Discussion Paper No. 915

Sequential Equilibria and Cheap Talk in Infinite Signaling Games

Draft version 2.6

Karl Iorio

Department of Medical Economics and Statistics
Kaiser Foundation Health Plan, Oakland, California

Alejandro M. Manelli

Dept. of Managerial Economics and Decision Sciences
Kellogg Graduate School of Management
Northwestern University

December 9, 1990

ABSTRACT

This paper shows the existence of sequential and weak-best-response equilibria for cheap-talk extensions of signaling games and for a class of signaling games called communication-impervious. An example shows there are well-behaved infinite signaling games with no sequential equilibria. The assumption that talk is cheap seems reasonable in many economic contexts and yields a very straightforward solution to the existence problem in infinite signaling games. The cheap-talk assumption opens the possibility of extending the methods of this paper to prove the existence of equilibrium in more-general extensive-form games with infinite action and information sets.

Sequential Equilibria and Cheap Talk in Infinite Signaling Games

Draft version 2.6

*Karl Iorio**

Department of Medical Economics and Statistics

Kaiser Foundation Health Plan, Oakland, California

*Alejandro M. Manelli**

Department of Managerial Economics and Decision Sciences

Kellogg Graduate School of Management

Northwestern University

I. Introduction

Most of the theory of extensive-form games requires that the players' choice variables have only a finite number of possible values (e.g., Kreps and Wilson 1982). In many economic applications of game theory, however, choices are more conveniently modeled as continuous variables. As a result, there is a growing literature extending the theory to infinite choice sets.[1] Our ultimate goal is to define sequential equilibrium for general extensive-form games with infinite information sets and action spaces and then to prove existence of equilibrium. As a first step, this paper concerns a special case, signaling games with infinite type and action spaces.

---

* We are grateful to Robert Anderson, Debra Aron, Eddie Dekel, Raymond Deneckere, Michael Kirscheneiter, Steven Matthews, Roger Myerson, Daniel Vincent and Robert Weber for comments on previous drafts of this paper.

[1] See, for example, Dasgupta and Maskin (1986), Harris (1985), Hellwig and Leininger (1987), Milgrom and Weber (1985), and Simon and Stinchcombe (1989).

In a signaling game, player 1 first learns some private information, called his type, and then sends a signal to player 2. Player 2 observes this signal, makes an inference about player 1's probable type, and then responds with an action. The game ends with the players receiving payoffs that in general depend on player 1's type and signal and player 2's action. Signaling games have generated considerable interest of their own. They have been extensively applied in economics and finance.[2] Several authors have used these games to analyze refinements of the sequential equilibrium concept.[3] There is also a developing literature on signaling games with cheap talk (costless signaling).[4]

In this paper, we first propose a definition of Sequential Equilibrium (SE) for infinite signaling games. We then prove general theorems concerning convergence and existence of SE for continuous games (in which the action and types spaces are compact metric spaces and the payoff functions are continuous). We are led to consider cheap talk by our convergence theorem. Define the outcome of a SE to be the probability distribution on the types and signals of player 1 and the responses of player 2 resulting from playing the SE strategies. Given a game, we consider a sequence of games that approximate it. Our convergence theorem says a limit of SE outcomes for the sequence of approximating games is a SE outcome, not for the limit game, but for a cheap-talk extension of the limit game. The cheap-talk extension modifies the original game by allowing player 1 to send an additional signal to player 2 that does not affect their payoffs. As part of the convergence theorem, we show how to obtain equilibrium strategies supporting the cheap-talk SE outcome for the limit game and how to derive player 2's equilibrium beliefs from her beliefs in the approximating games.

By approximating an infinite game with a sequence of finite games, the convergence theorem leads to a very general existence theorem: SE exist for cheap-talk extensions of continuous signaling games. An existence theorem of this generality is not possible for continuous signaling games directly; we give an example (due to Eric van Damme) of a simple continuous signaling game that has no subgame perfect equilibria.

The non-existence example and our existence theorem together show that adding cheap talk to a game can expand the set of equilibrium outcomes. This raises the question of whether adding cheap talk

[2] Examples of works using signaling games (or variants) include Bhattacharya (1978), Leland and Pyle (1977), Milgrom and Roberts (1982), Myers and Majluff (1984), Riley (1979), and Spence (1974).

[3] See Banks and Sobel (1987), Cho and Kreps (1987), and Cho and Sobel (1987).

[4] See, for example, Farrell and Gibbons (1986,1989), Matthews, Okuno-Fujiwara and Postlewaite (1990), Seidmann (1990), and Stein (1989).

fundamentally alters the nature of a signaling game. We prove that if the signal space for player 1 contains a sufficiently rich set of signals, then every cheap-talk SE outcome can be approximated arbitrarily closely by a sequential ε-equilibrium outcome for the original game. This result suggests to us that adding cheap talk to a rich signaling game does not alter the nature of the game in a significant way.

Now cheap talk may have no influence on the outcome of a cheap-talk SE; player 2 may ignore the talk (because it does not contain any information of value to her). In this case, we say talk is ineffective and we show for this case that a cheap-talk SE outcome must be a SE outcome for the original game. Games where every cheap-talk SE outcome is a standard SE outcome are called Communication-Impervious (CI). Since a cheap-talk SE exists for any continuous game, it follows immediately from the definition of CI that a SE exists for any CI game. Using our result on ineffective talk, we identify a class of games, which we call strongly monotonic games; that are CI. The non-existence example shows that not all games are CI.

In many finite games, a plethora of SE exist. To pare down the number of equilibria, a number of authors have proposed and analyzed refinements of the SE concept, as we mentioned above. We extend our convergence and existence theorems using one of the strongest of these refinements, the Weak Best Response test of Kohlberg and Mertens (1986).

As we said, we prove existence of SE for cheap-talk extensions of infinite signaling games using a sequence of finite approximating games and a convergence theorem on outcomes. One can envision at least three other approaches to proving existence of equilibrium. First, one could try using a fixed-point argument as Milgrom and Weber (1985) do for one-shot, simultaneous-move games with incomplete information. This does not seem possible with signaling games because even though payoffs are continuous, expected payoffs are not with respect to any common topology on strategies that makes the space of strategies compact (Iorio and Manelli 1990).

A second alternative is to use a sequence of finite approximating games and a convergence theorem on strategies (instead of outcomes). We pursued this approach in Iorio and Manelli (1990). One has to impose severe restrictions on a game to guarantee that a sequence of strategies will converge. We have abandoned this approach because it does not seem to be applicable to games of more-general forms, while the cheap-talk approach seems readily applicable.

A third alternative is to construct SE strategies using the equilibrium conditions of the game. Cho and Sobel (1987), Crawford and Sobel (1982), and Riley (1979) use this approach. Crawford and Sobel provide the original discussion of cheap talk. All these papers show the existence of SE for certain classes of signaling games. All impose strong assumptions on the players' payoff functions, e.g., differentiability and restrictions on the cross derivatives. Because of their strong assumptions, all can analyze the properties of equilibria, such as how much information player 1 will reveal. This type of approach will also be difficult to apply to more-general games.

The rest of this paper is organized as follows. In Section II, we introduce the games we consider, notation and definitions. In Section III, we informally discuss the non-existence example and our results. Section IV contains formal statements and proofs of our theorems. Section V concludes the paper. An Appendix contains statements and proofs of some general lemmas.

II. The Game

We consider signaling games of the following form. We summarize a game by $\Gamma = [(T,\rho),X,Y,U^1,U^2]$. In this game, player 1 first privately observes his type t from the set T of possible types and then sends a signal x from the set X. Player 2 observes this signal, infers player 1's probable types, and then selects an action y from the set Y. The game ends and each player i receives payoff $U^i(t,x,y)$. To complete the specification of the game, we assume that player 2 has prior beliefs ρ about the possible types t of player 1: ρ is a probability distribution on T that is common knowledge between the two players.

We will consider collections of games in subsequent sections of the paper. For the remainder of this section, fix a particular game $\Gamma = [(T,\rho),X,Y,U^1,U^2]$. We say that Γ is *continuous* if and only if T, X and Y are compact metric spaces and U^1 and U^2 are continuous. We shall be concerned in this paper only with continuous games.

We allow the players to use random strategies to select their actions in Γ. One type of random strategy is a behavioral strategy. For player 1, this is a function ξ from types T to the space $M(X)$ of probability distributions on X. Probability distributions on a space are measures defined on the events of the space, i.e., the Borel-measurable subsets. Given a type $t \in T$ and an event $B \subset X$, $\xi(t)(B)$ is the probability assigned to B

by the distribution $\xi(t)$.

A behavioral strategy represents a local view of the options of player 1. It will be convenient for the proofs to take a more global view of these options. The behavioral strategy ξ induces a distribution μ on $T \times X$ defined by integrating ξ with respect to ρ: for all event rectangles $A \times B \subset T \times X$,

$$\mu(A \times B) = \int_A \xi(t)(B)\ \rho(dt).$$

This definition has two consequences. First, for μ to be well-defined, the function $t \to \xi(t)(B)$ must be measurable for each fixed event $B \subset X$ (Billingsley 1979, p. 394). Second, the marginal distribution μ_T of μ must equal the prior distribution ρ. Following Milgrom and Weber (1985), we call μ a distributional strategy; it is simply another way of representing a behavioral (or mixed) strategy. We denote the set of possible strategies for player 1 in the game Γ by $\Sigma^1(\Gamma) = \{\ \mu \in M(T \times X)\ \mid\ \mu_T = \rho\ \}$. $\Sigma^1(\Gamma)$ is a compact metric space using the topology of weak convergence of measures. We use this topology on all spaces of distributions. .

We represent player 2's options using behavioral strategies. These are functions $\eta : X \to M(Y)$. Given this representation of a strategy, it will be convenient for us to work with the space $M(Y)$ instead of with the space Y directly. We extend the players' payoff functions U^i from Y to $M(Y)$ by taking expected values: for each $(t,x,\eta) \in T \times X \times M(Y)$, we let

$$U^i(t,x,\eta) = \int_Y U^i(t,x,y)\ \eta(dy).$$

U^i is a continuous function on $T \times X \times Y$ if and only if the extension of U^i to $T \times X \times M(Y)$ is continuous (Lemma A4 in the Appendix proves the non-trivial half of this statement).

We must impose measurability restrictions on player 2's strategies. We require that the function $x \to \eta(x)$ is measurable. The important consequence of this requirement is that player 2's strategies are all approximable using continuous strategies. We denote the set of possible strategies for player 2 in the game Γ as $\Sigma^2(\Gamma)$, the set of all measurable functions $\eta : X \to M(Y)$. As we saw for player 1, an alternate measurability requirement for a strategy is that the function $x \to \eta(x)(C)$ is measurable for all events C. These two requirements are equivalent (Bertsekas and Shreve 1978, Proposition 7.26), so we need not distinguish between them.

Given that player 1 plays a strategy $\mu \in \Sigma^1(\Gamma)$, a signal x will typically reveal to player 2 some information about the type of player 1. We represent player 2's posterior beliefs as a function $\beta : X \to M(T)$. We require that the beliefs of player 2 about the type of player 1 given signal x be consistent with the strategy of player 1 in that β must be a version of a regular conditional probability distribution of t given x derived from μ (see Parthasarathy 1967). Specifically, we require that (i) β is measurable, (ii) for all events $A \subset T$ and $B \subset X$, $\int_B \beta(x)(A)\ \mu_X(dx) = \mu(A \times B)$, and (iii) there exists an event $X^1 \subset X$ with $\mu_X(X^1) = 1$ such that $x \in X^1$ and $t \in \mathrm{supp}[\beta(x)]$ imply $(t,x) \in \mathrm{supp}[\mu]$. We define $\Sigma^3(\Gamma)$ as the set of pairs (β,μ) satisfying (i) through (iii). Regular conditional distributions always exist for distributions on compact metric spaces.

Define the mixed best-response correspondence for player 2 as

$$\begin{aligned} MBR(x,Y,T,U) = \{\ (\eta,\beta) \in M(Y) \times M(T) \\ \mid \int U(t,x,\eta)\ \beta(dt) \geq \int U(t,x,\eta')\ \beta(dt) \text{ for all } \eta' \in M(Y)\ \}. \end{aligned}$$

Given that player 1 signals x, $MBR(x,Y,T,U)$ is the set of pairs of a response η from $M(Y)$ and a belief β on T where η is a best response given the belief β and given that player 2's payoff function is U. This definition is a little different from the standard one in the literature: we associate to each best response the belief that supports it, and we allow the set Y and the function U to vary.

Kreps and Wilson (1982) defined sequential equilibrium for finite games. We adapt their definition to infinite signaling games as follows. A Sequential Equilibrium (SE) for $\Gamma = [(T,\rho),X,Y,U^1,U^2]$ is a triple $(\hat{\mu},\hat{\beta},\hat{\eta})$ such that

(S1) $\hat{\mu} \in \Sigma^1(\Gamma)$, $\hat{\eta} \in \Sigma^2(\Gamma)$;

(S2) for all $(t,x) \in \mathrm{supp}[\hat{\mu}]$, $U^1(t,x,\hat{\eta}(x)) \geq U^1(t,x',\hat{\eta}(x'))$ for all $x' \in X$;

(S3) $(\hat{\eta}(x),\hat{\beta}(x)) \in MBR(x,Y,T,U^2)$ for all $x \in X$;

(S4) $(\hat{\beta},\hat{\mu}) \in \Sigma^3(\Gamma)$.

Condition S1 requires that the players' strategies be valid ones. S2 and S3 are requirements for sequential rationality. S2 says that player 1 only sends signals x that will maximize his expected payoff given that his type is t and that player 2's strategy is $\hat{\eta}$. S3 says that player 2 responds with actions $\hat{\eta}(x)$ that maximize her expected payoff given her belief $\hat{\beta}(x)$ about player 1. S4 requires that the beliefs of player 2 be consistent with the strategy of player 1.

We define a sequential ε-equilibrium by replacing conditions S2 and S3 in the definition of SE with

(S2′) for all $(t,x) \in \text{supp}[\hat{\mu}]$, $U^1(t,x,\hat{\eta}(x)) \geq U^1(t,x',\hat{\eta}(x')) - \varepsilon$ for all $x' \in X$;

(S3′) for all $x \in X$, $\int_T U^2(t,x,\hat{\eta}(x))\ \hat{\beta}(x)(dt) \geq \int_T U^2(t,x,\eta)\ \hat{\beta}(x)(dt) - \varepsilon$ for all $\eta \in M(Y)$.

To prove the existence of a SE for Γ and to discuss refinements of the equilibrium concept, we will need a few more assumptions and definitions.

Cho and Sobel (1987) define a signaling game Γ to be *monotonic* if for all x in X, for all (η,β) and (η',β') in $MBR(x,Y,T,U^2)$, and for all t and t' in T,

$$U^1(t,x,\eta) \geq U^1(t,x,\eta') \text{ implies } U^1(t',x,\eta) \geq U^1(t',x,\eta').$$

Monotonicity holds in many applications of signaling games. For example, if y is player 2's choice of payment to player 1, monotonicity requires that player 1 prefers more money to less independently of t and x.

Monotonicity holds if Y is an interval in R^1, U^1 is increasing in y, and U^2 is strictly concave in y. We will need a stronger property also implied by these conditions.

We define Γ to be *strongly monotonic* if for all x in X, for all (η,β) and (η',β') in $MBR(x,Y,T,U^2)$, and for all t and t' in T,

$$U^1(t,x,\eta) \geq U^1(t,x,\eta') \text{ and } U^1(t',x,\eta') \geq U^1(t',x,\eta) \text{ imply } \eta = \eta'.$$

Strong monotonicity implies monotonicity. It also implies there is a linear ordering of the best responses of player 2 that reflects the preferences of all types of player 1.[5] We will show a strongly monotonic signaling game has a SE.

Many finite signaling games have a large number of sequential equilibria. Various authors have proposed refinements of the SE concept to reduce the number of equilibria by weeding out those that seem intuitively unreasonable (see Cho and Kreps 1987 and their references). We want to consider the strongest possible refinement criterion and show that SE exist that satisfy this criterion. The strongest criterion is Kohlberg and Mertens' (1986) strategic stability; it implies most of the weaker criteria that authors have proposed. It is

[5] A linear ordering is complete, transitive and antisymmetric.

not apparent to us how to apply this criterion directly to infinite games. Cho and Sobel (1987) show that for finite monotonic signaling games, the following criterion is generically equivalent to stability. Given a SE $(\hat{\mu},\hat{\beta},\hat{\eta})$, define

$$V^1(t) = U^1(t,x,\hat{\eta}(x)) \text{ if } (t,x) \in \text{supp}[\hat{\mu}].$$

These are the equilibrium payoffs to player 1 if his type is t. We say that $(\hat{\mu},\hat{\beta},\hat{\eta})$ satisfies the Weak Best Response (WBR) criterion if for all $x \in X \sim \text{supp}[\hat{\mu}_X]$, W1 implies W2, where

(W1) there exists $t \in T$ and $(\eta,\beta) \in MBR(x,Y,T,U^2)$ such that $U^1(t,x,\eta) \geq V^1(t)$;

(W2) for all $t \in \text{supp}[\hat{\beta}(x)]$, there exists $(\eta',\beta') \in MBR(x,Y,T,U^2)$ such that $U^1(t,x,\eta') = V^1(t)$, and for all $t' \in T$, $U^1(t',x,\eta') \leq V^1(t')$.

W2 restricts player 2's beliefs given x to those t for which x is a weak best response in some SE (with belief $\beta(x) = \beta'$ and response $\eta(x) = \eta'$) with the same outcome as $(\hat{\mu},\hat{\beta},\hat{\eta})$ (see Cho and Kreps 1987). W1 says such a restriction cannot be ruled out a priori. WBR is just a convenient restatement of Kohlberg and Mertens' Never a Weak Best Response test. It implies various other refinements of SE, e.g., the Intuitive Criterion of Cho and Kreps (1987), and the Universal Divinity test of Banks and Sobel (1987).

To show the existence of WBR equilibria, we will need one more assumption. Let d_T be a metric on the type space T. We say the payoff function of player 1 is *Lipschitz in t* if there exists a constant L such that for all t and t' in T, x in X, and y in Y,

$$|U^1(t,x,y) - U^1(t',x,y)| \leq Ld_T(t,t').$$

We say a family of functions is uniformly Lipschitz if each function in the family is Lipschitz using the same constant L. U^1 is Lipschitz in t if either T is finite or U^1 is continuously differentiable with respect to t. U^1 is Lipschitz in t on $T \times X \times Y$ if and only if the extension of U^1 to $T \times X \times M(Y)$ is Lipschitz in t.

III. Example and Discussion

The purpose of this section is to provide informal motivation for our results. We provide formal statements and proofs of the theorems in the next section. We begin the section by presenting an example of a game for which no sequential equilibrium exists. Our method of proving the existence of an equilibrium for a game takes a limit of the outcomes of SE for finite approximating games. A limit outcome always exists, but as the non-existence example shows, it may not be a SE outcome. It turns out, however, that the limit outcome is a SE outcome for a cheap-talk extension of the original game. This is our first result. Since we can always construct a limit outcome, we can always construct a cheap-talk SE. We use the example to illustrate how we do this.

Now cheap talk may have no influence on the outcome of a cheap-talk SE; player 2 may ignore the talk because it contains no valuable information. In this case we say talk is ineffective. Our next result is that when talk is ineffective, a cheap-talk SE outcome must be a SE outcome for the original game. Games where every cheap-talk SE outcome is a standard SE outcome are called Communication-Impervious. Any strongly monotonic game is CI. Since a cheap-talk SE exists for any game, it follows immediately from the definition of CI that a SE exists for any CI game.

As the non-existence example shows, adding cheap talk can expand the set of equilibrium outcomes of a non-CI game. Our final result indicates, however, that adding cheap talk does not fundamentally alter the nature of the game if the space X contains a sufficiently rich set of signals. We show in this case that every cheap-talk SE outcome can be approximated arbitrarily closely by a sequential ε-equilibrium outcome.

We will use the following example to introduce our theorems and illustrate the method of proof. The basic idea behind the example is due to Eric van Damme.[6]

Define the game Γ by

$$T = \{-1,1\},\ \rho(-1) = \rho(1) = ½,\ X = Y = [-1,1],$$
$$U^1(t,x,y) = -x^2 + ty,\ U^2(x,y) = xy.$$

To simplify notation, we write $\rho(-1)$ when we mean $\rho(\{-1\})$ since ρ is a measure. To further simplify matters, we will restrict ourselves in this section to pure strategies, and we will denote pure strategies by real-valued functions.

The game Γ is continuous, but it has no sequential equilibrium nor even a subgame perfect equilibrium. To see this, observe that since player 2's payoff function is independent of t, her best responses do not

[6] We thank Subir Chakrabarti for bringing an example of van Damme's to our attention.

depend on her beliefs concerning t. Letting the function $\tilde{y}$ denote player 2's strategy, in any potential sequential equilibrium we must have

$$\tilde{y}(x) = \begin{cases} -1 & \text{if } x < 0, \\ 1 & \text{if } x > 0. \end{cases} \tag{1}$$

Now player 1 will choose x given t to maximize his payoff $U^1(t,x,\tilde{y}(x))$. This amounts to player 1 choosing a maximizer for U^1 from the graph of the function $\tilde{y}$. On this graph, the payoff for either type of player 1 is strictly increasing as $x \to 0$ from either above or below. But no matter how $\tilde{y}(0)$ is chosen (even randomly), the graph of $\tilde{y}$ will not be closed at $x = 0$, so that a maximizer will not exist for at least one type of player 1. Hence, a sequential equilibrium cannot exist.

The addition of cheap talk to the game Γ can solve the problem of non-existence of equilibrium. Our method for locating an equilibrium with cheap talk is as follows. Let $<X^n>$ be a sequence of finite subsets of X that are increasingly fine approximations to X, say

$$X^n = \{\ (-1)^l k/n \mid l = 1, 2 \text{ and } k = 1, \ldots, n\ \}.$$

Let $<Y^n>$ be an analogous sequence for Y, say $Y^n = X^n$. The game $\Gamma^n = [(T,\rho),X^n,Y^n,U^1,U^2]$ is finite, so it has a SE. SE strategies are $\hat{x}^n(-1) = -1/n$, $\hat{x}^n(1) = 1/n$, and $\hat{y}^n(x) = \tilde{y}(x)$ given by (1).

Now look at the sequence of outcomes for these SE for Γ^n. This is a sequence of distributions $\hat{\lambda}^n$ on $T \times X^n \times Y^n$ given by $\hat{\lambda}^n(-1,-1/n,-1) = \hat{\lambda}^n(1,1/n,1) = ½$ (again simplifying the notation for measures).[7] This sequence will converge weakly to the distribution $\hat{\lambda}$ on $T \times X \times Y$ given by

$$\hat{\lambda}(-1,0,-1) = \hat{\lambda}(1,0,1) = ½. \tag{2}$$

This outcome cannot be a SE outcome because it cannot be realized by a pair of strategies. We can construct a pure strategy for player 1 out of $\hat{\lambda}$: it is $\hat{x}(-1) = \hat{x}(1) = 0$. We cannot, however, construct a response for player 2 because there is no unique conditional value of y given x: according to $\hat{\lambda}$, player 2 should respond to $x = 0$ sometimes with $y = -1$ and sometimes with $y = 1$. In each game Γ^n in this example, there is coordination between player 1 and player 2—player 2 plays high or low depending on whether player 1's type is

[7] In the next section, we use a slightly different definition of outcome to accommodate behavioral strategies.

high or low—but this coordination is lost in the limit game Γ.

We can solve this coordination problem by adding cheap talk to the game Γ. Let Y^* be a copy of the action space Y. Suppose player 1 can costlessly suggest a play $y^* \in Y^*$ to player 2 in addition to sending the costly signal x. The asterisk in y^* and Y^* distinguishes a signal that does not affect payoffs from a response y from Y that does affect payoffs. Adding this signaling capability creates a new game $\Gamma^* = [(T,\rho),X \times Y^*,Y,U^1,U^2]$ with a signal space $X \times Y^*$. A distributional strategy for player 1 is now a measure on $T \times X \times Y^*$ and a (pure) strategy for player 2 is a function from $X \times Y^*$ to Y. We call the game Γ^* the cheap-talk extension of Γ.

It is easy to construct a SE for Γ^* from the limit of the SE for the games Γ^n. The equilibrium distributional strategy for player 1 is just the limit distribution $\hat{\lambda}$ with y^* replacing y. For the example, this is given by (2). The equilibrium strategy for player 2 on the equilibrium path is $\hat{y}(x,y^*) = y^*$. The beliefs of player 2 on the equilibrium path are a version of the conditional distribution of t given (x,y^*) derived from $\hat{\lambda}$. (We must show that there exists a version such that $\hat{y}(x,y^*)$ is a best response to the belief $\hat{\beta}(x,y^*)$ for all (x,y^*) on the equilibrium path; this is the principal difficulty in this construction of an equilibrium.) For the example, $\hat{\lambda}$ implies a unique set of beliefs on the equilibrium path: the signal $(x,y^*) = (0,-1)$ comes from type -1 and the signal $(0,1)$ comes from type 1. To define $\hat{y}(x,y^*)$ and the beliefs $\hat{\beta}(x,y^*)$ off the equilibrium path, we take a limit of a subsequence $<x^n,\hat{y}^n(x^n),\hat{\beta}^n(x^n)>$ from the SE for $<\Gamma^n>$ such that $<x^n> \to x$. (We must select $\hat{y}$ and $\hat{\beta}$ so that they are measurable.) For the example, the beliefs of player 2 do not affect her actions, so we will just have $\hat{y}(x,y^*) = \tilde{y}(x)$ given by (1) and we can use any set of beliefs.

Generalizing this method to handle arbitrary behavioral strategies yields two convergence theorems concerning cheap-talk equilibria that we will state and prove in the next section. Letting $\Psi^* = M(Y)$, a corollary of these theorems is

Theorem A: Every continuous cheap-talk extension game $\Gamma^* = [(T,\rho),X \times \Psi^*,Y,U^1,U^2]$ has a SE. If U^1 is Lipschitz in t, then Γ^* has a SE satisfying the WBR criterion.

When U^1 is Lipschitz in t, the equilibrium payoffs to player 1 in approximating finite games converge continuously, and then a limit of WBR outcomes is a cheap-talk WBR outcome.

Theorem A is a general and simple result. It gives one good reason to consider allowing for cheap talk in applications of signaling games. Using $\Psi^* = M(Y)$ as the cheap-talk space is not very restrictive since it is an uncountable compact metric space and thus Borel equivalent to any other uncountable compact metric space (Parthasarathy 1967, Theorem I.2.12).[8]

To discuss the existence of SE for a game Γ without cheap talk, we adopt the framework provided by Matthews, Okuno-Fujiwara and Postlewaite (1989, 1990). We say a game Γ is *Communication-Impervious (CI)* if every SE (or WBR) outcome for the cheap-talk extension game Γ^* is also a SE (or WBR) outcome of the original game Γ. It is an immediate consequence of this definition and Theorem A that every CI game has a SE and every Lipschitz CI game has a SE satisfying WBR.

To identify CI games, we need another definition and result. We say a SE outcome $\hat{\lambda}$ of a cheap-talk extension game has *ineffective talk* if and only if[9]

(IT) $(t,x,y) \in \text{supp}[\hat{\lambda}]$ and $(t',x,y') \in \text{supp}[\hat{\lambda}]$ imply $y = y'$.

This says cheap talk does not influence the outcome $\hat{\lambda}$ because player 2's responses depend only on the signal x. Since the outcome is an equilibrium one, it must be that the cheap talk provides no information of value to player 2. In the next section we prove

Theorem B: Let Γ^* be the cheap-talk extension of a continuous game Γ. If $\hat{\lambda}$ is a SE outcome for Γ^* satisfying IT, then $\hat{\lambda}$ is a SE outcome for Γ. If $\hat{\lambda}$ is a SE outcome for Γ^* satisfying WBR and IT, then $\hat{\lambda}$ is a SE outcome for Γ satisfying WBR.

According to Theorem B, if every SE outcome for Γ^* satisfies IT, then Γ is CI and thus has a SE. In the next section we show that SE outcomes for strongly monotonic continuous games satisfy IT.[10]

---

[8] This statement requires some qualification. A SE under our definition may not remain a SE after a transformation using a Borel isomorphism because condition S2 may not be satisfied everywhere on the support of player 1's distributional strategy, only almost everywhere. Thus which cheap-talk space we use does matter unless we weaken the definition of SE.

[9] In the next section, we use a slightly different version of IT to accommodate behavioral strategies.

[10] Crawford and Sobel (1982) and Seidmann (1990) present conditions for pure cheap-talk games to be communication-impervious.

Another application of Theorem B is

Theorem C: If a continuous game $\Gamma = [(T,\rho),X,Y,U^1,U^2]$ has a finite X space, then Γ has a SE. If in addition U^1 is Lipschitz in t, then Γ has a SE satisfying the WBR criterion.

We prove Theorem C by considering a sequence of equilibrium outcomes for finite games that approximate Γ. These outcomes will satisfy IT. We select a subsequence of outcomes such that a limit outcome exists and also satisfies IT. We can do this when X is finite. This limit outcome will be an equilibrium outcome for Γ^*, and by Theorem B, also for Γ.

Since U^1 is Lipschitz when T is finite, Theorem C implies in particular that WBR equilibria exist for all finite signaling games, rather than just for generic finite signaling games as in Cho and Kreps (1987).

As the non-existence example shows, adding cheap talk can expand the set of equilibrium outcomes of a non-CI game. This raises the question of whether adding cheap talk fundamentally alters the nature of a signaling game. We will argue that it does not if the space X contains a sufficiently rich set of signals. Consider the following definition and theorem.

We say a signal space X is *rich* if and only if for all compact metric spaces Z and for all closed balls $B \subset X$, there exists a closed set $A \subset B$ and a continuous mapping from A onto Z. An example of a rich space is the interval [0,1] (Parthasarathy 1967, Theorem I.4.1).

Theorem D: Let $\Gamma = [(T,\rho),X,Y,U^1,U^2]$ be a continuous game with a rich signal space X, and let Γ^* be the cheap-talk extension of Γ. Then given $\varepsilon > 0$ and a $\hat{\lambda}$ that is a SE outcome for Γ^*, there exists a $\tilde{\lambda}$ that is an outcome of a sequential ε-equilibrium for Γ with Prohorov distance $p(\hat{\lambda},\tilde{\lambda}) < \varepsilon$.[11]

Theorem D says that when the signal space X is rich, every cheap-talk SE outcome can be approximated arbitrarily closely by a sequential ε-equilibrium outcome. This is done using a one-to-one mapping that replaces each signal (x,y^*) in $X \times Y^*$ (in the pure-strategy case) with a signal $\tilde{x}$ in X with $\tilde{x}$ close to x. Thus this mapping replaces cheap talk that uses a copy of the Y space with costly signaling that uses the X space. If player 2 responds to each signal $\tilde{x}$ as she would have responded to the corresponding (x,y^*) in a

[11] See Billingsley (1968) for a definition of the Prohorov distance on distributions.

cheap-talk SE, the result will be an ε-equilibrium since $\tilde{x}$ is close to x. Since the players can approximate a cheap-talk equilibrium outcome arbitrarily closely using costly signaling, this suggests to us that adding cheap talk to a rich signaling game does not alter the nature of the game in a significant way.

While the players can approximate cheap talk using the costly signal, doing so can be awkward. Consider what happens when we replace player 2's payoff function in the example game with $U^2(t,x,y) = -txy$. This new game has the same simple cheap-talk equilibrium outcome the original example game had: player 1 signals what response of player 2 that he would prefer by sending $(x,y^*) = (0,t)$ and player 2 follows along with $\hat{y}(x,y^*) = y^*$. Without cheap talk, a possible SE outcome has player 1 signaling $t\varepsilon$ with $0 < \varepsilon \leq 1$ and player 2 responding with

$$\tilde{y}(x) = \begin{cases} -1 & \text{if } x \leq -\varepsilon, \\ 0 & \text{if } -\varepsilon < x < \varepsilon, \\ 1 & \text{if } x \geq \varepsilon. \end{cases}$$

To support this response by player 2 requires that she believe $\hat{\beta}(x)(-1) = \hat{\beta}(x)(1) = \frac{1}{2}$ when $-\varepsilon < x < \varepsilon$. This equilibrium seems contrived: the cheap-talk equilibrium is simpler and more natural.

For finite signaling games, the set of SE outcomes coincides with the set of limits of sequences of sequential ε-equilibrium outcomes as $\varepsilon \to 0$. The non-existence example and Theorem D combine to show this is not true for infinite signaling games. Instead, the set of limits of sequential ε-equilibrium outcomes includes the cheap-talk SE outcomes when X is rich.

We conclude with some remarks on the relation of this paper to other work. Farrell and Gibbons (1986) argue that cheap talk can be credible, is ubiquitous, and economists and game theorists should give it more attention. We have found some more reasons for giving cheap talk attention in a non-CI game: it will simplify the question of existence of equilibrium and the analysis of the game without fundamentally altering the nature of the game.

Forges (1986) discusses various extensions of extensive-form games, including the use of cheap talk and the use of exogenous randomizing devices. In a game with simultaneous moves, however, these extensions can alter the nature of the game, as Aumann (1974) shows. He calls an equilibrium produced using an exogenous device a *correlated equilibrium*. He gives examples of finite, normal-form games for which the set of correlated equilibria is strictly larger than the set of Nash equilibria (and strictly larger than the set of limits of sequences of ε-equilibria as $\varepsilon \to 0$).

In considering cheap-talk extensions of a game, one has to decide whether the absence of cheap talk is essential to the economic analysis. We believe that in many situations this will not be the case. For example, one might model a game as a one-shot, simultaneous-move affair merely to abstract from time and the ordering of moves. Then it may be very reasonable and desirable to add cheap talk to the analysis.

IV. Theorems and Proofs

This section follows the basic outline of the previous section (although the statements of the theorems and their numbering are different). We first prove two convergence theorems that show that the limit of SE or WBR outcomes for approximating games is a SE or WBR outcome for the cheap-talk extension of the limit game. We show next that when talk is ineffective, a cheap-talk equilibrium may be converted to an equilibrium for the original game. We then show that talk is always ineffective for strongly monotonic games. Next we prove existence of equilibria as corollaries of the first three theorems. Lastly we show that cheap-talk SE outcomes may be approximated by sequential ε-equilibrium outcomes. We relegate to the Appendix statements and proofs of several general lemmas (they are numbered A1 through A6).

We begin with a few definitions and key lemmas. In discussing a game $\Gamma = [(T,\rho),X,Y,U^1,U^2]$, we will conserve notation by using Ψ in place of $M(Y)$. An $\eta \in \Psi$ denotes a generic response by player 2. We denote the cheap-talk extension of Γ by $\Gamma^* = [(T,\rho),X\times\Psi^*,Y,U^1,U^2]$, where the cheap-talk space Ψ^* is a copy of Ψ. The asterisk signifies that sending a signal $\eta^* \in \Psi^*$ does not affect the payoff of either player 1 or 2 and distinguishes the signal η^* from a response η.

Given a SE $(\hat{\mu},\hat{\beta},\hat{\eta})$ for Γ, we define the *outcome* $\hat{\lambda}$ of $(\hat{\mu},\hat{\beta},\hat{\eta})$ as a distribution on $T\times X\times\Psi$ (and not on $T\times X\times Y$ as in the previous section). This outcome is the distribution of the random element on $T\times X\times\Psi$ generated by the distributional strategy $\hat{\mu}$ and the function $f(t,x) = (t,x,\hat{\eta}(x))$. Thus, $\hat{\lambda} = \hat{\mu}\circ f^{-1}$. For a SE $(\hat{\nu},\hat{\gamma},\hat{\zeta})$ for the cheap-talk extension Γ^*, we again define the outcome as a distribution on $T\times X\times\Psi$, but now $\hat{\lambda} = \hat{\nu}\circ g^{-1}$, where $g(t,x,\eta^*) = (t,x,\hat{\zeta}(x,\eta^*))$. With these definitions, outcomes for the two types of games are comparable and depend only on payoff-relevant variables. We denote the set of SE for a game Γ by

$SE(\Gamma)$ and the set of SE outcomes by $SEO(\Gamma)$. We denote the set of SE satisfying WBR by $WBR(\Gamma)$ and the corresponding outcomes by $WBRO(\Gamma)$.

Given a SE $(\hat{\mu},\hat{\beta},\hat{\eta})$ for Γ, we define the equilibrium payoffs to player 1 of type t to be

$$V^1(t) = U^1(t,x,\hat{\eta}(x)) \text{ if } (t,x)\in \text{supp}[\hat{\mu}].$$

Condition S2 for a SE guarantees that this definition does not depend on which pairs $(t,x)\in\text{supp}[\hat{\mu}]$ that we choose. The lemma below shows that we can also derive player 1's equilibrium payoffs from the outcome $\hat{\lambda}$.

We will use the following fact in the lemma and repeatedly in the sequel: if a distribution such as $\hat{\mu}\in M(T\times X)$ has compact support, then $\text{supp}[\hat{\mu}_X] = \text{Proj}_X\,\text{supp}[\hat{\mu}]$, where Proj_X denotes the projection onto the X space.

Lemma 1: Let $\Gamma = [(T,\rho),X,Y,U^1,U^2]$ be continuous. Let $(\hat{\mu},\hat{\beta},\hat{\eta})$ be a SE for Γ with outcome $\hat{\lambda}$ and equilibrium payoff function V^1 for player 1. Then

(1) $(t,x,\eta)\in\text{supp}[\hat{\lambda}]$ implies $V^1(t) = U^1(t,x,\eta)$;

(2) $(t',x',\eta')\in\text{supp}[\hat{\lambda}]$ implies $V^1(t) \geq U^1(t,x',\eta')$ for all $t\in T$.

Proof: Applying Lemma A2 with $f(t,x) = (t,x,\hat{\eta}(x))$, $(t,x,\eta)\in\text{supp}[\hat{\lambda}]$ is in the closure of the set $f(\text{supp}[\hat{\mu}])$. Therefore there exists a sequence $<t^n,x^n,\hat{\eta}(x^n)> \to (t,x,\eta)$ with $(t^n,x^n)\in\text{supp}[\hat{\mu}]$ for all n. S2 implies

$$U^1(t^n,x^n,\hat{\eta}(x^n)) \geq U^1(t^n,x,\hat{\eta}(x)).$$

In a similar manner, since $\text{Proj}_{T\times X}\,\text{supp}[\hat{\lambda}] = \text{supp}[\hat{\mu}]$, $(t,x,\eta)\in\text{supp}[\hat{\lambda}]$ implies $(t,x)\in\text{supp}[\hat{\mu}]$. S2 now implies

$$U^1(t,x,\hat{\eta}(x)) \geq U^1(t,x^n,\hat{\eta}(x^n)).$$

Taking limits, we have

$$U^1(t,x,\eta) \geq U^1(t,x,\hat{\eta}(x)) \geq U^1(t,x,\eta).$$

Since $(t,x)\in\text{supp}[\hat{\mu}]$, we have

$$V^1(t) = U^1(t,x,\hat{\eta}(x)) = U^1(t,x,\eta),$$

which proves part (1).

To prove part (2), consider $(t',x',\eta') \in \mathrm{supp}[\hat{\lambda}]$. Just as before, there exists a sequence $<t^n,x^n,\hat{\eta}(x^n)> \to (t',x',\eta')$ with $(t^n,x^n) \in \mathrm{supp}[\hat{\mu}]$ for all n. S2 implies

$$V^1(t) \geq U^1(t,x^n,\hat{\eta}(x^n)),$$

and part (2) follows taking the limit. ❑❑

The following lemma and its application is an important technique employed in this paper. Given potential SE strategies $\hat{\mu}$ and $\hat{\eta}$, the lemma gives conditions enabling us to construct player 2's beliefs on the equilibrium path so that they satisfy S3 and S4.

Lemma 2: Let $\Gamma = [(T,\rho),X,Y,U^1,U^2]$ be continuous. Let $\hat{\mu} \in \Sigma^1(\Gamma)$ and $\hat{\eta} \in \Sigma^2(\Gamma)$. Let $\hat{\eta}$ be continuous on $\mathrm{supp}[\hat{\mu}_X]$ and let $\beta^0 : X \to M(T)$ be a measurable function satisfying $(\hat{\eta}(x),\beta^0(x)) \in MBR(x,Y,T,U^2)$ for all $x \in X \sim \mathrm{supp}[\hat{\mu}_X]$. Suppose for all continuous functions $\eta' : X \to \Psi$,

$$(1) \quad \int_{T \times X} U^2(t,x,\hat{\eta}(x))\, \hat{\mu}(dt \times dx) \geq \int_{T \times X} U^2(t,x,\eta'(x))\, \hat{\mu}(dt \times dx).$$

Then there exists $\hat{\beta}$ such that $(\hat{\beta},\hat{\mu}) \in \Sigma^3(\Gamma)$ and $(\hat{\eta}(x),\hat{\beta}(x)) \in MBR(x,Y,T,U^2)$ for all $x \in X$.

Proof: For $k = 0, 1, 2$, we define measurable sets X^k that together partition X and measurable functions $\beta^k : X^k \to M(T)$. We then set

$$\hat{\beta}(x) = \sum_{k=0}^{2} \beta^k(x)\mathbf{1}_{X^k}(x),$$

where $\mathbf{1}_{X^k}$ is the indicator function for the set X^k.

We take β^0 as given in the statement of the theorem and define $X^0 = X \sim \mathrm{supp}[\hat{\mu}_X]$. Let β^1 be any version of a regular conditional distribution of t given x derived from $\hat{\mu}$. Let

$$X^1 = \{\, x \in \mathrm{supp}[\hat{\mu}_X] \mid (\hat{\eta}(x),\beta^1(x)) \in MBR(x,Y,T,U^2) \,\},$$

and $X^2 = \mathrm{supp}[\hat{\mu}_X] \sim X^1$. Assume that X^2 is measurable and that $\hat{\mu}_X(X^2) = 0$. We will show this below. Then X^1 is dense in $X^1 \cup X^2$, and given this we can use β^1 on X^1 to define a measurable function β^2 on X^2

so that $(\hat{\eta}(x),\beta^2(x)) \in MBR(x,Y,T,U^2)$ for all $x \in X^2$.

To define β^2, let $C = \{ (x,\hat{\eta}(x),\beta^1(x)) \mid x \in X^1 \}$. Applying Lemma A6 to the closure $\bar{C}$, there exists a measurable function (η^2,β^2) such that $(x,\eta^2(x),\beta^2(x)) \in \bar{C}$ for all $x \in \hat{X} = \text{Proj}_X \bar{C}$. Since X^1 is dense in $X^1 \cup X^2$, $\hat{X} = X^1 \cup X^2$. By hypothesis, $\hat{\eta}$ is continuous on $X^1 \cup X^2$, so we have $\eta^2 = \hat{\eta}$. Since $(\hat{\eta}(x),\beta^1(x)) \in MBR(x,Y,T,U^2)$ for $x \in X^1$, we have $(\hat{\eta}(x),\beta^2(x)) \in MBR(x,Y,T,U^2)$ for $x \in X^2$ by Lemma A5.

Under our assumptions that X^2 is measurable and $\hat{\mu}_X(X^2) = 0$, we can thus now define a measurable function $\hat{\beta}(x) = \sum_k \beta^k(x)\mathbf{1}_{X^k}(x)$ such that $(\hat{\eta}(x),\hat{\beta}(x)) \in MBR(x,Y,T,U^2)$ for $x \in X$. $\hat{\beta}$ is a regular conditional distribution derived from $\hat{\mu}$ because $\hat{\beta}$ equals the regular conditional distribution β^1 on X^1 and $\hat{\mu}_X(X^1) = 1$. Hence, $(\hat{\beta},\hat{\mu}) \in \Sigma^3(\Gamma)$ and $\hat{\beta}$ satisfies the conclusions of the lemma.

To complete the proof, we now have to show that X^2 is measurable and $\hat{\mu}_X(X^2) = 0$. We sketch the proof of this. To show X^2 is measurable, define a function

$$h_\eta(x) = \int_T [U^2(t,x,\hat{\eta}(x)) - U^2(t,x,\eta)]\, \beta^1(x)(dt),$$

and let $\hat{\Psi}$ be a countable, dense subset of Ψ. The function h_η is a conditional expectation and thus is measurable (Chow and Teicher 1988, Theorem 7.2.1). The function h_η is also continuous in η for fixed x by Lemma A4. Then

$$X^2 = \bigcup_{\eta \in \hat{\Psi}} \{ x \mid h_\eta(x) < 0 \}$$

and this is a countable union of measurable sets and is therefore measurable.

To show $\hat{\mu}_X(X^2) = 0$, define a correspondence on X^2 by $H(x) = \{ \eta \in \hat{\Psi} \mid h_\eta(x) < 0 \}$. The graph of H is

$$\bigcup_{\eta' \in \hat{\Psi}} \{ (x,\eta) \mid h_\eta(x) < 0 \text{ and } \eta = \eta' \},$$

which is measurable. By the Measurable Selection Theorem (Hildenbrand 1974, p. 54), there exists a measurable function η^2 with $\eta^2(x) \in H(x)$ almost everywhere $[\hat{\mu}_X]$ on X^2. Set $\eta^2 = \hat{\eta}$ on $X \sim X^2$. Now if

$\hat{\mu}_X(X^2) > 0$, we can use Lusin's Theorem (Parthasarathy 1967, Lemma II.4.1) to find a continuous function η' that approximates η^2 closely enough so that η' violates condition (1) of the lemma. This contradiction implies we must have $\hat{\mu}_X(X^2) = 0$, and the proof is complete. □□

Our first two theorems concern convergence of a sequence of SE outcomes for a sequence of games. We have to define the types of convergence we use. We consider a sequence of games, say $\Gamma^n = [(T^n,\rho^n),X^n,Y^n,U^{1n},U^{2n}]$ for $n = 1, 2, \ldots$, and a limit game $\Gamma = [(T,\rho),X,Y,U^1,U^2]$. We assume that all the type and action spaces are compact subspaces of ambient compact metric spaces $\tilde{T}$, $\tilde{X}$ and $\tilde{Y}$. Convergence of any sequence is always relative to the relevant ambient space. For example, convergence for probability distributions on the type spaces means weak convergence in the space $M(\tilde{T})$. We write $<\rho^n> \Rightarrow \rho$ when a sequence of distributions with $\rho^n \in M(T^n)$ converges weakly to $\rho \in M(T)$.

Convergence for the payoff functions U^{in} means continuous convergence. We say that $<U^{in}> \to U^i$ continuously for $(t,x,y) \in T \times X \times Y$ when for all $(t,x,y) \in T \times X \times Y$, $<t^n,x^n,y^n> \to (t,x,y)$ implies $<U^{in}(t^n,x^n,y^n)> \to U^i(t,x,y)$.

Convergence for the type and action spaces means closed convergence or equivalently convergence using the Hausdorff metric on sets (see Hildenbrand 1974, p. 15). Given a sequence of sets $<T^n>$, define $\mathrm{Li}<T^n>$ as the set of limits of sequences $<t^n>$ with $t^n \in T^n$ for all n; define $\mathrm{Ls}<T^n>$ as the set of limits of subsequences. We say $<T^n> \to T$ if and only if $\mathrm{Li}<T^n> = T = \mathrm{Ls}<T^n>$, and we say T is the *closed limit* of the sequence $<T^n>$. One can show $<T^n> \to T$ if and only if $<M(T^n)> \to M(T)$.

We will need the following definitions to derive the strategies and beliefs for player 2 in SE for limit games. Given a SE $(\hat{\mu},\hat{\beta},\hat{\eta})$ for Γ, we say that $\hat{\beta}$ is *regular at x* if and only if

$$x \in \mathrm{supp}[\hat{\mu}_X] \text{ and } t \in \mathrm{supp}[\hat{\beta}(x)] \text{ imply } (t,x) \in \mathrm{supp}[\hat{\mu}].$$

When $(\hat{\beta},\hat{\mu}) \in \Sigma^3(\Gamma)$, then by definition $\hat{\beta}$ is regular at almost all x $[\hat{\mu}_X]$ and therefore is regular on a dense subset of X. Define

$$A(X,\hat{\mu},\hat{\beta},\hat{\eta}) = \{ (x,\hat{\eta}(x),\hat{\beta}(x)) \mid x \in X \text{ and } \hat{\beta} \text{ is regular at } x \}.$$

Given a sequence $<\hat{\mu}^n,\hat{\beta}^n,\hat{\eta}^n>$ of SE for $<\Gamma^n>$, let $B = \mathrm{Ls}<A(X^n,\hat{\mu}^n,\hat{\beta}^n,\hat{\eta}^n)>$. B is non-empty and compact.

B is equal to the set of all $(x,\eta,\beta)\in\tilde{X}\times M(\tilde{Y})\times M(\tilde{T})$ such that there exists a subsequence $<x^{n_k},\hat{\eta}^{n_k}(x^{n_k}),\hat{\beta}^{n_k}(x^{n_k})> \to (x,\eta,\beta)$ with $\hat{\beta}^{n_k}$ regular at x^{n_k}. We will use this characterization of B repeatedly.

Theorem 1: Consider a sequence of continuous games $\Gamma^n = [(T^n,\rho^n),X^n,Y^n,U^{1n},U^{2n}]$, $n = 1, 2, \ldots$, a continuous limit game $\Gamma = [(T,\rho),X,Y,U^1,U^2]$, and the cheap-talk extension game $\Gamma^* = [(T,\rho),X\times\Psi^*,Y,U^1,U^2]$. Suppose

(H1) there exists $<\hat{\lambda}^n> \Rightarrow \hat{\lambda}$ with $\hat{\lambda}^n\in SEO(\Gamma^n)$;

(H2) $<X^n> \to X$, $<Y^n> \to Y$, $<T^n> \to T$, $<\rho^n> \Rightarrow \rho$;

(H3) $<U^{in}> \to U^i$ continuously for $(t,x,y)\in T\times X\times Y$, $i = 1, 2$.

Then

(C1) $\hat{\lambda}\in SEO(\Gamma^*)$.

Let $<\hat{\mu}^n,\hat{\beta}^n,\hat{\eta}^n>$ be SE supporting $<\hat{\lambda}^n>$, and let $B = \mathrm{Ls}<A(X^n,\hat{\mu}^n,\hat{\beta}^n,\hat{\eta}^n)>$. If each X^n is finite, then

(C2) $(t,x,\eta)\in\mathrm{supp}[\hat{\lambda}]$ implies $(x,\eta,\beta)\in B$ for some β.

Remark: The outcome $\hat{\lambda}\in SEO(\Gamma^*)$ is supported by $(\hat{v},\hat{\gamma},\hat{\zeta})$ where $\hat{v} = \hat{\lambda}$, $\hat{\zeta}(x,\eta^*) = \eta^*$ on $\mathrm{supp}[\hat{v}_{X\times\Psi^*}]$, and on $X\times\Psi^*\sim\mathrm{supp}[\hat{v}_{X\times\Psi^*}]$, $(\hat{\zeta},\hat{\gamma}) = (\eta^0,\beta^0)$, which is any measurable selection from the set B. We have built B only from regular beliefs. This is not necessary for this theorem, but it will be necessary for Theorem 2.

Proof: We will construct the promised SE for Γ^* from $\hat{\lambda}$ and from the sequence of SE $<\hat{\mu}^n,\hat{\beta}^n,\hat{\eta}^n>$ supporting the outcomes $<\hat{\lambda}^n>$ for $<\Gamma^n>$. We begin with two lemmas.

Lemma 3: Let $(t,x,\eta)\in\mathrm{supp}[\hat{\lambda}]$ and let H1 and H3 hold. If either $(t',x',\eta')\in\mathrm{supp}[\hat{\lambda}]$ or $(x',\eta',\beta')\in B$, then $U^1(t,x,\eta) \geq U^1(t,x',\eta')$.

Proof: Let both (t,x,η) and (t',x',η') come from $\mathrm{supp}[\hat{\lambda}]$. Apply Lemma A3 with $f^n(t,x) = (t,x,\hat{\eta}^n(x))$. Then H1 implies there exists a sequence $<t^n,x^n,\hat{\eta}^n(x^n)> \to (t,x,\eta)$ with $(t^n,x^n)\in\mathrm{supp}[\hat{\mu}^n]$ for each n and a similar sequence $<t'^n,x'^n,\hat{\eta}^n(x'^n)> \to (t',x',\eta')$. Since $\hat{\mu}^n$ and $\hat{\eta}^n$ represent a SE, condition S2 implies

$$U^{1n}(t^n,x^n,\hat{\eta}^n(x^n)) \geq U^{1n}(t^n,x'^n,\hat{\eta}^n(x'^n)).$$

Taking limits using H3 and Lemma A4, we have

$$U^1(t,x,\eta) \geq U^1(t,x',\eta').$$

This proves the lemma when $(t',x',\eta') \in \text{supp}[\hat{\lambda}]$.

A similar argument establishes the case $(x',\eta',\beta') \in B$. Here we use a (sub)sequence $\langle x'^n,\hat{\eta}^n(x'^n),\hat{\beta}^n(x'^n)\rangle \to (x',\eta',\beta')$; such a sequence exists by definition of B. The rest of the argument proceeds as in the previous case. ❑❑

The next lemma shows how to construct player 2's strategy and beliefs off the equilibrium path.

Lemma 4: If H1-H3 hold, then there exists a measurable function $(\eta^0,\beta^0):X \to \Psi \times M(T)$ such that $(x,\eta^0(x),\beta^0(x)) \in B$ for all $x \in X$. Furthermore, $(x,\eta,\beta) \in B$ implies $(\eta,\beta) \in MBR(x,Y,T,U^2)$.

Proof: B as a closed limit is compact. By Lemma A6, there exists a measurable function $(\eta^0,\beta^0):\hat{X} \to \tilde{\Psi} \times M(\tilde{T})$ such that $(x,\eta^0(x),\beta^0(x)) \in B$ for all $x \in \hat{X} = \text{Proj}_{\tilde{X}}\, B$.

We must show that (η^0,β^0) is a function from X to $\Psi \times M(T)$. An element (x,η,β) of B is the limit of a (sub)sequence $\langle x^n,\hat{\eta}^n(x^n),\hat{\beta}^n(x^n)\rangle$ with $\hat{\beta}^n$ regular at x^n. This is true if and only if $x \in X$ since $\hat{\beta}^n$ is regular on a dense subset of X^n and $\langle X^n\rangle \to X$ by H2. Therefore $\hat{X} = X$. Since $(\hat{\eta}^n(x^n),\hat{\beta}^n(x^n)) \in MBR(x^n,Y^n,T^n,U^{2n})$ on the sequence, the limit $(\eta,\beta) \in MBR(x,Y,T,U^2)$ by Lemma A5. This implies in particular that $(\eta^0(x),\beta^0(x)) \in \Psi \times M(T)$ for all $x \in X$ and also proves the last statement of the lemma. ❑❑

We continue with the proof of Theorem 1. We first show conclusion C2. Let $(t,x,\eta) \in \text{supp}[\hat{\lambda}]$. Apply Lemma A3 with $f^n(t,x) = (t,x,\hat{\eta}^n(x))$. Then $\langle\hat{\lambda}^n\rangle \Rightarrow \hat{\lambda}$ implies there exists a sequence $\langle t^n,x^n,\hat{\eta}^n(x^n)\rangle \to (t,x,\eta)$ with $(t^n,x^n) \in \text{supp}[\hat{\mu}^n]$ for each n. Let $\beta = \lim_n \hat{\beta}^n(x^n)$ on a subsequence. Then $\langle x^n,\hat{\eta}^n(x^n),\hat{\beta}^n(x^n)\rangle \to (x,\eta,\beta)$, and since each X^n is finite, $\hat{\beta}^n$ is regular at x^n for all n. Therefore $(x,\eta,\beta) \in B$ by definition. This proves C2.

We now show C1, that $\hat{\lambda} \in SEO(\Gamma^*)$. We will construct a SE $(\hat{\nu},\hat{\gamma},\hat{\zeta})$ supporting the outcome $\hat{\lambda}$ in three steps. First, we define the strategy $\hat{\nu}$ of player 1 and show it satisfies S1. Second, we define a strategy $\hat{\zeta}$ for player 2 satisfying S1 and show that $\hat{\nu}$ and $\hat{\zeta}$ satisfy S2 and result in outcome $\hat{\lambda}$. Finally, we show that there exist beliefs $\hat{\gamma}$ such that $(\hat{\nu},\hat{\gamma},\hat{\zeta})$ satisfies S3 and S4.

Define $\hat{\nu} = \hat{\lambda}$. To show $\hat{\nu}$ satisfies S1, we have to show that $\hat{\lambda} \in M(T \times X \times \Psi)$ and $\hat{\lambda}_T = \rho$. Define $\Psi^n = M(Y^n)$. Now $<T^n \times X^n \times Y^n> \to T \times X \times Y$ only if $<M(T^n \times X^n \times \Psi^n)> \to M(T \times X \times \Psi)$. Therefore, H2 and $<\hat{\lambda}^n> \Rightarrow \hat{\lambda}$ imply $\hat{\lambda} \in M(T \times X \times \Psi)$. Similarly, since $<\hat{\lambda}^n> \Rightarrow \hat{\lambda}$ implies $<\hat{\lambda}^n_T> \Rightarrow \hat{\lambda}_T$, we have $<\hat{\lambda}^n_{T^n}> \Rightarrow \hat{\lambda}_T$. Since $\hat{\lambda}^n_{T^n} = \rho^n$ and $<\rho^n> \Rightarrow \rho$, $\hat{\lambda}_T = \rho$. Thus $\hat{\nu}$ satisfies S1.

We define player 2's strategy $\hat{\zeta}: X \times \Psi^* \to \Psi$ using the function $\eta^0: X \to \Psi$ derived in Lemma 4. Define

$$\hat{\zeta}(x,\eta^*) = \begin{cases} \eta^* & \text{if } (x,\eta^*) \in \text{supp}[\hat{\nu}_{X \times \Psi^*}], \\ \eta^0(x) & \text{otherwise.} \end{cases}$$

The strategy $\hat{\zeta}$ is a measurable since η^0 is measurable, so $\hat{\zeta}$ satisfies S1. By definition of $\hat{\zeta}$ on $\text{supp}[\hat{\nu}_{X \times \Psi^*}]$, $\hat{\lambda}$ is the outcome of playing the strategies $\hat{\nu}$ and $\hat{\zeta}$ as we require.

We show $\hat{\nu}$ and $\hat{\zeta}$ satisfy S2. Let $(t,x,\eta^*) \in \text{supp}[\hat{\nu}]$ and choose $(x',\eta^{*\prime}) \in X \times \Psi^*$ arbitrarily. First suppose that $(x',\eta^{*\prime}) \in \text{supp}[\hat{\nu}_{X \times \Psi^*}]$. Then $(t',x',\eta^{*\prime}) \in \text{supp}[\hat{\lambda}]$ for some t'. Applying Lemma 3 yields

$$U^1(t,x,\eta^*) \geq U^1(t,x',\eta^{*\prime}).$$

Using the definition of $\hat{\zeta}$ on $\hat{\nu}_{X \times \Psi^*}$, we have

$$U^1(t,x,\hat{\zeta}(x,\eta^*)) \geq U^1(t,x',\hat{\zeta}(x',\eta^{*\prime})),$$

which is S2 for this case. Now suppose that $(x',\eta^{*\prime}) \notin \text{supp}[\hat{\nu}_{X \times \Psi^*}]$. Then $\hat{\zeta}(x',\eta^{*\prime}) = \eta^0(x')$ and according to Lemma 4, $(x',\eta^0(x'),\beta^0(x')) \in B$. Applying Lemma 3 in this case yields

$$U^1(t,x,\eta^*) \geq U^1(t,x',\eta^0(x')),$$

which again implies

$$U^1(t,x,\hat{\zeta}(x,\eta^*)) \geq U^1(t,x',\hat{\zeta}(x',\eta^{*\prime})).$$

S2 follows.

The final step in the proof of C1 is to show that there exist beliefs $\hat{\gamma}$ for player 2 such that $(\hat{\nu},\hat{\gamma},\hat{\zeta})$ satisfies S3 and S4. According to Lemma 2, this will be true if several requirements hold. First, $\hat{\nu}$ and $\hat{\zeta}$ must satisfy S1. We have shown this. Second, $\hat{\zeta}$ must be continuous on $\text{supp}[\hat{\nu}_{X\times\Psi^*}]$. This holds by definition. Third, there must exist a measurable function $\gamma^0 : X\times\Psi^* \to M(T)$ such that $(\hat{\zeta}(x,\eta^*),\gamma^0(x,\eta^*)) \in MBR(x,Y,T,U^2)$ for all $(x,\eta^*) \in X\times\Psi^* - \text{supp}[\hat{\nu}_{X\times\Psi^*}]$. This holds by Lemma 4 and the definition of $\hat{\zeta}$ if we take $\gamma^0(x,\eta^*) = \beta^0(x)$.

The final requirement of Lemma 2 is that for any continuous function $\zeta : X\times\Psi^* \to \Psi$, we must have

$$\int_{T\times X\times\Psi^*} U^2(t,x,\hat{\zeta}(x,\eta^*))\ \hat{\nu}(dt\times dx\times d\eta^*) \geq \int_{T\times X\times\Psi^*} U^2(t,x,\zeta(x,\eta^*))\ \hat{\nu}(dt\times dx\times d\eta^*).$$

To demonstrate this, choose a continuous function ζ arbitrarily. Define $\Psi^n = M(Y^n)$. One can show that hypothesis H2 and the continuity of ζ imply there exists a sequence $<\zeta^n>$ of measurable functions $\zeta^n : X^n\times\Psi^n \to \Psi^n$ such that $<\zeta^n> \to \zeta$ continuously for $(x,\eta^*) \in X\times\Psi^*$. Since $<\hat{\mu}^n,\hat{\beta}^n,\hat{\eta}^n>$ is a SE for Γ^n, S3 implies

$$\int_{T^n} U^{2n}(t,x,\hat{\eta}^n(x))\ \hat{\beta}^n(x)(dt) \geq \int_{T^n} U^{2n}(t,x,\zeta^n(x,\hat{\eta}^n(x)))\ \hat{\beta}^n(x)(dt)$$

for all $x \in X$. Since $\hat{\beta}^n$ is a conditional distribution derived from $\hat{\mu}^n$, both sides of this inequality are conditional expectations derived from $\hat{\mu}^n$ (Chow and Teicher 1988 Theorem 7.2.1). Therefore, we may integrate both sides using $\hat{\mu}^n_{X^n}$ to get

$$\int_{X^n}\int_{T^n} U^{2n}(t,x,\hat{\eta}^n(x))\ \hat{\beta}^n(x)(dt)\hat{\mu}^n_{X^n}(dx) \geq \int_{X^n}\int_{T^n} U^{2n}(t,x,\zeta^n(x,\hat{\eta}^n(x)))\ \hat{\beta}^n(x)(dt)\hat{\mu}^n_{X^n}(dx),$$

and thus by definition of conditional expectation,

$$\int_{T^n\times X^n} U^{2n}(t,x,\hat{\eta}^n(x))\ \hat{\mu}^n(dt\times dx) \geq \int_{T^n\times X^n} U^{2n}(t,x,\zeta^n(x,\hat{\eta}^n(x)))\ \hat{\mu}^n(dt\times dx).$$

Since $\hat{\lambda}^n$ is the outcome of playing the strategies $\hat{\mu}^n$ and $\hat{\eta}^n$, it follows that

$$\int_{T^n\times X^n\times\Psi^n} U^{2n}(t,x,\eta)\ \hat{\lambda}^n(dt\times dx\times d\eta) \geq \int_{T^n\times X^n\times\Psi^n} U^{2n}(t,x,\zeta^n(x,\eta))\ \hat{\lambda}^n(dt\times dx\times d\eta).$$

Taking limits using Lemma A4, hypotheses H1-H3, and the assumption that $<\zeta^n> \rightarrow \zeta$ continuously,

$$\int_{T\times X\times\Psi} U^2(t,x,\eta)\ \hat{\lambda}(dt\times dx\times d\eta) \geq \int_{T\times X\times\Psi} U^2(t,x,\zeta(x,\eta))\ \hat{\lambda}(dt\times dx\times d\eta).$$

Finally, since $\hat{v} = \hat{\lambda}$ and since $\hat{\zeta}(x,\eta^*) = \eta^*$ when $(t,x,\eta^*)\in\text{supp}[\hat{v}]$, we have

$$\int_{T\times X\times\Psi^*} U^2(t,x,\hat{\zeta}(x,\eta^*))\ \hat{v}(dt\times dx\times d\eta^*) \geq \int_{T\times X\times\Psi^*} U^2(t,x,\zeta(x,\eta^*))\ \hat{v}(dt\times dx\times d\eta^*)$$

as required. Lemma 2 then asserts there exist beliefs $\hat{\gamma}$ such that $(\hat{v},\hat{\gamma},\hat{\zeta})$ satisfies S3 and S4. We have thus shown that $(\hat{v},\hat{\gamma},\hat{\zeta})\in SE(\Gamma^*)$ and the proof of the theorem is complete. □□

The next theorem concerns convergence of WBR equilibria.

Theorem 2: Consider a sequence of continuous games $\Gamma^n = [(T^n,\rho^n),X^n,Y^n,U^{1n},U^{2n}]$, $n = 1, 2, \ldots$, a continuous limit game $\Gamma = [(T,\rho),X,Y,U^1,U^2]$, and the cheap-talk extension game $\Gamma^* = [(T,\rho),X\times\Psi^*,Y,U^1,U^2]$. Suppose

(H1) there exists $<\hat{\lambda}^n> \Rightarrow \hat{\lambda}$ with $\hat{\lambda}^n\in WBRO(\Gamma^n)$;

(H2) $<X^n> \rightarrow X$, $<Y^n> \rightarrow Y$, $<T^n> \rightarrow T$ with $T^n\subset T^{n+1}$, $<\rho^n> \Rightarrow \rho$;

(H3) $<U^{in}> \rightarrow U^i$ continuously for $(t,x,y)\in T\times X\times Y$, $i = 1, 2$;

(H4) $<U^{1n}>$ is uniformly Lipschitz in t.

Then $\hat{\lambda}\in WBRO(\Gamma^*)$.

Remark: The strategies and beliefs supporting $\hat{\lambda}$ as an equilibrium outcome are the same as in Theorem 1 on the equilibrium path.

Proof: Let $<\hat{\mu}^n,\hat{\beta}^n,\hat{\eta}^n>$ be a sequence of SE supporting the outcomes $<\hat{\lambda}^n>$ of $<\Gamma^n>$. We show first that under H4 we may choose a subsequence of $<\hat{\mu}^n,\hat{\beta}^n,\hat{\eta}^n>$ so that the payoff functions V^{1n} converge continuously to a limit function V^1. Then we show that V^1 is an equilibrium payoff function for outcome $\hat{\lambda}$.

First note that by condition S2 in the definition of SE,

$$V^{1n}(t) \geq U^{1n}(t,x,\hat{\eta}^n(x))$$

holds for any $(t,x)\in T^n\times X^n$ and it holds with equality if $(t,x)\in\text{supp}[\hat{\mu}^n]$. Let both (t,x) and (t',x') come from $\mathrm{supp}[\hat{\mu}^n]$. Then

$$V^{1n}(t) - V^{1n}(t') \le U^{1n}(t,x,\hat{\eta}^n(x)) - U^{1n}(t',x,\hat{\eta}^n(x))$$
$$\le Ld_T(t,t')$$

for some L since $<U^{1n}>$ is uniformly Lipschitz in t. Similarly,

$$V^{1n}(t) - V^{1n}(t') \ge U^{1n}(t,x',\hat{\eta}^n(x')) - U^{1n}(t',x',\hat{\eta}^n(x'))$$
$$\ge -Ld_T(t,t').$$

Therefore the sequence of functions $V^{1n}:T^n \to R^1$ is equicontinuous. H2, H3 and the compactness of $T \times X \times \Psi$ imply $<V^{1n}>$ is bounded. By a minor extension of the Ascoli Theorem, a subsequence of $<V^{1n}>$ converges continuously to a continuous function $V^1:T \to R^1$.[12]

We now assume $<\hat{\lambda}^n>$ has been chosen so that the payoffs $<V^{1n}> \to V^1$ continuously. We must show that $(t,x,\eta) \in \mathrm{supp}[\hat{\lambda}]$ implies $V^1(t) = U^1(t,x,\eta)$, so that V^1 is an equilibrium payoff function for $\hat{\lambda}$. Let $(t,x,\eta) \in \mathrm{supp}[\hat{\lambda}]$. By Lemma A1, there exists $<t^n,x^n,\eta^n> \to (t,x,\eta)$ with $(t^n,x^n,\eta^n) \in \mathrm{supp}[\hat{\lambda}^n]$ for all n. Then $V^{1n}(t^n) = U^{1n}(t^n,x^n,\eta^n)$ by Lemma 1. We know that $V^{1n}(t^n) \to V^1(t)$ and $U^{1n}(t^n,x^n,\eta^n) \to U^1(t,x,\eta)$. Hence, $V^1(t) = U^1(t,x,\eta)$ as required.

We now show that $\hat{\lambda}$ is a WBR outcome for $\Gamma*$. We know from Theorem 1 that $\hat{\lambda}$ is a SE outcome. To support this outcome as a WBR outcome, we can use the Theorem 1 strategies for player 1 and the Theorem 1 strategies and beliefs for player 2 on the equilibrium path. We may need, however, to refine somewhat the strategies and beliefs of player 2 off the equilibrium path.

For Theorem 1, player 2's strategy/belief pair given signal $(x,\eta*)$ off the equilibrium path was $(\eta^0(x),\beta^0(x))$. Here we will define a new strategy/belief pair $(\tilde{\eta},\tilde{\beta})$ to replace (η^0,β^0). We will verify that this new pair will satisfy the WBR criterion and conditions S1, S3, and S4. It also will not upset the equilibrium strategy for player 1. Then $\hat{\lambda}$ will remain the equilibrium outcome, so the equilibrium payoff function

[12] We need to extend the Ascoli theorem because the V^{1n} functions are not defined on all of T. Using elements of the proof in Royden (1968,pp.177-9), one first shows that a subsequence of $<V^{1n}>$ converges uniformly to a continuous function V^1 defined on $\bigcup T^n$, which is dense in T. That $<T^n>$ in an increasing sequence is needed here. Then one shows that V^1 is uniformly continuous, so it has a unique continuous extension to T by Royden's Proposition 7.11. Finally, uniform convergence of $<V^{1n}>$ and continuity of V^1 imply continuous convergence of $<V^{1n}>$.

for player 1 will equal the limit payoff function V^1.

We now define a pair $(\tilde{\eta}(x,\eta^*),\tilde{\beta}(x,\eta^*))$ for each $x \in X$ with the understanding that $(\tilde{\eta}(x,\eta^*),\tilde{\beta}(x,\eta^*))$ will only be used to replace $(\eta^0(x),\beta^0(x))$ off the equilibrium path. For each $x \in X$, we consider a subsequence $<x^n,\hat{\eta}^n(x^n),\hat{\beta}^n(x^n)> \rightarrow (x,\eta^0(x),\beta^0(x))$. This sequence exists by definition of (η^0,β^0). There are four cases to consider. These cases are not all mutually exclusive, but they do not need to be.

Case 1: There is an infinite subsequence of $<x^n> \rightarrow x$ such that $x^n \in \text{supp}[\hat{\mu}^n_{X^n}]$. Restrict attention to this subsequence. We set $(\tilde{\eta}(x,\eta^*),\tilde{\beta}(x,\eta^*)) = (\eta^0(x),\beta^0(x))$ and show that we can use (η',β') set equal to the same $(\eta^0(x),\beta^0(x))$ to satisfy W2 for any $t \in \text{supp}[\tilde{\beta}(x,\eta^*)]$. First, $(\eta',\beta') \in MBR(x,Y,T,U^2)$ by Lemma A5 since it is the limit of $(\hat{\eta}^n(x^n),\hat{\beta}^n(x^n)) \in MBR(x^n,Y^n,T^n,U^{2n})$. Second, since $\hat{\beta}^n(x^n) \Rightarrow \beta'$, there exists $<t^n> \rightarrow t$ with $t^n \in \text{supp}[\hat{\beta}^n(x^n)]$ by Lemma A1. Since by construction $\beta' = \beta^0(x)$ is a limit of regular beliefs and since $x^n \in \text{supp}[\hat{\mu}^n_{X^n}]$, it follows that $(t^n,x^n) \in \text{supp}[\hat{\mu}^n_{X^n}]$. In this event, $U^{1n}(t^n,x^n,\hat{\eta}^n(x^n)) = V^{1n}(t^n)$. Taking limits yields $U^1(t,x,\eta') = V^1(t)$ since $<V^{1n}>$ converges continuously. Finally, for any $t' \in T$, take a sequence $<t'^n> \rightarrow t'$. By S2 for Γ^n, we have $U^{1n}(t'^n,x^n,\hat{\eta}^n(x^n)) \leq V^{1n}(t'^n)$, so that in the limit $U^1(t',x,\eta') \leq V^1(t')$. Hence W2 holds for $\tilde{\beta}(x,\eta^*)$.

Since we have set $(\tilde{\eta}(x,\eta^*),\tilde{\beta}(x,\eta^*)) = (\eta^0(x),\beta^0(x))$ and this is the SE strategy for player 2 off the equilibrium path, $(\tilde{\eta}(x,\eta^*),\tilde{\beta}(x,\eta^*))$ will automatically be an equilibrium strategy in the this case.

Case 2: There is an infinite subsequence of $<x^n> \rightarrow x$ such that $x^n \notin \text{supp}[\hat{\mu}^n_{X^n}]$ and W1 holds for $(\hat{\mu}^n,\hat{\beta}^n,\hat{\eta}^n)$ at x^n. Restrict attention to this subsequence. By H1, $(\hat{\mu}^n,\hat{\beta}^n,\hat{\eta}^n)$ satisfies WBR, so W2 holds. We set $(\tilde{\eta}(x,\eta^*),\tilde{\beta}(x,\eta^*)) = (\eta^0(x),\beta^0(x))$ and show that W2 holds for $\tilde{\beta}(x,\eta^*)$. Let $t \in \text{supp}[\tilde{\beta}(x,\eta^*)]$. Since $<\hat{\beta}^n(x^n)> \Rightarrow \tilde{\beta}(x,\eta^*)$, there exists $<t^n> \rightarrow t$ with $t^n \in \text{supp}[\hat{\beta}^n(x^n)]$. Let $(\eta^n,\beta^n) \in MBR(x^n,Y^n,T^n,U^{2n})$ satisfy W2 for t^n in the SE for Γ^n and let $<\eta^n,\beta^n> \Rightarrow (\eta',\beta')$ on a subsequence. Then $(\eta',\beta') \in MBR(x,Y,T,U^2)$ by Lemma A5. As in Case 1, the equation in W2 holds for η' and $V^1(t)$ since it holds for η^n and $V^{1n}(t^n)$. The inequality in W2 holds for any $t' \in T$ by a similar logic. Hence

W2 holds for $\widetilde{\beta}(x,\eta^*)$.

As in Case 1, we have set $(\widetilde{\eta}(x,\eta^*),\widetilde{\beta}(x,\eta^*)) = (\eta^0(x),\beta^0(x))$, so $(\widetilde{\eta}(x,\eta^*),\widetilde{\beta}(x,\eta^*))$ will automatically be an equilibrium strategy in the this case.

Case 3: There is an infinite subsequence of $<x^n> \to x$ such that W1 fails to hold both for Γ^n at x^n and for Γ^* at x. Then WBR holds vacuously using $(\widetilde{\eta}(x,\eta^*),\widetilde{\beta}(x,\eta^*)) = (\eta^0(x),\beta^0(x))$ and $(\widetilde{\eta}(x,\eta^*),\widetilde{\beta}(x,\eta^*))$ will automatically be an equilibrium strategy.

Case 4: There is an infinite subsequence of $<x^n> \to x$ such that W1 fails to hold with a strict inequality for Γ^n at x^n but W1 holds for Γ^* at x. Notice we require only that W1 fails to hold with a strict inequality at x^n, i.e., there does not exist $t \in T^n$ and $(\eta,\beta) \in MBR(x^n,Y^n,T^n,U^{2n})$ such that

$$U^{1n}(t,x^n,\eta) > V^{1n}(t).$$

This requirement is satisfied if W1 fails to hold with a weak inequality. The requirement guarantees that the set of x satisfying Case 4 is closed (and extends the case somewhat so it overlaps with Case 3).

Restrict attention to the given subsequence. We will show that there exists a $\widetilde{t} \in T$ such that if $\widetilde{\beta}(x,\eta^*)$ puts unit mass on $\widetilde{t}$, then $\widetilde{\beta}(x,\eta^*)$ will satisfy W2.

Since W1 holds for Γ, there exists t' and $(\eta',\beta') \in MBR(x,Y,T,U^2)$ satisfying

$$U^1(t',x,\eta') \geq V^1(t'). \tag{3}$$

Let $<\beta^n>$ be any sequence converging to β' with $\beta^n \in M(T^n)$ for all n. Take any convergent subsequence $<\eta^n>$ such that $(\eta^n,\beta^n) \in MBR(x^n,Y^n,T^n,U^{2n})$ and let $\eta'' = \lim_n \eta^n$. Since by hypothesis W1 fails to hold with strict inequality on the subsequences we are considering,

$$U^{1n}(t,x^n,\eta^n) \leq V^{1n}(t)$$

holds for every $t \in T^n$. Taking limits,

$$U^1(t,x,\eta'') \leq V^1(t) \tag{4}$$

holds for every $t \in T$.

For $s \in [0,1]$, let $\eta^s = s\eta' + (1-s)\eta''$. By Lemma A5, (η'',β') is in $MBR(x,Y,T,U^2)$, as is (η',β'), and this implies so is (η^s,β'). Define a function $m:[0,1] \to R^1$ by

$$m(s) = \max_{t \in T} [U^1(t,x,\eta^s) - V^1(t)].$$

The function m is well defined since V^1 and U^1 are continuous and T is compact. By the Theorem of the Maximum, m is continuous. Now by (3), $m(1) \geq 0$ and by (4), $m(0) \leq 0$, so there must be an $s \in [0,1]$ with $m(s) = 0$ by the Intermediate-Value Theorem. By definition of m, for this s there must exist a $\tilde{t} \in T$ such that

$$U^1(\tilde{t},x,\eta^s) = V^1(\tilde{t}) \tag{5}$$

and for all $t \in T$,

$$U^1(t,x,\eta^s) \leq V^1(t). \tag{6}$$

Let $\tilde{\beta}(x,\eta^*)$ put unit mass on $\tilde{t}$. This belief satisfies W2 because of (5) and (6) and because $(\eta^s,\beta') \in MBR(x,Y,T,U^2)$.

Let $\tilde{\eta}(x,\eta^*)$ be a best response by player 2 to x given the belief $\tilde{\beta}(x,\eta^*)$, so that $(\tilde{\eta}(x,\eta^*),\tilde{\beta}(x,\eta^*))$ will satisfy S3. The logic used to establish (4) implies we may assume that

$$U^1(t,x,\tilde{\eta}(x,\eta^*)) \leq V^1(t) \tag{7}$$

holds for all t. That is, some best responses given beliefs $\tilde{\beta}(x,\eta^*)$ may violate (7), but at least one will satisfy it. Given (7), playing $\tilde{\eta}(x,\eta^*)$ instead of $\eta''(x)$ will not upset the equilibrium strategy of player 1.

We have defined $(\tilde{\eta}(x,\eta^*),\tilde{\beta}(x,\eta^*))$ so that it satisfies the WBR criterion and conditions S2 and S3. We must now refine $(\tilde{\eta},\tilde{\beta})$ to make it a measurable function so that conditions S1 and S4 will be satisfied. Since $(\tilde{\eta},\tilde{\beta})$ equals the measurable function (η'',β'') except in Case 4, we only need to show that the set C of x where Case 4 holds is measurable and that we can refine $(\tilde{\eta},\tilde{\beta})$ so that it is measurable on C.

We defined Case 4 so that the set C is closed and thus measurable. Now define $D \subset C \times \Psi \times M(T)$ to be the set of triples (x,η,β) such that $(\eta,\beta) \in MBR(x,Y,T,U^2)$, x and η satisfy (7), and x and β satisfy W2. We showed in Case 4 that there exists such a pair (η,β) for each $x \in C$. The triples in D will satisfy WBR, S2 and S3. D is compact, so by Lemma A6 it has a measurable selection. If we replace $(\tilde{\eta}(x,\eta^*),\tilde{\beta}(x,\eta^*))$ on C with this selection, then $(\tilde{\eta},\tilde{\beta})$ will be measurable everywhere as required. This completes our proof. □□

The next theorem shows when a SE for the cheap-talk extension game Γ^* may be converted to a SE for Γ. We say a SE outcome $\hat{\lambda}$ of Γ^* has *ineffective talk* if and only if

(IT) $(t,x,\eta)\in \text{supp}[\hat{\lambda}]$ and $(t',x,\eta')\in \text{supp}[\hat{\lambda}]$ imply $\eta = \eta'$.

Theorem 3: Let $\Gamma = [(T,\rho),X,Y,U^1,U^2]$ be a continuous game with cheap-talk extension $\Gamma^* = [(T,\rho),X\times\Psi^*,Y,U^1,U^2]$. If $\hat{\lambda}\in SEO(\Gamma^*)$ and $\hat{\lambda}$ satisfies IT, then $\hat{\lambda}\in SEO(\Gamma)$. If $\hat{\lambda}\in WBRO(\Gamma^*)$ and $\hat{\lambda}$ satisfies IT, then $\hat{\lambda}\in WBRO(\Gamma)$.

Remark: Let the outcome $\hat{\lambda}$ for Γ^* be supported by the SE $(\hat{\nu},\hat{\gamma},\hat{\zeta})$. Then $\hat{\lambda}$ as an equilibrium outcome for Γ is supported by $(\hat{\mu},\hat{\beta},\hat{\eta})$ where $\hat{\mu} = \hat{\lambda}_{T\times X}$, $\hat{\eta}$ on supp$[\hat{\mu}_X]$ has graph supp$[\hat{\lambda}_{X\times\Psi}]$, and $(\hat{\eta},\hat{\beta})$ on $X\sim$supp$[\hat{\mu}_X]$ is any measurable selection from $(\hat{\zeta},\hat{\gamma})$.

Proof: We begin with a lemma that shows how to construct player 2's strategy on the equilibrium path.

Lemma 5: Let $\hat{\lambda}\in M(T\times X\times\Psi)$ and let $\hat{\lambda}$ satisfy IT. Then there exists a function $\eta^1{:}X \to \Psi$ such that

(1) $(t,x,\eta)\in \text{supp}[\hat{\lambda}]$ if and only if $\eta = \eta^1(x)$ and $(t,x)\in \text{supp}[\hat{\lambda}_{T\times X}]$;

(2) $\hat{\lambda} = \hat{\lambda}_{T\times X}\circ f^{-1}$ where $f(t,x) = (t,x,\eta^1(x))$;

(3) η^1 is continuous on supp$[\hat{\lambda}_X]$.

Proof: Since supp$[\hat{\lambda}_{T\times X}] = \text{Proj}_{T\times X}$ supp$[\hat{\lambda}]$, $(t,x)\in \text{supp}[\hat{\lambda}_{T\times X}]$ if and only if there exists η such that $(t,x,\eta)\in \text{supp}[\hat{\lambda}]$. By IT there is only one such η for each x with $(t,x)\in \text{supp}[\hat{\lambda}_{T\times X}]$. We set $\eta^1(x)$ equal to this η. This proves part (1) of the lemma.

Turn to part (2). We must show that for any event $E\subset T\times X\times\Psi$, $\hat{\lambda}(E) = \hat{\lambda}_{T\times X}(f^{-1}(E))$. Let R = supp$[\hat{\lambda}]$ and S = supp$[\hat{\lambda}_{T\times X}]$. We can write part (1) as

$(t,x,\eta)\in R$ if and only if $f(t,x) = (t,x,\eta)$ and $(t,x)\in S$.

This in turn implies

$$E \cap R = (f^{-1}(E) \cap S) \times \Psi \cap R.$$

Then

$$\hat{\lambda}(E) = \hat{\lambda}(E \cap R) = \hat{\lambda}((f^{-1}(E) \cap S) \times \Psi \cap R) = \hat{\lambda}((f^{-1}(E) \cap S) \times \Psi)$$
$$= \hat{\lambda}_{T \times X}(f^{-1}(E) \cap S) = \hat{\lambda}_{T \times X}(f^{-1}(E))$$

as we were to show.

For part (3), observe that part (1) implies that $x \in \text{supp}[\hat{\lambda}_X]$ if and only if $(x, \eta^1(x)) \in \text{supp}[\hat{\lambda}_{X \times \Psi}]$. Thus the graph of η^1 on $\text{supp}[\hat{\lambda}_X]$ is the closed set $\text{supp}[\hat{\lambda}_{X \times \Psi}]$, so η^1 is a continuous function on $\text{supp}[\hat{\lambda}_X]$. This proves part (3). Define η^1 on $X \sim \text{supp}[\hat{\lambda}_X]$ arbitrarily. This completes the proof of the lemma. □□

Proceeding with the proof of the theorem, let $(\hat{\nu}, \hat{\gamma}, \hat{\zeta})$ be a SE supporting $\hat{\lambda} \in SEO(\Gamma^*)$. We construct a SE $(\hat{\mu}, \hat{\beta}, \hat{\eta})$ supporting $\hat{\lambda} \in SEO(\Gamma)$ in three steps. First, we define a strategy $\hat{\mu}$ for player 1 that satisfies S1. Second, we define a strategy $\hat{\eta}$ for player 2 satisfying S1 and show that $\hat{\mu}$ and $\hat{\eta}$ satisfy S2 and result in outcome $\hat{\lambda}$. Finally, we show that there exist beliefs $\hat{\beta}$ such that $(\hat{\mu}, \hat{\beta}, \hat{\eta})$ satisfies S3 and S4.

We define $\hat{\mu} = \hat{\lambda}_{T \times X}$, so $\hat{\mu} \in \Sigma^1(\Gamma)$. To define $\hat{\eta}$, first let $\eta^1 : X \to \Psi$ be the function derived in Lemma 5; η^1 is continuous on $\text{supp}[\hat{\mu}_X]$. Now choose an arbitrary $\xi^* \in \Psi^*$ and define $\eta^0 : X \to \Psi$ by setting $\eta^0(x) = \hat{\zeta}(x, \xi^*)$. Then η^0 is measurable. Let the strategy $\hat{\eta}$ equal η^1 on $\text{supp}[\hat{\mu}_X]$ and η^0 on $X \sim \text{supp}[\hat{\mu}_X]$. We have $\hat{\eta} \in \Sigma^2(\Gamma)$ since it is measurable by construction. By Lemma 5, playing the strategies $\hat{\mu}$ and $\hat{\eta}$ results in outcome $\hat{\lambda}$ as required.

To show $\hat{\mu}$ and $\hat{\eta}$ satisfy S2, let $(t, x) \in \text{supp}[\hat{\mu}]$ and let $x' \in X$. By Lemma 5, $(t, x, \hat{\eta}(x)) \in \text{supp}[\hat{\lambda}]$, so by Lemma 1, $U^1(t, x, \hat{\eta}(x)) = V^1(t)$, the equilibrium payoff for type t in Γ^*. Hence, the equilibrium payoffs in the two games coincide. Suppose $x' \in \text{supp}[\hat{\mu}_X]$. Then $(t', x', \hat{\eta}(x')) \in \text{supp}[\hat{\lambda}]$ and $V^1(t) \geq U^1(t, x', \hat{\eta}(x'))$ by Lemma 1. Thus S2 holds in this case. Suppose now $x' \notin \text{supp}[\hat{\mu}_X]$. Then by S2 for Γ^*,

$$V^1(t) \geq U^1(t, x', \hat{\zeta}(x', \xi^*)) \equiv U^1(t, x', \hat{\eta}(x'))$$

and S2 holds for Γ in this case also.

The final step in the proof is to show that there exist beliefs $\hat{\beta}$ such that $(\hat{\mu},\hat{\beta},\hat{\eta})$ satisfies S3 and S4. According to Lemma 2, this will be true if several requirements hold. First, $\hat{\mu}$ and $\hat{\eta}$ must satisfy S1. We have shown this. Second, $\hat{\eta}$ must be continuous on $\hat{\mu}_X$. This holds by Lemma 5. Third, there must exist a measurable function $\beta^0:X \to M(T)$ such that $(\hat{\eta}(x),\beta^0(x)) \in MBR(x,Y,T,U^2)$ for all $x \in X \sim \mathrm{supp}[\hat{\mu}_X]$. Define β^0 by setting $\beta^0(x) = \hat{\gamma}(x,\xi^*)$ using the same $\xi^* \in \Psi^*$ defining η^0 above. Then β^0 is measurable. We also have $(\eta^0,\beta^0) \in MBR(x,Y,T,U^2)$ for all $x \in X \sim \mathrm{supp}[\hat{\mu}_X]$ because $(\eta^0(x),\beta^0(x)) = (\hat{\zeta}(x,\xi^*),\hat{\gamma}(x,\xi^*))$ and $\hat{\zeta}$ and $\hat{\gamma}$ are part of a SE.

The final requirement of Lemma 2 is that for any continuous function $\eta':X \to \Psi$, we must have

$$\int_{T\times X} U^2(t,x,\hat{\eta}(x))\ \hat{\mu}(dt\times dx) \geq \int_{T\times X} U^2(t,x,\eta'(x))\ \hat{\mu}(dt\times dx).$$

We will demonstrate this for an arbitrary measurable function η'. Since $(\hat{\nu},\hat{\gamma},\hat{\zeta})$ is a SE for Γ^*, S3 implies

$$\int_T U^2(t,x,\hat{\zeta}(x,\eta^*))\ \hat{\gamma}(x,\eta^*)(dt) \geq \int_T U^2(t,x,\eta'(x))\ \hat{\gamma}(x,\eta^*)(dt)$$

for all $(x,\eta^*) \in X\times\Psi^*$. Since $\hat{\gamma}$ is a conditional distribution derived from $\hat{\nu}$, both sides of this inequality are conditional expectations derived from $\hat{\nu}$ (Chow and Teicher 1988, Theorem 7.2.1). Therefore, we may integrate both sides using $\hat{\nu}_{X\times\Psi^*}$ to get

$$\int_{X\times\Psi^*}\int_T U^2(t,x,\hat{\zeta}(x,\eta^*))\ \hat{\gamma}(x,\eta^*)(dt)\hat{\nu}_{X\times\Psi^*}(dx\times d\eta^*) \geq \int_{X\times\Psi^*}\int_T U^2(t,x,\eta'(x))\ \hat{\gamma}(x,\eta^*)(dt)\hat{\nu}_{X\times\Psi^*}(dx\times d\eta^*),$$

and thus by definition of conditional expectation,

$$\int_{T\times X\times\Psi^*} U^2(t,x,\hat{\zeta}(x,\eta^*))\ \hat{\nu}(dt\times dx\times d\eta^*) \geq \int_{T\times X\times\Psi^*} U^2(t,x,\eta'(x))\ \hat{\nu}(dt\times dx\times d\eta^*).$$

Since $\hat{\lambda}$ is the outcome of playing the strategies $\hat{\nu}$ and $\hat{\zeta}$ and since $\hat{\mu} = \hat{\lambda}_{T\times X} = \hat{\nu}_{T\times X}$, it follows that

$$\int_{T\times X\times\Psi} U^2(t,x,\eta)\ \hat{\lambda}(dt\times dx\times d\eta) \geq \int_{T\times X\times\Psi} U^2(t,x,\eta'(x))\ \hat{\mu}(dt\times dx).$$

Finally, since $\hat{\lambda}$ is the outcome of playing strategies $\hat{\mu}$ and $\hat{\eta}$, we have

$$\int_{T\times X\times\Psi} U^2(t,x,\hat{\eta}(x))\ \hat{\mu}(dt\times dx) \geq \int_{T\times X\times\Psi} U^2(t,x,\eta'(x))\ \hat{\mu}(dt\times dx)$$

as required. Lemma 2 then asserts there exist beliefs $\hat{\beta}$ such that $(\hat{\mu},\hat{\beta},\hat{\eta})$ satisfies S3 and S4. Thus

$(\hat{\mu},\hat{\beta},\hat{\eta}) \in SE(\Gamma)$.

To complete the proof of the theorem, we claim that the preceding proof applies unchanged to WBR equilibria as well. To see this, note first that the equilibrium payoffs are the same in both games Γ and Γ^*. Second, the signals x off the equilibrium path are the same in both games. Third, the beliefs off the equilibrium path in Γ are just selections from the beliefs in Γ^*. Inspection of the WBR criterion shows that under these conditions, beliefs satisfying WBR in Γ^* will also satisfy WBR in Γ. Hence, we may use the preceding proof to convert a WBR equilibrium satisfying IT from Γ^* to Γ. □□

We constructed player 2's beliefs $\hat{\beta}$ for Γ indirectly using Lemma 2. It does not seem possible to define these beliefs directly using the beliefs $\hat{\gamma}$ for Γ^*.

Our next two existence results are corollaries of the preceding three theorems.

Proposition 1: If $\Gamma = [(T,\rho),X,Y,U^1,U^2]$ is continuous and X is finite, then Γ has a SE. If in addition U^1 is Lipschitz in t, then Γ has a SE satisfying the WBR criterion.

Proof: We show first that Γ has a SE. For $n = 1, 2, \ldots$, define finite games $\Gamma^n = [(T^n,\rho^n),X,Y^n,U^1,U^2]$ as follows. Let T^n be a finite subset of T with $T^n \subset T^{n+1}$ and $<T^n> \to T$. Let $\rho^n \in M(T^n)$ and let $<\rho^n> \Rightarrow \rho$. Define finite $Y^n \subset Y$ analogously to T^n.

Let $(\hat{\mu}^n,\hat{\beta}^n,\hat{\eta}^n)$ be a SE for the finite game Γ^n. Let $<\hat{\lambda}^n>$ be the sequence of outcomes of these equilibria. Choosing a subsequence if necessary for convergence, let $<\hat{\lambda}^n> \Rightarrow \hat{\lambda}$. Since X is finite, we may also require that the sequences $<\hat{\eta}^n(x)>$ and $<\hat{\beta}^n(x)>$ converge for each $x \in X$. This implies that for the set B derived from $<\hat{\mu}^n,\hat{\beta}^n,\hat{\eta}^n>$, there is only one element (x,η,β) in B corresponding to each $x \in X$.

Hypotheses H1-H3 of Theorem 1 apply to the sequence $<\hat{\lambda}^n>$. Theorem 1 thus implies $\hat{\lambda} \in SEO(\Gamma^*)$. We show that conclusion C2 of Theorem 1 implies $\hat{\lambda}$ satisfies IT. It will then follow from Theorem 3 that $\hat{\lambda} \in SEO(\Gamma)$. Let (t,x,η) and (t',x,η') both come from supp$[\hat{\lambda}]$. By conclusion C2, both (x,η,β) and (x',η',β') come from B for some β and β'. But by construction, B has only one element corresponding to each x, so we must have $\eta = \eta'$. Hence, IT holds and thus $\hat{\lambda} \in SEO(\Gamma)$.

We now show that Γ has a SE satisfying the WBR criterion if U^1 is Lipschitz in t. For $n = 1, 2, \ldots$, define finite games $\Gamma^n = [(T^n,\rho^n),X,Y^n,U^{1n},U^{2n}]$ as follows. Let T^n, ρ^n, and Y^n be defined as before. For each $i = 1, 2$, let $\varepsilon^{in}:T^n \times X \times Y^n \to R^1$ be functions such that $<\varepsilon^{in}> \to 0$ continuously and $<\varepsilon^{1n}>$ is uniformly Lipschitz in t. Define $U^{in}:T^n \times X \times Y^n \to R^1$ by setting

$$U^{in}(t,x,y) = U^i(t,x,y) + \varepsilon^{in}(t,x,y).$$

Choose each ε^{in} so that the game Γ^n has a SE $(\hat{\mu}^n,\hat{\beta}^n,\hat{\eta}^n)$ satisfying WBR. Such ε^{in} exist because WBR equilibria exist generically for finite games.

Let $<\hat{\lambda}^n>$ be the sequence of outcomes for $<\hat{\mu}^n,\hat{\beta}^n,\hat{\eta}^n>$, and let $<\hat{\lambda}^n> \Rightarrow \hat{\lambda}$ on a subsequence. As in the previous case, we also require that the sequences $<\hat{\eta}^n(x)>$ and $<\hat{\beta}^n(x)>$ converge for each $x \in X$. Hypotheses H1-H4 of Theorem 2 apply to this sequence. In particular, the way we chose ε^{in} implies $<U^{in}> \to U^i$ continuously and $<U^{1n}>$ is uniformly Lipschitz. Hence, $\hat{\lambda} \in WBRO(\Gamma^*)$. As we showed for the first part of the theorem, C2 of Theorem 1 implies IT holds as well. Thus $\hat{\lambda} \in WBRO(\Gamma)$ by Theorem 3. ☐☐

Proposition 2: Every continuous cheap-talk extension game $\Gamma^* = [(T,\rho),X \times \Psi^*,Y,U^1,U^2]$ has a SE. If U^1 is Lipschitz in t, then Γ^* has a SE satisfying the WBR criterion

Remark: This proposition follows directly from Theorems 1 and 2; however, we apply Proposition 1 for convenience.

Proof: Define for each $n = 1, 2, \ldots$, a game $\Gamma^n = [(T,\rho),X^n,Y,U^1,U^2]$ with X^n finite and $<X^n> \to X$. By Proposition 1, each game Γ^n has a SE $(\hat{\mu}^n,\hat{\beta}^n,\hat{\eta}^n)$, and when U^1 is Lipschitz in t, we may assume each SE satisfies the WBR criterion. Let $<\hat{\lambda}^n>$ be the sequence of outcomes of these equilibria with $<\hat{\lambda}^n> \Rightarrow \hat{\lambda}$. Hypotheses H1-H3 of Theorems 1 and 2 apply to this sequence, and when U^1 is Lipschitz, hypothesis H4 of Theorem 2 applies as well. By these theorems, the desired pair of SE for Γ^* exist. ☐☐

By definition, a cheap-talk equilibrium for a communication-impervious game is an equilibrium for the original game. Thus the previous proposition has an immediate

Corollary: Every communication-impervious, continuous game $\Gamma = [(T,\rho),X,Y,U^1,U^2]$ has a SE. If in addition U^1 is Lipschitz in t, then Γ has a SE satisfying the WBR criterion.

Our next result identifies a class of games that are CI and thus possess SE.

Proposition 3: A strongly monotonic, continuous game $\Gamma = [(T,\rho),X,Y,U^1,U^2]$ is communication-impervious.

Proof: Let Γ^* be the cheap-talk extension of Γ. Let $\hat{\lambda} \in SEO(\Gamma^*)$ (or $WBRO(\Gamma^*)$). We show that $\hat{\lambda}$ satisfies IT and thus by Theorem 3, $\hat{\lambda} \in SEO(\Gamma)$ (or $WBRO(\Gamma)$).

Let $(t,x,\eta) \in \mathrm{supp}[\hat{\lambda}]$ and $(t',x,\eta') \in \mathrm{supp}[\hat{\lambda}]$. By Lemma 1, $U^1(t,x,\eta) \geq U^1(t,x,\eta')$ and $U^1(t',x,\eta') \geq U^1(t',x,\eta)$. By strong monotonicity, $\eta = \eta'$ and thus $\hat{\lambda}$ satisfies IT as required. □□

Our last theorem shows that with a rich signal space, all cheap-talk SE outcomes can be approximated arbitrarily closely by sequential ε-equilibrium outcomes. Recall that a signal space X is rich if and only if for all compact metric spaces Z and for all closed balls $B \subset X$, there exists a closed set $A \subset B$ and a continuous mapping from A onto Z.

Theorem 4: Let $\Gamma = [(T,\rho),X,Y,U^1,U^2]$ be a continuous game with a rich signal space X. Let $\Gamma^* = [(T,\rho),X\times\Psi^*,Y,U^1,U^2]$ be the cheap-talk extension of Γ. Then given $\hat{\lambda} \in SEO(\Gamma^*)$ and $\varepsilon > 0$, there exists $\tilde{\lambda}$ that is an outcome of a sequential ε-equilibrium for Γ with Prohorov distance $p(\hat{\lambda},\tilde{\lambda}) < \varepsilon$.

Proof: We begin with a lemma giving us a function we will use to convert signals in $X\times\Psi^*$ for the game Γ^* to signals in X for the game Γ. Let d be the metric on $T\times X\times\Psi$ used to define the Prohorov distance p.

Lemma 6: If X is rich, then there exists a measurable set $\tilde{X} \subset X$ with closure $\hat{X}$ and a continuous function $f:\hat{X} \to X \times \Psi$ that is a bijection on $\tilde{X}$. Fixing $\varepsilon > 0$ and writing $f(x) = (\tilde{x}(x), \tilde{\eta}(x))$, we may choose f and $\tilde{X}$ so that for all $(t,x,\eta) \in T \times \hat{X} \times \Psi$,

(1) $d((t,x,\eta),(t,\tilde{x}(x),\eta)) < \varepsilon$, and

(2) $| U^i(t,x,\eta) - U^i(t,\tilde{x}(x),\eta) | < \varepsilon/2$ for $i = 1, 2$.

Proof: Choose δ with $0 < \delta < \varepsilon$ so that for all x and x' in X, t in T, η in Ψ, and $i = 1$ or 2,

$$d((t,x,\eta),(t,x',\eta)) < \delta \text{ implies } | U^i(t,x,\eta) - U^i(t,x',\eta)| < \varepsilon/2.$$

Such a δ exists because the functions U^i are uniformly continuous on the compact set $T \times X \times \Psi$. Define a metric d_X on X by setting

$$d_X(x,x') = \max_{t \in T, \eta \in \Psi} d((t,x,\eta),(t,x',\eta)).$$

Using the d_X metric on X, let $\{A_j\}_{j=1}^{N}$ be a covering of X by N closed balls of positive diameter less than δ. Let $B_1 = A_1$, and for $j = 2, \ldots, N$, let

$$B_j = A_j \sim \bigcup_{i=1}^{j-1} A_i .$$

Then $X = \bigcup B_j$ and each B_j is a measurable set disjoint from the others. Let M be the number of non-empty B_j and reindex these sets so that B_j is non-empty if $j \le M$. For each $j = 1, \ldots, M$, let C_j be a closed ball of positive diameter less than $\delta/2$ with $C_j \subset B_j$. Such C_j exist because each B_j is the intersection of a closed ball and an open set. Since X is rich, there exists a closed set $D_j \subset C_j$ and a continuous surjection $f_j : D_j \to \bar{B}_j \times \Psi$. By Theorem I.4.2 of Parthasarathy (1967), there exists a measurable set $E_j \subset D_j$ such that f_j restricted to E_j is a bijection from E_j onto $B_j \times \Psi$.

We can now define the sets $\hat{X}$ and $\tilde{X}$ and the function f given in the statement of the lemma. We let $\hat{X} = \bigcup \bar{E}_j$ and define $f:\hat{X} \to X \times \Psi$ by setting $f = f_j$ on $\bar{E}_j$. The function f is continuous since each f_j is continuous on the set $\bar{E}_j$ and the collection of these sets is disjoint by construction. Let $\tilde{X} = \bigcup E_j$. By a similar argument, f is a bijection on $\tilde{X}$.

Write $f(x) = (\tilde{x}(x), \tilde{\eta}(x))$. To complete the proof of the lemma, observe that $x \in \hat{X}$ implies $d_X(x, \tilde{x}(x)) < \varepsilon$ because by construction both x and $\tilde{x}(x)$ are contained in some ball A_j with diameter less than δ, which is less than ε. Then by definition of d_X, the function $\tilde{x}$ satisfies part (1) of the lemma. Part (1) and the definition of δ imply part (2). □□

Proceeding with the proof of the theorem, let $(\hat{\nu}, \hat{\gamma}, \hat{\zeta})$ be a SE for Γ^* that supports $\hat{\lambda}$. We first derive another, simpler SE for Γ^* with the same outcome $\hat{\lambda}$. The new SE will be denoted by $(\hat{\lambda}, \tilde{\gamma}, \tilde{\zeta})$. As this notation indicates, the strategy for player 1 in this new SE is just $\hat{\lambda}$. To define the strategy $\tilde{\zeta}$ for player 2, first choose an arbitrary $\xi^* \in \Psi^*$ and then set

$$\tilde{\zeta}(x, \eta^*) = \begin{cases} \eta^* & \text{if } (x, \eta^*) \in \operatorname{supp}[\hat{\lambda}_{X \times \Psi}], \\ \hat{\zeta}(x, \xi^*) & \text{otherwise.} \end{cases}$$

The pair $(\hat{\lambda}, \tilde{\zeta})$ will satisfy S1 and S2 (by Lemma 1) and the outcome of playing $(\hat{\lambda}, \tilde{\zeta})$ will be $\hat{\lambda}$. To define the beliefs $\tilde{\gamma}$ of player 2, apply Lemma 2 as in Theorem 1. Letting $\gamma^0(x) = \hat{\gamma}(x, \xi^*)$, $(\hat{\lambda}, \gamma^0, \tilde{\zeta})$ will satisfy the requirements of Lemma 2. Hence, there will exist beliefs $\tilde{\gamma}$ such that $(\hat{\lambda}, \tilde{\gamma}, \tilde{\zeta})$ will satisfy S3 and S4 and thus will be a SE for Γ^*.

To define an ε-equilibrium using $(\hat{\lambda}, \tilde{\gamma}, \tilde{\zeta})$, we will use the function f defined in Lemma 6. In the ε-equilibrium, player 1 is going to signal $f^{-1}(x, \eta^*)$ in place of signaling (x, η^*) in Γ^* and $\text{Proj}_X f^{-1}(x, \eta^*)$ will be close to x. Define $g: T \times \hat{X} \to T \times X \times \Psi$ by setting

$$g(t, x) = (t, f(x)) = (t, \tilde{x}(x), \tilde{\eta}(x)).$$

Then g is a continuous function and a bijection on $T \times \tilde{X}$. By the Kuratowski Theorem (Parthasarathy 1967, Corollary I.3.3), g has a measurable inverse on $T \times \tilde{X}$.

We now define an ε-equilibrium $(\hat{\mu}, \hat{\beta}, \hat{\eta})$ for Γ in four steps. First, we define a strategy $\hat{\mu}$ satisfying S1. Second, we define a strategy $\hat{\eta}$ also satisfying S1 and show that $\hat{\mu}$ and $\hat{\eta}$ result in an outcome $\tilde{\lambda}$ within ε of $\hat{\lambda}$. Third, we show that $\hat{\mu}$ and $\hat{\eta}$ satisfy S2′. Fourth, we define player 2's beliefs $\hat{\beta}$ and show that $(\hat{\mu}, \hat{\beta}, \hat{\eta})$ satisfies S3′ and S4.

Define player 1's strategy $\hat{\mu}$ by setting

$$\hat{\mu}(E) = \hat{\lambda}(g(E \cap T \times \tilde{X}))$$

for each event $E \subset T \times X$. Since g has a measurable inverse and $T \times \tilde{X}$ is measurable, $g(E \cap T \times \tilde{X})$ is measurable for all events E, so $\hat{\mu}$ is well-defined as a measure. Since $\hat{\mu}(T \times X) = \hat{\lambda}(T \times X \times \Psi) = 1$ and $\hat{\mu}_T = \hat{\lambda}_T = \rho$, we have $\hat{\mu} \in \Sigma^1(\Gamma)$, satisfying S1.

To define player 2's strategy $\hat{\eta}$, first choose an arbitrary $\xi^* \in \Psi^*$ and then set

$$\hat{\eta}(x) = \begin{cases} \tilde{\zeta}(f(x)) & \text{if } x \in \hat{X}, \\ \tilde{\zeta}(x,\xi^*) & \text{otherwise.} \end{cases}$$

Since $\tilde{\zeta}$ is measurable and f is continuous on the closed set $\hat{X}$, $\hat{\eta}$ is measurable and thus satisfies S1.

We show that playing $\hat{\mu}$ and $\hat{\eta}$ results in an outcome $\tilde{\lambda}$ with Prohorov distance $p(\tilde{\lambda},\hat{\lambda}) < \varepsilon$. Let $h(t,x) = (t,x,\hat{\eta}(x))$, so that $\tilde{\lambda} = \hat{\mu} \circ h^{-1}$. Given an event $F \subset T \times X \times \Psi$, define the set

$$\begin{aligned} G(F) &= g(h^{-1}(F) \cap T \times \tilde{X})) \cap \mathrm{supp}[\hat{\lambda}] \\ &= \{\, g(t,x) \mid x \in \tilde{X},\ h(t,x) \in F,\ g(t,x) \in \mathrm{supp}[\hat{\lambda}] \,\}. \end{aligned}$$

Then by definition

$$\tilde{\lambda}(F) = \hat{\mu}(h^{-1}(F)) = \hat{\lambda}(G(F)).$$

Let F_ε be the set of elements of $T \times X \times \Psi$ within distance ε of F. To show $p(\tilde{\lambda},\hat{\lambda}) < \varepsilon$, it suffices to show that $\tilde{\lambda}(F) \le \hat{\lambda}(F_\varepsilon)$ and $\hat{\lambda}(F) \le \tilde{\lambda}(F_\varepsilon)$. And for this it suffices to show that $G(F) \subset F_\varepsilon$ and $F \cap \mathrm{supp}[\hat{\lambda}] \subset G(F_\varepsilon)$.

To show $G(F) \subset F_\varepsilon$, let $g(t,x) \in G(F)$. We show that $g(t,x) \in F_\varepsilon$. Now by definition of $G(F)$, $g(t,x) \in \mathrm{supp}[\hat{\lambda}]$ and thus $f(x) = (\tilde{x}(x),\tilde{\eta}(x)) \in \mathrm{supp}[\hat{\lambda}_{X \times \Psi}]$. Then by definition of $\hat{\eta}$ and $\tilde{\zeta}$, $\hat{\eta}(x) = \tilde{\eta}(x)$. Therefore

$$d(h(t,x),g(t,x)) = d((t,x,\tilde{\eta}(x)),(t,\tilde{x}(x),\tilde{\eta}(x))) < \varepsilon$$

by Lemma 6. Since $h(t,x) \in F$ by definition of $G(F)$, we have $g(t,x) \in F_\varepsilon$ as we were to show.

To show $F \cap \text{supp}[\hat{\lambda}] \subset G(F_\varepsilon)$, let $g(t,x)$ be an arbitrary element of $F \cap \text{supp}[\hat{\lambda}]$ (since g is surjective, we can represent any element in F this way). Then as above $d(h(t,x),g(t,x)) < \varepsilon$, so $h(t,x) \in F_\varepsilon$, and thus $g(t,x) \in G(F_\varepsilon)$ as required. This shows that $p(\tilde{\lambda},\hat{\lambda}) < \varepsilon$.

We now show that $\hat{\mu}$ and $\hat{\eta}$ satisfy S2′. We must first show that $\text{supp}[\hat{\mu}] \subset g^{-1}(\text{supp}[\hat{\lambda}])$. To see this, note that because g is surjective on $T \times \tilde{X}$,

$$g(g^{-1}(\text{supp}[\hat{\lambda}]) \cap T \times \tilde{X}) = \text{supp}[\hat{\lambda}] \cap g(T \times \tilde{X}) = \text{supp}[\hat{\lambda}]$$

(Dugundji 1966, p. 12). Therefore by definition of $\hat{\mu}$,

$$\hat{\mu}(g^{-1}(\text{supp}[\hat{\lambda}])) = \hat{\lambda}(g(g^{-1}(\text{supp}[\hat{\lambda}]) \cap T \times \tilde{X})) = \hat{\lambda}(\text{supp}[\hat{\lambda}]) = 1.$$

Thus $g^{-1}(\text{supp}[\hat{\lambda}])$ is a set of full measure $[\hat{\mu}]$ and is a closed set by the continuity of g. It follows that $\text{supp}[\hat{\mu}] \subset g^{-1}(\text{supp}[\hat{\lambda}])$ since $\text{supp}[\hat{\mu}]$ is the smallest closed set of full measure $[\hat{\mu}]$.

To show S2′, let $(t,x) \in \text{supp}[\hat{\mu}]$ and choose $x' \in X$ arbitrarily. Since $\text{supp}[\hat{\mu}] \subset g^{-1}(\text{supp}[\hat{\lambda}])$, we have $x \in \hat{X}$ and

$$g(t,x) = (t,f(x)) = (t,\tilde{x}(x),\tilde{\eta}(x)) \in \text{supp}[\hat{\lambda}].$$

Suppose $x' \in \hat{X}$. By S2 for Γ^*,

$$U^1(t,\tilde{x}(x),\tilde{\zeta}(f(x))) \ge U^1(t,\tilde{x}(x'),\tilde{\zeta}(f(x'))).$$

Using part (2) of Lemma 6 and the definition of $\hat{\eta}$,

$$U^1(t,x,\hat{\eta}(x)) \ge U^1(t,x',\hat{\eta}(x')) - \varepsilon,$$

which establishes S2′ for this case. Now suppose $x' \notin \hat{X}$. By S2 for Γ^*,

$$U^1(t,\tilde{x}(x),\tilde{\zeta}(f(x))) \ge U^1(t,x',\tilde{\zeta}(x',\xi^*)).$$

Again using Lemma 6 and the definition of $\hat{\eta}$,

$$U^1(t,x,\hat{\eta}(x)) \ge U^1(t,x',\hat{\eta}(x')) - \varepsilon/2,$$

which establishes S2′ for this case also.

The final step in the proof is to show that there exist beliefs $\hat{\beta}$ for player 2 such that $(\hat{\mu},\hat{\beta},\hat{\eta})$ satisfies S3′ and S4. We define $\hat{\beta}$ in parallel with the definition of $\hat{\eta}$ by setting

$$\hat{\beta}(x) = \begin{cases} \tilde{\gamma}(f(x)) & \text{if } x \in \hat{X}, \\ \tilde{\gamma}(x,\xi^*) & \text{otherwise.} \end{cases}$$

Similarly to S2′, that $\hat{\eta}$ and $\hat{\beta}$ will satisfy S3′ follows from S3 for Γ^* and from Lemma 6 and the definitions of $\hat{\eta}$ and $\hat{\beta}$.

We now show $\hat{\beta}$ is a regular conditional distribution derived from $\hat{\mu}$. First, $\hat{\beta}$ is measurable. Second, we must show that for all events $A \subset T$ and $B \subset X$,

$$\int_B \hat{\beta}(x)(A)\, \hat{\mu}_X(dx) = \hat{\mu}(A \times B).$$

This will follow from the change-of-variable formula if we define $\tilde{f}$ as the restriction of f to $\tilde{X}$ and show that $\hat{\mu}_X \circ \tilde{f}^{-1} = \hat{\lambda}_{X \times \Psi}$.

By definition of $\hat{\mu}$ and g, for any event $B \subset X$ we have

$$\begin{aligned} \hat{\mu}_X(B) &= \hat{\mu}(T \times B) = \hat{\lambda}(g(T \times B \cap T \times \tilde{X})) \\ &= \hat{\lambda}(g(T \times B \cap \tilde{X})) = \hat{\lambda}(T \times \tilde{f}(B \cap \tilde{X})) \\ &= \hat{\lambda}_{X \times \Psi}(\tilde{f}(B \cap \tilde{X})). \end{aligned} \tag{8}$$

Let $D \subset X \times \Psi$ be an event and let $B = \tilde{f}^{-1}(D)$. Then since $\tilde{f}^{-1}(D) \subset \tilde{X}$ and $\tilde{f}$ is surjective, we have

$$\hat{\mu}_X(\tilde{f}^{-1}(D)) = \hat{\lambda}_{X \times \Psi}(\tilde{f}(\tilde{f}^{-1}(D) \cap \tilde{X})) = \hat{\lambda}_{X \times \Psi}(\tilde{f}(\tilde{f}^{-1}(D)) = \hat{\lambda}_{X \times \Psi}(D)$$

and thus $\hat{\mu}_X \circ \tilde{f}^{-1} = \hat{\lambda}_{X \times \Psi}$.

Let $A \subset T$ and $B \subset \tilde{X}$ be events. Since $\tilde{f}$ is also injective, we have $B = \tilde{f}^{-1}(\tilde{f}(B))$ and thus by the change-of-variable formula that

$$\begin{aligned} \int_B \tilde{\gamma}(\tilde{f}(x))(A)\, \hat{\mu}_X(dx) &= \int_{\tilde{f}(B)} \tilde{\gamma}(x,\eta)(A)\, \hat{\lambda}_{X \times \Psi}(dx \times d\eta) \\ &= \hat{\lambda}(A \times \tilde{f}(B)) = \hat{\lambda}(g(A \times B)) = \hat{\mu}(A \times B) \end{aligned}$$

by definition of $\tilde{\gamma}$ and $\hat{\mu}$. Now let $B \subset X$ be an arbitrary event and let $\tilde{B} = B \cap \tilde{X}$. Since $\hat{\mu}_X(B) = \hat{\mu}_X(\tilde{B})$ by (8),

$$\int_B \hat{\beta}(x)(A)\, \hat{\mu}_X(dx) = \int_{\tilde{B}} \tilde{\gamma}(\tilde{f}(x))(A)\, \hat{\mu}_X(dx) = \hat{\mu}(A\times\tilde{B}) = \hat{\mu}(A\times B),$$

and thus $\hat{\beta}$ is a conditional distribution derived from $\hat{\mu}$.

Lastly, we must show that $\hat{\beta}$ is regular on a set of full measure $[\hat{\mu}_X]$. Since $\tilde{\gamma}$ is a regular conditional distribution derived from $\hat{\lambda}$, there exists an event $D \subset X\times\Psi$ such that $\tilde{\gamma}$ is regular on D and $\hat{\lambda}_{X\times\Psi}(D) = 1$. Since $\hat{\beta}(x) = \hat{\gamma}(f(x))$ on $\tilde{f}^{-1}(D) \subset \tilde{X}$ and $f(\tilde{f}^{-1}(D)) = D$, $\hat{\beta}$ is regular on $\tilde{f}^{-1}(D)$. Since $\hat{\mu}_X \circ \tilde{f}^{-1} = \hat{\lambda}_{X\times\Psi}$, we have

$$\hat{\mu}_X(\tilde{f}^{-1}(D)) = \hat{\lambda}_{X\times\Psi}(D) = 1.$$

Hence $\hat{\beta}$ is regular on a set of full measure $[\hat{\mu}_X]$ as required. Thus $\hat{\beta}$ and $\hat{\mu}$ satisfy S4, which completes the proof that $(\hat{\mu},\hat{\beta},\hat{\eta})$ is a sequential ε-equilibrium. □□

V. Conclusion

This paper has shown the existence of sequential and weak-best-response equilibria for cheap-talk extensions of continuous signaling games and for communication-impervious continuous signaling games. The assumption that talk is cheap seems reasonable in many economic contexts and yields a very straightforward solution to the existence problem in infinite signaling games. The cheap-talk assumption opens the possibility of extending the methods of this paper to prove the existence of equilibrium in more-general extensive-form games with infinite action and information sets.

Appendix: Lemmas

Lemma A1: If $<\mu^n> \Rightarrow \mu$ and $x \in \text{supp}[\mu]$, then there exists a sequence $<x^n> \to x$ with $x^n \in \text{supp}[\mu^n]$ for each n.

Proof: Let B_ε be an open ball containing x and let B_ε^C be its complement. Then $\mu(B_\varepsilon^C) < 1$ since $\text{supp}[\mu]$ is the smallest closed set of full measure $[\mu]$. By Theorem 2.1 in Billingsley (1968),

$$\limsup \mu^n(B_\varepsilon^C) \le \mu(B_\varepsilon^C) < 1,$$

so $\text{supp}[\mu^n] \cap B_\varepsilon \neq \varnothing$ for all n sufficiently large. Therefore we can construct a sequence $<x^n> \to x$ by taking a sequence of B_ε containing x with $\varepsilon \to 0$. □□

Lemma A2: Let $\mu \in M(X)$, let $f: X \to Y$ be measurable, let $\lambda = \mu \circ f^{-1}$, and let $A = f(\text{supp}[\mu])$. Then $\text{supp}[\lambda] \subset \bar{A}$.

Proof: $\text{supp}[\mu] \subset f^{-1}(f(\text{supp}[\mu])) = f^{-1}(A)$, so

$$\lambda(\bar{A}) = \mu(f^{-1}(\bar{A})) \geq \mu(f^{-1}(A)) \geq \mu(\text{supp}[\mu]) = 1.$$

Hence, $\text{supp}[\lambda] \subset \bar{A}$ since $\bar{A}$ is closed and $\text{supp}[\lambda]$ is the smallest closed set of full measure $[\lambda]$. □□

Lemma A3: For each $n = 1, 2, \ldots$, let $\mu^n \in M(X)$ and $f^n: X \to Y$ be measurable. Suppose $\lambda^n = \mu^n \circ (f^n)^{-1}$ and $<\lambda^n> \Rightarrow \lambda$. Then $y \in \text{supp}[\lambda]$ implies there exists a sequence $<x^n>$ such that $x^n \in \text{supp}[\mu^n]$ for all n and $<f^n(x^n)> \to y$.

Proof: By Lemma A1, there exists $<y^n> \to y$ with $y^n \in \text{supp}[\lambda^n]$. By Lemma A2, for each n there exists an $x^n \in \text{supp}[\mu^n]$ such that $d_Y(f^n(x^n), y^n) < d_Y(y^n, y)$. Then for each n $d_Y(f^n(x^n), y) < 2d_Y(y^n, y)$, so $<f^n(x^n)> \to y$. □□

Lemma A4: Let $U: X \times Y \to R^1$ be a measurable function. For $n = 1, 2, \ldots$, let $\eta^n \in M(Y)$, let $x^n \in X$ and let $U^n: X \times Y \to R^1$ be measurable. Assume $<\eta^n> \Rightarrow \eta$, $<x^n> \to x$ and $<U^n> \to U$ continuously for $(x,y) \in X \times Y$. Then $<\int_Y U^n(x^n,y)\ \eta^n(dy)> \to \int_Y U(x,y)\ \eta(dy)$.

Proof: Let $V^n(y) = U^n(x^n,y)$ and $V(y) = U(x,y)$. The continuous convergence of $<U^n>$ to U implies the continuous convergence of $<V^n>$ to V. This and $<\eta^n> \Rightarrow \eta$ implies

$$<\int_Y V^n(y)\ \eta^n(dy)> \to \int_Y V(y)\ \eta(dy)$$

by Theorem 5.5 of Billingsley (1968). □□

Lemma A5: The MBR correspondence is upper hemi-continuous in the following sense. For $n = 1, 2, \ldots$, let $(\eta^n,\beta^n) \in MBR(x^n,Y^n,T^n,U^n)$. Let $<x^n> \to x$, $<Y^n> \to Y$, $<T^n> \to T$, and $<U^n> \to U$ continuously for $(t,x,y) \in T \times X \times Y$. Then $<\eta^n,\beta^n> \Rightarrow (\hat{\eta},\hat{\beta})$ implies that $(\hat{\eta},\hat{\beta}) \in MBR(x,Y,T,U)$.

Proof: First $<Y^n> \to Y$ and $<T^n> \to T$ imply $<M(Y^n) \times M(T^n)> \to M(Y) \times M(T)$. This with $(\eta^n,\beta^n) \in M(Y^n) \times M(T^n)$ and $<\eta^n,\beta^n> \Rightarrow (\hat{\eta},\hat{\beta})$ imply $(\hat{\eta},\hat{\beta}) \in M(Y) \times M(T)$ by the definition of closed convergence.

The proof of upper hemi-continuity is the standard one of the Theorem of the Maximum (Hildenbrand 1974, Theorem B.III.3). It uses the continuity established in Lemma A4 of the integral $\int U^n(t,x^n,\eta)\ \beta^n(dt)$ as a function of n and η. □□

Lemma A6: Let $B \subset X \times Y$ and let $\hat{X} = \text{Proj}_X B$. If B is compact, then there exists a measurable function $\beta : \hat{X} \to Y$ such that $(x, \beta(x)) \in B$ for all $x \in \hat{X}$.

Proof: The set B is the graph of a correspondence $\hat{B} : \hat{X} \to Y$ defined by $\hat{B}(x) = \{\ y \in Y \mid (x,y) \in B\ \}$. Since B is compact, $\hat{B}$ is a closed correspondence with a compact range and thus is upper-hemicontinuous. Therefore by Proposition B.III.1 in Hildenbrand (1974), for each closed set $F \subset Y$ the set $\{\ x \mid \hat{B}(x) \cap F \neq \varnothing\ \}$ is closed, hence measurable. It then follows from Hildenbrand's Lemma D.II.2.1 that $\hat{B}$ has a measurable selection, i.e., a measurable function $\beta : \hat{X} \to Y$ such that for every x, $\beta(x) \in \hat{B}(x)$ and thus $(x, \beta(x)) \in B$. This proves the lemma. □□

References

Aumann, R.J. "Subjectivity and Correlation in Randomized Strategies." *Journal of Mathematical Economics* 1 (1974): 67-96.

Banks, J., and J. Sobel. "Equilibrium Selection in Signaling Games." *Econometrica* 55 (1987): 647-61.

Bertsekas, D., and S. Shreve. *Stochastic Optimal Control: The Discrete-Time Case.* New York: Academic Press, 1978.

Bhattacharya, S. "Imperfect Information, Dividend Policy, and the 'Bird in the Hand' Fallacy." *Bell Journal of Economics* 70 (1978): 259-70.

Billingsley, P. *Convergence of Probability Measures.* New York: John Wiley, 1968.

———. *Probability and Measure.* New York: John Wiley, 1979.

Cho, I., and D. Kreps. "Signaling Games and Stable Equilibria." *Quarterly Journal of Economics* 102 (1987):179-221.

Cho, I., and J. Sobel. "Strategic Stability and Uniqueness in Signaling Games." Unpublished, 1987.

Chow, Y.S., and H. Teicher. *Probability Theory.* 2nd ed. New York: Springer-Verlag, 1988.

Crawford, V.P., and J. Sobel. "Strategic Information Transmission." *Econometrica* 50 (1982): 1431-9.

Dasgupta, P., and E. Maskin. "The Existence of Equilibria in Discontinuous Games, 1: Theory" *Review of Economics and Statistics* 53 (1986): 1-27.

Dugundji, J. *Topology*. Boston: Allyn and Bacon, 1966.

Farrell, J., and R. Gibbons. "Cheap Talk in Bargaining Games." Unpublished, MIT, 1986.

———. "Cheap Talk with Two Audiences." *American Economic Review* 79 (1989): 1214-23.

Forges, F. "An Approach to Communication Equilibria." *Econometrica* 54 (1986): 1375-86.

Harris, C. "Existence and Characterization of Perfect Equilibrium in Games of Perfect Information." *Econometrica* 53 (1985): 613-28.

Hellwig, M., and W. Leininger. "On the Existence of Subgame-Perfect Equilibrium in Infinite-Action Games of Perfect Information." *Journal of Economic Theory* 43 (1987): 55-75.

Hildenbrand, W. *Core and Equilibria of a Large Economy*. Princeton: Princeton University Press, 1974.

Iorio, K., and A. Manelli. "Existence of Equilibrium in Infinite Signaling Games." Unpublished, Northwestern University, 1990.

Kohlberg, E., and J.F. Mertens. "On the Strategic Stability of Equilibria." *Econometrica* 54 (1986): 1003-37.

Kreps, D., and R. Wilson. "Sequential Equilibria." *Econometrica* 50 (1982): 863-94.

Leland, H., and S. Pyle. "Informational Asymmetries, Financial Structure, and Financial Intermediation." *Journal of Financial Economics* (1977): 371-87.

Matthews, S., M. Okuno-Fujiwara, and A. Postlewaite. "Communication in Bayesian Games: Issues and Problems." Unpublished, May 1989.

———. "Refining Cheap-Talk Equilibria." CMSEMS D.P. No. 892, Northwestern University, June 1990.

Milgrom, P., and J. Roberts. "Limit Pricing and Entry under Incomplete Information: An Equilibrium Analysis." *Econometrica* 50 (1982): 443-59

Milgrom, P., and R.J. Weber. "Distributional Strategies for Games with Incomplete Information." *Mathematics of Operations Research* 10 (1985): 619-32

Myers, S., and N. Majluff. "Corporate Financing and Investment Decisions when Firms Have Information that Investors Do Not Have." *Journal of Financial Economics* (1984): 187-221.

Myerson, R. "Multistage Games with Communication." *Econometrica* 54 (1986): 323-58.

Parthasarathy, K. *Probability Measures on Metric Spaces*. New York: Academic Press, 1967.

Riley, J. "Informational Equilibrium." *Econometrica* 47 (1979): 331-59.

Royden, H.L. *Real Analysis*. New York: Macmillan, 1968.

Seidmann, D. "Effective Cheap Talk with Conflicting Interests." *Journal of Economic Theory* 50 (1990): 445-58.

Simon, L., and M. Stinchcombe. "Extensive Form Games in Continuous Time: Pure Strategies." *Econometrica* 57 (1989): 1171-214.

Simon, L., and W. Zame. "Subgame Perfect and Subgame Correlated Equilibria of Hotelling and Other Infinite Multi-Stage Games." Unpublished, 1990.

Spence, A.M. *Market Signalling*. Cambridge: Harvard University Press, 1974.

Stein, J.C. "Cheap Talk and the Fed: A Theory of Imprecise Policy Announcements." *American Economic Review*. 79 (1989): 32-42.