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CONVERGENCE TO PRICE-TAKING BEHAVIOR IN A SIMPLE MARKET

by

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Abstract

An independent private values model of trade with m buyers and m sellers is considered in which price is chosen to equate revealed demand and supply. In every symmetric Bayesian Nash equilibrium, each trader does not act as a price-taker, but instead strategically misrepresents his true demand/supply to influence price in his favor. This misrepresentation causes inefficiency. It is shown that the amount by which a trader misreports is $O(1/m)$ and the corresponding inefficiency is $O(1/m^2)$. Price-taking behavior and its associated efficiency thus quickly emerges despite the asymmetric information and the noncooperative behavior of traders.

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1. INTRODUCTION

A trader who privately knows his own preferences tends to demand more favorable terms than he is in truth willing to accept. Such behavior, which is the essence of bargaining, may lead to an impasse that delays or prevents the realization of gains from trade. Price-taking behavior by contrast means reacting to prices rather than trying to manipulate them in one's favor. The two standard assumptions within price theory that justify price-taking behavior are many traders and complete information, for the existence of a large number of traders diminishes any single trader's impact on prices and complete information makes plausible the discovery of market clearing prices (e.g., by a Walrasian auctioneer). If, however, either of these assumptions were necessary for price-taking, then price theory would have little descriptive value because most markets have a limited number of traders whose preferences are private. Nevertheless price theory does provide insight into a far wider variety of situations than the strength of these assumptions would suggest. The problem is to explain how this can be.

Our contribution is to consider a finite market in which the rules for price formation and trading are explicit, the number of traders is small, and each trader privately knows his own preferences. We show that strategic noncooperative behavior converges rapidly to price-taking behavior as the number of traders increases. The inefficiency this behavior causes vanishes quickly. Numerical evidence suggests that the convergence is so fast that markets with as few as twelve traders can be almost fully efficient. Consequently even small markets can be successful in eliciting enough private information from traders about their preferences to reallocate the traded good to those who most highly value it.

Model. The market consists of m buyers, each of whom wants to buy one unit of the good at a price less than his reservation value v, and n sellers, each of whom wants to sell one unit of the good at a price greater than his reservation value c. The reservation value of every buyer is independently drawn from a commonly known distribution G on the unit interval. Sellers' reservation values are similarly drawn from a distribution F. Each trader privately observes the draw of his own value.

The class of price formation rules we study is the family of k-double auctions. A given $k \in [0,1]$ specifies a member of this family. Every trader submits a bid/offer. These bids and offers are aggregated to define demand and supply functions, both of which are step functions because each trader's demand/supply is unitary. These functions determine an interval [a,b] from which a market clearing price may be selected. The price the k-double auction picks is $kb + (1-k)a$. The market then clears at that price and disperses.

Note that the endpoints of the interval from which the price is chosen are themselves bids/offers. Price is thus a convex combination of particular bids and offers. Given the bids and offers of all other traders, a trader whose bid/offer receives positive weight in this formula therefore may perturb the price by altering his bid/offer up or down. Of course, a trader at the time he submits his bid/offer does not know the bids/offers of the other traders. Therefore in choosing his bid/offer he weighs the likelihood that it will affect price.

In submitting bids and offers traders are assumed to maximize their expected utility. Following Harsanyi's (1967-68) notion of Bayesian Nash equilibrium within a game of incomplete information, an equilibrium is a pair of functions $\langle S,B \rangle$, each defined on the unit interval. For a seller with reservation value c, the offer S(c) maximizes his expected utility given that every other seller uses the strategy S to select his offer and every buyer uses the strategy B to select his bid. Analogously, the bid B(v) maximizes the expected utility of a buyer with reservation value of v.

<u>Results</u>. In any equilibrium $\langle S,B \rangle$ each trader misreports his reservation value in order to influence price in his favor. Specifically, buyers generally under report and sellers over report, i.e., $B(v) < v$ and $S(c) > c$. Our first convergence result, stated for the simplest case in which $m = n$, is that the maximal amount by which any trader distorts his reservation value is $O(1/m)$: given F and G, a κ that is independent of m exists such that

$$v - B(v) < \kappa/m \quad \text{and} \quad S(c) - c < \kappa/m \tag{1.1}$$

for any equilibrium $\langle S,B \rangle$ in the market of size $m = n$. Numerical calculation in the case of uniform F and G indicates that for m as small as eight is close to reporting true valuations.

The meaning of price-taking behavior in this model is subtle. Price is determined endogenously by all bids/offers; no price exists when traders choose their bids/offers. Nevertheless the likelihood of a particular trader affecting price is small within any but the smallest markets. If a trader decides to ignore the small probability event that he might affect price, then the best bid/offer for him to submit is his reservation value, for that bid/offer guarantees he will trade whenever the realized price--which he takes as exogenous to his own actions--yields him gains from trade. This is exactly analogous to a trader in a competitive market who takes the market price as given and chooses his purchase to maximize his utility without taking into account the very small effect his purchase has on price. Therefore within the k-double auction price-taking behavior is honest reporting of one's reservation value. The result (1.1) thus describes the speed with which price-taking behavior emerges as the market increases in size.

Our second result describes the rate at which the market approaches efficiency as it becomes large. The cause of inefficiency in the k-double auction is the traders' strategic misrepresentation. Though the market always clears in the k-double auction, it clears on the basis of reported valuations, not true valuations. Thus, a buyer's true valuation v may exceed some seller's true valuation c with neither trading at the realized price p: $v > p > c$, $S(c) > p > B(v)$, and the gain $v - c$ is left unrealized. A missed opportunity of this kind is ex post classically inefficient in the sense of Holmstrom and Myerson's (1983) taxonomy. Stated here for the simple case of $m = n$, our second convergence result is that the fraction of the expected potential gains from trade that are unrealized is no more than ξ/m^2 where ξ is a positive number that is independent of m. Numerical investigation for the case of uniform F and G not only confirms the ξ/m^2 rate of convergence but also shows that the relative loss is inconsequential even in small markets. For example, the relative loss is much less than 1% for $m = 6$.

Antecedent Work. Myerson and Satterthwaite (1983) showed for the bilateral case (m = n = 1) that the k-double auction, or any other set of trading rules in which trade is voluntary, can not be ex post classically efficient. Gresik and Satterthwaite (1989) in the context of optimal mechanisms and Satterthwaite and Williams (1989b) and Williams (1990) for the buyer's bid double auction (BBDA) explored the rate at which this ex post inefficiency vanishes as the market grows. As with the latter two papers mentioned, the present paper improves on Gresik and Satterthwaite (1989) by (i) obtaining a faster rate of convergence and (ii) considering k-double auctions, which are more realistic than optimal mechanisms because they are not defined in terms of the distributions F and G from which reservation values are drawn.[1]

In addition this paper improves on Satterthwaite and Williams (1989b) and Williams (1990) in four ways. First, the k-double auction is a richer set of rules in that the BBDA is essentially identical to the 1-double auction. This paper's result on a rate of convergence to truthful revelation therefore generalizes those of the earlier papers, and its proof is much shorter. Second, this paper establishes a rate of convergence to ex post classical efficiency that had been numerically demonstrated but not previously proven. Third, while the earlier papers assumed risk neutral traders, here we allow traders to be risk averse.

Fourth, from both theoretical and practical perspectives the k-double auction (for k strictly in the unit interval) is more interesting than the

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[1] Wilson (1987, p. 36) critiques trading rules that are defined in terms of traders' beliefs about each other's private information. Such rules change whenever beliefs change, and hence provide little insight into the persistence of the specific market institutions we observe in reality.

BBDA because both buyers and sellers strategically misreport. In the BBDA sellers have the dominant strategy of honestly reporting their reservation values; only buyers strategically misreport their reservation values. In our observation strategic misreporting on both sides is the norm in real and experimental markets. This generalization is also important because it removes the suspicion that rapid convergence to truthful revelation requires a delicate selection of rules, e.g., that at least one side of the market have truthful reporting as their equilibrium strategy.

Several additional papers should be mentioned because they have played an important role in our thinking about the equilibria and efficiency of k-double auctions. Chatterjee and Samuelson (1983), Leininger, Linhart, and Radner (1989), and Satterthwaite and Williams (1989a) characterized equilibria of the k-double auction in the bilateral case. Wilson (1985) initiated study of the multilateral case and showed that the k-double auction is interim incentive efficient when the market is sufficiently large. McAfee (1989) defined a dominant strategy mechanism and established a convergence rate that is almost quadratic.

Limitations. Our model is restrictive in its trading environment, its informational assumptions, and in the role of time. Two aspects of the trading environment should be noted. First, the unitary supply/demand of any trader means that he becomes small as the market becomes large. If a trader could trade many units, then he might persist in trying to manipulate price despite the presence of many traders. Second, our convergence results only hold for sequences of markets in which m/n is bounded above and away from zero. We do not fully understand what happens when one side of the market becomes increasingly larger than the other side.

With respect to information, we assume (i) each trader privately knows his reservation value and (ii) these values are statistically independent. Two examples show the restrictiveness of these assumptions. First, the value a trader places on a share of stock depends on what he knows about the company that issued it. Contrary to the first assumption, he might revise his value if he learned another trader's private information. Second, in the residential real estate market, a buyer may believe that others share his tastes. Therefore, contrary to the second assumption, if he admires a particular property, then he may infer that others are also likely to admire it. Milgrom and Weber (1982, p. 1095) have analyzed auctions within a general model that relaxes both these assumptions; it would be highly desirable to do the same for double auctions.

The k-double auction is a static procedure. Among static procedures it is compelling because it is the natural implementation of trading at a price determined endogenously to equate reported supply with reported demand. Many real trading procedures, however, operate through time. Experimental work suggests that time plays a key role in facilitating trade because the prices at which trades occur communicate information about preferences that is useful to traders who intend to trade subsequently. Wilson (1986) and Wolinsky (1990) represent steps towards embedding trade in a dynamic context.

Within our model there are several limitations of our analysis. We only consider equilibria that are symmetric in the sense that all buyers use one strategy and all sellers use some other strategy. Another limitation is that we do not prove existence of equilibria. Two lines of research, however, suggest that they do exist. First, Williams (1990) proved that for a generic choice of distributions F and G a piecewise smooth equilibrium exists in the 1- and the 0-double auctions. This proof appears as if it could be generalized to the case of $k \in (0,1)$. It is a difficult proof in the $k = 0$ and $k = 1$ cases, however, and the case of $k \in (0,1)$ is much more complex. To our knowledge, the difficulty of this task is the only obstacle. Second, Satterthwaite and Williams (1989a) developed a numerical approach that has made it relatively easy to compute a range of equilibria in the k-double auction for $k \in (0,1)$. The approach and some equilibria that were computed using it are discussed in Section 5. Thus, while we surely recognize the value of a general existence theorem, the proof in Williams (1990) and our ability to actually compute a multiplicity of equilibria suggests that the value of our results does not hinge upon the proof of such a theorem.

2. THE MODEL

<u>The rules of the k-double auction</u>. As stated in the Introduction, we consider a market with $n \geq 2$ sellers, each of whom has one item to sell, and $m \geq 2$ buyers, each of whom wants to buy at most one item.[2] Trade is organized with the following rules. Each seller submits an offer while each buyer submits a bid. An offer/bid can be any real number. The bids and offers are organized in a list $s_{(1)} \leq s_{(2)} \leq \ldots \leq s_{(n+m)}$.[3] For $k \in [0,1]$, the <u>k-double auction</u> chooses $p = (1-k)s_{(m)} + ks_{(m+1)}$ as the market price.

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[2] The bilateral case ($m = n = 1$) is considered in Satterthwaite and Williams (1989a).

[3] Throughout the paper, $s_{(q)}$ denotes the qth smallest value in a specified sample of offers/bids.

Trade occurs between buyers whose bids were at least p and sellers whose offers were no more than this price.

Table 2.1 is used to explain exactly who trades at this price. Because there are m buyers,

$$w + x + z = m. \tag{2.1}$$

Also, if $s_{(m)} \neq s_{(m+1)}$, then

$$t + z = m. \tag{2.2}$$

The supply t at price $p = (1-k)s_{(m)} + ks_{(m+1)}$ therefore equals the demand w+x (i.e., p is a market-clearing price) when $s_{(m)} \neq s_{(m+1)}$. When $s_{(m)} = s_{(m+1)}$, shortages or surpluses may exist at this price. The allocation in such a case is then carried out as far as possible by assigning priority in trade according to the size of offers/bids, with sellers whose offers were smallest and buyers whose bids were largest receiving priority. If this does not complete the allocation, then a fair lottery is used to determine which of the remaining traders on the short side of the market get to trade. This completes the rules of the k-double auction.

For future reference, we now show that every seller whose offer was less than p and every buyer whose bid exceeded p surely trades. This is proven above for the case of $s_{(m)} \neq s_{(m+1)}$; we therefore assume that $s_{(m)} = s_{(m+1)}$. Note first that

$$s + t + x + z > m. \tag{2.3}$$

The inequalities (2.1) and (2.3) together imply that the s+t items available at the price $p = s_{(m)} = s_{(m+1)}$ are sufficient to supply each of the w buyers who bid more than $s_{(m+1)}$. Because they have priority in trading, these buyers surely trade. A similar argument shows that t < w+x, so every seller whose offer is less than p surely sells. When necessary, random allocation is therefore carried out over a subset of the s+x traders whose offers/bids exactly equal p.

The complexity of the allocation rule when ties occur should not obscure the fact that a k-double auction is a rule for selecting a market-clearing price from revealed supply and demand curves (as expressed by the offers and bids). We show in the next section that in a Bayesian game model ties are a probability zero event and the market almost always clears at the chosen price. The proof of this, however, requires the above, explicit definition of how ties are resolved.

The Bayesian model. We use Harsanyi's (1967-68) Bayesian model to predict the outcome of trade. A trader's preferences are determined by his reservation value and his utility function. Each seller's reservation value is independently drawn from a distribution F and each buyer's reservation value is independently drawn from a distribution G. We use v (for "value") to denote a buyer's reservation value and c (for "cost") to denote a seller's value. While each trader privately observes his own reservation value, the distributions F and G are common knowledge. Each of the distributions F and G is a C^1 function on [0,1] whose density function is positive on [0,1]. Let f denote the density of F and g the density of G.

Each trader on a given side of the market has the same utility function. The function may exhibit either risk aversion or risk neutrality. Specifically, a seller's utility when he trades is a function C(p-c) of the difference between the price that he receives and his cost. A buyer's utility when he trades is a function V(v-p) of the difference between his value and the price. A trader who fails to trade has utility equal to zero.

The functions C, V are increasing, concave, differentiable, and normalized so that $C(0) = V(0) = 0$.

A trader's <u>strategy</u> is a Lebesgue measurable function that specifies an offer/bid for each of his reservation values. A set of strategies, one for each trader, defines a <u>Bayesian-Nash equilibrium</u> if, at each reservation value of each trader, the offer/bid specified by his strategy maximizes his conditional expected utility given that the other traders are using their specified strategies.

<u>Restrictions on equilibria</u>. We only consider equilibria that are <u>symmetric</u> in the sense that all traders on the same side of the market use the same strategy. Let S be the common strategy of sellers, B the common strategy of buyers, and let $\langle S,B \rangle$ denote the use of S by each seller and the use of B by each buyer. Additionally we insist that equilibrium strategies S and B satisfy the following:

(i) $\{c \mid S(c) < 1\}$ has positive F-measure and $\{v \mid B(v) > 0\}$ has positive G-measure; (2.4)

(ii) at every $c, v \in [0,1]$, $S(c) \geq c$ and $B(v) \leq v$. (2.5)

Assumption (2.4) states that it is a positive probability event that traders on one side of the market make offers/bids at which traders on the other side can profitably trade. This rules out "no-trade" equilibria, e.g., $B(v) = 0$ and $S(c) = 1$ for all $c, v \in [0,1]$. Assumption (2.5) rules out equilibria in which traders use dominated strategies. Note that these assumptions do not restrict the strategies that are available to any trader as he attempts to maximize his conditional expected utility; rather they restrict the equilibria for which we prove results. In the remainder of the paper, "equilibrium" means a pair <S,B> that satisfies (2.4) and (2.5).

3. ELEMENTARY PROPERTIES OF EQUILIBRIUM STRATEGIES

Let λ denote an offer/bid of a trader. Given <S,B>, define the following notation:

$P_b(\lambda)$ ≡ probability that a buyer trades when he bids λ, all sellers use S, and the other m-1 buyers use B;

$P_s(\lambda)$ ≡ probability that a seller trades when λ is his offer, all buyers use B, and the other n-1 sellers use S;

$EV(v,\lambda)$ ≡ a buyer's expected utility when v is his reservation value, λ is his bid, all sellers use S, and the other m-1 buyers use B;

$EC(c,\lambda)$ ≡ a seller's expected utility when c is his reservation value, λ is his offer, all buyers use B, and the other n-1 sellers use S;

$\underline{v} \equiv \inf \{v \mid P_b(B(v)) > 0\}$;

$\bar{c} \equiv \sup \{c \mid P_s(S(c)) > 0\}$;

$\bar{b} \equiv \sup \{B(v) \mid v < 1\}$;

$\underline{s} \equiv \inf \{S(c) \mid c > 0\}$.

Note that $P_b(\lambda)$ is nondecreasing and $P_s(\lambda)$ is nonincreasing in λ. Assumption (2.4) implies that $P_b(B(v)) > 0$ near $v = 1$ and $P_s(S(c)) > 0$ near $c = 0$. The values $\underline{v}$, $\bar{c}$ are thus well-defined and satisfy $\underline{v} < 1$, $\bar{c} > 0$.

As explained below, statement (3.1) of the following theorem implies that supply almost always equals demand in an equilibrium of a k-double auction. Statement (3.2) is useful in explaining the significance of the values $\underline{v}$ and $\bar{c}$. The proofs of all theorems are in the Appendix.

Theorem 3.1. The following statements hold for an equilibrium <S,B> that satisfies (2.4) and (2.5):

(i) if $P_b(\lambda) > 0$, then $B^{-1}(\lambda) \equiv \{v \mid B(v) = \lambda\}$ has G-measure zero, (3.1)
and if $P_s(\lambda) > 0$, then $S^{-1}(\lambda) \equiv \{c \mid S(c) = \lambda\}$ has F-measure zero;

(ii) the function $P_s \cdot S$ is nonincreasing and the function $P_b \cdot B$ is (3.2)
nondecreasing on $[0,1]$.

Random allocation is necessary only if (i) $s_{(m)} = s_{(m+1)} = p$, (ii) some offers are no more than p, and (iii) some bids are as large as p. Statement (3.1) implies that this is a probability zero event. The possibility of random allocation is therefore ignored in the remainder of the paper. Statement (3.2) implies that $P_s \cdot S$ is positive on $[0,\bar{c})$ and zero on $(\bar{c},1]$, while $P_b \cdot B$ is zero on $[0,\underline{v})$ and positive on $(\underline{v},1]$. A seller's offer cannot exceed $\bar{b}$ if he is to have a positive probability of trading; it follows that $S(c) \le \bar{b} \le 1$ for $c \le \bar{c}$. A similar argument shows that $0 \le \underline{s} \le B(v)$ for $v > \underline{v}$. Because a seller whose value c is above $\bar{c}$ almost never trades, the definition of an equilibrium does not prevent him from submitting a very large number as his offer; similarly, a buyer whose value is below $\underline{v}$ may in equilibrium bid a negative number. Misrepresentation thus cannot be bounded over $[0,\underline{v}]$ and $[\bar{c},1]$. We call $[0,\bar{c})$ and $(\underline{v},1]$ the <u>intervals over which</u>

serious offers/bids are made. It is over these intervals that the definition of an equilibrium entails properties of the strategies S and B.

Suppose that <S,B> is an equilibrium that satisfies (2.4) and (2.5). Theorem 3.2 states that: (i) S is increasing and hence differentiable almost everywhere in the interval $[0,\bar{c})$, while B is increasing and differentiable almost everywhere in $(\underline{v},1]$; (ii) $\underline{v} = \underline{s}$ and $\bar{c} = \bar{b}$.[4] Figure 3.1 illustrates this geometric relationship between a pair of equilibrium strategies. The first result justifies the first order study of equilibria that we develop in the next section and leads to bounds on misrepresentation over $[0,\bar{c})$ and $(\underline{v},1]$. The second result leads to bounds on the intervals $[\bar{c},1]$, $[0,\underline{v}]$. Misrepresentation itself can not be bounded over these intervals, but the bounds upon them implies a bound on the loss in expected profit this misrepresentation causes.

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[4] This is an appropriate point to clarify a slight discrepancy in the rules of the 1-double auction and the BBDA. The BBDA is defined in Satterthwaite and Williams (1989b) as the procedure in which the price is set equal to $s_{(m+1)}$ and trade occurs between buyers who bid at least $s_{(m+1)}$ and sellers whose offers were strictly less than this value. This is different from the 1-double auction in that a seller whose offer equals $s_{(m+1)}$ never trades in the BBDA, while he may trade when $s_{(m)} = s_{(m+1)}$ in the 1-double auction. For the BBDA-Th. 2.1 in Satterthwaite and Williams (1989b) shows that each seller has $S(c) \equiv c$ as his unique dominant strategy; for the 1-double auction a similar proof is easily constructed. Th. 2.2 in Satterthwaite-and Williams (1989b) establishes that a strategy B(v) in any equilibrium <S,B> of the BBDA is increasing in v; Th. 3.2 here establishes the same for the 1-double auction. In both procedures ties among bids/offers that require random tie-breaking occur with probability zero. The difference in the rules is therefore inconsequential and all remaining results of this paper apply directly to the BBDA.

Theorem 3.2. For $c' < c''$, $v' < v''$ in $[0,1]$, the following statements are true for an equilibrium $\langle S,B \rangle$ that satisfies (2.4) and (2.5):

(i) if $c' < \bar{c}$, then $S(c') < S(c'')$; (3.3)

(ii) if $\underline{v} < v''$, then $B(v') < B(v'')$; (3.4)

(iii) $\lim_{v \downarrow \underline{v}} B(v) = \underline{v} = \underline{s}$ and $\lim_{c \uparrow \bar{c}} S(c) = \bar{c} = \bar{b}$; (3.5)

As a consequence of (3.3) and (3.4), S is differentiable almost everywhere in the interval $[0,\bar{c})$ and B is differentiable almost everywhere in $(\underline{v},1]$.

Table 3.1 defines the dual market to the market defined thus far in the paper. Its middle column summarizes our notation. The final theorem of this section establishes a symmetry between equilibria of a market and its dual.

Theorem 3.3. If $\langle S,B \rangle$ is an equilibrium satisfying (2.4) and (2.5) in the given market, then $\langle S^*,B^* \rangle$ is an equilibrium in the dual market that also satisfies (2.4) and (2.5).

This symmetry between buyers and sellers means that results proven for the case of buyers have direct analogues for sellers. Several of our proofs therefore require an argument about only one side of the market. The theorem is particularly strong when $k = 0.5$, $m = n$, and both F and G are uniform. In this case, $k^* = 1-k$, $F^* = G^* = F = G$ and equilibria come in pairs in the sense that every equilibrium sellers' strategy defines an equilibrium buyers' strategy and vice versa.

4. THE FIRST ORDER APPROACH

The first order conditions that any equilibrium <S,B> must satisfy almost everywhere serve two purposes in our exposition. First, they provide the bound (4.3) derived below that underlies the convergence results. Second, they enable us to illustrate convergence in Section 5 by calculating the set of smooth equilibria.

We begin by focusing on the first order conditions in the case of risk neutral traders at some $v \in (\underline{v},1)$ and $c \in (0,\bar{c})$ at which $B'(v)$ and $S'(c)$ both exist and $B(v) = S(c) \equiv \lambda$. Let $\dot{v} \equiv dB^{-1}(\lambda)/d\lambda = 1/B'(v)$ and $\dot{c} \equiv dS^{-1}(\lambda)/d\lambda = 1/S'(c)$. Buyer and seller first order conditions at such a (c,λ,v) triple are

$$\begin{aligned} 0 &= \frac{\partial EV(v,\lambda)}{\partial \lambda} \\ &= (v-\lambda)\left[nK_{n,m}(\lambda)f(c)\dot{c} + (m-1)L_{n,m}(\lambda)g(v)\dot{v}\right] - kM_{n,m}(\lambda) \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} 0 &= \frac{\partial EC(c,\lambda)}{\partial \lambda} \\ &= -(\lambda-c)\left[(n-1)J_{n,m}(\lambda)f(c)\dot{c} + mK_{n,m}(\lambda)g(v)\dot{v}\right] + (1-k)N_{n,m}(\lambda) \end{aligned} \tag{4.2}$$

respectively, where:

$K_{n,m}(\lambda) \equiv$ the probability that offer/bid λ lies between $s_{(m-1)}$ and $s_{(m)}$ in a sample of m-1 buyers using strategy B and n-1 sellers using S;

$L_{n,m}(\lambda) \equiv$ the probability that bid λ lies between $s_{(m-1)}$ and $s_{(m)}$ in a sample of m-2 buyers using strategy B and n sellers using S;

$M_{n,m}(\lambda) \equiv$ the probability that bid λ lies between $s_{(m)}$ and $s_{(m+1)}$ in a sample of m-1 buyers using strategy B and n sellers using S;

$J_{n,m}(\lambda) \equiv$ the probability that offer λ lies between $s_{(m-1)}$ and $s_{(m)}$ in a sample of m buyers using strategy B and n-2 sellers using S;

$N_{n,m}(\lambda) \equiv$ the probability that offer λ lies between $s_{(m-1)}$ and $s_{(m)}$ in a sample of m buyers using strategy B and n-1 sellers using S.

Formulas (A.20-A.24) for these probabilities are in the Appendix.

Without going into too much detail it is useful to develop some intuition concerning a risk neutral buyer's first order condition (4.1). The symmetry of a buyer's and a seller's situations extends this intuition to a seller's first order condition (4.2). The first order condition (4.1) can be interpreted as equating a buyer's marginal expected gain from changing his bid with the marginal expected cost. An incremental increase $\Delta\lambda$ in the buyer's bid can have two effects. First, if the bid λ is insufficient to include him among the buyers who trade, then by increasing his bid to $\lambda+\Delta\lambda$ he may surpass other bids and offers and move into the set of buyers who trade. Second, if the bid λ is sufficient to include him among those who trade, then increasing his bid by $\Delta\lambda$ may simply increase the price he pays by $k\Delta\lambda$ through the price-setting rule $(1-k)s_{(m)} + ks_{(m+1)}$.

The sum in brackets times $\Delta\lambda$ is the probability that the buyer enters the set of buyers who trade as he incrementally raises his bid by $\Delta\lambda$. The first term in the sum is the marginal probability of acquiring an item by passing a seller's offer and the second term is the marginal probability of acquiring an item by passing another buyer's bid. The profit from such a trade is between $(v-\lambda)$ and $(v-\lambda-\Delta\lambda)$. Therefore the marginal expected profit for a buyer who raises his bid is $(v-\lambda)$ times the term in brackets. On the other side of the ledger, $M_{n,m}$ is the probability that a buyer who increases his bid by $\Delta\lambda$ simply increases the price he pays by $k\Delta\lambda$. Therefore $kM_{n,m}$ is the buyer's marginal expected loss from increasing his bid above λ.

Formal derivation and lengthy discussion of this first order condition can be found in Satterthwaite and Williams (1989b), which concerns the $k = 1$ double auction (or BBDA).[5] There are two changes in that paper's formula (3.2) for the buyer's marginal expected payoff as one moves to the k-double auction from the BBDA. First, since a seller no longer has the dominant strategy of truthfully revealing his cost, the $f(c)$ term in eq. (3.2) of Satterthwaite and Williams (1989b) is replaced here by $f(c)\dot{c}$ to account for the manner in which S distorts the density of costs into the density of offers. Second, the $M_{n,m}$ term in (3.2) becomes $kM_{n,m}$ to account for the diminished effect that a change in the buyer's bid can have on price.

It is interesting that a seller's first order condition (4.2) continues to hold when $k = 1$. Inspection shows that the truthful strategy $\lambda = c$ solves this equation when k assumes this value. Moreover, it can be shown that if the probability of trade $P_s(\lambda)$ is positive, then so is the expression in brackets in (4.2) and $\lambda = c$ is the unique solution. This captures the intuition of a proof that $\tilde{S}(c) \equiv c$ is a seller's unique

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[5] A minor flaw of that derivation is its assumption that the joint distribution $P(\cdot)$ of $x = s_{(m)}$ and $y = s_{(m+1)}$ has a density $e(\cdot)$. This is easily corrected by simply replacing $e(x,y)dxdy$ with $dP(x,y)$. The argument is then correct for a value of $b = B(v)$ at which $B'(v)$ exists.

dominant strategy in the 1-double auction. Similar remarks concerning a buyer's strategy hold for the 0-double auction.

For $v \in [\underline{v},1]$ the first order condition (4.1) of a buyer may not hold when either (i) $B(v)$ is outside of the range of S or (ii) $S'(c)$ does not exist for the value of c that solves $S(c) = B(v)$. Nevertheless as long as $B'(v)$ exists the inequality

$$(v-\lambda)(m-1)L_{n,m}(\lambda)g(v)\dot{v} - kM_{n,m}(\lambda) \leq 0 \qquad (4.3)$$

holds because in equilibrium the marginal expected gain from passing a buyer (disregarding the possibility of passing a seller) as a result of raising one's bid surely can not exceed the marginal expected cost from raising the price.

We now argue that (4.3) continues to hold when a buyer's utility is $V(v-\lambda)$ rather than simply $v-\lambda$, i.e., when buyers are risk averse rather than risk neutral. The marginal expected gain to such a buyer from raising his bid is at least $V(v-\lambda)(m-1)L_{n,m}(\lambda)g(v)\dot{v}$. For $0 \leq p \leq \lambda$, assume now that the price would be p if the selected buyer bid λ. The buyer's marginal loss from raising his bid above λ in this case is $kV'(v-p)$, which is no more than $kV'(v-\lambda)$ because V is concave. The buyer's marginal expected loss from raising his bid is therefore no more than $kM_{n,m}(\lambda)V'(v-\lambda)$. At $B(v) = \lambda$, it follows that

$$V(v-\lambda)(m-1)L_{n,m}(\lambda)g(v)\dot{v} - kM_{n,m}(\lambda)V'(v-\lambda) \leq 0. \qquad (4.4)$$

Because V is concave and $V(0) = 0$,

$$V(v-\lambda) \leq (v-\lambda)\ V'(v-\lambda). \qquad (4.5)$$

The inequalities (4.4) and (4.5) imply (4.3).[6]

Theorem 3.2's result that B' exists almost everywhere in the interval $[\underline{v},1]$ therefore implies that the inequality (4.3) holds almost everywhere in that interval. This inequality forms the basis for our convergence result below.

5. CONVERGENCE TO TRUTHFUL REVELATION

Define $q(n,m)$ as the value

$$q(n,m) \equiv \max \left\{ \frac{1}{n} \left[1 + \frac{m}{n} \right], \frac{1}{m} \left[1 + \frac{n}{m} \right] \right\} .$$

The main results of this section are: (i) a trader's equilibrium misrepresentation is at most $O(q(n,m))$ on the interval over which he makes serious offers/bids; (ii) the complement of this interval has length $O(q(n,m))$.

Theorem 5.1. Suppose F and G are C^1 distributions on $[0,1]$ with positive densities over this interval and $k \in [0,1]$. Consider any equilibrium $\langle S,B \rangle$ satisfying (2.4) and (2.5) of a k-double auction in a market with m buyers and n sellers. There exists a constant $\kappa(F,G) > 0$, which is independent of $\langle S,B \rangle$, m, and n, such that

$$v - B(v) \leq \kappa q(n,m) \tag{5.1}$$

for all $v \in (\underline{v},1]$,

$$S(c) - c \leq \kappa q(n,m) \tag{5.2}$$

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[6] This is the only argument in the paper that uses the differentiability of V. Its use here could be avoided by interpreting $V'(v-\lambda)$ in (4.4) and (4.5) as the supremum of the supergradient of V at $v-\lambda$. See Ekeland and Turnbull (1983, p.110-111) for a discussion of this property of concave functions.

for all $c \in [0,\bar{c}]$, and

$$\underline{v},\ 1 - \bar{c} \le \kappa q(n,m). \tag{5.3}$$

To develop some intuition concerning the theorem and the function $q(n,m)$, consider a sequence of markets in which n/m is bounded both above and away from zero. When n/m is bounded in this way, the equality $O(q(n,m)) = O(1/n) = O(1/m)$ holds and describes the rate at which (i) misrepresentation vanishes on the intervals over which serious offers/bids are made and (ii) these intervals grow to include the entire range $[0,1]$ of possible reservation values. As a point of comparison, Williams (1990) showed that misrepresentation by a buyer in the BBDA is $O(1/\min(m,n))$, which equals both $O(1/m)$ and $O(1/n)$ when n/m is bounded. Theorem 5.1 thus extends this earlier result to k-double auctions for sequences in which n/m is bounded both above and away from zero. Note, however, that $q(n,m)$ becomes infinite and Theorem 5.1's bound becomes ineffective as n/m approaches either zero or infinity. Williams' result (1990) holds for all n and m, and in this sense it is a stronger result.

<u>Computation of the set of smooth equilibria</u>. We illustrate this convergence result by computing the set of smooth equilibria for each of several different sizes of markets. We use the method Satterthwaite and Williams (1989a) devised for computing smooth equilibria of bilateral k-double auctions. The method is only outlined here; see this earlier paper for a complete discussion. The first order conditions (4.1-4.2) form a linear system in $\dot{c}$ and $\dot{v}$ whose coefficients are functions of triples (c,λ,v) satisfying $0 \le c \le \lambda \le v \le 1$. These inequalities define the tetrahedron ABCD in Figure 5.1. This linear system is nondegenerate at all points in

the tetrahedron except at its vertices and on its edges. It determines a unique value for $(\dot{c},\dot{v})$ at each point (c,λ,v) of the tetrahedron at which it is nondegenerate. Adding the tautology $\dot{\lambda} \equiv d\lambda/d\lambda \equiv 1$ defines a vector $(\dot{c},\dot{\lambda},\dot{v})$ at every such point in the tetrahedron. A solution curve to this vector field is a curve within the tetrahedron whose tangent at any point is the vector $(\dot{c},\dot{\lambda},\dot{v})$. Such a curve is easily computed by starting at any point within the tetrahedron and then iteratively stepping both forwards and backwards with the vector field directing each step.

Every smooth equilibrium is represented by a solution curve to the vector field within the tetrahedron. Specifically, consider an equilibrium <S,B> such that S is differentiable on $[0,\bar{c}]$ and B is differentiable on $[\underline{v},1]$. This equilibrium necessarily satisfies the two first order conditions (4.1-4.2) and the tautology $\dot{\lambda} = 1$. Consequently for $\lambda \in [S(0), B(1)]$ the parametric equations $\lambda = \lambda$, $v = B^{-1}(\lambda)$, and $c = S^{-1}(\lambda)$ define a solution curve. Figure 5.2 illustrates such a curve with m = n = 2, F and G both uniform, and risk neutral traders.[7] The solution curve starts at point $E = (c=0,\ \lambda=S(0)=B(\underline{v})=\underline{v},\ v=\underline{v})$ and proceeds through the tetrahedron to point $F = (c=\bar{c},\ \lambda=\bar{c}=S(\bar{c})=B(1),\ v=1)$. The graph of a buyer's strategy B can be recovered by projecting the curve leftward onto the face ABC of the tetrahedron. Similarly, the graph of a seller's strategy is the projection of the curve onto the top face BCD. The two strategies determined by the solution curve in Figure 5.2 are shown in Figure 3.1. The tetrahedron's edge AD, given by the equalities $v = \lambda = c$, corresponds to

[7] All examples in the remainder of this section are for F,G uniform, risk neutral traders, and m = n.

truthful revelation by each trader. The gap between a solution curve and this edge therefore represents equilibrium misrepresentation.

Conversely, if F/f and (1-G)/g are increasing functions on [0,1], then a solution curve $(c(\lambda),\lambda,v(\lambda))$ defines an equilibrium in the k-double auction provided the functions $c(\cdot)$ and $v(\cdot)$ are increasing.[8] These sufficient conditions are satisfied in the case of uniform F and G. Inspection of the formulas for the vector field on the faces and edges of the tetrahedron implies that any equilibrium solution curve must enter from a point $(c,\lambda,v) = (0,\underline{v},\underline{v})$ on the edge AC and exit through a point $(c,\lambda,v) = (\bar{c},\bar{c},1)$ on the edge BD. The curve thus defines B over the interval $[\underline{v},1]$ and S over $[0,\bar{c}]$. We complete their definitions by setting $B(v) \equiv v$ for $v \in [0,\underline{v}]$ and $S(c) \equiv c$ for $c \in [\bar{c},1]$.

Theorem 3.2 establishes that smooth strategies S and B define an equilibrium only if $S'(c) = 1/\dot{c} \geq 0$ for all $c \in [0,\bar{c}]$ and $B'(v) = \dot{v} \geq 0$ for all $v \in [\underline{v},1]$. Examination of the formulas for $\dot{c}$ and $\dot{v}$ shows that there are regions within the tetrahedron in which one or the other of these derivatives is negative.[9] Solution curves that enter these regions thus do not define equilibria. Figure 5.3 shows the projections of such a solution curve. Let Ω_m denote the set of smooth equilibria for a given m. A graphic representation of this set is easily computed by (i) choosing a grid of points in the tetrahedron that lie on a plane separating edges AC and BD,

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[8] The proof of sufficiency is straightforward. The method of proof is illustrated for the bilateral k-double auction in Satterthwaite and Williams (1989a, p.129-130) and for the buyer's bid double auction in Williams (1990, Thm. 5.1).

[9] This is true for $m, n \geq 2$. Such regions do not exist for the bilateral case. See Leininger, Linhart, and Radner (1989) and Satterthwaite and Williams (1989a) for characterizations of the equilibrium set in the bilateral case.

(ii) computing the solution curve through each point in the grid, and (iii) graphing only those curves along which both $\dot{c}$ and $\dot{v}$ remain nonnegative.

<u>Shape of the equilibrium set</u>. Figures 5.4-5.8 depict the results of this procedure when m takes the values 2, 4 and 8. Five aspects of these figures merit discussion. First, the movement of Ω_m towards the edge AD as m increases illustrates Theorem 5.1's convergence statement. Inspection of Figures 5.4, 5.6 and 5.8 with the aid of a ruler confirms that the maximal distance of Ω_m from edge AD is O(1/m).

Second, the set Ω_m also shrinks as it converges towards AD. Figures 5.5 and 5.7 dramatically show this shrinkage by depicting the cross-section of Ω_m. Figure 5.7, which shows the cross-section of Ω_m for m=4, suggests that all symmetric, smooth equilibria are essentially identical in a C^0 sense for relatively small m. This contrasts starkly with the bilateral case in which equilibria fill the tetrahedron and the traders' problem of coordinating their choices of strategies is severe. The coordination problem thus rapidly becomes inconsequential as m increases.

Third, Figures 5.4, 5.6 and 5.8 show that equilibria wind around a line through the center of Ω_m that is not itself a solution curve.[10] Figure 5.9, which shows the same solution curve as Figure 5.2, reveals that the curve winds counterclockwise as it proceeds from E to F. As noted before, Figure 3.1 shows the equilibrium strategies <S,B> that are determined by this solution curve. Note that S'(c) and B'(v) can be either large or small, depending on the values of c and v. This oscillation in the values of S' and B' occurs as the solution curve winds around the line through the

[10] It can be shown when F and G are uniform that no linear equilibrium exists in the 0.5-double auction for $m \geq 2$. Thus the linear equilibrium that Chatterjee and Samuelson (1983) derived in the m = 1 case is exceptional.

tetrahedron. Figures 5.6 and 5.8 show that this oscillation persists as m increases. Equilibrium strategies therefore converge to truthful revelation in a C^0 sense, but not in a C^1 sense, as m increases.[11]

Fourth, if traders truthfully revealed their reservation values, then the expected price would be 0.5. It can be shown that for m sufficiently large the distribution of prices in the market of size m is nearly normal with mean 0.5, regardless of the choice of equilibrium. The variance of this distribution goes to zero as m approaches infinity, which means that in a large market a trader is only likely to affect price if his reservation value is near 0.5. Our pictures reflect this, for as m increases the "action" in the strategies persists only near v = c = 0.5.

Fifth, in Figures 5.4, 5.6 and 5.8 all the curves in Ω_m appear to enter from a very small interval--almost a point to the eye--on edge AC and exit from a similar interval on edge BD. This suggests that $\underline{v} = \underline{s}$ and $\overline{c} = \overline{b}$ are essentially constants in a given size of market regardless of which smooth equilibrium is considered. This regularity is surprising because (i) a large number of equilibria exist in a market of given size, (ii) these equilibria exhibit a wide variety of behavior around $\lambda = .5$, and (iii) $\underline{v}$, $\underline{s}$, $\overline{c}$, and $\overline{b}$ are defined in Section 3 as functions of the equilibrium strategies. Curves enter and exit from other points on the edges AC and BD, but each of these fails to define an equilibrium because at some point along it $\dot{c}$ or $\dot{v}$ turns negative.

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[11] The set Ω_m is constructed by eliminating those solution curves along which $\dot{c}$ or $\dot{v}$ turns negative. Thus some of the curves left are ones for which $\dot{c}$ or $\dot{v}$ is close to zero at some point. At such a point S' or B' is therefore very large. Consequently as m increases and Ω_m shrinks the derivatives of equilibrium strategies need not approach one as would be required for C^1 convergence.

6. CONVERGENCE TO EFFICIENCY

Theorem 5.1's rate of convergence to truthtelling suggests that the loss in efficiency caused by misrepresentation vanishes rapidly as the market becomes large. The next theorem makes this precise. For any draw of the traders' reservation values, the potential gain from trade is the maximum total profit that can be achieved through trading. This is the amount that would be achieved in a k-double auction if each trader honestly revealed his reservation value. The expected potential gain from trade is the expected value of this random variable. Given an equilibrium $<S,B>$, the expected loss due to misrepresentation is the expected potential gain from trade minus the expected total profit given this equilibrium. The relative inefficiency is the expected loss due to misrepresentation as a fraction of the expected potential gain from trade. Theorem 6.1 states that as the relative inefficiency of the k-double auction is $O(1/m^2)$ for sequences of markets in which n/m is bounded above and away from zero.

Theorem 6.1. Suppose F and G are C^1 distributions on [0,1] with positive densities over this interval and let K be some fixed positive number. For m and n such that $1/K < n/m < K$, consider equilibria $<S,B>$ of the k-double auction with m buyers and n sellers that satisfy (2.4-2.5). There exists a number $\xi(K,F,G)$, independent of m and n, such that the relative inefficiency of any such equilibrium is no more than ξ/m^2.

Let $n \wedge m \equiv \min(n,m)$. At most $n \wedge m$ trades of value one can be made, so the expected potential gain from trade is no more than $n \wedge m$. A lower bound on the expected potential gain from trade when there are m buyers and n sellers can be computed by pairing off each of $n \wedge m$ buyers with a seller and noting: (i) the expected potential gain from trade within each pair is some positive number η; (ii) the expected potential gain from trade among all n+m traders is at least as large as the amount that can be achieved through pairwise trading. The expected potential gain from trade is therefore at least $\eta(n \wedge m)$. It follows that the expected potential gain from trade is exactly $O(n \wedge m)$, which by the bounds on n/m is the same as O(m). The proof of Theorem 6.1 is thus reduced to showing that the expected loss due to misrepresentation is O(1/m). In the Appendix, Theorem 5.1's bounds are used to show that the value of any trade that ought to be made but is not is O(1/m). The difficult part of the proof is then to show that the expected number of trades that inefficiently fail to occur is bounded.

<u>Intuition for this rate of convergence</u>. This square relationship between the amount of misrepresentation and the inefficiency it causes is suggested by the simple supply and demand curve diagram of Figure 6.1. Abstracting from both the uncertainty and the indivisibilities of our double auction model, let SS' and DD' be "true" supply and demand curves. If misrepresentation did not occur, then the market-clearing price would be p_C, the quantity q_C would be traded, and the entire potential gain from trade would be achieved. This gain is equal to the area of the triangle DAS, which remains constant in this example. Now suppose that, as in equilibria of a k-double auction, buyers and sellers do misrepresent, though by no more than κ/m. Let dd' and ss' be the "strategic" demand and supply curves; each is offset in the appropriate direction from the true curve by the vertical distance κ/m. Given these strategic curves, the actual market-clearing

quantity is q_S. The gain from trade actually realized is the area of the trapezoid DBCS and the unrealized potential gain from trade is the area of the shaded triangle BAC. This area BAC is $O(1/m^2)$ because the length of its base BC is $2\kappa/m$ and its height is thus also $O(1/m)$.

A numerical illustration. Returning to the example of uniform F and G, risk neutral traders, and m = n, Table 6.1 tabulates the relative inefficiencies of the 0.5-double auction for values of m ranging from 1 to 8. Also included for comparison are the relative inefficiencies of the optimal mechanism and the BBDA. For a given m, let:

T_0 ≡ the expected potential gain from trade;

T^* ≡ the expected gain from trade in the optimal mechanism, i.e., the mechanism that maximizes the expected gain from trade subject to the incentive constraints and interim individual rationality;[12]

$Q(S,B)$ ≡ the expected total gain from trade given the equilibrium $\langle S,B\rangle$ of the 0.5-double auction;

Q_{max} ≡ $\sup_{\langle S,B\rangle \in \Omega_m} Q(S,B)$;

Q_{min} ≡ $\inf_{\langle S,B\rangle \in \Omega_m} Q(S,B)$;

Q_{BBDA} ≡ the expected total profit from trade generated by the equilibrium $S(c) = c$, $B(v) = mv/(m+1)$ of the BBDA.[13]

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[12] This mechanism is derived using the revelation principle in Gresik and Satterthwaite (1989).

[13] This equilibrium is discussed in Satterthwaite and Williams (1989b) and in Williams (1990). For uniform F and G, it is the only smooth, symmetric equilibrium of the BBDA that satisfies (2.4) and (2.5).

In this notation Theorem 6.1 states that $1-(Q_{min}/T_0)$, the relative inefficiency of 0.5-double auction's least efficient smooth equilibrium, is $O(1/m^2)$.

Four features of Table 6.1 are noteworthy. First, convergence to efficiency (i.e., a relative inefficiency of zero) as m increases is very fast in the 0.5-double auction, even for the sequence of the least efficient equilibria. Specifically, consistent with Theorem 6.1, $1-(Q_{min}/T_0)$ follows the $O(1/m^2)$ law, and for m as small as six is inconsequential. Second, for all values of m, the best equilibrium's relative inefficiency, $1-(Q_{max}/T_0)$, is so close to achieving the relative inefficiency of the optimal mechanism that to the degree of accuracy of the table they are the same. Third, except for the m = 1 case, smooth equilibria of the 0.5-double auction are uniformly more efficient than the smooth equilibrium of the BBDA. Strategic misrepresentation by both sides of the market in the 0.5-double auction is thus less costly in terms of ex ante efficiency than strategic misrepresentation only by buyers in the BBDA. Fourth, as m increases, the gap between the relative inefficiency of the best and the worst equilibrium of the 0.5-double auction rapidly decreases from large to small as m increases, reflecting the shrinkage of Ω_m as m increases that is so obvious in Figures 5.3-5.8. Measured either in terms of efficiency or by a trader's choice of his offer/bid, equilibrium selection becomes inconsequential as m increases.

7. PRICE-TAKING AS AN ALMOST OPTIMAL STRATEGY

The strength of this paper is its focus upon equilibrium behavior in which each trader strategically seeks to affect price. This contrasts with the classical model of a competitive market in which each trader takes price as given and maximizes in response to it. In this section we measure in terms of a trader's payoff the difference between equilibrium behavior and price-taking behavior. Specifically, we calculate (i) the cost to a trader of deviating from an equilibrium by acting as a price-taker when all others play their equilibrium strategies and (ii) the benefit to a trader of optimizing when all others act as price-takers. Our consideration of nonequilibrium behavior is motivated by the intuition that optimizing is complex and hence costly to a trader while price-taking is straightforward and costless. Our goal is to demonstrate that the cost of price-taking in a market of equilibrium players and the benefit of optimizing against price-takers are both inconsequential in all but the smallest of markets. A trader for whom optimizing is costly and price-taking is free thus might rationally adopt price-taking as his strategy even in a finite market.

Consider first a buyer who acts as a price-taker when all other traders play some equilibrium $\langle S,B\rangle$. Deviating from $B(v)$ to v can affect this buyer's expected payoff in two ways: (i) he may gain a profitable trade that he would miss by bidding $B(v)$; (ii) he may lose by raising the price that he pays. Because $B(v)$ is a best response to the behavior of the other traders, the expected loss from increasing the price outweighs the expected addition of profitable trades. The net loss is clearly no more than $O(q(n,m))$ because he increases the price by no more than $v - B(v)$, which Theorem 5.1

states is $O(q(n,m))$. A trader in a reasonably large market thus loses little by price-taking instead of optimizing.

Table 7.1 shows for varying values of $m = n$ in the case of uniform F and G the expected gain from trade a buyer forgoes by price-taking when all other traders play equilibrium strategies.[14] The table indicates that the relative expected loss is $O(1/m^2)$; for $m \geq 6$ the buyer's relative loss is in fact inconsequential. This $O(1/m^2)$ rate is faster than the $O(1/m)$ rate derived above because it takes into account the diminishing likelihood that a buyer raises the price as he raises his bid. Thus in all but the smallest markets a trader does essentially as well employing the simple rule of price-taking as by optimizing.

Conversely a trader can profit by optimizing in response to price-taking behavior by other traders.[15] The expected gain of deviating from price-taking to a best response vanish at the same rate as the gain of deviating from equilibrium behavior to price-taking. Specifically, for the example of $m = n$ and uniform F and G, Table 7.2 indicates that the relative

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[14] Because a buyer knows his reservation value when he chooses his bid, one can argue that Tables 7.1 and 7.2 should list a buyer's interim expected gain from trade (i.e., expected gain conditional on the buyer's realized reservation value) rather than his ex ante gain from trade. For a given value of $m = n$, the ex ante gain from trade reported in a table is the average of the interim gains taken with respect to a buyer's possible reservation values. Numerical investigation shows that the ex ante gains that are listed meaningfully summarize the underlying interim gains.

[15] For $F = G$ Satterthwaite and Williams (1989b, p. 493) showed that a buyer's misrepresentation in the BBDA is $O(1/m)$ when he optimally responds to price-taking behavior by the other traders. Earlier Roberts and Postlewaite (1976) asked the same kind of question within a complete information general equilibrium model. They proved for a generic economy that if all agents except one are price-takers, then that particular agent's incentive to misrepresent strategically vanishes as the market becomes large.

expected gain from optimizing against price-takers is $O(1/m^2)$ and for $m \geq 6$ is essentially zero.

In the general model of this paper the gain from this optimizing behavior is no more than $O(q(n,m))$. Consider a buyer who chooses a bid λ below his reservation value v to maximize his expected utility given that all other traders act as price-takers. This buyer gains as he lowers his bid below v by decreasing the price that he pays when he trades. As in the argument made above, the $O(q(n,m))$ rate we seek follows from proving that the difference between v and the optimal λ is no more than $O(q(n,m))$, for $v - \lambda$ bounds the effect upon price of changing from bid v to bid λ. Return to formula (4.1) and substitute $f(\lambda)$ for $f(c)\dot{c} = f(c)/S'$ and $g(\lambda)$ for $g(v)\dot{v} = g(v)/B'$ so that its right-hand side represents the buyer's marginal expected payoff when all other traders honestly report their reservation values as their offers/bids. As in the proof of Theorem 5.1, the first order condition for the optimal λ implies the bound $v - \lambda \leq kM_{n,m}(\lambda)/(m-1)L_{n,m}(\lambda)$. The proof that this ratio of probabilities is $O(q(n,m))$ is a key part of the proof of Theorem 5.1. The bound on this ratio provides the desired bound on $v - \lambda$.

8. CONCLUDING REMARKS

The origins of price-taking behavior is the main theme of our paper. Two other themes that are implicit can be brought out by relating our results to two problems, one posed by Wilson and the other by Hayek. In reviewing game-theoretic analyses of trading Wilson emphasized that a basic problem for theory is

> explaining the prevalence of a few simple trading rules in most of the commerce conducted via organized exchanges. A short list--

including auctions, double auctions, bid-ask markets, and specialist trading--accounts for most organized exchange. Indeed, bid-ask markets (such as those conducted in the commodities pits) have long been economists' paradigms for the nearly perfect markets addressed by the Walrasian theory of general equilibrium. The rules of these markets are not changed daily as the environment changes; rather they persist as stable, viable institutions. As a believer that practice advances before theory, and that the task of theory is to explain how it is that practitioners are (usually) right, I see a plausible conjecture: These institutions survive because they employ trading rules that are efficient for a wide class of environments (1986, p. 36-37).

Our results support his conjecture: for a wide class of underlying beliefs F and G the simple, invariant rules of the k-double auction lead to almost efficient trade in all but the smallest markets.

Hayek emphasized the dual task for a market of simultaneously eliciting information and allocating goods. The resource allocation problem

is thus in no way solved if we can show that all the facts, if they were known to a single mind (as we hypothetically assume them to be given to the observing economist), would uniquely determine the solution; instead we must show how a solution is produced by the interactions of people each of whom possesses only partial knowledge (1945, p. 530).

Because the Bayesian game model explicitly models private information, our results show how an allocation problem can be solved despite the dispersion of information among traders and each one's attempts to use his own information strategically.

APPENDIX

Proof of Theorem 3.1. Only the parts of the theorem concerning P_b and B are proven here; similar arguments prove the statements concerning P_s and S. We begin by showing that there is no bid λ^* such that $P_b(\lambda^*) > 0$ and

$$B^{-1}(\lambda^*) \equiv \{v \mid B(v) = \lambda^*\}$$

is a set of positive G-measure.

Select a buyer. Let $x \equiv s_{(m)}$ and $y \equiv s_{(m+1)}$ in the sample of offers/bids from the other m+n-1 traders and let $P(x,y)$ denote the joint distribution of x and y. Let $p(x,y,\lambda)$ denote the price at which he trades when λ is his bid:

$$p(x,y,\lambda) \equiv \begin{cases} (1-k)x + k\lambda & \text{if } x \le \lambda \le y \\ (1-k)x + ky & \text{if } \lambda > y \end{cases} .$$

Define $\theta(x,y,\lambda)$ as the probability that the selected buyer trades given x, y and his bid λ. As explained in section 2, $\theta(x,y,\lambda) = 1$ if $\lambda > x$, $\theta(x,y,\lambda) = 0$ if $\lambda < x$, and $\theta(x,y,\lambda)$ may be between zero and one if $x = \lambda$. The selected buyer's expected utility when he bids λ is

$$EV(v,\lambda) = \int_{\{x \le \lambda\}} V(v-p(x,y,\lambda))\ \theta(x,y,\lambda)\ dP(x,y). \tag{A.1}$$

As explained after the statement of Theorem 3.1, the function θ can be omitted from (A.1) once (3.1) is established.

Choose v' such that $B(v') = \lambda^*$ and $B^{-1}(\lambda^*) \cap (0,v') \neq \emptyset$. Because $B(v) = \lambda^* \le v$ for some $v < v'$ and $P_b(\lambda^*) > 0$, it is clear that $0 < \lambda^* < v' \le 1$. As in Satterthwaite and Williams (1989b, Thm. 2.2), we now argue that $P_b(\lambda)$ has a jump discontinuity at $\lambda = \lambda^*$, from which a contradiction easily follows.

Because $P_b(\lambda^*) > 0$, the set $\{c \mid S(c) \le \lambda^*\}$ has positive F-measure. Because $S(c) \ge c$ and $\lambda^* < 1$, the set $\{c \mid S(c) > \lambda^*\}$ also has positive F-measure. It is therefore a positive probability event that (i) all other buyers bid λ^*, (ii) one offer is no more than λ^* and the remaining n-1 offers are strictly more.[16] In this event, $x = \lambda^*$, and if the selected buyer also bids λ^*, then the market price is λ^* and the available supply of one item must be randomly allocated among the m buyers. By raising his bid above λ^* the selected buyer obtains an item with probability one in this event rather than with probability less than one due to rationing, which is shown below to increase his expected utility. Formally, this discussion implies that $x = \lambda^*$ is a positive probability event, i.e.,

$$\int_{\{x=\lambda^*\}} dP(x,y) > 0,$$

and the selected buyer's conditional probability of trading given $x = \lambda^*$ and his bid λ^* is less than one,

$$\frac{\int_{\{x=\lambda^*\}} \theta(x,y,\lambda^*)\ dP(x,y)}{\int_{\{x=\lambda^*\}} dP(x,y)} < 1. \tag{A.2}$$

For $\lambda > \lambda^*$ we have

$$EV(v',\lambda) - EV(v',\lambda^*) =$$

$$\int_{\{x=\lambda^*\}} V(v'-p(x,y,\lambda)) - V(v'-p(x,y,\lambda^*))\theta(x,y,\lambda^*)\ dP(x,y) \tag{A.3}$$

$$+ \int_{\{x<\lambda^*<y\}} V(v'-p(x,y,\lambda)) - V(v'-p(x,y,\lambda^*))\ dP(x,y) \tag{A.4}$$

$$+ \int_{\{\lambda^*<x\leq\lambda\}} V(v'-p(x,y,\lambda))\theta(x,y,\lambda)\ dP(x,y). \tag{A.5}$$

The integral in (A.5) is nonnegative. As λ decreases to λ^*, the integrand in (A.4) converges uniformly to zero, so in the limit this integral is zero. The integrand in (A.3) converges uniformly to

$$V(v'-p(\lambda^*,y,\lambda^*))(1-\theta(\lambda^*,y,\lambda^*)) = V(v'-\lambda^*)(1-\theta(\lambda^*,y,\lambda^*)).$$

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[16] The assumption that $n,\ m \geq 2$ is needed here.

The integral in (A.3) therefore converges to

$$V(v'-\lambda^*) \int_{\{x=\lambda^*\}} 1-\theta(\lambda^*,y,\lambda^*)\ dP(x,y),$$

which by (A.2) is positive. Therefore, for λ near λ^*,

$$EV(v',\lambda) - EV(v',\lambda^*) > 0,$$

which contradicts the optimality of $\lambda^* = B(v')$ for the buyer with reservation value v'.

An argument of Satterthwaite and Williams (1989b, Thm. 2.2) for the BBDA (which was inspired by Theorem 1 of Chatterjee and Samuelson (1983)) generalizes to prove (3.2). Several functions are needed for our proof that $P_b \circ B$ is nondecreasing. Consider any $v' < v''$. For $0 \le p \le v'$ define

$$\Delta V(p) \equiv V(v''-p) - V(v'-p).$$

Because V is increasing and concave, ΔV is nondecreasing and bounded below by $V(v'')-V(v')$, which is positive. Define

$$H(\lambda) \equiv \int_{\{x\le\lambda\}} \Delta V(p(x,y,\lambda))\ dP(x,y).$$

The function H is nondecreasing.

We now prove that $P_b(B(v')) \le P_b(B(v''))$. The definition of an equilibrium implies that

$$EV(v'',B(v'')) - EV(v'',B(v')) \ge 0, \text{ and} \tag{A.6}$$

$$EV(v',B(v')) - EV(v',B(v'')) \ge 0. \tag{A.7}$$

Adding (A.6) and (A.7) and then rearranging terms produces

$$H(B(v'')) - H(B(v')) \ge 0. \tag{A.8}$$

If $B(v") \geq B(v')$, then it is obvious that $P_b(B(v")) \geq P_b(B(v'))$. We therefore assume that $B(v") < B(v')$. Because H is nondecreasing, (A.8) implies that

$$H(B(v')) = H(B(v")).$$

The definition of H then implies

$$0 = H(B(v')) - H(B(v")) =$$

$$\int_{\{B(v")<x\leq B(v')\}} \Delta V(p(x,y,B(v'))\ dP(x,y) \tag{A.9}$$

$$+ \int_{\{x\leq B(v")\}} \Delta V(p(x,y,B(v')) - \Delta V(p(x,y,B(v"))\ dP(x,y). \tag{A.10}$$

The integral in (A.10) is nonnegative because (i) $p(x,y,\lambda)$ is nondecreasing in λ and (ii) $\Delta V(p)$ is nondecreasing in p. The integral in (A.9) is therefore nonpositive. Because ΔV is bounded below by a positive number, this implies

$$\int_{\{B(v")<x\leq B(v')\}} dP(x,y) = 0.$$

This last integral equals $P_b(B(v')) - P_b(B(v"))$, which gives the desired result. Q.E.D.

<u>Proof of Theorem 3.2</u>. Differentiability follows from monotonicity by a well-known theorem in analysis (e.g., see Royden (1968, p.96)). Because the proofs of parts (3.3) and (3.4) are so similar, only (3.4) is proven here. We first suppose that $B(v') > B(v")$ and derive a contradiction. The strategy B is therefore nondecreasing over $[\underline{v},1]$. Statement (2.1) of Theorem 3.1 then implies that B is increasing over this interval.

Because P_b and $P_b \cdot B$ are both nondecreasing, it must be true that $P_b(B(v")) = P_b(B(v'))$, i.e., a lower type buyer (v') bids more than the higher type buyer (v") even though it doesn't increase the probability that he trades. We show that this contradicts the assumption that $B(v')$ is an optimal bid for a selected buyer with reservation value v'. The argument rests upon the following facts:

(i) the set $\{c \mid S(c) \leq B(v'')\}$ has positive measure; (A.11)

(ii) for $\lambda \in (B(v''),B(v'))$, the set $\{c \mid S(c) > \lambda\}$ has positive measure; (A.12)

(iii) the set $\{v \mid B(v) < B(v'')\}$ has positive measure. (A.13)

Statement (A.11) is true because $P_b(B(v'')) > 0$, statement (A.12) is true because $S(c) \geq c$ (by (2.5)) and $\lambda < B(v') \leq 1$, and statement (A.13) is true because $B(v'') > 0$ and $B(v) \leq v$ (by (2.5)). Statements (A.11-A.13) imply that for a bid $\lambda \in (B(v''),B(v'))$ by the selected buyer, it is a positive probability event that $x < \lambda < y$, where (as in the proof of Theorem 3.1) $x \equiv s_{(m)}$ and $y \equiv s_{(m+1)}$ in the sample of offers and bids from the other m+n-1 traders. In this event, the selected buyer affects the price at which he trades, which we now show implies that his expected utility decreases as he raises his bid over the interval $(B(v''),B(v'))$. We have

$$EV(v',B(v'')) - EV(v',B(v')) = \tag{A.14}$$

$$\int_{\{x\leq B(v'')\}} V(v'-p(x,y,B(v''))) - V(v'-p(x,y,B(v')))\, dP(x,y) \tag{A.15}$$

$$- \int_{\{B(v'')<x\leq B(v')\}} V(v'-p(x,y,B(v')))\, dP(x,y). \tag{A.16}$$

Because $P_b(B(v'')) = P_b(B(v'))$, the integral (A.16) is computed over a set of measure zero and is therefore zero. The integrand in (A.15) is nonnegative. For a fixed $\lambda \in (B(v''),B(v'))$, it is positive over the set

$\{x \le B(v'') < \lambda \le y\}$, which (as argued above) is a set of positive P-measure. The difference (A.14) is therefore strictly positive, which contradicts the optimality of $B(v')$ for the buyer with reservation value v'. We conclude that $B(v') \le B(v'')$.

We now prove that $\lim_{v \downarrow \underline{v}} B(v) = \underline{v} = \underline{s}$; the proof that $\lim_{c \uparrow \bar{c}} S(c) = \bar{c} = \bar{b}$ is omitted because it is so similar to this argument. The equality $\underline{v} = \underline{s}$ is established by proving that both of the inequalities $\underline{v} < \underline{s}$, $\underline{s} < \underline{v}$ lead to contradictions. If $\underline{v} < \underline{s}$, then $B(v) \le v < \underline{s}$ for values of v that are greater than but sufficiently near $\underline{v}$. As a consequence, $P_b(B(v)) = 0$ at such values of v, which contradicts the definition of $\underline{v}$. If $\underline{s} < \underline{v}$, then $c \le S(c) < v < \underline{v}$ for c near zero and v less than but sufficiently near $\underline{v}$. Trading opportunities therefore exist for a buyer with such a reservation value v, which implies that $P_b(B(v))$ must be positive. This also contradicts the definition of $\underline{v}$.

The equality $\lim_{v \downarrow \underline{v}} B(v) = \underline{v}$ is established by recalling that $v \ge B(v) \ge \underline{s}$ for $v > \underline{v}$, which implies $\underline{v} \ge \lim_{v \downarrow \underline{v}} B(v) \ge \underline{s}$. The desired equality now follows from $\underline{v} = \underline{s}$. Q.E.D.

<u>Proof of Theorem 3.3</u>. It is clear that S^* and B^* satisfy (2.4) and (2.5). Because the dual of the dual market is the given market, it is sufficient to show that the expected utility of a buyer with reservation value v who bids $\lambda \le v$ in the given market is the same as the expected utility of a seller with value $c^* = 1-v$ whose offer is $\lambda^* \equiv 1-\lambda$ in the dual market. This follows directly from a change of variable in an integral representation of expected utility. The buyer's expected utility is

$$\int_{x \le \lambda < y} V(v - (1-k)x - k\lambda)\, dP(x,y) + \int_{y \le \lambda} V(v - (1-k)x - ky)\, dP(x,y), \tag{A.17}$$

where x, y and P(x,y) are defined in the proof of Theorem 3.1. Because the transformation to the dual market reverses the order of offers/bids without any shuffling, we have $1-y^* \to x$ and $1-x^* \to y$, where, in a sample of the offers/bids of n^*-1 sellers using S^* and m^* buyers using B^*, $x^* \equiv s_{(m^*-1)}$ and $y^* \equiv s_{(m^*)}$. Note also that the joint distribution P^* of x^* and y^* satisfies the equality $P(1-y^*,1-x^*) = 1-P^*(x^*,y^*)$. After the change of variable and some simplification, (A.17) reduces to

$$\int_{x^*<\lambda^*\le y^*} C^*((1-k^*)\lambda^*+k^*y^*-c^*)\,dP^*(x^*,y^*)$$

$$+ \int_{\lambda^*\le x^*} C^*((1-k^*)x^*+k^*y^*-c^*)\,dP^*(x^*,y^*),$$

which is the expected utility of a seller in the dual market whose offer is λ^*. Q.E.D.

<u>Probabilities in the First Order Conditions (4.1) and (4.2)</u>. For $\lambda \in (\underline{v},1]$, define the function $S^{-1}(\lambda)$ by the formula

$$S^{-1}(\lambda) \equiv \inf\{c \mid S(c) \ge \lambda\}, \qquad \text{(A.18)}$$

and for $\lambda \in [0,\bar{c})$ define the function $B^{-1}(\lambda)$ by

$$B^{-1}(\lambda) \equiv \sup\{v \mid B(v) \le \lambda\}. \qquad \text{(A.19)}$$

Using $S^{-1}(\lambda) = c$ and $B^{-1}(\lambda) = v$, the probabilities $K_{n,m}(\lambda)$, $L_{n,m}(\lambda)$, $M_{n,m}(\lambda)$, $J_{n,m}(\lambda)$, and $N_{n,m}(\lambda)$ can be written as functions of c and v:

$$K_{n,m}(\lambda) = \sum_{\substack{i+j=m-1\\ 0\le i\le m-1\\ 0\le j\le n-1}} \binom{m-1}{i}\binom{n-1}{j} G(v)^i F(c)^j (1-G(v))^{m-1-i}(1-F(c))^{n-1-j}, \qquad \text{(A.20)}$$

$$L_{n,m}(\lambda) = \sum_{\substack{i+j=m-1 \\ 0\le i\le m-2 \\ 0\le j\le n}} \binom{m-2}{i}\binom{n}{j} G(v)^i F(c)^j (1-G(v))^{m-2-i}(1-F(c))^{n-j}, \qquad (A.21)$$

$$M_{n,m}(\lambda) = \sum_{\substack{i+j=m \\ 0\le i\le m-1 \\ 0\le j\le n}} \binom{m-1}{i}\binom{n}{j} G(v)^i F(c)^j (1-G(v))^{m-1-i}(1-F(c))^{n-j}, \qquad (A.22)$$

$$J_{n,m}(\lambda) = \sum_{\substack{i+j=m-1 \\ 0\le i\le m \\ 0\le j\le n-2}} \binom{m}{i}\binom{n-2}{j} G(v)^i F(c)^j (1-G(v))^{m-i}(1-F(c))^{n-2-j}, \qquad (A.23)$$

$$N_{n,m}(\lambda) = \sum_{\substack{i+j=m-1 \\ 0\le i\le m \\ 0\le j\le n-1}} \binom{m}{i}\binom{n-1}{j} G(v)^i F(c)^j (1-G(v))^{m-i}(1-F(c))^{n-1-j}. \qquad (A.24)$$

A detailed explanation of these formulas can be found in Satterthwaite and Williams (1989b, p.485).

Proof of Theorem 5.1. We prove below that $v-B(v) \le \kappa q(n,m)$ for $v \in (\underline{v},1]$. With this in hand, the rest of the theorem follows easily. The symmetry result established in Theorem 3.3 implies that $S(c)-c \le \kappa q(n,m)$ for all $c \in [0,\bar{c})$ because if an equilibrium S existed that violated the bound, then it could be used to construct an equilibrium B* for the dual market that violates the bound upon it.[17] To bound $\underline{v}$, we apply Theorem 3.2 and the bound just established to deduce that $\underline{v} = \underline{s} \equiv \lim_{c\downarrow 0} S(c) = \lim_{c\downarrow 0}(S(c)-c) \le$

[17] Different values of κ may result here and later in the proof. This problem is rectified by choosing the largest of the values.

$\kappa q(n,m)$. Finally, using the symmetry result again, the bound $1-\bar{c} \leq \kappa q(n,m)$ follows from the bound on $\underline{v}$.

For $v \in (\underline{v},1]$, let $\lambda = B(v)$ and $c = S^{-1}(\lambda)$. Solving (4.3) for v-B(v) gives

$$v-B(v) \leq \frac{kM_{n,m}(\lambda)}{(m-1)L_{n,m}(\lambda)g(v)\dot{v}}, \tag{A.25}$$

which holds for almost all $v \in (\underline{v},1]$. The following bound on the right-hand side of (A.25) is proven below using a straightforward combinatorial analysis:

$$\frac{kM_{n,m}(\lambda)}{(m-1)L_{n,m}(\lambda)g(v)\dot{v}} \leq \frac{2k}{g(v)\dot{v}}\,\frac{1}{m-1}\left[\,G(v) + \frac{n}{m}\,\frac{F(v)(1-G(v))}{(1-F(v))}\,\right]. \tag{A.26}$$

Because the density g is continuous and positive on [0,1], L'Hopital's rule implies that G(v)/g(v) and (1-G(v))F(v)/g(v)(1-F(v)) are continuous functions on this interval. A $\kappa > 0$ therefore exists such that

$$(v-B(v)) \leq \frac{\kappa}{m\,\dot{v}}\left[\,1 + \frac{n}{m}\,\right] \leq \frac{\kappa}{\dot{v}}\,q(n,m)$$

for all $v \in (\underline{v},1]$ at which $\dot{v}$ exists. This is our bound when $\dot{v} \geq 1$. The remainder of our proof is a demonstration that the bound continues to apply for values of $v \in (\underline{v},1]$ at which either (i) $\dot{v} < 1$ or (ii) $\dot{v}$ fails to exist.

We first note the following. Consider an increasing sequence $\{v_i\}$ that has as its limit $v \in (\underline{v},1]$. Suppose $v_i-B(v_i) \leq \kappa q(n,m)$ for each element of the sequence. Then $v-B(v) \leq \kappa q(n,m)$ because $\lim_{i\to\infty} v_i = v$ and, since B is increasing, $B(v) \geq \lim_{i\to\infty} B(v_i)$.

Consider now a $v' \in (\underline{v},1]$ at which $\dot{v} < 1$. At some value v in the interval $[\underline{v},v')$ the derivative $\dot{v} = 1/B'(v)$ exists and is at least one. To see this, recall from Theorem 3.2 that $\lim_{v \downarrow \underline{v}} B(v) = \underline{v}$. If at almost all $v \in (\underline{v}, v')$ the derivative $\dot{v}$ were less than one, then at all v in this interval B(v) would exceed v, which violates the constraint (2.5) that $B(v) \leq v$. The value $v^* \equiv \sup\{v \in (\underline{v}, v') \mid \dot{v} = 1/B'(v) \geq 1\}$ therefore exists. Our result from immediately above implies that $v^*-B(v^*) \leq \kappa q(n,m)$. Almost everywhere on the interval (v^*, v') the derivative $B'(v) = 1/\dot{v}$ exists and exceeds one. Since B is increasing, it follows that

$$v'-v^* \leq \int_{v^*}^{v'} B'(v)dv \leq B(v') - B(v^*),$$

which upon rearrangement gives the desired result: $v'-B(v') \leq v^*-B(v^*) \leq \kappa q(n,m)$.

Next consider values of $v \in (\underline{v}, 1]$ at which $\dot{v}$ does not exist. Because B is differentiable almost everywhere, any such value is a limit point of an increasing sequence of values at which $\dot{v}$ exists. The desired bound holds wherever $\dot{v}$ exists and hence at every value in the sequence. As argued above it therefore holds at v. Q.E.D.

<u>Proof of (A.26)</u>. We now show that the ratio $M_{n,m}(\lambda)/L_{n,m}(\lambda)$ satisfies the bound

$$\frac{M_{n,m}(\lambda)}{L_{n,m}(\lambda)} \leq 2G(v) + \frac{2}{m}\frac{n}{} \frac{(1-G(v))F(c)}{1-F(c)} \tag{A.27}$$

$$\leq 2G(v) + \frac{2\ n}{m} \frac{(1-G(v))F(v)}{1-F(v)}, \tag{A.28}$$

where (A.28) implies (A.26). Inequality (A.28) follows from (A.27) because $c \leq \lambda \leq v$ and $F(c)/(1-F(c))$ is increasing in c. We therefore focus upon (A.27).

Define

$Y_{n,m}(\lambda) \equiv$ the probability that the bid λ lies between $s_{(m)}$ and $s_{(m+1)}$ in a sample of m-2 buyers using the strategy B and n sellers using S.

We first show that

$$M_{n,m} = (1 - G(v))Y_{n,m} + G(v)L_{n,m}, \tag{A.29}$$

or equivalently

$$\frac{M_{n,m}}{L_{n,m}} = (1 - G(v)) \frac{Y_{n,m}}{L_{n,m}} + G(v). \tag{A.30}$$

The bound (A.27) will be obtained by bounding $Y_{n,m}/L_{n,m}$ and then substituting into (A.30). The probability $M_{n,m}$ is defined for a sample of offers/bids from m-1 buyers using the strategy B and n sellers using the strategy S. Select a buyer. The event that defines $M_{n,m}$ is the disjoint union of the following two events:

(i) the selected buyer bids at least λ and λ lies between $s_{(m)}$ and $s_{(m+1)}$ in the sample of offers/bids from the remaining m-2 buyers and n sellers;

(ii) the selected buyer bids less than λ and λ lies between $s_{(m-1)}$ and $s_{(m)}$ in the sample of offers/bids from the remaining m-2 buyers and n sellers.

The selected buyer bids at least λ with probability 1-G(v) and less than λ with probability G(v). Equation (A.29) then follows from the definitions of $Y_{n,m}$ and $L_{n,m}$.

To bound $Y_{n,m}/L_{n,m}$, we partition the events that define these probabilities according to the number i of buyers' bids that are no more than λ. For $0 \le i \le m-2$, define

$Y^i_{n,m}(\lambda) \equiv$ the probability that the bid λ lies between $s_{(m)}$ and $s_{(m+1)}$ in a sample of m-2 buyers using the strategy B and n sellers using S, and exactly i of the offers/bids at or below λ are buyers' bids;

$L^i_{n,m}(\lambda) \equiv$ the probability that the bid λ lies between $s_{(m-1)}$ and $s_{(m)}$ in a sample of m-2 buyers using strategy B and n sellers using S, and exactly i of the offers/bids at or below λ are buyers' bids.

It is clear that

$$Y_{n,m} = \Sigma_{i=0}^{m-2} \; Y^i_{n,m}, \tag{A.31}$$

$$Y^i_{n,m} = \binom{m-2}{i} \binom{n}{m-i} G(v)^i F(c)^{m-i} (1-G(v))^{m-2-i} (1-F(c))^{n-m+i}, \tag{A.32}$$

$$L_{n,m} = \Sigma_{i=0}^{m-2} \; L^i_{n,m}, \text{ and} \tag{A.33}$$

$$L^i_{n,m} = \binom{m-2}{i} \binom{n}{m-1-i} G(v)^i F(c)^{m-1-i} (1-G(v))^{m-2-i} (1-F(c))^{n-m+i+1}. \tag{A.34}$$

The identity

$$i \binom{m-2}{i} \equiv (m-1-i) \binom{m-2}{i-1}$$

and formulas (A.32) and (A.34) imply that

$$\frac{Y^i_{n,m}}{L^{i-1}_{n,m}} \le \frac{G(v)}{1-G(v)} \quad \text{for } (m-1)/2 \le i \le m-2. \tag{A.35}$$

The identity

$$(m-i)\binom{n}{m-i} \equiv [(n+1)-(m-i)]\binom{n}{m-1-i}$$

and the bound

$$\frac{(n+1)-(m-i)}{(m-i)} \le \frac{2\ n}{m} \quad \text{for } 0 \le i \le (m-2)/2$$

together with formulas (A.32) and (A.34) imply that

$$\frac{Y^{i}_{n,m}}{L^{i}_{n,m}} \le \frac{2\ n}{m}\ \frac{F(c)}{1-F(c)} \quad \text{for } 0 \le i \le (m-2)/2. \tag{A.36}$$

It follows that

$$\frac{Y_{n,m}}{L_{n,m}} = \frac{\Sigma^{m-2}_{i=0}\ Y^{i}_{n,m}}{\Sigma^{m-2}_{i=0}\ L^{i}_{n,m}} \tag{A.37}$$

$$= \frac{\Sigma_{0\le i\le (m-2)/2}\ Y^{i}_{n,m}}{\Sigma^{m-2}_{i=0}\ L^{i}_{n,m}} + \frac{\Sigma_{i\ge (m-1)/2}\ Y^{i}_{n,m}}{\Sigma^{m-2}_{i=0}\ L^{i}_{n,m}} \tag{A.38}$$

$$\le \frac{\Sigma_{0\le i\le (m-2)/2}\ Y^{i}_{n,m}}{\Sigma_{0\le i\le (m-2)/2}\ L^{i}_{n,m}} + \frac{\Sigma_{i\ge (m-1)/2}\ Y^{i}_{n,m}}{\Sigma_{i\ge (m-1)/2}\ L^{i-1}_{n,m}} \tag{A.39}$$

$$\le \frac{2\ n}{m}\ \frac{F(c)}{1-F(c)} + \frac{G(v)}{1-G(v)}, \tag{A.40}$$

where the left and right terms in (A.39) are bounded by firs rewriting (A.36) and (A.35) as upper bounds on $Y^{i}_{n,m}$ and then substituting into the numerators. Q.E.D.

<u>Proof of Theorem 6.1</u>. As explained after the statement of the theorem, we must show here that the expected loss due to misrepresentation is at most O(1/m). Let μ denote the distribution of the market price p when traders use the equilibrium <S,B>. We bound the expected loss due to

misrepresentation by integrating the expected value of trades that inefficiently fail to be made at the price p with respect to $d\mu(p)$:

$$\text{expected loss} = \int_{p=-\infty}^{p=\infty} (\text{expected loss when price} = p)\, d\mu(p).$$

The bound is derived in two steps. For a properly chosen value of $\varepsilon > 0$, we first show that the probability that p lies below ε or above $1-\varepsilon$ is $O(2^{-m\wedge n})$. Because the total value of missed trades is no more than $m\wedge n$, it follows immediately that the integral over these intervals is no more than $O(1/m)$. The second step is then to show that the integral over $[\varepsilon, 1-\varepsilon]$ is also $O(1/m)$.

A preliminary step is to show two inequalities:

$$\max(m,n) \le \min(Km,Kn) \tag{A.41}$$

where K is the number in the bound $1/K < n/m < K$, and

$$\binom{m+n}{m} \le 4^{\max(m,n)}. \tag{A.42}$$

The inequality (A.41) follows from the bound on n/m. Inequality (A.42) is established in the case of m = n by

$$\binom{2m}{m} = \frac{\Pi_{i=1}^{m} 2i \cdot \Pi_{i=1}^{m}(2i-1)}{\Pi_{i=1}^{m} i \cdot \Pi_{i=1}^{m} i} \le 4^{m}. \tag{A.43}$$

Note the symmetry of (A.42) in m and n. Consequently we need only establish it for the case $n \ge m$. We do this by induction on n: assuming (A.42) is true for a particular n and m with $n \ge m$, then

$$\binom{n+1+m}{m} = \frac{n+1+m}{n+1}\binom{n+m}{m} \le 2 \cdot 4^{\max(m,n)}$$

since $(n+1+m)/(n+1) < 2$ whenever $n \ge m$.

The value $\varepsilon > 0$ is chosen so that $F(\varepsilon)$, $G(2\varepsilon)$, $1-F(1-2\varepsilon)$, and $1-G(1-\varepsilon)$ are all less than $1/2^{2K+1}$. Given the bound on n/m, Theorem 5.1 states the existence of a constant κ such that

$$v - B(v) < \kappa/m \text{ for } v \in (\underline{v},1],$$

$$S(c) - c < \kappa/m \text{ for } c \in [0,\bar{c}), \text{ and}$$

$$\underline{v},\ 1-\bar{c} \le \kappa/m$$

for any equilibrium <S,B> in the market with n sellers and m buyers that satisfies (2.4) and (2.5). We restrict our attention to values of m such that $\kappa/m < \varepsilon/2$, which implies that $\underline{v} \in [0,\varepsilon/2)$ and $\bar{c} \in (1-\varepsilon/2,1]$.

The bound on the probability that p is below ε is derived as follows. If p is the price, then at least m offers/bids must be no greater than p. Consequently, the probability that p is below ε is bounded above by the probability that at least m offers/bids are less than ε. Because $S(c) \ge c$, the probability that a seller's offer is below ε is no more than $F(\varepsilon)$. A buyer's bid may be less than his reservation value v, but by no more than κ/m when $v > \underline{v}$; consequently, the probability that a buyer's bid is in $[0,\varepsilon)$ is no more than $G(\varepsilon + \kappa/m)$, which is less than $G(2\varepsilon)$. The probability that the market price is below ε is therefore no more than

$$\binom{m+n}{m} [\max\{G(2\varepsilon),\ F(\varepsilon)\}]^m < 4^{\max(m,n)} \frac{1}{2^{(2K+1)m}}$$

$$= \frac{4^{\max(m,n)}}{4^{Km}} 2^{-m} \le 2^{-m}$$

where the first inequality follows from (A.42) and the choice of ε and the last inequality follows from (A.41). A similar argument shows that the probability that p lies above $1-\varepsilon$ is $O(2^{-n})$.

We next bound the expected loss given a particular value of p in $[\varepsilon, 1-\varepsilon]$. Note first that $S^{-1}(p)$ and $B^{-1}(p)$ are well-defined by (A.18) and (A.19) at such a p and satisfy the bounds

$$p - S^{-1}(p) \le \lim_{c \downarrow S^{-1}(p)} [S(c) - c] \le \kappa/m, \text{ and}$$

$$B^{-1}(p) - p \le \lim_{v \uparrow B^{-1}(p)} [v - B(v)] \le \kappa/m.$$

It follows that for $p \in [\varepsilon, 1-\varepsilon]$,

$$B^{-1}(p) - S^{-1}(p) \le 2\kappa/m. \tag{A.44}$$

It is also true that for p in this interval

$$p \le \lim_{v \downarrow B^{-1}(p)} B(v) \le \lim_{v \downarrow B^{-1}(p)} v = B^{-1}(p), \text{ and}$$

$$p \ge \lim_{c \uparrow S^{-1}(p)} S(c) \ge \lim_{c \uparrow S^{-1}(p)} c = S^{-1}(p),$$

from which it follows that $B^{-1}(p) \ge \varepsilon$ and $S^{-1}(p) \le 1-\varepsilon$ for $p \in [\varepsilon, 1-\varepsilon]$.

Consider now the loss at any draw of reservation values for which $p \in [\varepsilon, 1-\varepsilon]$ is the market price and i profitable trades fail to be made. This is possible only if there are i buyers and i sellers such that:

(i) each of the i sellers asks for at least p; (A.45)

(ii) each of the i buyers bids no more than p; (A.46)

(iii) the i buyers can be paired with the i sellers so that in each pair the reservation value of the buyer exceeds the reservation value of the seller. (A.47)

Condition (A.45) implies that the reservation value of each of the i sellers is at least $S^{-1}(p)$ and (A.46) implies that the value of each of the i buyers is no more than $B^{-1}(p)$. Condition (A.47) then implies that each of these 2i values lies in the interval $[S^{-1}(p), B^{-1}(p)]$. The loss when p is the market price and i profitable trades fail to take place is therefore no more than $i[B^{-1}(p) - S^{-1}(p)]$.

We are now ready to bound the integral

$$\int_{\varepsilon}^{1-\varepsilon} (\text{expected loss when price} = p)\, d\mu(p). \tag{A.48}$$

We rewrite the expected loss when p is the price by considering separately for $0 \le j \le n\wedge m$ and $1 \le i \le (n\wedge m)-j$ the case of j successful trades and i profitable trades that weren't made when p is the price. This approach and the bound obtained above on the loss when i profitable trades fail to be made at the price p show that (A.48) is no more than

$$\int_{\varepsilon}^{1-\varepsilon} \Sigma_{j=0}^{n\wedge m}\, q_j(p) \left[\Sigma_{i=1}^{(n\wedge m)-j}\, p_{ij}(p)\; i\; [B^{-1}(p) - S^{-1}(p)] \right] d\mu(p), \tag{A.49}$$

where $q_j(p)$ is the conditional probability that j trades are made given that p is the price and $p_{ij}(p)$ is the conditional probability that i profitable trade aren't made given that p is the price and j trades are made. The conditional probability $p_{ij}(p)$ satisfies the bound

$$p_{ij}(p) \le \max \left\{ \left[\frac{\gamma^{i}}{i!} \right]^2, \left[\frac{\gamma^{i-1}}{(i-1)!} \right]^2 \right\}, \tag{A.50}$$

for some constant γ. This bound, which is established below, does not depend upon either j, n or m. Assuming (A.50), we now complete the proof.

Let $\zeta(i)$ denote the number on the right-hand side of the inequality (A.50). For each p, the probabilities $(q_j(p))_{1\le j\le (n\wedge m)}$ sum to one. This fact and the bounds (A.44), (A.50) imply that (A.49) is no greater than

$$\frac{2\kappa}{m} \int_{\varepsilon}^{1-\varepsilon} \Sigma_{i=1}^{n \wedge m} i\zeta(i)\ d\mu(p).$$

It is sufficient to show that the integrand converges as $n \wedge m$ goes to infinity. For large i, $\zeta(i) = [\gamma^{(i-1)}/(i-1)!]^2$. It is therefore sufficient to show that the series

$$\Sigma_{i=1}^{\infty}\ i\left[\frac{\gamma^{i-1}}{(i-1)!}\right]^2$$

converges, which follows from the ratio test. Q.E.D.

Proof of (A.50). The proof of (A.50) requires some thought about how knowing p and the number j of successful trades affects what one knows about the distribution of the offers/bids of the (m+n-2j) traders who don't trade. In particular, we are interested in how it affects the likelihood that the reservation values of i sellers and i buyers of the remaining traders lie in the interval $[S^{-1}(p), B^{-1}(p)]$, which is a necessary condition for i profitable trades to have been missed. Our discussion centers upon which traders' reservation values are "linked" by virtue of the fact that their offers/bids determine the price p.

Given j trades at the price p, we partition the event in which i profitable trades fail to occur according to whose offers/bids determine the price by equaling either $s_{(m)}$ or $s_{(m+1)}$:

(i) a seller and a buyer who trade; (A.51)

(ii) a seller and a buyer who don't trade; (A.52)

(iii) a buyer who trades and a buyer who doesn't; (A.53)

(iv) a seller who trades and a seller who doesn't; (A.54)

(v) a buyer who trades and a seller who doesn't; (A.55)

(vi) a seller who trades and a buyer who doesn't. (A.56)

Event (A.55) occurs with probability zero, for a buyer who trades and a seller who doesn't can each affect the price only when the offer/bid of each of these traders equals $s_{(m+1)}$. A similar argument rules out (A.56). Given the price p, j successful trades, and one of the remaining four events, we show that the conditional probability that i profitable trades are missed is bounded above by the right-hand side of (A.50). Because the conditional probability $p_{ij}(p)$ is a convex combination of these four probabilities, this establishes (A.50). Events (A.53) and (A.54) are symmetric, so we shall only consider (A.51-A.53) here.

Let $x \equiv s_{(m)}$ in the sample of offers/bids of the m+n traders. We bound each of these three conditional probabilities with an integral computed with respect to the conditional distribution $\phi(\cdot)$ of x in the given event.[18] For each $x \in [0,p]$, define t(x) as the value of $s_{(m+1)}$ that solves the equation $p = (1-k)x + ks_{(m+1)}$. The reservation value of a buyer who doesn't trade must lie in $[0,B^{-1}(x)]$ and the reservation value of a seller who doesn't trade lies in $[S^{-1}(t(x)),1]$. Note that $S^{-1}(p) \le S^{-1}(t(x))$ and $B^{-1}(x) \le B^{-1}(p)$. Recall the discussion of (A.45-47). If $S^{-1}(t(x)) >$ $B^{-1}(x)$, then all profitable trades are made. If $S^{-1}(t(x)) \le B^{-1}(x)$, then the reservation value of any trader who is inefficiently excluded from trade must lie in $[S^{-1}(t(x)),B^{-1}(x)]$. Rather than considering all values of $x \in$ $[0,p]$, we are thus instead interested in the subset

$$\Gamma_p \equiv \{x \in [0,p] \mid S^{-1}(t(x)) \le B^{-1}(x)\}, \tag{A.57}$$

which is just those values of x for which the interval $[S^{-1}(t(x)),B^{-1}(x)]$ is nondegenerate.

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[18] For convenience, the dependence of $\phi(\cdot)$ upon the price p, the number j of trades, and which of the events (A.48-A.50) holds is suppressed.

The events (A.51-A.53) are distinguished by whether or not the traders whose offers/bids determine price are among the 2i traders who inefficiently fail to trade. Consider first event (A.51). Given that $p \in \Gamma_p$, the probability that i profitable trades are missed is bounded above by the probability that the reservation values of at least i of the m-j buyers who do not trade and at least i of the n-j sellers who do not trade also lie in $[S^{-1}(t(x)),B^{-1}(x)]$. Recall that, given x, the reservation value of a buyer who does not trade lies in $[0,B^{-1}(x)]$ and the reservation value of a seller who doesn't trade lies in $[S^{-1}(t(x)),1]$. This then implies the following upper bound on the conditional probability in question:

$$\int_{\Gamma_p} \binom{m-j}{i}\binom{n-j}{i} \left[\frac{G(B^{-1}(x)) - G(S^{-1}(t(x)))}{G(B^{-1}(x))} \right]^i \cdot \left[\frac{F(B^{-1}(x)) - F(S^{-1}(t(x))}{1 - F(S^{-1}(t(x)))} \right]^i d\phi(x). \tag{A.58}$$

In event (A.52) a buyer bids x and a seller offers t(x). If any trades are inefficiently excluded, then this particular buyer-seller pair is inefficiently excluded. Therefore an upper bound on the conditional probability given j, p, and the event (A.52) that at least i profitable trades are missed is

$$\int_{\Gamma_p} \binom{m-j-1}{i-1}\binom{n-j-1}{i-1} \left[\frac{G(B^{-1}(x)) - G(S^{-1}(t(x)))}{G(B^{-1}(x))} \right]^{i-1} \cdot \left[\frac{F(B^{-1}(x)) - F(S^{-1}(t(x))}{1 - F(S^{-1}(t(x)))} \right]^{i-1} d\phi(x). \tag{A.59}$$

Finally, in event (A.53) a buyer bids x and does not trade. If any buyer is inefficiently excluded from trade, then the buyer who bids x is inefficiently excluded. The conditional probability given j, p, and event (A.53) that at least i trades are inefficiently excluded is therefore bounded above by

$$\int_{\Gamma_p} \binom{m-j-1}{i-1}\binom{n-j}{i}\left[\frac{G(B^{-1}(x)) - G(S^{-1}(t(x)))}{G(B^{-1}(x))}\right]^{i-1} \cdot \left[\frac{F(B^{-1}(x)) - F(S^{-1}(t(x))}{1 - F(S^{-1}(t(x)))}\right]^{i} d\phi(x). \tag{A.60}$$

To obtain (A.50) from the bounds (A.58-A.60), we first substitute $S^{-1}(p)$ for $S^{-1}(t(x))$ and $B^{-1}(p)$ for $B^{-1}(x)$, which only loosens these bounds. Because F and G are C^1 functions on [0,1], a number τ exists such that for $p \in [\varepsilon, 1-\varepsilon]$,

$$\frac{F(B^{-1}(p)) - F(S^{-1}(p))}{1 - F(S^{-1}(p))} \leq \frac{\tau}{m}, \text{ and}$$

$$\frac{G(B^{-1}(p)) - G(S^{-1}(p))}{G(B^{-1}(p))} \leq \frac{\tau}{m}.$$

These inequalities are used to bound the fractions in brackets within these three integrals. Finally, note the inequalities

$$\binom{m-j}{i}\left[\frac{\tau}{m}\right]^{i} \leq \binom{m}{i}\left[\frac{\tau}{m}\right]^{i} = \frac{m!}{i!\,(m-i)!}\frac{\tau^{i}}{m^{i}} \leq \frac{(\tau K)^{i}}{i!}, \tag{A.61}$$

$$\binom{m-j-1}{i-1}\left[\frac{\tau}{m}\right]^{i-1} \leq \binom{m-1}{i-1}\left[\frac{\tau}{m}\right]^{i-1} = \frac{(m-1)!}{(i-1)!(m-i)!}\frac{\tau^{i-1}}{m^{i-1}} \leq \frac{(\tau K)^{i-1}}{(i-1)!} \tag{A.62}$$

$$\binom{n-j}{i}\left[\frac{\tau}{m}\right]^{i} \leq \binom{n}{i}\left[\frac{\tau}{m}\right]^{i} = \frac{n!}{i!\,(n-i)!}\frac{\tau^{i}}{m^{i}} \leq \frac{(\tau K)^{i}}{i!}, \text{ and} \tag{A.63}$$

$$\binom{n-j-1}{i-1}\left[\frac{\tau}{m}\right]^{i-1} \leq \binom{n-1}{i-1}\left[\frac{\tau}{m}\right]^{i-1} = \frac{(n-1)!}{(i-1)!(n-i)!}\frac{\tau^{i-1}}{m^{i-1}} \leq \frac{(\tau K)^{i-1}}{(i-1)!}. \tag{A.64}$$

Setting $\gamma \equiv \tau K$ and applying (A.61-A.64) then produces the desired bound. Q.E.D.

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Table 2.1

The array of offers and bids in a k-double auction.

	offers	bids
No. > $s_{(m+1)}$	r	w
No. = $s_{(m+1)}$	s	x
No. < $s_{(m+1)}$	t	z

Table 3.1

The dual market to a given market.

	given market	dual market
number of sellers	n	$n^* \equiv m$
number of buyers	m	$m^* \equiv n$
seller's reservation value	c	$c^* \equiv 1-v$
buyer's reservation value	v	$v^* \equiv 1-c$
seller's utility function	C	$C^* \equiv V$
buyer's utility function	V	$V^* \equiv C$
seller's distribution	F	$F^*(c^*) \equiv 1-G(1-c^*)$
buyer's distribution	G	$G^*(v^*) \equiv 1-F(1-v^*)$
seller's strategy	S	$S^*(c^*) \equiv 1-B(1-c^*)$
buyer's strategy	B	$B^*(v^*) \equiv 1-S(1-v^*)$
double auction	k	$k^* \equiv 1-k$

Table 6.1.

Relative inefficiencies of the optimal mechanism, the 0.5-double auction, and the buyer's bid double auction for different market sizes in the case of risk neutral traders and uniform F and G.

m=n	$1-(T^*/T_0)$	$1-(Q_{max}/T_0)$	$1-(Q_{min}/T_0)$	$1-(Q_{BBDA}/T_0)$
1	0.16	0.16	1.00	0.25
2	0.056	0.056	0.063	0.074
4	0.015	0.015	0.016	0.017
6	0.0069	0.0069	0.0070	0.0075
8	0.0039		0.0039	0.0042

<u>Notes</u>. For m = 1, Myerson and Satterthwaite (1983) calculated T^* and established the equality $T^* = Q_{max}$, Williams (1987) computed the smooth equilibrium for the BBDA from which Q_{BBDA} is calculated, and Satterthwaite and Williams (1989a) showed that arbitrarily inefficient smooth equilibria exist. For $m \geq 2$, the values for T^* and T_0 are from Gresik and Satterthwaite (1989, Table 1) and the values for Q_{BBDA} are from Satterthwaite and Williams (1989b).

The values for Q_{max} and Q_{min} are estimated using the following procedure. A grid of initial points in the tetrahedron determined a sample of solution curves. Solution curves within Ω_m determine equilibria while those outside Ω_m were discarded. The relative inefficiency of each of these equilibria was computed; Q_{max} and Q_{min} are approximated by the max and min of these computed inefficiencies. Note that these values provide respectively a lower bound on Q_{max} and an upper bound on Q_{min}.

Calculation of these values posed numerical difficulties. Consequently results are reported to only two significant digits and for m = 8 no value was obtained for $1-(Q_{max}/T_0)$, though we expect it is essentially equal 0.0039.

Table 7.1

Relative loss to a buyer in a 0.5-double auction from deviating from an equilibrium strategy to price-taking behavior in the case of uniform F and G and risk neutral traders.

m	Equilibrium expected profit	Loss from price-taking	Relative loss
2	0.094	0.0055	0.059
4	0.109	0.0017	0.016
6	0.115	0.00080	0.0070
8	0.117	0.00046	0.0039

Notes. For each m the values in the table were computed as follows. For a sample of equilibria $\langle S,B \rangle \in \Omega_m$ we computed (i) the ex ante gain from trade a buyer receives in the equilibrium $\langle S,B \rangle$, (ii) the loss he would suffer from unilaterally switching from B to price-taking behavior. We then averaged across the sample to obtain estimates of expected ex ante gain and expected ex ante loss from truthtelling. The relative loss was then computed by dividing the expected ex ante loss by the expected ex ante gain.

Table 7.2

Relative gain to a buyer in the 0.5-double auction from deviating from price-taking behavior to a best response in the case of uniform F and G and risk neutral traders.

m	Expected profit from price-taking	Gain from best response	Relative gain
2	0.100	0.0088	0.088
4	0.111	0.0022	0.020
6	0.115	0.00093	0.0081
8	0.118	0.00052	0.0044

Notes. For each m the values in the table were computed as follows. The second column's value is the ex ante profit a buyer realizes if he and all other traders honestly report. The third column's value is the increase in ex ante profit the buyer obtains by choosing a best response against the price-taking behavior of the other traders. The fourth column's value is column three's value divided by column two's value.

Figure 3.1. A pair of equilibrium strategies in the case of m = n = 2, risk neutral traders, and uniform F and G. These strategies were computed using the method described in Section 5. The figure illustrates the equalities $\underline{v} = \underline{c}$ and $\bar{c} = \bar{b}$ that are established in Thm. 3.2.

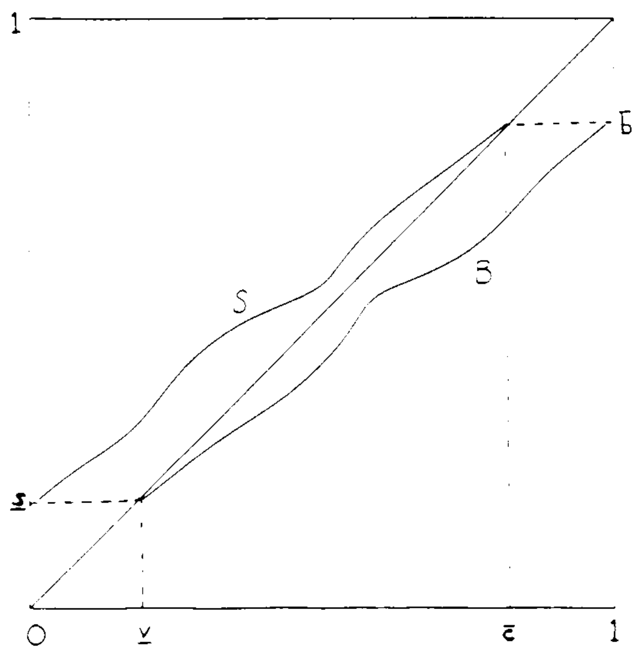

Figure 5.1. The first order conditions (4.1-4.2) for an equilibrium define a vector field on the tetrahedron ABCD. This figure describes the orientation of the tetrahedrons in the figures that follow.

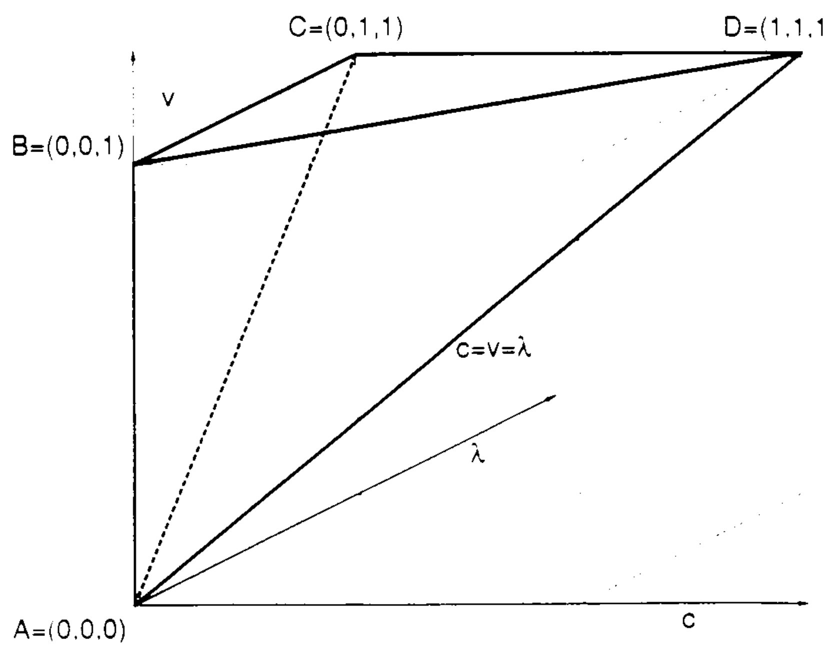

Figure 5.2. The solution curve to the vector field through the point $(c,\lambda,v) = (.395,.5,.565)$ for the case of $m = n = 2$, $k=.5$, and uniform F and G. Appropriate projections of this curve give the equilibrium strategies that are graphed in Figure 3.1.

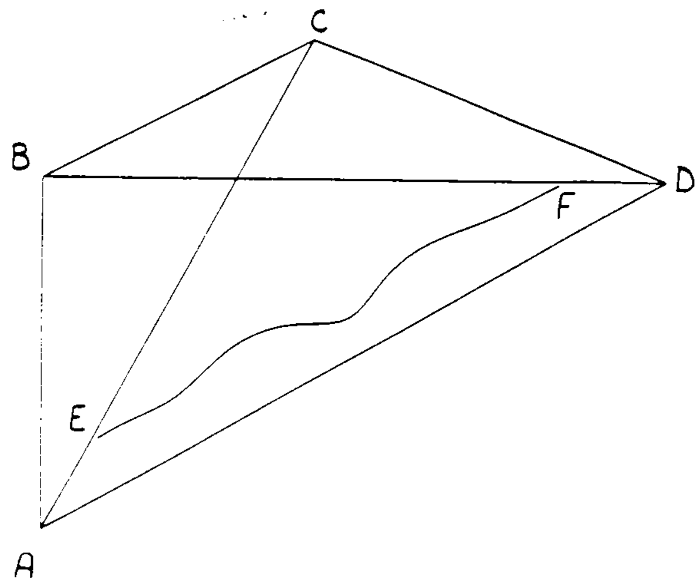

Figure 5.3. In the case of $m = n = 2$, $k=.5$, and uniform F and G, the values $\dot{v}$ and $\dot{c}$ turn negative along the solution curve through $(c,\lambda,v) = (.325,.5,.665)$. The bottom curve B is the (λ,v) projection of this curve and the top curve S is the (c,λ) projection. This solution curve does not represent an equilibrium in the $k=.5$-double auction because the curves S and B do not define λ as increasing functions of c and v respectively.

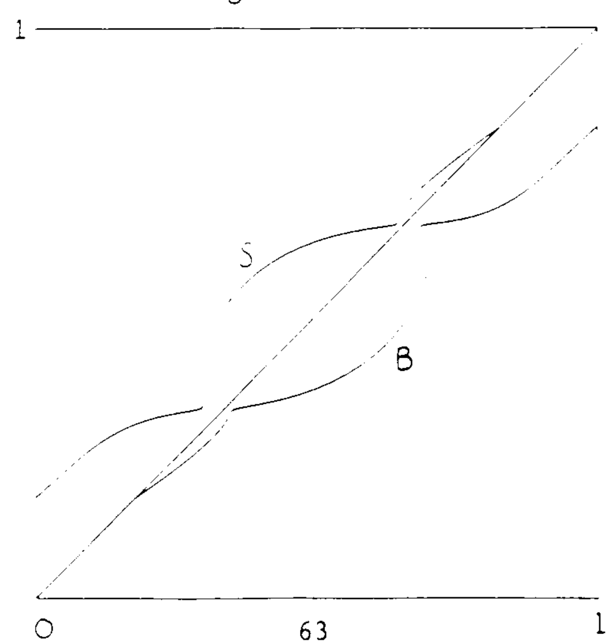

Figure 5.4. The set of smooth equilibria for the case of m = n = 2, k=.5, and uniform F and G.

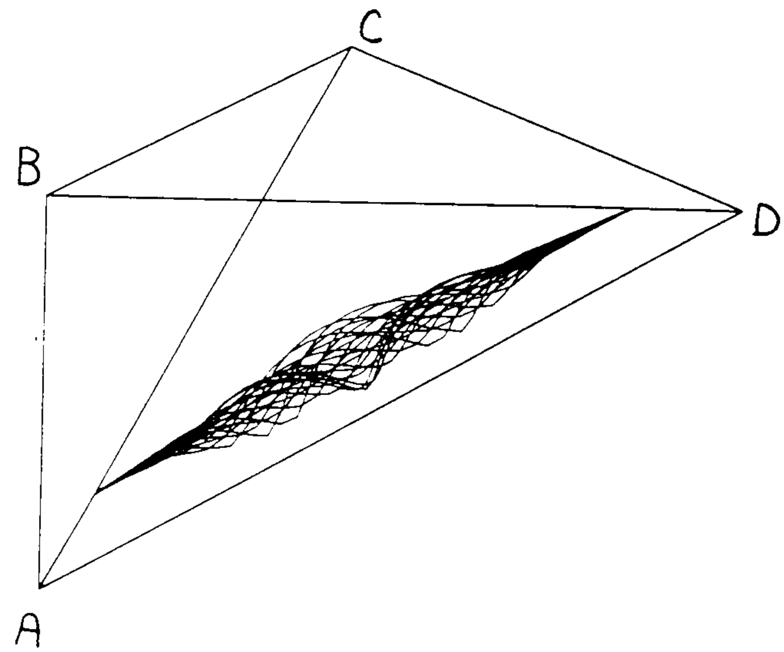

Figure 5.5. The cross-section of the set of smooth equilibria shown in Figure 5.4.

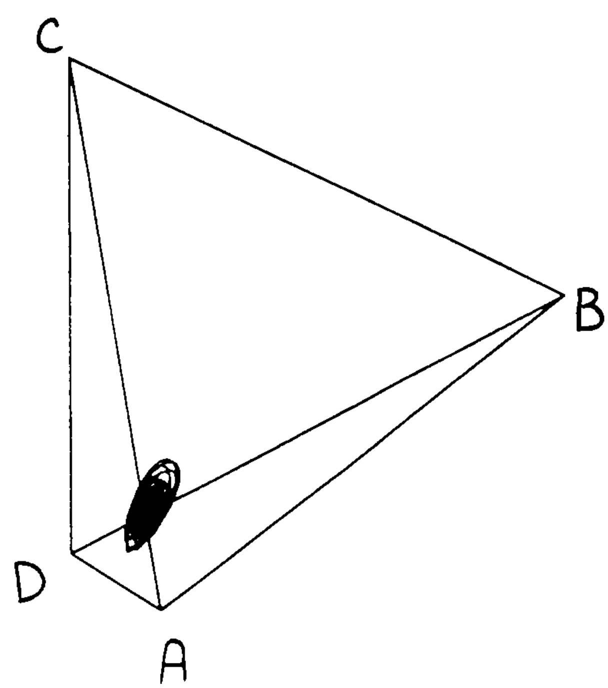

Figure 5.6. The set of smooth equilibria for the case of m = n = 4, k=.5, and uniform F and G.

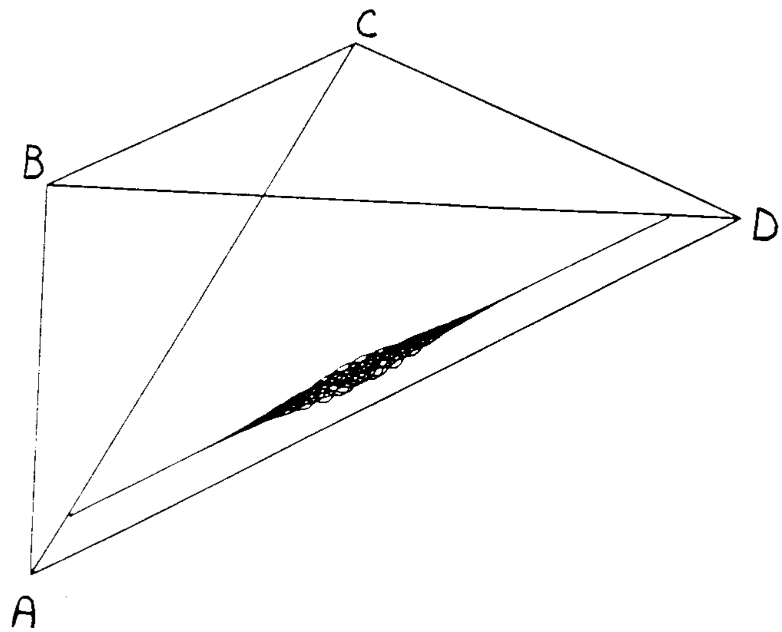

Figure 5.7. The cross-section of the set of smooth equilibria shown in Figure 5.6.

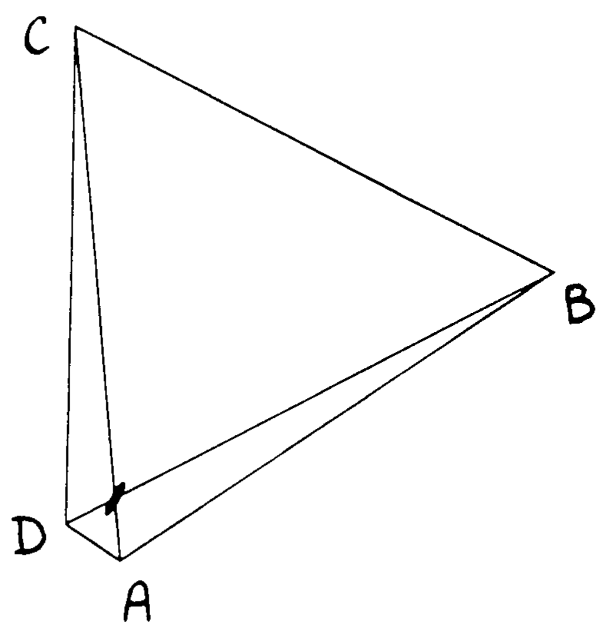

Figure 5.8. The set of smooth equilibria for the case of m = n = 8, k=.5 and uniform F and G.

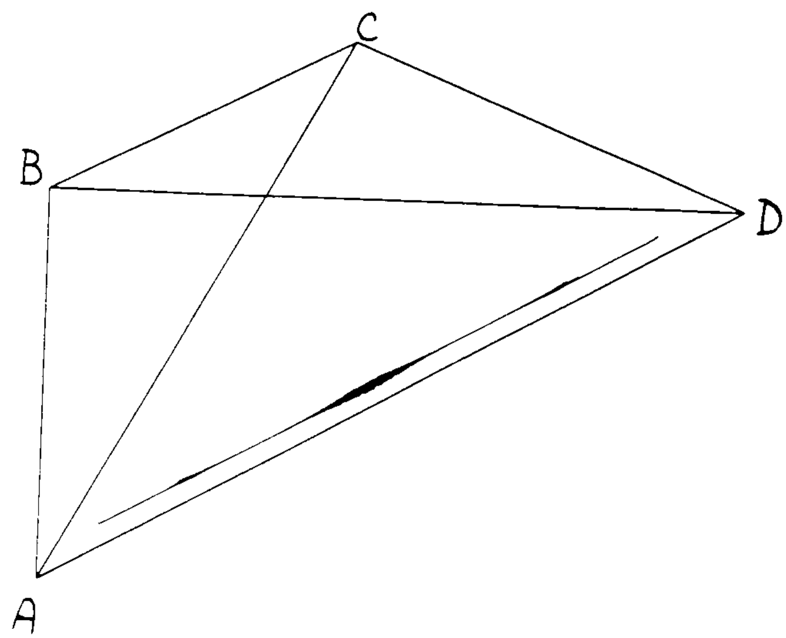

Figure 5.9. The solution curve from Figure 5.2, as it proceeds from E to F, winds counterclockwise around the line EF.

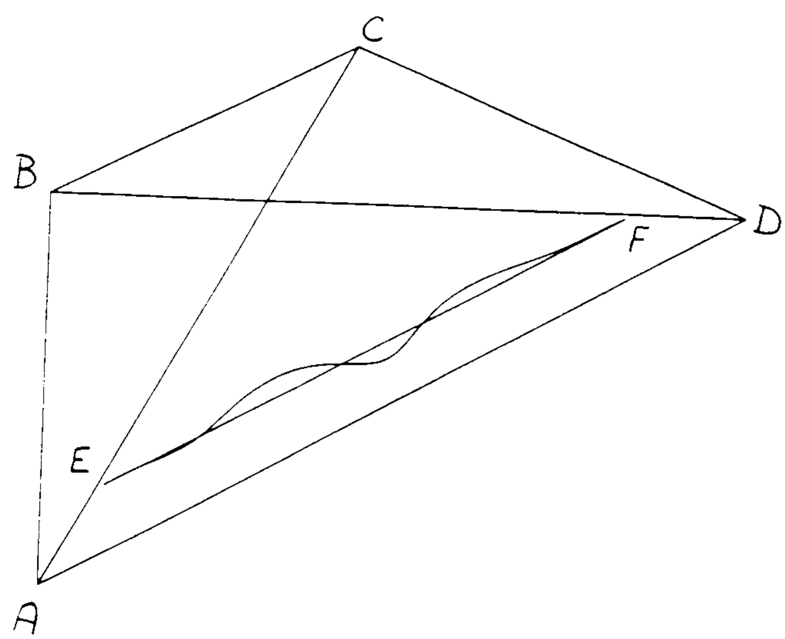

Figure 6.1. In a standard supply-demand diagram, there is a square relationship between the amount κ/m by which demand is underreported and supply is overreported and the corresponding loss in gains from trade (given by the area of the triangle ABC). This is consistent with our bounds in Thms. 5.1 and 6.1.