ell \le k$. (That is, $A_1,\ldots,A_k$ are the non-singleton coalitions in P.) Obviously, $i \in A_{k+1}$ is a dummy player in g_P. Similarly, every $i,j \in \cup_{\ell=1}^k A_\ell$ are interchangeable since for every $Q \in \mathcal{P}$

$$g_P(Q \wedge (\{\{i\}\} \cup P_c^{N\setminus\{i\}})) = g_P(Q \wedge (\{\{j\}\} \cup P_c^{N\setminus\{j\}})) = 0.$$

Hence, Ψ has to satisfy

$$(\Psi g_P)(i) = \begin{cases} 1/\Sigma_{\ell=1}^k\ |A_\ell| & \text{for } i \in \cup_{\ell=1}^k A_\ell \\ 0 & \text{otherwise} \end{cases}$$

Thus a Shapley value, if such exists, is unique and the formulae above may be used to compute it via the coefficients $\{\alpha_P(f)\}_P$. Since existence was established earlier, $\Psi f = \Phi(v_f)$ is proved. (Notice that this equality is also simple to verify directly: for g_P with $P = \{A_1, \ldots, A_k\} \cup P_f^{A_{k+1}}$ as above.

$$v_{g_P} = u_{\cup_{\ell=1}^{k} A_\ell}$$

where u_S is the unanimity game on S.) //

<u>Remark 4.1.2</u>: It may seem surprising that the Shapley value of f does not depend on all of the numbers $\{f(P)\}_{P \in \mathcal{P}}$. As a matter of fact, the (small) subset

$$\{f(P_c^A \cup P_f^{A^c})\}_{A \subseteq N}$$

i.e., the value of f on "all-or-none" partitions alone determines Ψf, while the value of f on partitions which are not of this form is immaterial.

An attempt to understand this phenomenon may be the following: the axiom which should be held responsible for it is the interchangeability axiom (3): it focuses on the damage that a player may cause by deserting his/her coalition, and should two such players have the same "threat" power, they are given the same payoff. In a way, this axiom simply distinguishes between those players who do cooperate in some way (i.e., in some nontrivial coalition) and those who do not (singletons). The former have a viable threat, the latter do not. The precise way in which the "cooperative" players cooperate--i.e., via which coalition--does not matter; it only matters that they do. Hence, the payoff depends only on the best that the "cooperative" players may obtain--$f(P_c^A \cup P_f^{A^c})$--where A is the set of "cooperative" ones.

Whether this property is desirable or not is debatable. We believe that in some situations it will be quite intuitive and will capture the essence of the cooperative global game, while in others it may well be inappropriate. Since axiom (3) seems innocent, yet guarantees uniqueness, we chose it to define "the Shapley value." However, one may certainly wish to consider other solution concepts, as suggested below.

Remark 4.1.3: One obvious alternative to the interchangeability axiom is the good old-fashioned symmetry: for a permutation π: $N \to N$ and $f \in F_0(\mathcal{P})$ deinfe $\pi f \in F_0(\mathcal{P})$ by $(\pi f)(\{A_1,\dots,A_k\}) = f(\{\pi A_1,\dots,\pi A_k\})$, and for $x \in \mathbb{R}^N$ define $\pi x \in \mathbb{R}^N$ by $\pi x(i) = x(\pi i)$. Then we may define

(3') Symmetry: for every permutation π: $N \to N$, $\psi(\pi f) = \pi(\psi f)$.

It is easy to check that, in the presence of (1), (2) and (4), this axiom is strictly weaker than (3). More specifically, when defining ψ_{g_P}, one is restricted to assign $(\psi_{g_P})(i) = (\psi_{g_P})(j)$ if $|P(i)| = |P(j)|$ but players in coalitions of different sizes may get different payoffs.

4.2 The Core

For ordinary games, the core is a set of allocations $x \in \mathbb{R}^N$ no coalition may (unilaterally) improve upon. For global games, however, it is not entirely clear what is meant by "improve upon," since a coalition cannot act alone.

However, suppose that in a global environment clean-up game a certain set of countries is assigned a share in the total cost which exceeds what it would cost this set to perform the clean-up on its own. That is, suppose that

$$\Sigma_{i \in A} x(i) < f(P_c^A \cup P_f^{A^c}).$$

In such a case it would make sense for the coalition A to undertake the whole project by itself, and the allocation x is not "stable."

Next, suppose that two coalitions may get more if they operate alone, while the others do not cooperate. I.e.,

$$\Sigma_{i \in A} x(i) + \Sigma_{i \in B} x(i) < f\ (P_c^A \cup P_c^B \cup P_f^{(A \cup B)^c}).$$

And a similar argument excludes such an allocation x. Thus we are led to the following definition: $x \in \mathbb{R}^N$ is in the <u>core</u> of $f \in F_0(\mathcal{P})$ iff for every $P = \{A_1, A_2, \ldots, A_k\} \cup P_f^{A_{k+1}} \in \mathcal{P}$ with $|A_\ell| > 1$, $\ell = 1, \ldots, k$, the following condition holds:

$$\Sigma_{i \in \cup_{\ell=1}^k A_i} x(i) \geq f(P)$$

with equality for $P = \{N\}$.

<u>Observation 4.2.1</u>: If $f \in F_0(\mathcal{P})$ is monotone, then

$$\text{core}(f) = \text{core}(v_f)$$

(where the core of an ordinary game is defined in the usual sense, i.e., $x \in \text{core}(v_f)$ if $\sum_{i \in S} x(i) \geq v_f(S)$ for all $S \subseteq N$ with equality for $S = N$.)

Proof: The inclusion $\text{core}(f) \subseteq \text{core}(v_f)$ is immediate, while the converse inclusion is trivial (in the presence of monotonicity of f). //

Thus, the Shapley-Bondareva conditions for non-emptiness of the core of ordinary games (Bondareva (1963), Shapley (1967)), also characterize non-emptiness of the core of global games. Moreover,

Observation 4.2.2: If a global game f is convex and monotone it has a non-empty core and its Shapley value is included in it.

Proof: See Shapley (1971) and Proposition 3.3 above. //

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