Discussion Paper No. 918

DYNAMIC EXTERNALITIES, MULTIPLE EQUILIBRIA AND GROWTH*

by

Michele Boldrin
MEDS-KGSM
Northwestern University

First Version: October 1988,
Revised : December 1990.

Preliminary Version. Comments Welcome.

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* The first version of this paper was prepared in Fall 1988 while I was with the UCLA Economics Department. I would like to thank the Sante Fé Institute in Sante Fé, New Mexico and the Institut d'Analisi Economica at the Universitat Autonoma de Barcelona for their kind hosptiality and financial support. I am grateful to Tim Kehoe, José Scheinkman, Michael Woodford and seminar participants at the 1990 Northwestern Summer Conference, the Theoretical Division at Los Alamos National Laboratory, UCLA, Bocconi University, Universitat Autonoma de Barcelona and the University of Chicago for comments and suggestions.

Abstract

In this paper I consider an OLG model with production and a single commodity. I show that in such an environment unbounded growth of income per capita is not possible if the aggregate technology is of the usual constant returns to scale type. This is not due to lack of productivity of the capital stock in the long run but, rather, to inappropriate distribution of income across generations, which makes it impossible for the young savers to afford buying the existing stock of capital,

I then introduce an external effect, due to the stock of capital, in the aggregate production function and derive conditions under which persistent growth is an equilibrium outcome. I also show that the introduction of an external effect, while making growth feasible, also creates "poverty traps" and open sets of initial conditions for which there exists an infinite multiplicity of equilibria. I also show that, when such a multiplicity exists, equilibria with the very same initial position, will display remarkably different asymptotic behaviors. Finally I show that by introducing appropriate tax schemes such multiplicity can be eliminated but that the same is not true for poverty traps, which appears to be quite robust with respect to policy interventions.

1. Introduction.

I consider a standard overlapping generations model with one sector production, with and without a positive external effect induced by the aggregate stock of capital. I show that, for different ranges of parameter values, persistent growth, poverty traps, multiple equilibria and oscillatory equilibria may result when the externality is present.

This paper belongs to the crowded area of research which has developed in the footsteps of Romer [1986] and Lucas [1988]. Contrary to others I have very few novelties to offer the reader; still I find the exercise worthy of some attention for at least three motives.

The model is very simple and easy to handle analytically . In particular one can derive from it all the properties that this kind of externalities are known to induce on competitive equilibria. It is very difficult, if not impossible, to do the same in the other existing theoretical models, which have typically been built with one very specific feature in mind.

Secondly I find this a useful framework to assess a couple of open problems in the theory of economic development. They are: (a) modeling the historically undeniable fact that countries with the same technology and in very similar economic conditions a century or less ago have now reached very different levels of economic development; (b) the consistency between persistent growth and the assumption of decreasing returns within the standard one sector neoclassical model of aggregate production.

Point (a) has been the object of research since about fifty years ago and has been recently reconsidered by Azariadis and Drazen [1990] and Murphy, Shleifer and Vishny [1989]. In one case (Murphy et al.) underdevelopment results from lack of coordination in the adoption of production techniques across industries. If two methods of production are available, one of which exhibits increasing returns and therefore lower unit costs at high output level, then picking the latter induces a "good equilibrium" which the authors identify with development whereas the choice of the other more primitive technology yields the low output equilibrium which they associate with underdevelopment. The framework adopted is completely static, which makes it difficult to understand how a country could possibly move in time from one position to the other. If one takes the model literally, the jump from rags to riches occurs in just one period, almost by fiat, i.e. by coordinating the choice of the production processes across industries. The economy may equally move back to underdevelopment at any time in the future if such coordination fails.

The Azariadis and Drazen model is a dynamic one, and quite close to mine as they also adopt the OLG-cum-production setup. Intuitions and results are nevertheless quite distinct from those I present here. Two sources of growth are suggested: a pure external effect coming from the stock of capital and human capital which is produced by means of labor time and already existent human capital. In the first case in order to get the desired result the authors appeal to the rather special assumption that the external effect is described by a step function. Of the second case the authors provide only a local analysis around the stationary states. While they can show that a (continuum of) "bad" stationary positions with no human capital accumulation exist which are also locally stable, the existence and local stability of a "good" steady state with persistent growth is left vague. Even under the additional assumption that such a position exists the model would still be mute with regard to how and when a country could get there as its global dynamic is not discussed nor it appears easy to figure out.

In the model I study the external effect is only required to increase productivity as the capital stock increases. Nevertheless a poverty trap, in the form of a low-income locally stable stationary state, always arises whenever persistent growth is possible. Furthermore when the external effect is very strong there exists an open set of initial conditions giving rise to an infinity of equilibrium paths, some of which oscillate forever within a trapping region of low income levels whereas others take off to sustained growth.

As for (b) it is important because a convincing reconciliation between growth and constant returns would allow the study of long run phenomena within the same theoretical frame adopted in the study of business cycles. Jones and Manuelli [1990a] have suggested that within the standard one sector model of optimal growth such a reconciliation could take place. I think it is relevant to ask if the result is robust with respect to changes in the demographic assumptions underlying the infinite horizon framework. In light of the finiteness of our lives the OLG model of capital accumulation appears to be the obvious alternative candidate. The two are notoriously equivalent if intergenerational altruism is operative and bequests are passed over from one generation to the next. The empirical question of whether this occurs in reality is not easy to answer, which makes it important to keep the OLG model on equal theoretical footing with the infinite horizon one for the study of capital accumulation problems.

The present exercise confirms the critical role played by bequests: persistent growth in the presence of a convex aggregate technology set is not feasible in the OLG framework without substantial intergenerational transfers from the old's to the young's. This is because, even if the expected rate of return on capital remains high enough to motivate young people to invest out of their labour income, the latter becomes a negligible fraction of the stock of capital when this grows unbounded. This problem is easily solved when the external effect from the stock of capital induces enough asymptotic nonconvexity to allow the wage rate to grow at least as fast as the capital stock.

Finally the model lends itself very easily to carrying on economic policy exercises. The present paper is maintained completely within purely theoretical boundaries, but I do provide a preliminary discussion of the lines along which fiscal policy may or may not be used to eliminate poverty traps and the multiplicity of equilibria due to external effects. A full study of these matters will have to be pursued on another occasion.

In the next section I introduce the notation and derive the equilibria of the model without external effects. Section 3 introduces the external effect and characterizes the set of equilibria in this case. In Section 4 some examples illustrate the working of the model. Section 5 discusses the policy implications just mentioned and concludes the paper. A few proofs are collected in the Appendix.

2. Accumulation without an External Effect.

2.1 The Basic Model.

Each generation lives for two periods and is of a constant size equal to one. The economic life of one within the continuum of identical agents born at any time $t = 0,1,2,...$ is all in the programming problem:

$$\begin{aligned} &\text{Max } u(c_t, c_{t+1}) \\ &\text{s.t. } c_t + s_t \le w_t \\ &c_{t+1} \le s_t \cdot \pi_{t+1} \\ &c_i \ge 0, \; i = t, \; t+1. \end{aligned} \tag{1}$$

where c is consumption, w is the wage rate, s is saving and π is the net of depreciation rate of return on capital. Under the usual regularity conditions on u (i.e. strict quasi-concavity, continuity and monotonicity) the unique solution to (1) can be expressed by means of the saving function:

$$S(w_t, \pi_{t+1}) = \underset{0 \le s \le w_t}{\text{Arg Max}} \; u(w_t - s, s \cdot \pi_{t+1}) \tag{2}$$

It is very well known that, even under the stated regularity conditions, the saving function S, which is monotone increasing in w, may fail to be so in π. One way out is to assume that consumption levels in the two periods of life are gross substitutes. In fact, given that I want to rule out complications coming from the preferences side I will also assume the utility function to be homothetic. This yields a very simple and well behaved saving function. I will impose the formal assumptions directly on $S(w,\pi)$.

(H.1) The function $S: \Re_+^2 \to \Re_+$ satisfies:

- there exists a function $g:\Re_+ \to [0,1]$ of class C^1 and increasing, with $g(\pi) = 0$ for all $\pi \le \varepsilon$ some $0 \le \varepsilon < \infty$, $\lim g(\pi) = 1$ as $\pi \to \infty$, and such that S can be written as $S(w,\pi) = w \cdot g(\pi)$.

On the technological side I assume aggregate production is representable as:

$$Y_t = A_t \cdot F(K_t, L_t) \tag{3}$$

where A_t represents the time t level of an aggregate scaling factor. This will be endogenized in Section 3 through the external effect. In the present section it is taken as exogenous to the model and normalized to $A_t = 1$ for all t. Firms last one period only (they are managed by the old half of the population) and maximize their profits period by period:

$$\begin{aligned} &\text{Max } Y - w \cdot L - \pi \cdot K \\ &\text{s.t. } Y \le F(K,L) \\ &K \ge 0,\ L \ge 0. \end{aligned} \tag{4}$$

where all variables should be understood as of time t. Once again the usual assumptions of continuity and concavity are placed over F so that (4) has a (unique) solution for all pairs $(w,\pi)\in\mathbb{R}_+^2$, which is characterized by the first order conditions: $\partial F(K,L)/\partial K = \pi$ and $\partial F(K,L)/\partial L = w$. F is also assumed homogeneous of degree one, hence we can define $f(x) = F(K/L,1)$ upon which we will now impose the following restrictive set of hypotheses:

(H.2) The function $f:\mathbb{R}_+ \to \mathbb{R}_+$ is of class C^2 and satisfies:

a) $f(x) > 0$ for $x>0$, $f(0) \geq 0$ and, $f'(x) > 0$, $f''(x) < 0$ for all x.

b) $\lim f'(x) = \infty$ and $\lim x\cdot f'(x) = 0$ as $x\to 0$.

c) The wage/capital ratio $\omega(x) = [f(x)/x - f'(x)]$ is decreasing in x and satisfies $\lim \omega(x) \geq 1$ for $x\to 0$.

While a) and b) are standard neoclassical assumptions, c) has been introduced to guarantee that a unique, strictly positive stationary state exists for the "uncorrupted" model and to assure its dynamic asymptotic stability.

Equilibrium in the labor market implies $L_t = 1$ and $w_t = [f(x_t) - x_t\cdot f'(x_t)]$ for all t, while the equilibria of the output and capital markets are summarized by $\pi_t = f'(x_t)$ and the following implicit function which equates tomorrow's total demand for capital to the level of saving obtained today:

$$G(x,y) = y - [f(x) - x\cdot f'(x)]\cdot g[f'(y)] = 0 \tag{5}$$

where $y = x_{t+1}$ and $x = x_t$. The following is true about (5):

Proposition 1. Under hypotheses (H.1) and (H.2) there exists a function τ: $\mathbb{R}_+ \to \mathbb{R}_+$ solving (5) and such that for any initial condition $x_0 > 0$ the unique equilibrium path of capital accumulation $x_{t+1} = \tau(x_t)$, converges asymptotically to the unique stationary position $x^* = \tau(x^*)$.

Proof. See Appendix.

2.2 Persistent Growth is not Possible.

I will now address the following question: is persistent growth possible within the basic model and in the absence of any external effect if some of the restrictive assumptions imposed by (H.1) and especially (H.2) are relaxed? The answer is contained in Proposition 2 and it is negative.

It should be clear that (H.1) does not really impose any limit on attainable growth rates and it just simplifies the algebra. In any case in the present subsection I will work under the extreme hypothesis that individuals care only for their consumption when retired and therefore save and invest all their incomes in every period of their lives but the last one. Given that growth coincides with the accumulation of capital this is the most "growth oriented" among all the preference structures one may think of. The restrictions imposed on the production function by (H.2) are more demanding: again I will retain only continuity, concavity and positive monotonicity. Furthermore I will, following along the lines of

Jones and Manuelli [1990a], impose a positive lower bound on the rate of return on capital.

Other special assumptions may also hide in the demographic structure adopted: the OLG model is meant to capture the finiteness of individual lifetimes, and there is no reason to believe that a two-period model is the best way of representing the life cycle. I will therefore assume that each generation lives for (T+1) periods, where T is any positive and finite integer number, and is endowed with one unit of labor for the first T periods and none in the last. To simplify matters I will also assume that their saving behavior in all periods but the (T+1st) satisfies the requirements discussed in the previous paragraph. Once again this simplifies the proof and, if anything, creates a bias in favor of sustained growth in the capital stock. Still the following negative result is true:[1]

Proposition 2. Assume every individual lives for (T+1) periods and has a utility function defined as $u(c_t,c_{t+1},\ldots,c_{t+T}) = c_{t+T}$. Assume furthermore that the production function $f(x)$ is of class C^2, monotone increasing, concave and such that $\lim f'(x) = b > 1$ for $x\to\infty$.
Then every equilibrium path $\{x_t\}_{t=0}^{\infty}$ beginning at a given initial condition $x_0 \geq 0$ is bounded above for all $t\geq 0$.
Proof. See Appendix.

The economic intuition is, indeed, very simple: capital accumulation is always carried on by the younger generations and their sources of income are current wages plus the previously earned wage bills accumulated (for at most (T-1) periods) at the equilibrium rates of return. As the capital stock grows infinitely larger the wage bill becomes negligible (because f(x) is concave) and therefore the base of accumulation becomes negligible. No matter how high is the rate at which compounding occurs it occurs only for a finite number of period and it cannot compensate for the fact that the ratio between the wage bill and the stock of capital is going to zero. This is always true, as long as T is finite. When T converges to infinity the compounding factor $\Pi_{i=1}^{T-1} f'(x_{t-i})$ may grow faster than the base $(1/T)\cdot\omega(x_{t-T})$ shrinks to zero, thereby permitting unbounded growth. This argument points to a strong discontinuity "at infinity" in the qualitative properties of the model, something that should be taken in proper account in evaluating the predictions of the infinite horizon model.

Proposition 2 raises an important empirical question: taking for granted that growth is generated by capital accumulation what sources of income are used to finance it and, in particular, how does productive capital get transferred across generations? It is apparent from the proof of Proposition 2 that even if we force the members of the oldest generation to transfer their depreciated stock of capital to the people in the generation immediately after them and therefore consume only the net return on their life-time saving, persistent growth would still be impossible. If increasing returns are not the key to sustained growth and if we accept the one sector growth model without any government intervention as the appropriate analytical tool, then bequests are relevant and they must go over and above depreciated capital to include a portion of the net return on the stock owned by the oldest generation.[2]

3. The Model with Positive Externalities.

I now internalize the scale factor A_t in (3) by making it dependent on the aggregate stock of capital. I posiit $A_t = \psi(x_t)$ and interpret the function ψ as the formal description of an economy-wide external effect. This representation can be associated to a variety of observable phenomena. Beside the pure external benefits (if any) that an individual firm may derive from the aggregate capital stock (my phone is quite more useful if there are many other phones to which it is connected, my software is much more valuable if it is widely adopted, etc.), one may also think of a learning-by-doing mechanism in which the achieved degree of social experience is measured by the accumulated stock of capital as in Arrow's original contribution. Alternatively one may consider ψ as a measure of social knowledge as initially proposed by Romer [1986]; or a reduced form representation of the type of pecuniary externalities investigated by Murphy et al. [1989]. Finally, if none of the former suggestions is convincing, there remains the option of interpreting ψ as an interesting theoretical device whose consequences should be investigated.

The dynamic equilibrium condition (5) becomes:

$$(6) \qquad G(x,y) = y - \psi(x)[f(x)-x\cdot f'(x)]\cdot g[\psi(y)f'(y)] = 0$$

Inspection of (6) and in particular of the partial derivatives $\partial G(x,y)/\partial x$ and $\partial G(x,y)/\partial y$ will show why the simple properties of (5) may now be violated. In fact one may conceive of so many different ways in which things can go wrong, that additional restrictive hypotheses are needed to make the ensuing discussion economically relevant. Before doing this let me illustrate an especially "abnormal" equilibrium that may often occur. Recall from (H.1) that $\varepsilon \geq 0$ is the largest expected rate of return at which young people are not willing to save any portion of their wage income. It will often be $\varepsilon = 0$, but $\varepsilon > 0$ cannot be excluded if the slope of the utility function is appropriately limited on the boundary of $\mathbb{R}^2_+$. Inspection of (6) reveals that the following is true.

<u>Proposition 3.</u> Assume (H.1)-(H.2) are true and assume that ψ and f satisfy: $\psi(0)f'(0) \leq \varepsilon$. Then: for all $x \geq 0$ the path $\{x,0,0,\ldots\}$ is an equilibrium.

The intuition is quite simple: if the external effect is strong enough to reduce the private rate of return to its minimum when the aggregate stock of capital is zero then, no matter what the capital stock is today, it does not pay private individuals to invest if they expect that all the other agents will not. The example with a Cobb-Douglas production function discussed in section 4 illustrates this possibility.

Abnormal equilibria apart there are two important channels through which the external factor ψ may affect the aggregate production function $\phi(x) = \psi(x)f(x)$. It can make the private rate of return $\pi(x) = \psi(x)f'(x)$ increasing in x (at least over a certain range) and bounded away from ε even when x gets infinitely big. It can turn the ratio between total wages and the aggregate stock of capital $\omega(x) = \psi(x)[f(x)/x - f'(x)]$ into a non decreasing function of the stock of capital itself. While both phenomena are somewhat linked to the fact that ϕ may now be convex (at least over certain subsets of $\Re_+$), exact equivalence holds only for particular functional forms like the exponential one I use for the Cobb-Douglas example. This is easily verified by working out the algebra of other admissible pairs of production and externality functions. For example: (1) $f(x) = \ell n(\alpha+x)$ $(\alpha>1)$ and $\psi(x) = x^\beta$ $(\beta<1)$ give ϕ neither concave nor convex, π increasing first and then decreasing and ω decreasing first, then increasing, then decreasing again; (2) $f(x) = a + bx$ and $\psi(x) = x^\beta$ $(\beta<1)$ give ϕ neither concave nor convex, π increasing and ω decreasing for all x; (3) $f(x) = x^\beta$ $(\beta<1)$ and $\psi(x) = a + bx$ give ϕ as in example (2) but now both π and ω are decreasing for small values of x and increasing for larger ones! It is therefore opportune to make explicit assumptions about each one of the three effects separately.

The exact behavior of $\pi(x) = \psi(x)f'(x)$ for large values of x is hard to theorize upon as the anedoctal evidence yields conflicting suggestions. Learning-by-doing seems to end after a while; network externalities may transform in (negative) congestion effects when total investment exceeds certain levels, etc. On the other hand there is no evidence that the positive benefits from the accumulation and dispersion of knowledge in society are decreasing with its quantity and that private returns from investment are being negatively affected by increases in the aggregate stock of capital. Whichever of the two tendencies actually dominates in reality is fortunately of no concern in this model as long as $\pi(x)$ remains appropriately bounded below. This is all I will assume:

(H.3) The function $\psi: \Re_+ \to \Re_+$ is of class C^2, monotone increasing and such that:

(i) $\pi(x) = \psi(x)f'(x) > \varepsilon$ for all $x \geq 0$.

(ii) $\lim w(x) = \psi(x)[f(x) - x \cdot f'(x)] = 1$ as $x \to 0$.

Assumptions (i) rules also out the type of equilibria described in Proposition 3, while (ii) has only the technical function of removing the steady state from the origin. The case in which the latter exists is described with enough generality by the first example in Section 4.

I find reasonable to argue that external effects tend to make the private rate of return on investment an increasing function of the aggregate capital stock at low levels of the latter. This fact may have some striking consequences on the structure of the equilibrium set whenever the utility function is not very concave, i.e. whenever the function $g(\pi)$ is very elastic. Define the new function:

$$J(y) = \{g[\pi(y)]\}^{-1} \tag{7}$$

J is decreasing in y if π is increasing in it and $\partial G/\partial y$ is just $(J+yJ')$. $\partial G/\partial y$ may therefore vanish somewhere in its domain and have opposite signs over different subsets of the positive orthant. This occurs if J's degree of elasticity is not uniformly below or above one. In these circumstances there will exist some open subset of the real line over which a function solving (6) is not defined. Most typically this will destroy uniqueness of equilibrium for that set of initial conditions. As the examples illustrate this is a far from remote possibility. It is formally stated in my next hypothesis:

(H.4) The social production function $\phi(x)$ and the saving function $S(w,\pi)$ are such that:

(i) there exists a pair of values $0 \leq \underline{x} < \bar{x} \leq \infty$ such that $\pi(x) = \psi(x)f'(x)$ is increasing in x for all $x \in [\underline{x},\bar{x}]$;

(ii) there exists a pair of values $\underline{x} < y^1 < y^2 < \bar{x}$ such that the function $J(x)$ defined in (7) is more than unitary elastic for $y^1 < x < y^2$ and less than unitary elastic for $x < y^1$ and $x > y^2$.

Proposition 2 has already proved that a wage rate with larger than unitary elasticity to variations in the stock of capital is a necessary condition for persistent growth. This is now a possible feature of the aggregate production function ϕ:

(H.5) The production function ϕ is such that for all $x \geq \tilde{x} > 0$ the function $\omega(x) = \psi(x)[f(x)/x - f'(x)]$ is increasing with $\lim \omega(x) = \infty$ as $x \to \infty$.

The last part of this hypothesis is stronger than necessary. To guarantee persistent growth, an asymptotic wage/investment ratio larger than $\{\lim J(x),\ x\to\infty\}$ would suffice. I have chosen the form (H.5) only because it minimizes notation, even if it may imply an ever increasing growth rate of the aggregate capital stock.

From now on I will assume that (H.1)-(H.3) are always true. Thus, according to the arguments I have discussed so far, the introduction of ψ can produce either of the following four different scenarios:

CASE 1: neither (H.4), nor (H.5) applies, i.e. $\omega(x)$ and $\pi(x)$ are still decreasing in x.

CASE 2: (H.4) applies but (H.5) does not.

CASE 3: (H.5) applies but (H.4) does not.

CASE 4: both (H.4) and (H.5) apply.

Case 1 is not interesting: equilibrium paths still behave as in Proposition 1. Case 2 may induce multiple equilibria, but none of them display persistent growth and it can be easily understood from the study of Case 4. I will therefore examine only the latter and Case 3.

For a given stock of capital $x_t = x \geq 0$ the value $x_{t+1} = y \geq 0$ is an equilibrium choice if:

$$yJ(y) = x\omega(x) \tag{8}$$

is satisfied. (H.3)(ii) has ruled out steady states at the origin, hence $x^* > 0$ is a steady state if:

$$J(x^*) = \omega(x^*) \tag{9}$$

holds. I indicate the set of solutions to (9) as Fix(τ); here τ denotes the application solving (8): it can be either a function from $\mathbb{R}_+$ into itself or a correspondence from $\mathbb{R}_+$ into $\mathcal{P}(\mathbb{R}_+)$, depending on (H.4). Local stability of the steady states can be characterized by means of the two functions ω and J. Let $x^{*}\in Fix(\tau)$ and assume the implicit function theorem is satisfied in a neighborhood U×U of (x^{*},x^{*}). With a small abuse of notation I denote also with τ the locally well defined C^1 function from U into U solving (8). One has:

$$(10) \qquad \tau'(x) = [\omega(x)+x\omega'(x)]\cdot[J(\tau(x))+\tau(x)J'(\tau(x))]^{-1}$$

for all $(x,\tau(x))\in U\times U$. One can check that $[\omega(x)+x\omega'(x)] > 0$ for all x's and independently from (H.4) and (H.5). Clearly $\omega(x) \approx J(\tau(x))$ for $(x,\tau(x))\in U\times U$. This proves:

<u>Proposition 4.</u> Assume (H.1)-(H.3). Let $x^{*}\in Fix(\tau)$ and assume there exists an open neighborhood U of x^{*} such that $[J(y)+yJ'(y)]\neq 0$ for $y\in U$. Let $\tau:U\to U$ solve $\tau(x)J(\tau(x)) = x\omega(x)$ for all $x\in U$. Then τ is C^1 and satisfies:

a) τ is decreasing on U if $[J(\tau(x))+\tau(x)\cdot J(\tau(x))] < 0$ and increasing otherwise;

b) for $x\in U$ the slope of τ is determined according to:

- $0 < \tau'(x) < 1$ when $0 < \omega'(x) < J'(\tau(x))$,
- $1 < \tau'(x)$ when $-J(\tau(x))/x < J'(\tau(x)) < \omega'(x)$,
- $-1 < \tau'(x) < 0$ when $[2\omega(x)+x\omega'(x)] < -\tau(x)J'(\tau(x))$,
- $\tau'(x) < -1$ when $J(\tau(x)) < -\tau(x)J'(\tau(x)) < [2\omega(x)+x\omega'(x)]$.

The next Proposition characterizes the equilibria for Case 3:

<u>Proposition 5.</u> Assume (H.1)-(H.3) and (H.5) are true. Then there exists a C^1 function $\tau:\Re_+ \to \Re_+$ solving (8) and such that, for any initial condition x_0, the unique equilibrium is described as $x_{t+1} = \tau(x_t)$. The set $Fix(\tau)$ generically contains an even number of elements and it is non-empty whenever $J(x) > \omega(x)$ for some $x \in \Re_+$. Denote with x_{max} the largest element of $Fix(\tau)$, ($x_{max} = 0$ if $Fix(\tau)$ is empty). Then:

a) all paths $x_{t+1} = \tau(x_t)$ with $x_0 \leq x_{max}$ are uniformly bounded by x_{max} and converge to some element of $Fix(\tau)$;

b) all paths $x_{t+1} = \tau(x_t)$ with $x_0 > x_{max}$ are monotonically increasing and unbounded. If $\lim \omega(x) \cdot g[\pi(x)] = \lambda > 1$ then they grow at the constant rate λ.

<u>Proof.</u> See Appendix.

A typical configuration is described in Figure 1. In Case 4 instead we have a more complicate picture.

<u>Proposition 6.</u> Assume (H.1)-(H.5) are true. Then $\tau:\Re_+ \to \wp(\Re_+)$ is a correspondence over $[x^1, x^2]$, ($x^1 = \tau^{-1}(y^2)$ and $x^2 = \tau^{-1}(y^1)$), and a function everywhere else. The set $Fix(\tau)$ generically contains an even number of elements and it is non-empty whenever $J(x) > \omega(x)$ for some $x \in \Re_+$. Denote again with x_{max} its largest element. Two cases are possible:

a) $(x^1, x^2) \cap Fix(\tau) = \emptyset$. Then:

for every initial condition $x_0 \in \Re_+$ there exist a finite number of equilibria $\{x_t\}_{t=0}^{\infty}$ solving (8), they are bounded and convergent to some element in $Fix(\tau)$ for $x_0 \leq x_{max}$ and grow asymptotically unbounded if $x_0 > x_{max}$;

b) $(x^1, x^2) \cap Fix(\tau) \neq \emptyset$.

b.1) If there exists an interval $\Delta = [a, x^2] \subset [x^1, x^2]$ and a selection $\chi(x) \subset \tau(x)$, such that $\chi(\Delta) \subset \Delta$, and for all $x \in \Delta$ $\chi(x)$ is not a singleton, then:

- for every initial condition $x_0 < a$ there exists a finite number of equilibrium paths $x_{t+1} \in \tau(x_t)$ all of which converge to an element $x^* \in Fix(\tau)$;

- for every initial condition $x_0 \in \Delta$ there exists a countable infinity of equilibrium paths $x_{t+1} \in \tau(x_t)$ which remain in Δ for all t, and a finite number of equilibrium paths $x_{t+1} \in \tau(x_t)$ leaving Δ after a finite number of periods: some of them converge to an element $x^* \in Fix(\tau)$ while some other may grow unbounded;

- for every initial condition $x_0 > x^2$ there exists a unique equilibrium path $x_{t+1} = \tau(x_t)$ converging to some $x^* \in Fix(\tau)$ if $x_0 \leq x_{max}$ and growing unbounded otherwise.

b.2) If either the interval Δ or the selection χ as in b.1) do not exist then:

- for every $x_0 < x^1$ there exists a unique equilibrium $x_{t+1} = \tau(x_t)$ converging to some $x^* \in Fix(\tau)$;

- for every $x_0 \in [x^1, x^2]$ there exist a finite number of equilibria, some of which converge to $x^* \in Fix(\tau)$ while some others grow unbounded;

- for every $x_0 > x^2$ there exists a unique equilibrium converging to $x^* \in Fix(\tau)$ if $x_0 \leq x_{max}$ and growing unbounded otherwise.

Proof. See Appendix.

The phrasing of Proposition 6 is certainly not very attractive. I hope that Figures 2.a and 2.b, portraying cases b.1) and b.2) will help the reader. Notice that, in order to get an infinity of equilibria, the downward sloping portion of τ has to cut the diagonal with a relatively flat slope in order to make it possible for the interval Δ to contain its own image $\chi(\Delta)$ under the selection. Multiplicity arises because, at each t = 1,2,3,... there is more than one x_{t+1} to be selected from $\chi(x_t)$, and randomization should occur in each period thereby creating a countable infinity of different paths.

The economic content of Propositions 5 and 6 is very simple: a strong enough external effect makes persistent growth possible if initial conditions are appropriate. It also makes a poverty trap an unavoidable outcome at others initial conditions. What I find striking is that, in Proposition 6, the set of initial conditions for which growth is an equilibrium is not disjoint from the set of initial conditions for which permanent stagnation is also an equilibrium. Contrary to many known cases, here the multiplicity of equilibria implies completely different asymptotic behaviors for accumulation paths beginning at the very same initial condition.

There is a large debate concerning the causes of cross countries differences in growth and development performances and the remarkable disparities of current incomes per capita between countries that were very similar fifty or a hundred years ago. In Boldrin and Scheinkman [1988] we advanced an interpretation of this phenomenon, attributing it to the interplay between a learning by doing external effect and international comparative advantages. The present model may be used to suggest a different explanation. The (formally random) selection of one of the possible equilibria can be interpreted as if it is carried on by unspecified institutions. The theory says that it is the interaction between the external effect and the different institutional mechanisms chosen to handle the multiplicity of equilibria that induces different development histories. This is clearly a very weak argument as "institutional mechanisms" is just an empty expression, (or an exogenous factor if you like). In the last section I try to show that it may be given some meaning by looking at governments' fiscal policies.

4. Examples.

I have chosen very familiar functional forms not because I find them more interesting or empirically relevant but only to show that the assumptions of Propositions 5 and 6 are not particularly extravagant.

4.1 Cobb-Douglas Technology.

Preferences are given by: $u(c,c') = [(c)^{1-\sigma} + (c')^{1-\sigma}]/(1-\sigma)$; the technology is $Y = A\cdot(K)^{\alpha}(L)^{1-\alpha}$, with $A = (K/L)^{\beta}$. We have: $g(\pi) = [1 + \pi^{\gamma}]^{-1}$, $\pi(x) = \alpha\cdot x^{\alpha+\beta-1}$ and $\omega(x) = (1-\alpha)\cdot x^{\alpha+\beta-1}$. For $\alpha\in(0,1)$, $\sigma\in[0,1)$ and $\underline{\beta = 0}$ assumptions (H.1)-(H.2) are satisfied and Propositions 1 and 2 apply. When $\underline{\beta\in(0,1-\alpha)}$ we obtain Case 1, to which Propositions 1 and 2 still apply. Case 2 cannot arise in this example as long as σ is less than one.

When $\underline{\beta \geq 1-\alpha}$, the social production function $\phi(x) = x^{\alpha+\beta}$ exhibits increasing returns in capital alone and hypothesis (H.3)(i) is satisfied; (H.3)(ii) has been violated on purpose as I want a stationary state at the origin. The implicit function $G(x,y) = 0$ describing the equilibrium paths is:

$$y - (1-\alpha)x^{\alpha+\beta}\cdot[1 + by^{\rho}]^{-1} = 0 \tag{11}$$

where I have set: $\gamma = -(1-\sigma)/\sigma < 0$, $b = \alpha^{\gamma}\in(1,\infty)$, $\rho = (\alpha+\beta-1)\gamma < 0$. As $\psi(0)f'(0) = 0$, the example satisfies Proposition 3 with $\varepsilon = 0$: for every $x > 0$ $\{x,0,0,0,\ldots\}$ is always an equilibrium in this economy. To illustrate Propositions 5 and 6 we have to distinguish between two cases.

When $\underline{(1-\alpha) < \beta < (1-\sigma)^{-1}}$ Proposition 5 applies. The origin is always a stationary state. The only other point in Fix(τ) is strictly positive and satisfies:

$$(1-\alpha)\cdot x^{\alpha+\beta-1} = 1 + b\cdot x^{\rho} \tag{12}$$

It is easy to prove that it is unstable because when x satisfies (12) the following inequalities are true:

$$(1-\alpha)(\alpha+\beta)\cdot x^{\alpha+\beta-1} > (1-\alpha)\cdot x^{\alpha+\beta-1} = 1 + b\cdot x^{\rho} > 1 + b\cdot(1+\rho)x^{\rho} \tag{13}$$

and the slope of the function τ solving (11) is:

$$(1-\alpha)(a+\beta)\cdot x^{\alpha+\beta-1}\cdot[1+b(1+\rho)x^{\rho}]^{-1} \tag{14}$$

at $x \in \text{Fix}(\tau)$. Formula (14) also shows that $\tau'(x) \to 0$ for $x \to 0^{+}$. The function $\tau(x)$ is therefore as in Figure 3, where we have used the parameter values $\alpha=.4$, $\sigma=.5$ and $\beta=.7,.9$ and 1.1 respectively. Notice that the interior stationary state converges to zero as the magnitude of the external effect increases, thereby reducing the size of the set of initial conditions that would trap the economy into permanent underdevelopment.

Finally when $\underline{\alpha+\beta > (1-\sigma)^{-1}}$, one has $(1+\rho) < 0$ and τ is now a correspondence from x into y as in Case 4. In order to graph it we can study the function $x = \theta(y)$ which solves G(x,y) and is still well defined as $\partial G/\partial x \neq 0$ for all x and y. This is:

$$x = \{[y + b\cdot y^{1+\rho}]\cdot 1/(1-\alpha)\}^{1/(\alpha+\beta)} \tag{15}$$

With some algebra it can be verified that $\theta(y)$ has a unique critical point at $\bar{y} = [-b(1+\rho)]^{1/|\rho|}$ and a unique interior fixed point y^{*}. It is decreasing for $y < \bar{y}$ and increasing for $y > \bar{y}$. The situation described in Proposition 6,b.1) arises here whenever y^{*} is less than $\bar{y}$. This is represented in Figure 4. While it is impossible to compute analytically the set of parameters at which multiplicity occurs, it is easy to do so numerically.

For example all the following triples have open neighborhood in which (15) is as in Figure 4: $\{\alpha=.5,\ \beta=2,\ \sigma=.1\}$, $\{\alpha=.4,\ \beta=2,\ \sigma=.2\}$, $\{\alpha=.35,\ \beta=2.15,\ \sigma=.3\}$.

4.2 Other Examples.

The choice of the Cobb-Douglas production function makes analytical derivation simple and feasible but, obviously, introduces special features that one may not find desirable. In particular: the aggregate production function is a convex function of the stock of capital, the degenerate equilibrium $\{x,0,0,\ldots\}$ is present and the "underdeveloped" stationary state is at the origin. I will now show that all the relevant predictions of the theory are retained also in a model economy where the special characteristics of the Cobb Douglas production function are not present.

I select the production function to be CES, i.e. $Y = A[K^{\rho} + L^{\rho}]^{1/\rho}$, with $\rho\in(0,1)$, and the externality function to be $A = (x^{\rho} + 1)^{\beta}$ with $\beta\in\mathbb{R}_{+}$. One has: $\pi(x) = x^{\rho-1}\cdot(x^{\rho} + 1)^{\theta}$ with $\theta = (\beta + (1-\rho)/\rho) > 0$ and $\omega(x) = (x^{\rho} + 1)^{\theta}/x$. The utility function is as in the previous example.

Once again for $\underline{\beta = 0}$ (H.1) and (H.2) apply and the equilibria are as in Propositions 1 and 2. Also, no matter how big β is, the limit of $\pi(x)$ for x going to zero is always infinity which implies that Proposition 3 cannot be applied to this example. In fact it is easy to verify, by inspection of G(x,y) as defined below in (16), that $\{x,0,0,\ldots\}$ is never an equilibrium.

When $\underline{\beta > 0}$ the external effect plays a role. Let me notice first that, no matter how large β is the aggregate production function never exhibits global increasing returns: it is always concave for $0 \leq x \leq [(1-\rho)/\beta\rho]^{1/\rho}$ and convex otherwise. Instead, for any $\beta > 0$, both parts of hypothesis (H.3) are satisfied. The implicit function G(x,y) is given by:

$$y - (x^{\rho} + 1)^{\theta} \cdot [1 + y^{\gamma(\rho-1)} \cdot (y^{\rho}+1)^{\theta\gamma}]^{-1} \tag{16}$$

which is, unfortunately, impossible to solve explicitly in the form $y = \tau(x)$. The "backward" equilibrium map $x = \theta(y)$ can be computed but turns out to be quite more complicated than in the first example. In particular I cannot obtain closed form solutions for its fixed points and critical points. It is possible to see, nevertheless, that for $\underline{\beta \in (0,1)}$ the function $\omega(x)$ is downward sloping with $\lim \omega(x) = \infty$ for $x \to 0$ and $\lim \omega(x) = 0$ for $x \to \infty$: persistent growth is therefore not possible in this case.

When $\underline{\beta > 1}$ the wage rate becomes more than unitary elastic at high levels of the stock of capital. In fact the function $\omega(x)$ is now shaped like a parabola, decreasing first and then increasing with $\lim \omega(x) = \infty$ for both $x \to 0$ and $x \to \infty$. Hypothesis (H.5) is therefore verified and either Proposition 5 or Proposition 6 apply. It is remarkably difficult, though, to characterize analytically the sets of parameter values at which the two different propositions apply. This is because, while it is easy to see that $\pi(x)$ satisfies assumption (H.4)(i) for all $\beta > 0$, with $\underline{x} > 0$ and $\bar{x} = \infty$, it is very difficult to compute the parameter values at which J(y) satisfies the portion (H.4)(ii). Graphical inspection and numerical simulations show, nevertheless, that $J(0) = 1$ and that $\lim J(y) = 1$ for $y \to \infty$. J(y) is therefore increasing for $0 \leq y \leq \underline{x}$ and decreasing thereafter. Even if $\underline{x}$ can be computed analytically the two points y^1 and y^2 cannot; I have only been able to show that they exists at certain sets of parameter values. In such cases

Proposition 6 applies exactly to this model and equilibrium correspondences as depicted in Figure 2 emerge.

5. Policy Implications and Conclusions.

The purpose of this section is very modest: I try to see if there are fiscal policies that may eliminate the two undesirable features that the external effect induces on the equilibria of this economy, i.e. the existence of a poverty trap and the multiplicity of alternative paths from a common initial condition. I will not attempt to investigate the implications of any particular welfare function nor will I address the existence of Pareto improving reallocations. I simply assume that growth is good and stagnation and indeterminacy of equilibria are bad.

This is only apparently similar to the exercise carried on in Jones and Manuelli (1990b): in that instance they start from a situation in which, (for technological reasons), growth is not feasible no matter what the initial conditions are and ask if there are distributive policies that may induce it. In this contest growth is feasible, the externality being its engine as opposed to a redistributive fiscal policy: I simply want to see if the externality can be controlled and persistent growth retained.

5.1 Can Fiscal Policy Eliminate Poverty Traps?

I will consider first the situation described by Proposition 5. In this case the poverty trap is composed of two pieces: (1) a purely technological one, say $[0,\kappa]$, where κ is such that $\phi(x) < x$ for all $x < \kappa$ and $\phi(x) > x$ for all $x > \kappa$; and (2) a "market driven" piece $[\kappa, x_{max}]$ where growth would be feasible (i.e. $\phi(x) > x$ for $x \in [\kappa, x_{max}]$) but the distribution of income between labor and capital is not the appropriate one.

Clearly $\kappa = 0$ would often occur (e.g. in the CES case of section 4.2 for values of ρ close enough to 1) but one cannot exclude $\kappa > 0$, as the Cobb-Douglas case confirms. In the latter technological environment the best that a government may hope to achieve is to move x_{max} as close as possible to κ. I will assume the central authority can use income taxes and lump-sum transfers and is forced to balance payments and receipts period by period. What should be taxed and what should be subsidized? and is the objective achievable? Well it depends, and it depends.

Given the nature of the externality it may seem obvious to expect that investments should be subsidized. When growth, (as opposed to allocational efficiency) is the target, this is not necessarily true because: i) in order to subsidize investments the government will have to tax labor income, thereby reducing the base of accumulation, ii) the receipts from current labor income $\{\phi(x)[f(x)-xf'(x)]\}$ may not be enough to make up for the required amount $\{\phi'(x)f(x)x\}$ of investment subsidies if the social production function is convex. Let me try to put things a bit more formally.

I will assume the government has the following instruments: it can tax or subsidize either type of incomes at the constant rates θ_w and θ_π and can lump-sum transfer income to the youngs, for an amount V. The balanced budget implies $V = \theta\psi(x)f'(x)x$, where θ is the rate of capital income taxation and the taxes on wages have been incorporated in V. This is because labor supply is assumed exogenous in the model. Also, I am not considering the case in which the government can lump-sum transfer income to capital owners because it is easy to see that it will never helps to increase growth rates. The new dynamic equilibrium condition is:

$$y = [\psi(x)f(x) - (1-\theta)x\psi(x)f'(x)]\cdot g[\psi(y)f'(y)(1-\theta)] \tag{17}$$

Inspection of (17) reveals that it is most difficult that (except for trivial cases like the logarithmic utility function) some real value of θ exists such that κ and x_{max} coincide: a high positive value of θ pushes up the $\omega(x)$ function but, contrary to what would be needed, increases also the function J(y). Which one of the two effects is dominant depends on the relative elasticities. As I will show in a moment by using the Cobb-Douglas example, it is unlikely that for standard functional forms a negative value of θ would be recommended. On the other hand the numerical example also seems to suggest that to adopt an active fiscal policy does not really make a big difference in this situation, as the reduction in the size of the poverty trap that can be achieved is quite small. For the Cobb-Douglas case the equilibrium condition (17) reads:

$$y = [1 - \alpha(1-\theta)]\cdot x^{\alpha+\beta}\cdot[1 + b(1-\theta)^{\epsilon}y^{\rho}]^{-1} \tag{18}$$

The behavior of (18) at the chosen set of parameter values is typical of most others, less easy to compute cases. I have set α=.5, β=1.5, σ=.5. An interior steady state solves: $[1-.5(1-\theta)]\cdot x^2 - x - 2/(1-\theta) = 0$. Here $\kappa = 1$ and for θ=0 one has x_{max} = 3.23. For negative values of θ the value of x_{max} increases, thereby worsening the poverty trap. For positive values of θ instead x_{max} decreases up to θ = .3, where x_{max} = 3.00. At larger values of θ the negative impact on π of additional taxation more than compensates the positive impact on wages and income redistribution is not beneficial for growth. All in all a taxation of about 30% on capital income with relative transfer of the receipts to wage earners reduces the size of the poverty trap of only 10%. Not a startling results. Indeed when the size of the external effect is brought within more empirically reasonable bounds the positive impact of taxation is even smaller.

Obviously the result is purely suggestive as I am not imposing empirically relevant restrictions on parameter values nor I am taking into account the distortions that taxation would induce when labor supply is endogenous and the consequential effects on the utility index[3].

In the next subsection I show that, at least in the Cobb-Douglas case there is a relatively simple nonlinear tax scheme that eliminates the multiplicity of equilibria. It is a matter of algebra to apply it to the present case to verify that also that tax scheme does not do any better than the simple one I have just examined. Notice that in the nonlinear tax scheme labour income is taxed at low level of x to subsidize the return on capital.

Naturally I cannot exclude that other, more sophisticated fiscal policies may be able to do wonder in this model. I have only pointed out that simple ones do not seem to.

5.2 Can Fiscal Policy Eliminate Indeterminacy?

I will now ask if a fiscal policy with the same characteristics of the one just studied may be useful to eliminate the indeterminacy of equilibria described in Proposition 6. Assume therefore that case b.1) of that proposition is true, the tax scheme will be effective if the elasticity of the function J(y) is reduced (in absolute value) by the introduction of the tax. Let g_θ denote the function $g[\psi(y)f'(y)(1-\theta)]$ for $\theta \neq 0$. Then by manipulating the first derivative of J(y) one can see that the elasticity will be reduced whenever the following inequality holds over the appropriate interval:

$$(19) \qquad (g_\theta'/g')\cdot(1-\theta) < (g_\theta/g)^3$$

Again this condition is not easily satisfied, in general, for real values of θ as can be seen by considering the often occurring case in which $g(\pi)$ is a concave function. In the Cobb-Douglas case, one can use (19) to check when $\partial G/\partial y$ would be uniformly positive. The condition is $y^{|\rho|} > [b(|1+\rho|)]\cdot(1-\theta)^{\epsilon}$. In this case a <u>negative value</u> of θ would help.

A more careful examination of the behavior of J(y), though, shows that what really matters is the "threat" associated to the choice of the "wrong" equilibrium. Assume the government has the informational ability to charge a nonlinear tax rate, dependent on the level of the stock of capital in existence when the tax is actually levied, say $\theta = \theta(x)$. Then if: $\theta'(y)/[1-\theta(y)] > \pi'(y)/\pi(y)$ is satisfied for all the relevant y's the sign of $\partial J/\partial y$ will always be positive. While finding a $\theta(x)$ that satisfies the former requisite is a very difficult problem the general case, it has a simple solution for the Cobb-Douglas economy. The following tax rule would do: $\theta(y) = 1 - \gamma\cdot x^{-\nu}$ with the two parameters γ and ν appropriately chosen in order to satisfy the following two restrictions:

1) set γ such that $(\alpha\gamma)^{1/\nu} < x_0$;

2) set ν such that $\alpha + \beta - 1 < \nu < (\alpha + \beta - 1)/(1-\sigma)$.

I have imposed condition 1) to avoid an infinitely large and negative tax for small values of x. Nevertheless 1) will work only when the accumulation path is monotone increasing. If x_0 is so small that the equilibrium is in any case going to zero, than be it! The left hand side of condition 2) guarantees that $\partial J/\partial y$ is positive, i.e. it realizes the threat, while the right hand side makes sure that the threat is not too strong, i.e. it guarantees that permanent growth is possible ($\omega(x)$ asymptotically dominates $J(x)$). Obviously there is no reasons to expect that the suggested scheme would eliminate the poverty trap together with multiplicity. Even worse it may add other fixed points to (6). Nevertheless my numerical exercises suggest that it does not and, indeed, it may somewhat reduce the size of the poverty trap when γ is chosen according to rule 1) and ν is set close to its lower bound[4].

5.3. Conclusions.

I have shown that, in models where agents live for a finite number of periods and leave no bequests, persistent growth is not possible with non-increasing returns in production. I have also shown that instead the introduction of an external effect of the type suggested in Romer [1986] makes persistent growth possible. Strictly speaking this does not require the aggregate production function to display increasing returns in the stock of capital: a non decreasing wage/investment ratio and a sufficiently bounded rate of return on investment are the necessary and sufficient requisites.

I have also considered other implications of the external effect for the set of dynamic equilibria. Under reasonably general assumptions on technology and preferences I am able to fully characterize the set of equilibria for this model economy. I have shown accordingly that, together with the growth equilibria, it contains stagnation equilibria converging to some low level stationary state and oscillatory equilibria moving more or less randomly in some bounded region. In the latter case I have in fact proved that, for a given initial condition, there exists an infinity of such equilibria. Finally I have shown that there exist non trivial sets of initial conditions to which one can associate an infinity of equilibrium paths. The equilibria belonging to this infinite set may have remarkably distinct asymptotic behaviors; in particular, for a given initial condition, there may be some equilibria converging to a low income stationary state and some others displaying persistent growth. I have suggested that this phenomenon may be associated to the existence of countries that starting from very similar initial conditions have subsequently followed very distinct development paths.

Finally I have briefly studied the possibility of using a redistributive fiscal policy to alleviate the negative effects of the externality. I have shown that, apparently, little can be done to eliminate poverty traps and that external financing or aids might therefore be appropriate, whereas fiscal policy has some power to eliminate multiplicity of equilibria. In particular I show that, for a parametric class of economies, there exists a nonlinear tax scheme that eliminates the multiplicity of equilibria.

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Appendix.

Proof of Proposition 1.

To show the existence of τ apply the implicit function theorem to $G(x,y)$ as defined in (5). Both $\partial G/\partial x$ and $\partial G/\partial y$ are continuous and never vanish for all x and $y \geq 0$. They are of opposite signs which also implies that τ is monotone increasing. Given x_0 the equilibrium path is then unique and described by the iterates of τ.
If $f(0) = 0$ the origin is a stationary point, otherwise it is not. From (5) it is easy to see that any strictly positive stationary state has to satisfy $J(x) = \omega(x)$. The l.h.s of the latter has been defined in (7) and is monotone increasing (as $\pi(x)$ is decreasing when f is concave) and has a value of at most one at the origin. Part c) of (H.2) together with the fact that $\lim \omega(x) = 0$ as $x\to\infty$ (again, because f is concave) imply that $\omega(x)$ intersects $J(x)$ at one and only one value $x^*>0$.
As τ is differentiable we can compute its derivative at the point $x=x^*$, this is:

$$\tau'(x) = -x\cdot f''\cdot g/[1-x\cdot f''\cdot g'/g]$$

where all functions on the r.h.s. are evaluated at x^* and use has been made of the fact that x^* is the interior steady state. As $g'>0$ and $f''<0$ the formula gives:

$$\tau'(x) < -x\cdot f''\cdot g = -x\cdot f''\cdot[\omega(x)]^{-1} < 1$$

where the last inequality follows from the hypothesis that $\omega(x)$ is decreasing. This proves that the interior stationary state is locally asymptotically stable. Given that it is unique and τ is increasing we have $\tau(x)>x$ for all $x<x^*$ and $\tau(x)<x$ for all $x>x^*$, which proves that x^* is globally stable. Q.E.D.

Proof of Proposition 2.

Let $\{x_t\}_{t=0}^{\infty}$ be an equilibrium sequence along which $x_t\to\infty$ with $t\to\infty$. We need to prove that its feasibility leads to a contradiction. Feasibility implies that, for large enough t, $x_{t+1}\geq x_t$ must hold and therefore one can always extract from $\{x_t\}_{t=0}^{\infty}$ a monotone increasing subsequence. If we define $\lambda_t=x_{t+1}/x_t$ the bounds $1\leq\lambda_t\leq b$ will always hold. The dynamic equilibrium condition at time $t\geq T-1$ is:

$$x_{t+1} = w_t + ((T-1)/T)\cdot w_{t-1}\cdot f'(x_t) + \ldots\ldots+(1/T)\cdot w_{t-T+1}\cdot(\Pi_{h=0}^{T-2} f'(x_{t-h}))$$

Dividing both sides by x_t and rearranging one gets:

$$x_{t+1}/x_t = \omega(x_t) + \textstyle\sum_{i=1}^{T-1}((T-i)/T)\cdot(\Pi_{j=1}^{i}\lambda_{t-j})^{-1}\cdot(\Pi_{h=0}^{i-1} f'(x_{t-h})\cdot\omega(x_{t-i}).$$

Before taking the limit of the r.h.s. for $x_{t-i}\to\infty$, for $i = 0,1,\ldots,T-1$, notice that as x gets large the following bounds hold:

$$(\Pi_{j=1}^{i}\lambda_{t-j})^{-1} \leq 1 \text{ for all } i;$$
$$(\Pi_{h=0}^{i-1}f'(x_{t-h})) \leq b^{i} < \infty \text{ for all } i.$$

Therefore the limit of the r.h.s. for $x_{t-i}\to\infty$ is bounded above by:

$$\omega(x_t) + \sum_{i=1}^{T-1}((T-i)/T)\cdot b^{i}\cdot\omega(x_{t-i})$$

Finally, using the fact that $\omega(x)\to 0$ as $x\to\infty$ and the assumption that $\lambda_t \geq 1$ we have:

$$1 \leq \lambda_t \leq \lim_{x_t\to\infty} \omega(x_t) + \sum_{i=1}^{T-1}((T-i)/T)\cdot b^{i}\cdot\lim_{x_{t-i}\to\infty} \omega(x_{t-i}) = 0$$

Q.E.D.

Proof of Proposition 5.

Our assumptions imply that $\partial G/\partial x$ and $\partial G/\partial y$ are as in Proposition 1, τ therefore exists, is differentiable and monotone increasing. Also $\tau(0) > 0$ because of (H.3)(ii). Any $x\in Fix(\tau)$ has to satisfy $J(x) = \omega(x)$. We have assumed ((H.3)(ii)) that $x\omega(x) \to 1$ for $x\to 0$, hence $\lim \omega(x) = \infty$ for $x\to 0$, and $\omega(x)$ has to be decreasing in a neighborhood of the origin. (H.5) says it is increasing for $x > \bar{x}$, thus it must have at least one critical point between zero and $\bar{x}$. As for $J(x)$ it is bounded below by one and above by some finite number because of (H.3)(i): hence either it always lies below $\omega(x)$ or it crosses it at least twice if, for some $x>0$ $J(x) > \omega(x)$. To see that the number of elements in $Fix(\tau)$ is even one can use standard results from index theory. More plainly consider the following three facts: 1) at x_{min} τ has to have a slope less than one because $\tau(0) > 0$, 2) at x_{max} τ must have a slope larger than one because $\omega(x)$ is larger than $J(x)$ for all $x > x_{max}$ (use Proposition 4), 3) τ is continuous (in fact differentiable).

Q.E.D.

Proof of Proposition 6.

Again it is trivial to apply the implicit function theorem and verify that τ is not a function between x^1 and x^2 and a function everywhere else. The same arguments used in the proof to Proposition 5 can be used to show that either $Fix(\tau)$ is empty or it contains an even number of elements. In fact part (i) of assumption (H.4) does not introduce any modification in the asymptotic behavior of $J(x)$ and $\omega(x)$, which is all that matter for $Fix(\tau)$. The simplest way to prove that for $x\in[x^1,x^2]$ $\tau(x)$ has to contain at least three elements, is to consider the function $\theta(y) = x$ which solves (6) and is well defined and continuously differentiable over the whole real line. Then $\theta'(y^1) = \theta'(y^2) = 0$ and $\theta(y)$ is increasing for values of y outside $[y^1,y^2]$ and decreasing inside it. Given that $\tau^{-1}(y) = \theta(y)$ the statement is proved.
The details of the statements in a) and b) are trivial and tedious to prove. I will only sketch the essentials.
That $x_0 > x_{max}$ is required in a) to get persistent growth is due to the

fact that, otherwise, $\tau(x) < x$ will hold for some $x \in [x_0, x_{max}]$.
To see that in b.1) the requirements imposed on the interval Δ and the selection χ are sufficient to produce an infinity of equilibria consider the procedure with which the equilibria are constructed. For $x_t \in \Delta$ $\chi(x_t)$ contains at least two elements belonging to Δ, to each one of them associate a new equilibrium (say $x^i{}_{t+1}$ for $i = 1,2$). Then $\chi(x^i_{t+1})$ has the same properties as $\chi(x_t)$ and the same procedure can be repeated for all $t = t+1, t+2, \ldots$. The number of equilibria at each t is therefore bounded below by 2^t which goes to infinity with t.
This procedure cannot be replicated whenever one of the two conditions is violated, either because the equilibrium would leave the interval $[x^1, x^2]$ after a finite number of period or because $\chi(x)$ would contain only one element from a certain period onward. This yields the implications given in b.2) as $\tau(x)$ is monotone increasing outside the interval $[x^1, x^2]$. Finally one can see that, exactly because the selection $\chi(x)$ may be a strict subset of $\tau(x)$, for $x \in [x^1, x^2]$ there may be $y \in \tau(x)$ which does not belong to Δ. The equilibria departing from such values would also leave the interval $[x^1, x^2]$ after a finite number of period. In that case they either converge to some element in $Fix(\tau)$ or grow unbounded. That the latter is a possibility follows from the fact that $x_{max} \in [x^1, x^2]$ may obtain, in which case the highest among the upward sloping branches of $\tau(x)$ must satisfy $\tau(x) > x$ for all $x \in [x^1, x^2]$. This is exactly the case I have portrayed in Figure 2.

Q.E.D.

1. Exception made for the extension to the general T-period model, Proposition 2 is identical to a result reported in a recent paper by Jones and Manuelli [1990b] of which I was unaware until July 1990, when a version of the present paper was presented at the Northwestern summer conference.

2. The ifs are needed in force of the arguments discussed in Jones and Manuelli [1990b]. Beside bequests they study other two ways out of the no-growth result for convex OLG models: income transfers from the young to the old by means of income taxation and a two sector model where the relative price of capital decreases asymptotically enough to compensate for the slow growth rate of the wage bill.

3. A more detailed analysis is being carried on in Boldrin [1991].

4. I have omitted this part together with a more extended discussion of taxation issues to reduce an already lengthy paper. Boldrin [1991] will contain these and other results.

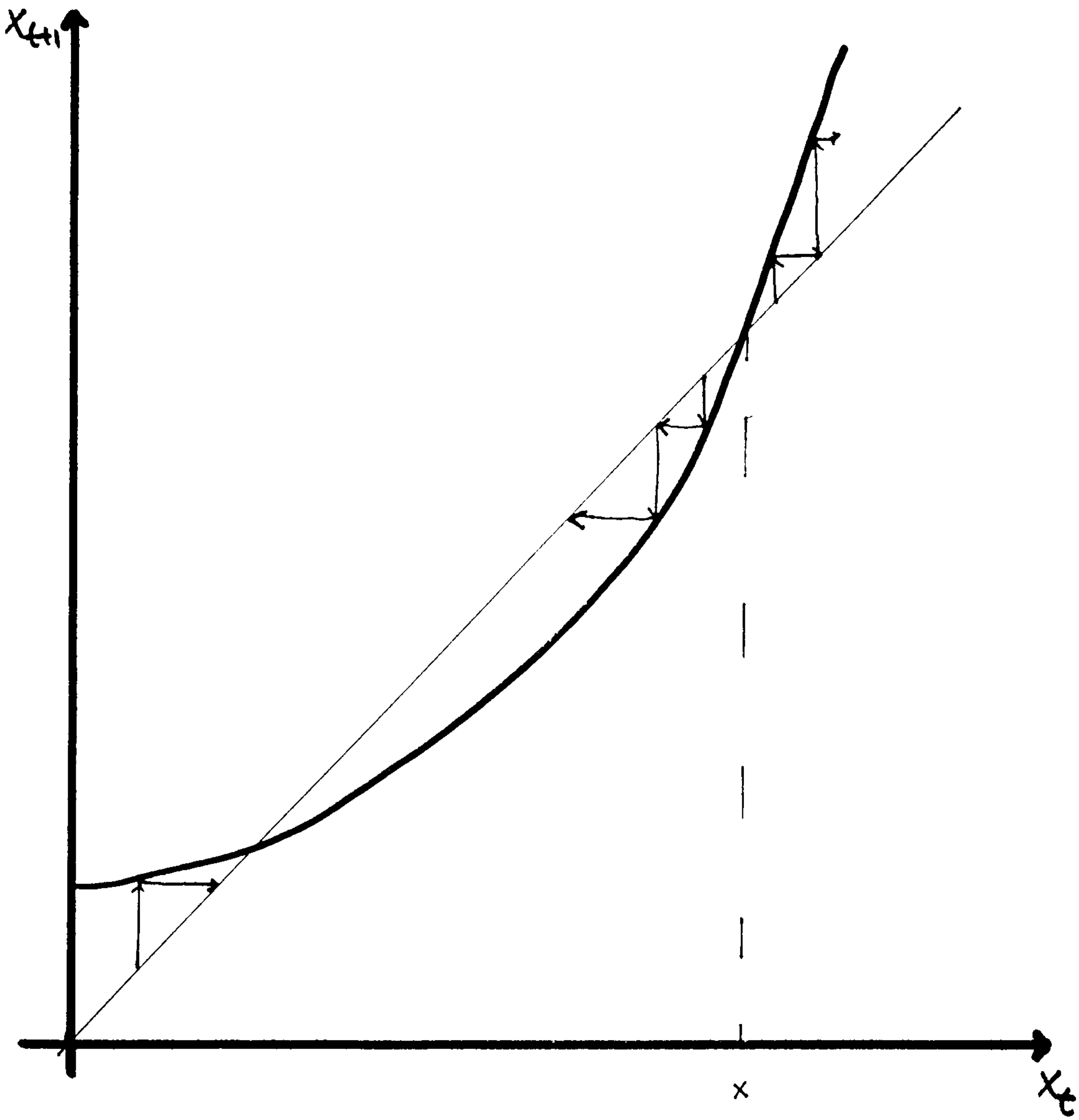

FIGURE 1

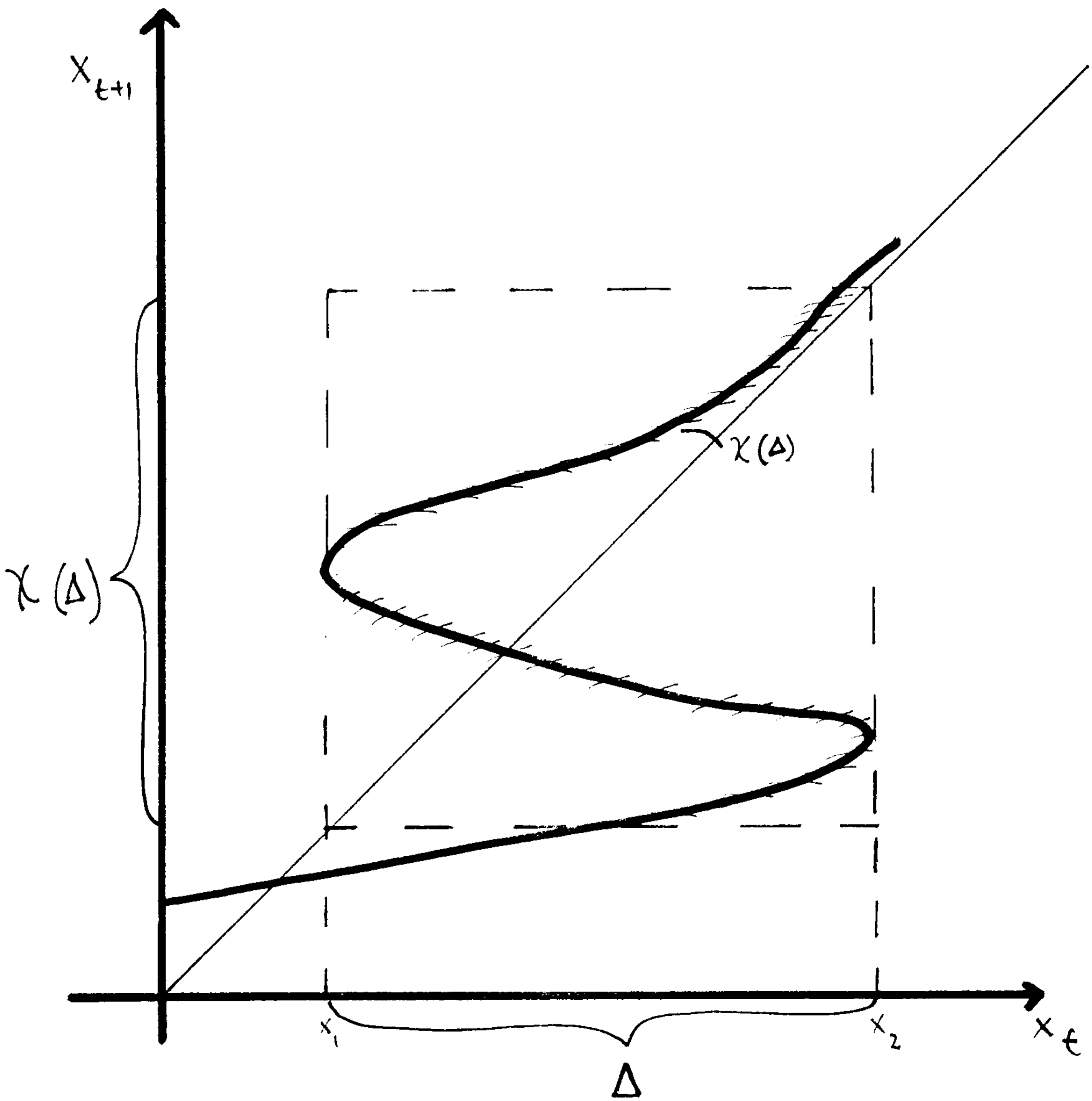

FIGURE 2.a

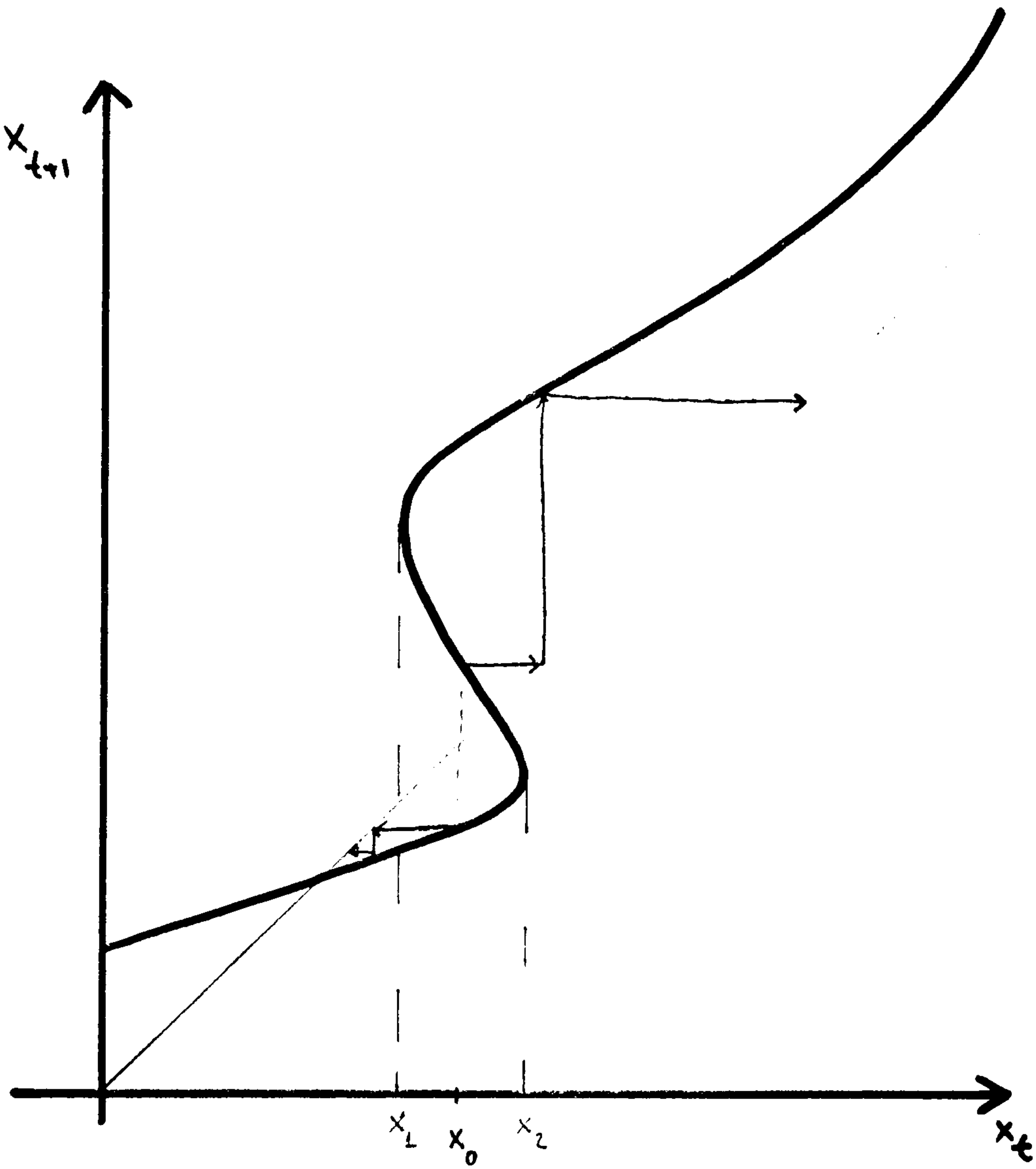

FIGURE 2.b

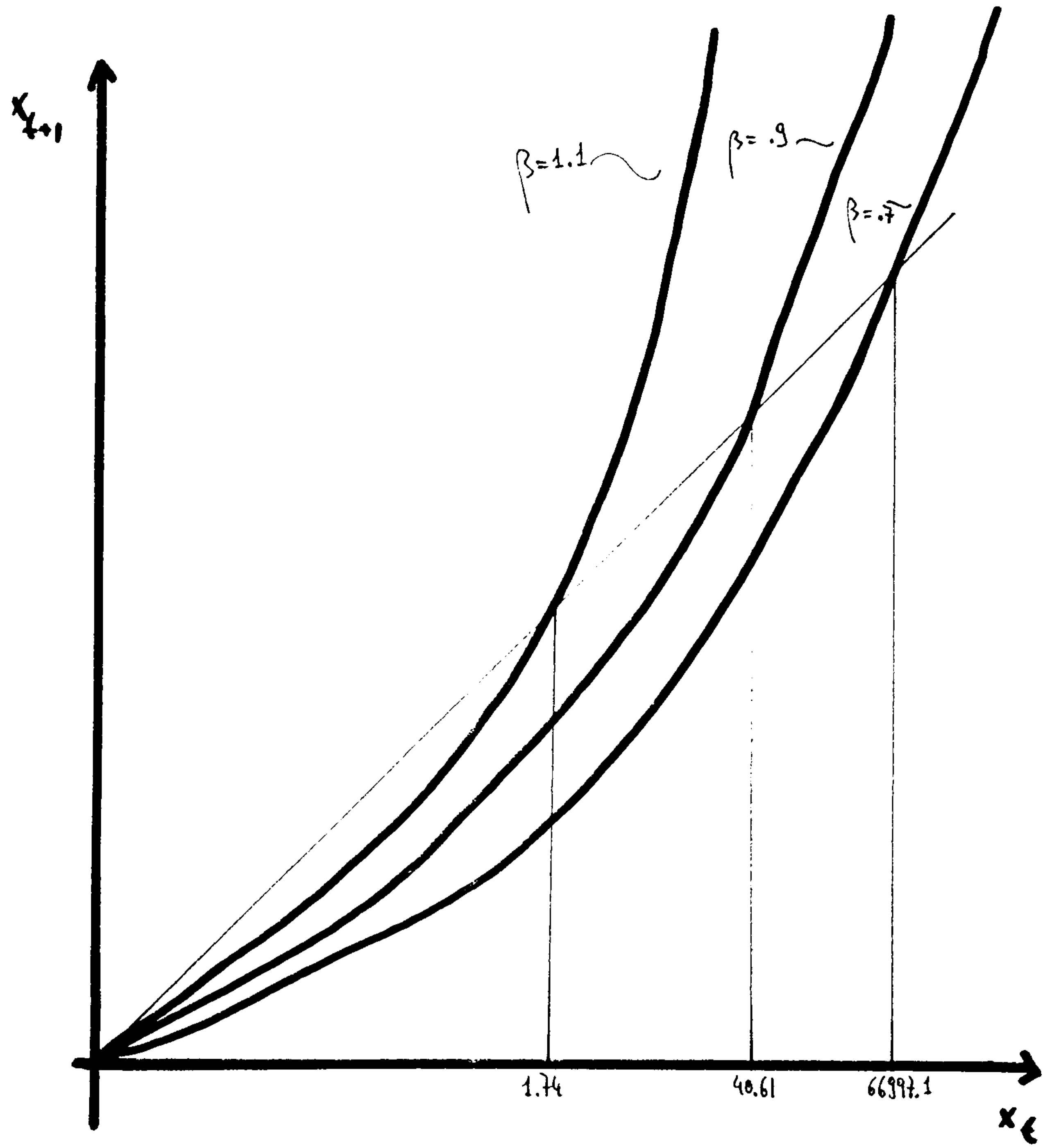

FIGURE 3

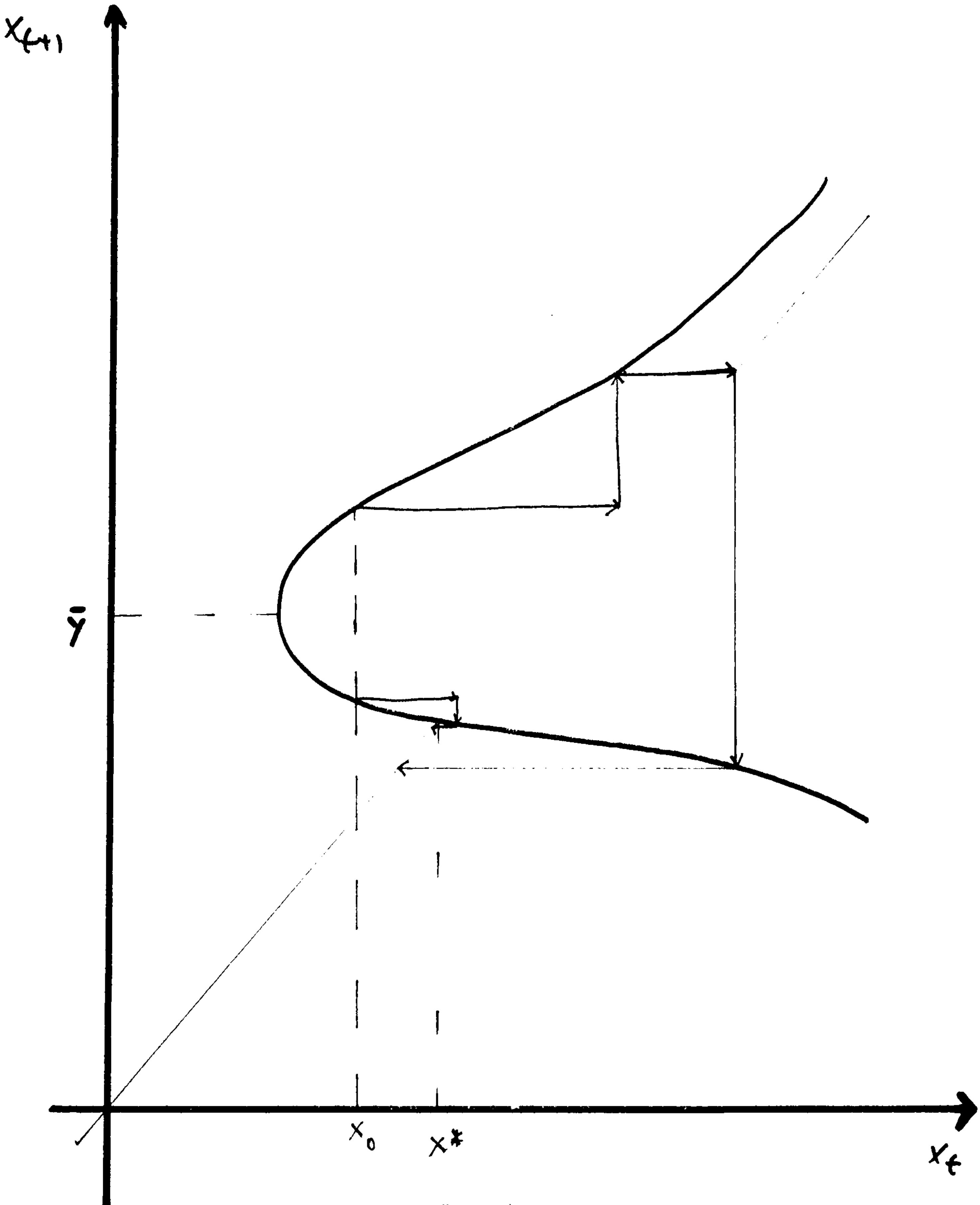

FIGURE 4