Discussion Paper No. 913

The Logit as a Model of Product Differentiation: Further Results and Extensions

by

Simon P. Anderson
University of Virginia

and

André de Palma
Northwestern University

December 1989
revised December 1990

JEL Classification Numbers: 022, 611

Keywords. Discrete choice model, CES model oligopolistic competition, multiple product firms, nested logit model.

This paper was initiated when the authors were visiting CEME, Universite Libre de Bruxelles, and the research was partially supported by the "Projet d'Action de Recherche Concertée" of the Belgian Government under contract 84/8965. The second author's research was supported by the National Science Foundation under Award NO. SES-89097[illegible]. We should like to thank two anonymous referees for their comments, and Jacques Thisse for suggestions on the multiproduct firms analysis.

Abstract

There is a growing interest in using the discrete choice approach to study oligopolistic competition under product differentiation, and a prominent discrete choice model is the multinomial logit. Here we analyze various aspects of the logit in this context. We first show that the predictions of the logit are very similar to those of the well-known CES model and explain why this is so. We next discuss the existence and uniqueness of a price equilibrium for a general version of the logit. We then illustrate (using the logit) the flexibility and tractability of the discrete choice approach for two different problems. The first of these finds a free-entry equilibrium with multiproduct firms. The second uses the logit to construct a simple search model. We also introduce the nested logit as an oligopoly model and apply it to these two problems.

I. Introduction

There is a growing interest in the discrete choice approach to modelling product differentiation under oligopoly. Under the discrete choice approach each consumer chooses one option from a number of alternatives. In the original formulation of discrete choice models, a stochastic utility is associated to each alternative, and consumers choose the one (here a variant of the differentiated product) for which this utility is greatest. The distribution of the random variables which describe the valuations of alternatives expresses the distribution of individual preferences. These models (see for example Perloff and Salop (1985) and Sattinger (1984)) are appealing because they start at the level of individual preferences and make assumptions on the distribution of tastes across individuals to construct an aggregate demand system. In recent work, Anderson, de Palma and Thisse (1989) have shown an explicit relation between discrete choice models and characteristics (or address) models in which consumer preferences are continuously distributed over the characteristics space in which products are located. Essentially, the distribution of consumers over the characteristics space plays the role of the probability density in the discrete choice model. The connections between the original formulation of discrete choice models and the characteristics approach-which has yielded many insights in Industrial Organization - is but one reason for studying discrete choice models in this context. Another obvious reason is the rising use of discrete choice models, in particular the logit and the probit, in econometric analysis (see for example Train, McFadden and Ben-Akiva (1987), McFadden (1989) and Feenstra and Levinsohn (1989)). Furthermore, the existence of a price equilibrium has been shown under very general conditions on the distribution of consumer preferences, by Caplin and Nalebuff (1990).

Among the possible discrete choice models the logit plays a key role because of its closed analytical form, its tractability and its econometric properties. For all the reasons above (theoretical foundations, econometric properties, existence results) such a discrete choice model seems to us to provide a very useful tool as a model of product differentiation. However, few authors have studied how a model such as the logit can be widely applied in Industrial Organization.[1] The purpose of this paper is to show how the multinomial logit model can be used and extended to deal with various problems in Industrial Organization. In a previous paper (Anderson and de Palma (1986)) we have used a version of the logit in the oligopoly context to look at the question of whether the market provides the efficient variety of products. A somewhat different version has been used in a similar context by Besanko, Perry and Spady (1990).

In this paper we first recall (in Section II) the logit model used by Anderson and de Palma (1986). Since one of the most commonly used formulations of product differentiation is the well-known CES model, it is a natural idea to compare the results provided by these two models. It is shown that in their simplest versions both models give similar (but not identical) results. We explain the reasons for the similarity by reference to previous results by Anderson, de Palma and Thisse (1987) which show how the CES representative consumer model can be derived from a discrete choice approach. This section then should serve to allay any misgivings about the logit model by placing it in the mainstream of (differentiated products) oligopoly theory.

[1] The logit model has also been usefully employed to extend spatial competition theory by (for example) de Palma et al. (1985), Anderson and de Palma (1988), Braid (1988) and Besanko and Perry (1989).

There follows a series of extensions of the logit model of Section II. In Section III we extend the basic model of Anderson and de Palma (1986) to allow for different classes of consumer and provide sufficient conditions for existence of a price equilibrium. We also show there may exist no (pure strategy) equilibrium by providing a simple example in which the existence results of Caplin and Nalebuff (1990) do not apply.

Because of the explicit microeconomic foundations of the logit it is easy to generalize it in different directions. One useful extension is described in Section IV where we introduce the nested logit model. The nested model allows us to treat the intuitive idea that products may be grouped into clusters (or "nests") with the degree of substitutability of products within a given cluster being higher than for products in different clusters.

In Section V we relax the assumption that each firm produces a single product to consider multiproduct oligopoly. One of the benefits of the explicit functional form of the logit manifests itself here - the equilibrium prices, number of firms and number of products per firm can all be easily computed explicitly. Moreover, these benefits extend to the nested logit model, which can be directly applied to analyze multiple product firms. Section VI explores the properties of the logit and nested logit in the context of consumer search and imperfect information. Finally, Section VII presents our concluding comments.

II. Comparison Between the Logit and CES Demand Systems.

Let us first describe the logit demand functions we shall use for the comparison when there are n products sold at prices $p_1 \ldots p_n$. The multinomial logit demand we consider for good i is

$$X_i = \frac{N \exp(-p_i/\mu)}{\sum_{j=1}^{n} \exp(-p_j/\mu)} \tag{II.1}$$

where N is the number of consumers and μ is a positive parameter to be interpreted as the degree of product heterogeneity.[2] When all products are priced at p, the own price elasticity of demand is given by $\frac{-p}{\mu}\left(\frac{n-1}{n}\right)$ which shows higher μ implies more inelastic demand (greater differentiation between products). Each firm produces one of the n products and profit of firm i is given by

$$\Pi_i = (p_i - c)X_i - K \tag{II.2}$$

where c is marginal cost and K is fixed cost.

In Anderson and de Palma (1986) we worked out the explicit values of price, output per firm, profit per firm and number of firms for four different situations. These values are reported in Table 1a below. The first column describes short-run (or fixed numbers) equilibrium under the Bertrand-Nash assumption. The next column endogenizes the number of firms in the long-run via a zero-profit condition. We then consider two concepts of social optimality, the first-best where the objective is maximization of social surplus (consumer surplus plus profit), and the second-best which maximizes social surplus subject to the constraint that firms make non-negative profits. These are the two social objectives considered by Dixit and Stiglitz (1977)

[2] We are assuming that the prices p_j are low enough for consumers to be able to afford all variants. When this constraint is satisfied the demand for each product in the differentiated goods sector under study is independent of income.

among others. As we noted in our previous paper, the comparative static properties of Table 1.a can be explained intuitively.

The benchmark demand model against which we wish to compare the logit (II.1) is the well-known CES formulation.[3]

The CES representative consumer's utility function is assumed to be given by

$$U = \left(\sum_{i=1}^{n} X_i^{\rho}\right)^{1/\rho} X_0^{\alpha} \tag{II.3}$$

with $\rho \in (0,1)$ and $\alpha > 0$, X_0 the numeraire and X_i, $i=1\ldots n$ the variants of the differentiated product. Maximizing (II.3) subject to the budget constraint $\sum_{i=0}^{n} p_i X_i = Y$ and defining $m \equiv (1-\rho)/\rho \in (0,\infty)$ yields the demand functions

$$X_i = \tilde{Y} \frac{p_i^{-1-1/m}}{\sum_{j=1}^{n} p_j^{-1/m}}; \qquad i = 1\ldots n, \tag{II.4}$$

so that the total spending on the differentiated products is $\tilde{Y} \equiv \frac{Y}{1+\alpha}$. The demand curves have slopes

$$\frac{\partial X_i}{\partial p_i} = \left(-\frac{1}{m} - 1\right)\frac{X_i}{p_i} + \left(\frac{X_i^2}{\tilde{Y}m}\right) \tag{II.5}$$

---

[3] The CES has been used in oligopoly theory by, for example, Blanchard and Kiyotaki (1987), Dixit and Stiglitz (1977), Feenstra and Judd (1982), Flam and Helpman (1987), Koenker and Perry (1981), Krugman (1982), Lawrence and Spiller (1983), Raubitschek (1987) and Spence (1976).

Profit is given by (II.2) and the profit maximizing price by $X_i + (p_i - c)\partial X_i/\partial p_i = 0$. Note that a necessary condition for nonnegative profit is $\tilde{Y} \geq K$. We assume henceforth that this condition holds.

In a symmetric equilibrium, $X = \tilde{Y}/np$. Using this in the first order condition yields an equilibrium price (for a fixed number of firms) as

$$p^* = c + cnm/(n-1) \tag{II.6}$$

with associated output per firm

$$X^* = \frac{\tilde{Y}}{nc}\,\frac{n-1}{(n(1+m)-1)} \tag{II.7}$$

and equilibrium per firm profit

$$\Pi^* = \frac{\tilde{Y}m}{(n(1+m)-1)} \tag{II.8}$$

From (II.8) we can directly compute the free entry equilibrium number of firms such that profits are zero. Substitution in (II.6) and (II.7) then yields the zero profit equilibrium price and output per firm. These values are reported in Table 1b below.

For the first best problem, utility (II.3) is maximized subject to the resource constraint $Y = nK + c\sum_{i=1}^{n} X_i + X_0$, where the social cost of producing the numeraire is normalized to unity. At a symmetric solution, the utility function incorporating the constraint is

$$U^w = nX^\rho[Y - nK - ncX]^{\alpha\rho} \tag{II.9}$$

Maximizing (II.9) with respect to X and n gives the values in Table 1b.

To solve the second best problem, utility (II.3) is maximized subject to consumer reactions (the demand functions (II.4)) and the zero profit constraint. At a symmetric solution, these constraints are $X = \tilde{Y}/np$ and $p = c\tilde{Y}/(\tilde{Y}-nK)$ respectively. Substituting into (II.3) yields

$$U^S = n \left[\frac{\tilde{Y}-nK}{nc}\right]^{\rho} (\tilde{Y}\alpha)^{\alpha\rho} \qquad (II.10)$$

Maximization with respect to n enables us to find the second best optimal number of firms; substitution into the constraints finds the corresponding price and output per firm.

	Fixed Numbers	Free entry Equilibrium	First best Optimum	Second best Optimum
price	$c + \frac{\mu n}{n-1}$	$c + \mu + \frac{K}{N}$	c	$c + \mu$
output per firm	$\frac{N}{n}$	$\frac{KN}{K+\mu N}$	$\frac{K}{\mu}$	$\frac{K}{\mu}$
profit per firm	$\frac{\mu N}{n-1} - K$	0	-K	0
number of firms	n	$1 + \frac{N\mu}{K}$	$\frac{N\mu}{K}$	$\frac{N\mu}{K}$

Table 1a. Comparison of equilibrium and optimum for the logit model (II.1)

	Fixed Numbers	Free entry Equilibrium	First best Optimum	Second best Optimum
price	$c + \frac{cmn}{n-1}$	$\frac{c\tilde{Y}(1+m)}{\tilde{Y}-K}$	c	$c(1+m)$
output per firm	$\frac{\tilde{Y}(n-1)}{nc[n(1+m)-1]}$	$(\frac{\tilde{Y}-K}{\tilde{Y}m+K})\frac{K}{c}$	K/mc	K/mc
profit per firm	$\frac{\tilde{Y}m}{[n(1+m)-1]}-K$	0	$-K$	0
number of firms	n	$\frac{\tilde{Y}m}{(1+m)K}+\frac{1}{1+m}$	$\frac{\tilde{Y}m(1+\alpha)}{(1+\alpha+m)K}$	$\frac{\tilde{Y}m}{(1+m)K}$

Table 1b. Comparison of equilibrium and optimum for the CES representative consumer (II.4).

Qualitatively, there is much similarity between the two models. K plays a similar role in both, and $\tilde{Y}$ and N play roughly the same roles (the reason is given below). Likewise μ and $m(= \frac{1-\rho}{\rho})$ act very similarly (these are the "preference for diversity" parameters). Also, first and second best optimal outputs are equal in both models. We find a one-firm difference between the entry equilibrium and constrained optimal number of firms for the logit; for the CES the difference is $0 < \rho < 1$ firms. For the CES the first best optimum number of firms exceeds the second best one, although they are equal for the logit.[4] It is not our intention here to discuss the problem of the optimal number firms (for an insightful analysis of this question, see Deneckere and Rothschild (1986)). Rather we wish to highlight the similarities between two models.

[4] When the logit is extended to allow for outside alternatives, the logit has the same property as the CES (see Anderson and de Palma (1986)).

All in all, the logit and the CES yield very similar predictions (although they do not exactly twin with each other). We can explain this similarity by returning to the derivation of the logit model provided by McFadden (1973). This author considers a population of N statistically independent and identical consumers, each of whom will purchase one unit of the product which yields the greatest utility. The conditional indirect utility of a consumer choosing product i is given by

$$u_i = a+y - p_i + \mu\epsilon_i \qquad i = 1 \ldots n \tag{II.11}$$

where y is consumer income, a is a quality index, and the ϵ_i's are random variables which are ii double exponentially distributed, i.e.

$$\text{Prob } (\epsilon_i < x) = \exp\ (-\exp - x). \tag{II.12}$$

The expected demand for product i is given by $X_i = N$ Prob ($u_i = \max u_j$, $j = 1 \ldots n$), which under the above assumptions yields the logit form (II.1).

A similar disaggregated discrete choice foundation can be provided for the CES demands (II.4). Instead of the form (II.11), suppose the conditional indirect utility is given by

$$u_i = a + (1 + \alpha)\ln y - \ln p_i + \mu\epsilon_i \qquad i = 1 \ldots n. \tag{II.13}$$

Using Roy's lemma, the demand of a consumer choosing product i is $\tilde{y}/p_i$, where $\tilde{y} = y/(1+\alpha)$. Under the assumption that ϵ_i is ii double exponentially distributed, Anderson et al. (1987) have shown the resulting demand functions are given by the CES form (II.4) with $Ny = Y$ and $\mu = m(-\frac{1-\rho}{\rho})$.

Viewed in this light, i.e. from the perspective of possible microeconomic foundations, we see the reasons for the similarity in predictions between the two models. Both can be seen as the aggregation of the demand of N individuals making discrete choices of products. Moreover, in both models individual preferences are distributed according to the same distribution (the double exponential). However there is an important difference in the models. In casting the CES as a discrete choice model, it is assumed each consumer spends a *fixed money amount* on the good selected. By contrast, in the logit model, each consumer is assumed to buy a *single unit* of the good chosen.

The comparison in this section shows the logit gives qualitatively similar results to the CES and can be derived from similar preference foundations. In the rest of this paper we consider various extensions to the basic logit model (II.1). For the most part these extensions exploit the individual choice formulation (II.11).

III. Existence and Non-Existence of Equilibrium

We now investigate the conditions under which a pure strategy Bertrand-Nash equilibrium exists for a fixed number of firms, n, producing at constant marginal cost c_i, $i = 1...n$. We shall consider a more general demand function than (II.1), and use the discrete choice model (II.11) extended to encompass both different consumer types and the possibility of a consumer choosing not to purchase one of the n products. That is, we allow the N consumers to be (possibly) of different types and allow also for non-purchase. This is close to the use of the logit in econometrics, and we shall interpret the discrete choice model below in that vein. Formally, each consumer is to choose one unit of one of n+1 possible alternatives. Without loss of generality we let option zero be the null option of not purchasing within the group. The conditional indirect utility of a consumer h purchasing option i at price p_i is given by[5]

$$U_{ih} = a_{ih} - p_i + \mu\epsilon_{ih}; \quad i = 0,1,\ldots n; \quad h = 1,\ldots N. \qquad \text{(III.1)}$$

where the ϵ_{ih} are ii double exponentially distributed random variables. The term a_{ih} represents observable characteristics and may further be split (if desired) into consumer attributes (such as income) and pure product attributes (such as mileage per gallon) see Amemiya (1981) for example. The term $a_{ih} - p_i$ is frequently called the measured utility. The random term represents unobservable taste differences, unobservable product attributes, etc., (see eg., Ben Akiva and Lerman (1986)); different consumers evaluate differently the large number of characteristics inherent in any product. The parameter $\mu > 0$ measures the relative importance of this effect.

Each agent chooses the option i for which the conditional utility is greatest. The probability - as modelled by the firm - of consumer type h purchasing option i is given by:

$$P_{ih} = \frac{\exp((a_{ih} - p_i)/\mu)}{\sum_{j=0}^{n} \exp((a_{jh} - p_j)/\mu)}; \quad i = 0,1,\ldots n, \quad h = 1\ldots N \qquad \text{(III.2)}$$

[5] Without restriction, it may be that $p_0 = 0$ (there may not always be a price associated with the outside option).

Two limit cases are worthy of note. For $\mu \to 0$, consumer choice is perfectly predictable - the option for which $(a_{jh} - p_j)$ is greatest is chosen with certainty by consumer h. Secondly, for $\mu \to \infty$, $\mathbb{P}_{ih} \to \frac{1}{n+1}$, and differences in product and consumer attributes have no predictive power. The demand functions (III.2) have two important derivative properties[6] used below:

$$\frac{\partial \mathbb{P}_{ih}}{\partial p_i} = \frac{\mathbb{P}_{ih}(\mathbb{P}_{ih} - 1)}{\mu} \qquad i = 0,1,\ldots n, \qquad \text{(III.3)}$$

and

$$\frac{\partial \mathbb{P}_{ih}}{\partial p_j} = \frac{\mathbb{P}_{ih}\mathbb{P}_{jh}}{\mu} \qquad i,\ j = 0,1\ldots n, \quad j \neq i\ . \qquad \text{(III.4)}$$

Hence products are symmetric substitutes. We can write firm i's expected profit as

$$\Pi_i = (p_i - c_i) \sum_{h=1}^{N} \mathbb{P}_{ih}; \qquad i = 1\ldots n\ . \qquad \text{(III.5)}$$

The profit derivative is

$$\frac{d\Pi_i}{dp_i} = \sum_{h=1}^{N} \mathbb{P}_{ih} + (p_i - c_i) \sum_{h=1}^{N} \mathbb{P}_{ih}(\mathbb{P}_{ih} - 1)/\mu; \qquad i = 1\ldots n \qquad \text{(III.6)}$$

---

[6] The logit model has sometimes been criticised for exhibiting the Independence of Irrelevant Alternatives (IIA) property, as illustrated by the famous red bus/blue bus example of Debreu (1960). However, as noted by Train (1986) (p. 22), this can be partially dealt with once we allow for differences in the parameters a_{ih}.

Now, noting that $P_{ih} \in (0,1)$ for all finite prices, the derivative (III.6) is positive when evaluated at $p_i = c_i$, at which point profit is zero. Furthermore the profit function is twice continuously differentiable and $\lim_{p_i \to \infty} \Pi_i = 0$ (by l'Hospital's rule). Hence a maximum in Π_i is characterized by (III.6) identically zero, or

$$\frac{p_i - c_i}{\mu} = \frac{\sum_{h=1}^{N} P_{ih}}{\sum_{h=1}^{N} P_{ih}(1 - P_{ih})} > 0 \; ; \quad i = 1 \ldots n \, . \qquad \text{(III.7)}$$

Notice that when an equilibrium exists, price exceeds marginal cost regardless of the level of cost, as long as the number of firms is finite and $\mu > 0$. It is shown in Appendix 1 that any solution to (III.7) is a local maximum if and only if

$$\sum_{h=1}^{N} \sum_{k=1}^{N} P_{ih} P_{ik} \{(P_{ik} - P_{ih})^2 - (1 - P_{ih})\} < 0; \quad i = 1 \ldots n \quad \text{(III.8)}$$

at that solution. For instance, if all consumers are of the same type then $P_{ik} = P_{ih}$ for all h, k = 1...N (as $a_{ik} = a_{ih}$) and (III.8) is guaranteed to hold. In this case then the profit functions are necessarily strictly quasi-concave in own price (although note that they are not concave). Equilibrium existence then follows from a standard fixed point argument[7] (see

[7] We also require the strategy spaces to be bounded. Here we can use the same trick as Caplin and Nalebuff (1990): no consumer will buy if price exceeds income so that demand falls to zero when a firm sets a price greater than income. Price strategies are bounded below by c and above by income.

Friedman (1986) for example). Hence *there exists a pure strategy price equilibrium when all consumers are of the same type*. It is further shown in Appendix 2 that this equilibrium is unique.

Consider now the case when consumers are of different types (that is, $a_{ih} \neq a_{ik}$ for some h, k = 1...N). From (III.8), a stronger sufficiency condition is

$$2(\mathbb{P}_{ik} - \mathbb{P}_{ih})^2 - (2 - \mathbb{P}_{ih} - \mathbb{P}_{ik}) < 0 \text{ ; for all h, k = 1...N} \quad \text{(III.9)}$$

This in turn is guaranteed to be satisfied if

$$|\mathbb{P}_{ih} - \mathbb{P}_{ik}| < 1/2; \quad h, k = 1...N \qquad \text{(III.10)}$$

Thus equilibrium existence is guaranteed unless there is a large difference in consumer types (leading to large differences in purchase probabilities and refuting (III.10)). The role of μ is important here. The larger is μ the smaller is the relative influence exerted by the a_{ih} terms which differentiate types. This increases the likelihood (III.10) will hold, and for μ large enough it will necessarily hold.[8]

To illustrate what may go awry here, let us consider the limit case $\mu = 0$. Now each consumer will purchase the product for which $a_{ih} - p_i$ is greatest (see II.11)). In the event of a tie, demand is presumed to be split equally among tying firms. Let there be two consumer types in numbers N_A and N_B, and two firms, i = 1,2. Furthermore, let there be no outside option

[8] A similar role is played by μ in the Hotelling model of spatial competition when extended to allow for product differentiation à la logit. There existence is guaranteed for μ large enough, but there is no pure strategy equilibrium for μ sufficiently small (de Palma et al. (1985)).

($a_0 \to -\infty$), and let us label firms and products such that $a_{1A} > a_{2A}$ and $a_{1A} - a_{2A} > a_{1B} - a_{2B}$. Hence type A's have a relative preference for good 1 and the B's have a weaker relative preference - indeed, they may prefer good 2. The demand addressed to firm 1 is then given by

$$X_1 = \begin{cases} 0 & \text{for } a_{1A} - p_1 < a_{2A} - p_2 \\ N_A/2 & \text{for } a_{1A} - p_1 = a_{2A} - p_2 \text{ and } a_{1B} - p_1 < a_{2B} - p_2 \\ N_A & \text{for } a_{1A} - p_1 > a_{2A} - p_2 \text{ and } a_{1B} - p_1 < a_{2B} - p_2 \\ N_A + \frac{N_B}{2} & \text{for } a_{1A} - p_1 > a_{2A} - p_2 \text{ and } a_{1B} - p_2 = a_{2B} - p_2 \\ N_A + N_B & \text{for } a_{1B} - p_1 > a_{2B} - p_2 \end{cases} \quad \text{(III.11)}$$

and $X_2 = N_A + N_B - X_1$. This demand function is illustrated in Figure 1.

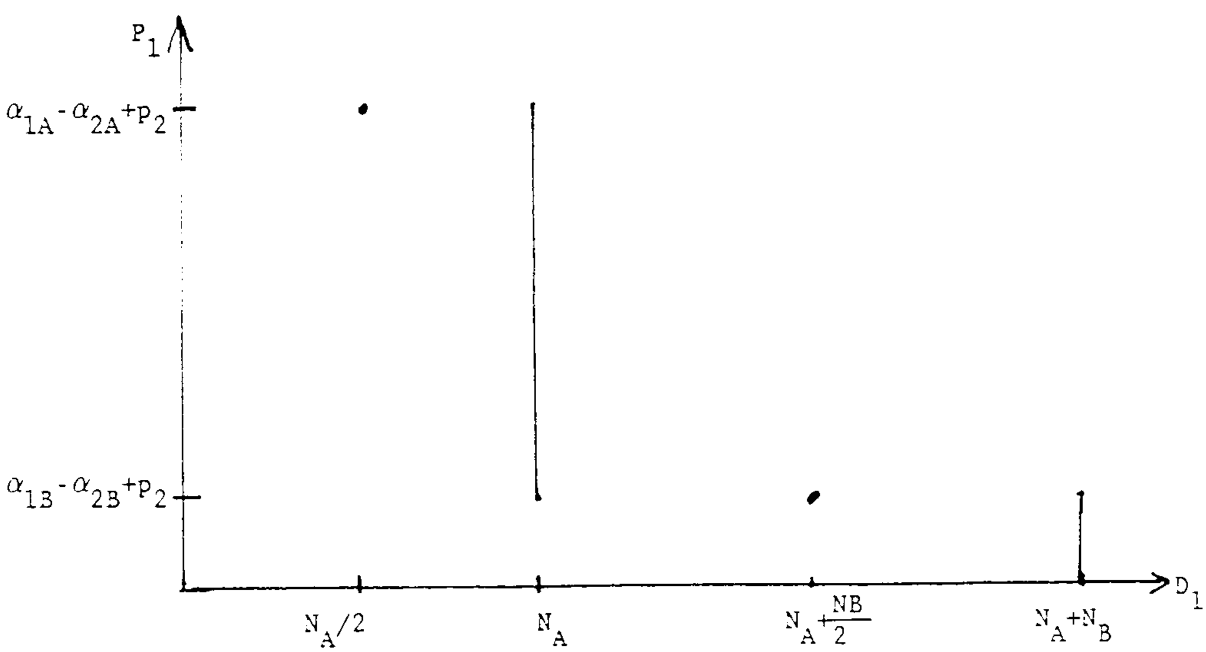

Figure 1. The Demand Function for the Case $\mu = 0$ and Two Consumer Types

Supposing production costs to be zero, it is straightforward to show (see Appendix 3) that no equilibrium in pure price strategies exists for $(N_A + N_B)(a_{1B} - a_{2B}) < N_A(a_{1A} - a_{2A})$. As previously noted, introducing further consumer heterogeneity into the model via the parameter μ will serve to smooth the discontinuities in the profit functions and to increase the likelihood of finding an equilibrium.

IV. The Nested Logit Model

The logit model has some very strong properties, especially in the form (II.1) when there is only one type of consumer. In particular, when all prices and qualities are the same, the cross price elasticities of demand are all equal (to $\frac{p}{n\mu}$ when there is no outside option). The nested logit allows for products to be clustered in groups in the sense that cross-price elasticities within a group will exceed those for products in different groups. However, the nested version still retains (as we show in the next sections) the tractability of the simple logit formulation, and allows for the latter as a special case.

To illustrate, suppose there are n product groups, product group i consisting of m_i products, $i = 1 \ldots n$. The demand for product k belonging to product group i is

$$X_{ik} = NP_{ik} = N\, P_i\, P_{k|i} \tag{IV.1}$$

where

$$P_{k|i} = \frac{\exp[(a_{ik} - p_{ik})/\mu_2]}{\sum_{\ell=1}^{m_i} \exp[(a_{i\ell} - p_{i\ell})/\mu_2]} \qquad (IV.2)$$

and

$$P_i = \frac{\exp(A_i/\mu_1)}{\sum_{j=1}^{n} \exp(A_j/\mu_1)}, \qquad (IV.3)$$

with

$$A_j = \mu_2 \ln \sum_{\ell=1}^{m_j} \exp[(a_{j\ell} - p_{j\ell})/\mu_2] \qquad (IV.4)$$

and $\mu_1 \geq \mu_2 \geq 0$.

This formulation can be understood intuitively as follows (see Ben Akiva (1973)): consumer choice can be seen as a sequential process in which first a group (or "nest) is chosen, and then, conditional on the choice of nest, a particular product is selected. The two choice processes are made according to a logit model. The choice within a nest is described by (IV.2). The choice within a nest is described by (IV.3), where A_j (see (IV.4)) measures the expected benefit of choosing nest j. Indeed, it can be shown that $A_j = E[\max_{\ell} (a_{j\ell} - p_{j\ell} + \mu_2 \epsilon_{j\ell})]$ where $e_{j\ell}$ are i.i. double exponentially distributed.

McFadden (1978) has shown that the nested logit model is consistent with the maximization of a random utility function of the the form (II.11) and arises from a specific correlation between the ϵ. The distribution function depends on two parameters, μ_1 and μ_2, with $\mu_1 \geq \mu_2$.[9]

The parameter μ_1 can be viewed as a measure of heterogeneity of groups; whereas μ_2 represents intergroup heterogeneity. The condition $\mu_1 \geq \mu_2$, has an intuitive interpretation: the products within a group are more similar than products belonging to different groups. Note that the cross price derivative for products in the same nest is $\partial P_{ik}/\partial p_{i\ell} = P_i P_{k|i} P_{\ell|i} [\frac{1}{\mu_2} - \frac{(1 - P_i)}{\mu_1}]$ with $\ell \neq k$; therefore the cross price derivatives are always positive provided that $\mu_1 \geq \mu_2$. When $\mu_1 = \mu_2$, all products are equally "close" and groups therefore cannot be distinguished: it can easily be verified that (IV.1) through (IV.4) reduce to the simple logit model with $\sum_{j=1}^{n} m_j$ products. On the other hand, when $\mu_2 = 0$, the products within each group are perfect substitutes, and only the product with the largest value of $a_{j\ell} - p_{j\ell}$ will be selected in any given nest j.

To see the role of μ_1 and μ_2, suppose all products are priced at p and have the same quality a; moreover each nest comprises m products so that there are nm products in total. The cross price elasticity of demand for products in different nests is $\frac{p}{nm\mu_1}$, while for products within the same nest it is $\frac{p}{nm}(\frac{n}{\mu_2} - \frac{n - 1}{\mu_1})$. Clearly the latter exceeds the former for $\mu_1 > \mu_2$, and they are equal for $\mu_1 = \mu_2$.

[9] The distribution function for the nested logit model is given by $F(x_1 \ldots x_n) = \exp[-H(e^{-x_1} \ldots e^{-x_n})]$ with $H(x_1 \ldots x_n) = \sum_{j=1}^{n} \left[\sum_{\ell=1}^{m_j} x_\ell^{1/\mu_2} \right]^{\mu_2/\mu_1}$.

The consumer surplus associated with the nested logit model is

$$CS = \mu_1 \ln \sum_{j=1}^{n} \exp[A_j/\mu_1] + Y ; \qquad \text{(IV.5)}$$

it can be verified that using Roy's lemma that the choice probabilities P_{ik} can be recovered. When prices are all equal to p, qualities are all equal to a, and the number of products in each nest is m, (IV.5) reduces to

$$CS = \mu_1 \ln(n) + \mu_2 \ln(m) + a - p + y \qquad \text{(IV.6)}$$

i.e., consumer surplus increases as the number of clusters, n, or the number of products per cluster, m, increases.

V. Multiproduct Firms

The analysis of multiproduct firms under product differentiation is as rare in economic theory as it is common in actual markets. This is presumably because such analysis is exceedingly complex - although we should note recent work by Raubitschek (1987) has succeeded in shedding some light on the problem using the CES model.[10] We show now that the logit model can readily be extended to deal with situations where firms sell several products.

[10] Raubitschek (1987) assumes that all products owned by a firm are operated individually by independent profit-maximizing managers. In what follows, we assume instead that all pricing decisions are coordinated by the firm to maximize overall profit.

(a) The multinomial logit.

Suppose there are n firms and firm i produces m_i products, $\ell = 1 \ldots m_i$, with qualities $a_{i\ell}$ and marginal costs $c_{i\ell}$ per product, and prices $p_{i\ell}$. Its net revenue is given by

$$R_i = N \sum_{\ell=1}^{m_i} (p_{i\ell} - c_{i\ell}) \mathbb{P}_{i\ell} \tag{V.1}$$

where

$$\mathbb{P}_{ik} = \frac{\exp\left[(a_{ik} - p_{ik})/\mu\right]}{\sum_{j=1}^{n} \sum_{\ell=1}^{m_j} \exp\left[(a_{j\ell} - p_{j\ell})/\mu\right]} . \tag{V.2}$$

which corresponds to the nested logit of Section IV with $\mu_1 = \mu_2 = \mu$. The first order condition with respect to p_{i1} is

$$\frac{dR_i}{dp_{i1}} = \frac{N\mathbb{P}_{i1}}{\mu} \left\{ \mu - (p_{i1} - c_{i1}) + \sum_{\ell=1}^{m_i} (p_{i\ell} - c_{i\ell}) \ \mathbb{P}_{i\ell} \right\} \tag{V.3}$$

Setting this equal to zero yields

$$p_{i1} - c_{i1} = \sum_{\ell=1}^{m_i} (p_{i\ell} - c_{i\ell}) \ \mathbb{P}_{i\ell} + \mu \equiv k_i \tag{V.4}$$

so that the mark up per product, k_i, is the same for all products sold by firm i. This is a rather restrictive property (which also holds for the nested logit model) which depends on the fact that individuals care only about price and quality differences. It is not such a limitation in the analysis below which considers a symmetric environment.

Substituting k_i back in (V.4) gives

$$k_i = \frac{\mu}{1 - \sum_{\ell=1}^{m_i} \mathbb{P}_{i\ell}} \qquad i = 1\ldots m \tag{V.5}$$

We now assume the a's and the c's are identical for all products. Fixed cost per product is F, set-up cost per firm is K (abusing slightly our previous usage). We shall find a symmetric free entry equilibrium where each firm produces the same number of products at the same price. In this case when each of n firms produces m products (V.5) becomes

$$p - c = k = \frac{\mu n}{n - 1} , \tag{V.6}$$

which is independent of m (see also Table 1a). There are two effects at work when m rises. On the one hand, more products on the market tends to depress the market price. On the other hand each firm internalizes the effect of price changes on its own products, which in itself is a force which leads prices to rise. Here these effects exactly cancel.[11]

The equilibrium concept we use to fully solve the model can be viewed as a three-stage process. In the first stage a number of firms decide to enter the market. In the second stage they decide how many products to produce. The

---

[11] We expect that introducting outside alternatives (for example as in Section III) will cause equilibrium mark-up to rise with m. In this case more consumers will be drawn from the outside alternative as there is greater product variety as m rises.

last stage is the price equilibrium described above. We want to find a symmetric equilibrium in the second stage, for a given number of entrants n. To do so, we suppose n-1 of the firms choose m_2 products each, and look at the decision problem of the first firm, which is to choose m_1 products. Hence from (V.4) and (V.5) we have the price charged by firm 1 at the last-stage price equilibrium as

$$p_1 - c = \frac{\mu}{1 - \mathbb{P}_1} = \frac{\mu}{(n-1)\mathbb{P}_2} \tag{V.7}$$

and for all other firms we have

$$p_2 - c = \frac{\mu}{1 - \mathbb{P}_2} \tag{V.8}$$

where (from (V.2))

$$\mathbb{P}_1 = \frac{m_1 \exp(-p_1/\mu)}{m_1 \exp(-p_1/\mu) + (n-1)m_2 \exp(-p_2/\mu)} \tag{V.9}$$

which is the probability a consumer buys any one of firm 1's products, and $\mathbb{P}_2$ is similarly defined (using $\mathbb{P}_1 + (n - 1)\mathbb{P}_2 = 1$). The proof that the prices p_1 and p_2 are indeed an equilibrium to the price subgame contingent upon m_1 and m_2 is provided in Appendix 4 and is discussed further below.

Introducing the subgame equilibrium prices into the profit function, firm 1's problem in the second (product range) stage of the game is thus to maximize

$$\Pi_1 = \frac{N\mu}{(n-1)P_2} P_1 - m_1 F - K$$

$$= \frac{N\mu m_1}{(n-1)m_2} \exp\chi - m_1 F - K \tag{V.10}$$

with respect to m_1, where $\chi \equiv \dfrac{p_2 - p_1}{\mu}$ and p_1 and p_2 are given by (V.7) and (V.8). The derivative of (V.10) is

$$\frac{d\Pi_1}{dm_1} = \frac{N\mu \exp\chi}{(n-1)m_2}\left(1 + m_1 \frac{d\chi}{dm_1}\right) - F \tag{V.11}$$

where, evaluating at a symmetric equilibrium, $m_1 = m_2 = m^*$, and $\chi = 0$, $\dfrac{d\chi}{dm_1}$ is equal to[12] $\dfrac{-n}{(n^2 - n + 1)\, m^*}$ and the solution to (V.11) is

$$m^* = \frac{N\mu}{F} \frac{n - 1}{(n^2 - n + 1)} . \tag{V.12}$$

Clearly, for any n, there exist values of N, μ and F such that each firm only chooses a single product. The greater is the preference for differentiated products μ, the greater is the variety provided by each firm. Furthermore, (V.12) is decreasing in n for $n \geq 2$, as expected. It is proved in Appendix 5 that $m_1 = m^*$ does indeed maximize (V.10) when $m_2 = m^*$. The equilibrium price is given by (V.6). The first (entry) stage is now simple. If there are n firms in the market, each producing m^* products, the profit per firm is

[12] Using $\chi = \dfrac{[m_2 - m_1\exp\chi][(n-1)m_2 + m_1\exp\chi]}{(n-1)m_2[(n-2)m_2 + m_1\exp\chi]}$.

$$\Pi(n) = \frac{N\mu n}{(n-1)(n^2 - n + 1)} - K \qquad (V.13)$$

It is readily shown that this is decreasing in n so that there is a single solution n^* to $\Pi(n) = 0$. When $n = n^*$, no firm wishes to enter the market, and no firm wishes to leave. Note that n^* is independent of F, is decreasing with K, and increasing in μ. Hence higher μ leads both to more firms and to more products per firm. The case where each firm chooses a single product arises either if K is sufficiently small (so there are many firms) or if F is sufficiently large.

(b) *The Nested Logit*

We now turn to the application of the nested logit to the multiple product firm problem. We shall assume that the products produced by any firm are closer substitutes for each other than they are for products produced by different firms.[13] That is, μ_2 is the degree of heterogeneity across products of any given firm, whereas μ_1 is the heterogeneity across firms, with $\mu_1 \geq \mu_2$.[14] The demand system is then given by (IV.1) through (IV.4) where n is the number of firms and m_j is the number of products produced by firm j, $j = 1 \ldots n$. We can now retrace the steps for the simple logit model since the structure of the argument is similar.

We again consider a three stage entry - product range - price equilibrium defined as above, and solve recursively. Given n firms, and supposing firm 1 has m_1 products, with the remaining n-1 firms having m_2 products each, we show

[13] An alternative assumption, which may fit some markets better, is to suppose product groups (nests) consist of products produced by different firms.

[14] In the context of the two stage decision process envisaged by Ben-Akiva (1973) and described in Section IV, consumers can be viewed as first selecting a firm, and then selecting one of its products.

in Appendix 4 that firm 1 will sell all its products at the same price, p_1, and all other firms will set the same price, p_2, for all their products. Hence firm 1's profit is

$$\Pi_1 = N(p_1 - c)\mathbb{P}_1 - m_1F - K \qquad \text{(V.14)}$$

with

$$\mathbb{P}_1 = \frac{\exp[\frac{\mu_2}{\mu_1}\ln(m_1)]}{(n-1)\exp[\frac{\mu_2}{\mu_1}\ln(m_2) - \chi) + \exp[\frac{\mu_2}{\mu_1}\ln(m_1)]} \qquad \text{(V.15)}$$

where $\chi = \dfrac{p_2 - p_1}{\mu_1}$. Solving the first order conditions to find the price equilibrium we obtain the same equations (V.7) and (V.8) as before, except with μ_1 now replacing μ (this is because μ_1 represents interfirm heterogeneity - although note that μ_2 appears in the expressions for $\mathbb{P}_1$ and $\mathbb{P}_2$, as per (V.15)).

It has still to be shown that the prices p_1 and p_2 as given above do constitute a price equilibrium for given m_1 and m_2. It seems intractable to address this problem by showing the Hessian of the profit function is negative definite at the first order conditions. Likewise, the results of Caplin and Nalebuff (1990) apply only to single product firms. In Appendix 4 we use an alternative method of proof. This first shows that there is a unique solution to the first order conditions for each individual firm as well as a unique solution to the whole system of first order conditions. We then construct a hypercube $[c,\bar{p}]^{m_i}$, in firm i's price strategy space and show that i's profit is maximized in the interior of the hypercube. Firm i's profit therefore has a unique global maximizer which is in the interior of $[c,\bar{p}]^{m_i}$. Since this global maximizer satisfies i's first order conditions, there is a unique Nash equilibrium for the price subgame. Using the price equilibrium expression in (V.14) we have

$$\Pi_1 = \frac{N\mu_1}{n-1}\left(\frac{m_1}{m_2}\right)^{\frac{\mu_2}{\mu_1}} \exp\gamma - m_1 F - K , \tag{V.16}$$

which is the generalization of (V.10). Now,

$$\frac{d\Pi_1}{dm_1} = \frac{N\mu_1}{n-1}\left(\frac{m_1}{m_2}\right)^{\frac{\mu_2}{\mu_1}} \exp\gamma\left[\frac{\mu_2}{\mu_1}\frac{1}{m_1} + \frac{d\gamma}{dm_1}\right] - F \tag{V.17}$$

and, at a symmetric equilibrium with $m_1 = m_2 = m^*$, $\frac{d\gamma}{dm_1}$ equals $\frac{-\mu_2}{\mu_1}\frac{1}{m_1}\frac{n}{n^2 - n + 1}$ so that (V.17) is solved by

$$m^* = N\frac{\mu_2}{F}\frac{n-1}{n^2 - n + 1} . \tag{V.18}$$

The corresponding equilibrium prices are

$$p^* = c + \frac{\mu_1 n}{n-1} \tag{V.19}$$

Note that (V.18) is the same as (V.12) except the relevant μ is now μ_2. Loosely, μ_2 matters since it measures the relative attractiveness of the product ranges per se (see (IV.6)). The proof of existence of the symmetric equilibrium for the product range game is given in Appendix 5.

Substituting (V.18) into (V.16) and setting equal to zero yields the free entry equilibrium number of firms as the solution to

$$\frac{N[(\mu_1 - \mu_2)(n-1)^2 + \mu_1 n]}{(n-1)(n^2 - n + 1)} = K \, , \tag{V.20}$$

where the left-hand side is decreasing in n (since $\mu_1 \geq \mu_2$) so that there is a unique solution (we also use the condition $n \geq 2$ in the comparative static results that follow).

The equilibrium number of firms is decreasing with μ_2, since a higher value of μ_2 causes firms to offer wider product ranges (for n fixed) - in a symmetric equilibrium, price doesn't change so that profits are eaten away via this extra competition for customers. As μ_1 rises, on the other hand, holding n fixed, consumer loyalty to firms rises so prices rise, and product ranges are unchanged. The resulting higher profitability leads to more entry: as further firms enter, firms now start to offer smaller product ranges. That is, *an increase in μ_1 ultimately causes more firms, each offering fewer products; whereas a rise in μ_2 leads to the opposite result.*

We can now derive the first best social optimum configuration of n and m. Note the first best is equivalent to the second best subject to a zero profit constraint (the other case in Section II) since adding a constant to all prices serves only to convert consumer surplus to profits.

Maximization of the surplus function for the nested logit given in (IV.6) gives (for interior solutions):

$$m^0 = N \frac{\mu_2}{nF} \tag{V.21}$$

for given n, and

$$m^0 = \frac{\mu_2 K}{F(\mu_1 - \mu_2)} \tag{V.22}$$

when $n = n^0$, given by

$$n^0 = \frac{N(\mu_1 - \mu_2)}{K}. \tag{V.23}$$

Note that $n^0 = 1$ when $\mu_1 = \mu_2$ (the simple logit case treated earlier) since extra firms bring no variety benefit per se. In this case from (V.21) m^0 becomes $N\mu_2/F$ (c.f. Table 1a).

The comparative statics are qualitatively similar to those of the free entry equilibrium. The comparison between optimum and equilibrium shows that the market solution provides *too many firms* but that there are *too few products per firm*. However, the *total number of products* produced, $n^* m^*$, *is too few*.

As far as we know, the comparison between equilibrium and optimum has not been addressed in the literature for anything other than single-product firms. Raubitschek (1987) goes part of the way by allowing for multiple products, but she assumes the number of firms is fixed. She concludes that since there are too few products in the Spence (1976) model on which here analysis is based, there will a fortiori be too few products in her set-up since, when deciding whether to introduce a new product, a firm will internalize part of the "business stealing" (or cannibalization) effect, the business stealing effect being a force toward having too many products. In this light, a comparison of our results here with those of the single product model described in Table 1a is fruitful. There we found there to be too many products. Here, for fixed n, there are too few products. This indicates that the internalization of business stealing in the multiple product context suffices to bring about the opposite conclusion.

For the problem we consider, with endogenous firm numbers as well as product ranges, an entrant firm is associated with three types of externality. There is the standard business stealing externality (an entrant does not account for the detrimental effect on existing firms' profits) which is a tendency for overentry. Then there is the consumer surplus externality whereby an entrant cannot extract the whole surplus associated with producing its product range, and this is a tendency toward underentry. In our context there is an *additional negative externality* which has to do with total variety. An entrant also leads existing firms to contract their product ranges (at least in our model), reducing the variety offered by them. For firms, at the margin the net value of an additional product in the range is zero (via the first order condition for profit maximizing choice of product range). However, the net social value of these lost products is positive due to the consumer surplus associated with them. This suggests this is an additional force tending toward insufficient product variety and overentry of firms. Indeed, our final result, the net effect, is overentry of firms, and this is combined with product ranges which are not broad enough.

This result is also interesting because the previous literature has measured diversity only as the number of products, which there is the same as the number of firms. In the nested logit formulation, there are two measures of taste for variety, μ_1 and μ_2; and there are two dimensions to the diversity provided by the market, the number of firms and the product range per firm.

VI. Logit Models with Search.

In this section we exploit the structure of the discrete choice model (II.11) to describe a simple search model based on the logit. We start with the single product firm model. Suppose consumers do not know the value of $\mu\epsilon_i$ associated with a particular product i, but know the distribution of ϵ_i is given by (II.12). However, if they incur a search cost of γ, the value of the random variable for product i is revealed. We can view the random term as a "match value" expressing the benefit of a particular product to a particular consumer. Consumers are assumed to sample s of the available n products (produced by n independent firms) and then to choose the one (of the s sampled) yielding greatest utility (II.11).

For example, an individual wishing to have an extension built on his/her house may invite several builders to offer tenders, i.e. provide cost estimates and design plans. We assume the number of builders sampled is determined before having received any of their bids. Another example may be a student who applies to several universities at the same time and then after receiving more information about them chooses the one whose characteristics best mesh with his/her own ideals. We model the interaction between consumers and firms as a simultaneous Nash equilibrium. That is, given the prices set by firms, consumers choose how many products to sample; given how many products are sampled and the prices set by all other firms, each firm chooses its own price. We shall look for a symmetric equilibrium where all firms set the same price p^*. Hence consumer expectations that firms charge p^* are fulfilled in equilibrium. We first find the number of searches made by consumers.

For the logit model, the expected net benefit (i.e., the expected maximum of u_i in (II.11)) for a consumer sampling s products all priced at p^* with common quality a is given by $\mu\ln[s\ \exp(a-p^*/\mu)]$ (see (IV.4)) so that the consumer's problem is to choose s so as to maximize surplus given by

$$a-p^* + \mu\ln s - \gamma s$$

where we assume it costs γ to sample a product. The solution is given by

$$s^* = \min[n,\ \max[1,\frac{\mu}{\gamma}]] \quad , \tag{VI.1}$$

so that the number of searches is independent of the common price p^*. We shall henceforth assume $\gamma n > \mu > \gamma$ so that we have the interior solution $n^* = \mu/\gamma$. Higher search costs mean less search but greater variation in match values (μ) leads to more search.

Firms take s^* as given so that if all firms other than 1 charge p^*, 1's expected profit is

$$\Pi_1 = (p_1 - c)\ N\mathbb{P}_1 \cdot \frac{s^*}{n} \tag{VI.2}$$

where we note s^*/n is the probability of being sampled by a given consumer and

$$\mathbb{P}_1 = \frac{\exp\ (-p_1/\mu)}{\exp(-p_1/\mu) + (s^*-1)\exp(-p^*/\mu)} \quad . \tag{VI.3}$$

The standard first-order condition is $p_1 = c + \mu/(1-\mathbb{P}_1)$, which in the symmetric equilibrium implies $p^* = c + \frac{\mu n^*}{n-1}$, or, using (VI.1),

$$p^* = c + \frac{\mu^2}{\mu - \gamma} \qquad \text{(VI.4)}$$

Here the equilibrium price increases with search costs because fewer firms are sampled and competitive pressures are reduced. For $\mu \geq 2\gamma$ (which implies $s^* \geq 2$) the price is increasing with μ - despite the fact that higher μ leads to more sampling which increases competition, the larger μ means also that customers have more intense preferences (enabling firms to have higher prices) and the latter effect dominates.

Finally, equilibrium profits are given by

$$\Pi^* = \frac{N\mu^2}{n(\mu - \gamma)} \qquad \text{(VI.5)}$$

which is clearly decreasing in n so that a unique free entry equilibrium is ensured.[15] For $\gamma \leq \frac{\mu}{n}$, all the solutions above are given by the same expressions as in Table 1a, since in this case consumers search over all firms.

We now extend the model above to the nested logit context with multiple product firms described in the previous section. We assume now that the search cost γ enables a consumer to sample a firm and the match value of *all* the products it produces. Again consumers must decide in advance the number of firms to sample, and then pick a product from the produce range provided by one of these.

In a symmetric situation, consumers take m (products per firm) and the common price level p as given, and, using (IV.6), the consumer problem is to maximize expected surplus, i.e.,

---

[15] The free entry equilibrium number of firms is $\frac{N\mu^2}{K(\mu - c)}$ where K is set-up cost per firm.

$$\max_{s}\ a-p+\mu_1\ln(s) + \mu_2\ln(m) - \gamma s \ .$$

The solution to this problem, which assume is interior, is

$$s^* = \mu_1/\gamma \qquad \text{(VI.6)}$$

Firms take s^* as given, and (VI.2) becomes (for firm 1)

$$\Pi_1 = N(p_1 - c)\mathbb{P}_1 \frac{s^*}{n} - m_1F - K \qquad \text{(VI.7)}$$

(cf. (V.14)), with $\mathbb{P}_1$ given by

$$\mathbb{P}_1 = \frac{\exp[\frac{\mu_2}{\mu_1}\ln(m_1)]}{(s^*-1)\exp[\frac{\mu_2}{\mu_1}\ln(m_2) - \chi] + \exp[\frac{\mu_2}{\mu_1}\ln(m_1)]} \ . \qquad \text{(VI.8)}$$

so the only difference with (V.15) is that n is replaced by s^*. Likewise, the only difference between (VI.7) and (VI.4) is that N is replaced by Ns^*/n. Using these observations, we can use the analysis of Section V to find the equilibrium product range per firm (using (V.18)), given n, as

$$m^* = \frac{N}{n} s^* \frac{\mu_2}{F} \frac{s^* - 1}{(s^* - 1)^2 + s^*} \ . \qquad \text{(VI.9)}$$

We also note the equilibrium price is

$$p^* - c = \frac{\mu_1 s^*}{s^* - 1}. \tag{VI.10}$$

Equilibrium profits, given n, are then, from (VI.7)

$$\Pi^* = \frac{Ns^*}{n}\left[\frac{\mu_1}{s^* - 1} \quad \frac{\mu_2(s^* - 1)}{(s^* - 1)^2 + s^*}\right] - K, \tag{VI.11}$$

from which the unique free entry equilibrium can be found.

The comparative statics of the free entry equilibrium can be determined from the last three equations. As γ rises (recall $s^* = \mu_1/\gamma$), equilibrium price rises since firms compete with fewer rivals. This price effect tends to increase the marginal profitability of adding products, but the same time each firm competes with fewer other firms which reduces marginal profitability. The net effect (seen from (VI.9)) is for m^* to fall. Profits (and ultimately the number of firms) then rise for two reasons - higher prices and lower costs through smaller product ranges.

If μ_2 rises, search and price stay the same. However, m^* rises, which cuts profits since costs rise.

Finally, we can decompose an increase in μ_1 into two effects, one through an increase in search alone - which works in the opposite direction to the increase in γ considered above; the second through a rise in μ_1 holding s^* constant. The latter effect raises equilibrium price and profits with m^* constant. However the former effect decreases profits. The net effect on price (from (VI.10) is to unambiguously raise it. However, the search effect through m^* can outweigh this, and the result for profit is ambiguous.

To see this, first note if $\mu_2 = 0$, (VI.11) is rising with μ_1. On the other hand, if $\mu_1 = \mu_2$, we can write (VI.11) as $\Pi^* = \frac{N}{\gamma n}\left\{\frac{s^{*3}}{(s^* - 1)[(s^* - 1)^2 + s^*]}\right\}$, which is decreasing in s^* and hence in μ_1.

This latter result implies the number of firms may fall as *both* measures of product heterogeneity rise. The reason (for μ_1) is that greater heterogeneity elicits more search for a good match, and the extra competition via product ranges dominates the price-raising effect larger heterogeneity otherwise induces. This particular result does not hold true in the single product per firm search model (given above - see (VI.5)) and can be ascribed to the added insight provided by considering multiple product firms. Likewise, increased μ_1, will raise firm numbers (see Section V) in the absence of search.

The model presented above assumed firms do not take into account the possibility that a price change may alter the number of searches made by consumers. To see the importance of this assumption, suppose now that consumers are ex-ante perfectly informed as to product prices (but not match values). Then if firm i charges a price lower than all others, it will be sampled by all consumers because any product is as likely as any other to yield the highest match value and the product with the lowest price is ex-ante most attractive. In this case competition over price will drive equilibrium price to marginal cost as each firm strives (à la Bertrand) to be the lowest price firm.[16]

[16] Alternatively, if consumers were only imperfectly informed about prices of products and did not know with certainty which firm was charging the lowest price, a price-cutter will not be sampled by all consumers and we should expect and equilibrium with p > c.

VII. Conclusions

We have argued that the logit model is a very convenient and tractable model of product-differentiation. To substantiate this claim we have explored how it can be used in different contexts in Industrial Organization. The specific problems with which we dealt would *a priori* require specific models, yet the logit provides a basis for analysis in each case. We started off by showing the basic logit applied to oligopoly with product differentiation is qualitatively similar (with respect to its predictions) to the CES model which has found favor with many authors in the field. We explained these results by referring to the similarity of the roots of both models.

In Section III we examined the existence of equilibrium in which prices are chosen endogenously taking as given consumer and product characteristics. Section IV provides a generalization of the simple logit model to a nested logit, which is itself a special case of the Generalized Extreme Value model of McFadden (1978). The nested logit, and indeed the GEV, would seem to be especially appropriate in the study of multiple product firms. This indeed was the topic taken up in Section V. Among other things, we showed in that section that market equilibrium shows a tendency toward having too many firms, although each produces too narrow a range of products.

Section VI then presents a simple search model using the logit, and we pursue further the issue of multiple product firms in this context.

This paper is intended to show various properties of the logit and suggest ways it can be extended. In the applications we have taken simple forms and still (for the most part) obtained simple closed form solutions. This means we expect further aspects could be introduced into the model, while still retaining a tractable analytic framework. For example, several of the parameters of the model, such as N, F, K, a, c, could be assumed to depend on other factors. Quality choice by firms could, for instance, reasonably influence any or all of these (except perhaps N). Likewise the multiple product firm framework could be used to address questions concerning mergers and acquisitions. One advantage of having explicit closed form solutions is that they provide a benchmark case, indicating at least some of the relevant trade-offs at work. Thus, although some of the results are particular to the framework used (e.g., that price is independent of product range in the multi product firm analysis), we hope that this will help point out directions for further research.

References

Amemiya, T. (1981), "Qualitative Response Models: A Survey", *Journal of Economic Literature* 19, 1483-1536.

Anderson, S. P. and A. de Palma (1986), "Market Equilibrium and Optimal Product Diversity: A Logit Specification" CORE D. P. 8608, Université Catholique de Louvain, Louvain-la-Neuve, Belgium.

Anderson, S. P. and A. de Palma (1988), "Spatial Price Discrimination with Heterogeneous Products", *Review of Economic Studies*, 55, 573-592.

Anderson, S. P., A. de Palma and J.-F. Thisse (1987), "The CES is a Discrete Choice Model?" *Economics Letters*, 24, 139-140.

Anderson, S. P., A. de Palma and J.-F. Thisse (1988), "A Representative Consumer Theory of the Logit Model", *International Economic Review*, 29, 461-466.

Anderson, S. P., A. de Palma and J.-F. Thisse (1989), "Demand for Differentiated Products, Discrete Choice Models, and the Characteristics Approach", *Review of Economic Studies*, 56, 21-35.

Ben-Akiva, M. (1973), *Structure of Passenger Travel Demand Models*, Ph.D. Dissertation, Department of Civil Engineering, MIT, Cambridge, Mass.

Ben-Akiva, M. and S. Lerman (1986), *Discrete Choice Analysis*, MIT Press, Cambridge, Mass.

Besanko, D. and M. K. Perry (1989), "Exclusive Dealing in a Spatial Model of Retail Competition", mimeo, Bellcore.

Besanko, D., M. K. Perry and R. H. Spady (1990), "The Logit Model of Monopolistic Competition", *Journal of Industrial Economics*, forthcoming.

Blanchard, O. J. and N. Kiyotaki (1987), "Monopolistic Competition and the Effects of Aggregate Demand", *American Economic Review*, 77, 647-666.

Braid, R. (1988), "Heterogeneous Preferences and Non-Central Agglomeration of Firms", *Regional Science and Urban Economics*, 18, 57-68.

Caplin, A. and B. Nalebuff (1990), "Aggregation and Imperfect Competition: On the Existence of Equilibrium", *Econometrica*, forthcoming.

Debreu, G. (1960), "Review of R. Luce *Individual Choice Behaviour*," *American Economic Review*, 50, 1986-88.

De Palma, A., V. Ginsburgh, Y. Y. Papageorgiou and J.-F Thisse (1985), "The Principle of Minimum Differentiation Holds Under Sufficient Heterogeneity", *Econometrica*, 53, 767-781.

Deneckere, R. and M. Rothschild (1986), "Monopolistic Competition and Preference Diversity", CMSEMS, D. P. #684, Northwestern University.

Dixit, A. K. and J. E. Stiglitz (1977), "Monopolistic Competition and Optimal Product Diversity", *American Economic Review*, 67, 297-308.

Feenstra, R. C. and K. Judd (1982), "Tariffs, Technology Transfer and Welfare", *Journal of Political Economy*, 90, 1142-1165.

Feenstra, R. C. and J. A. Levinsohn (1989), "Distance, Demand and Oligopoly Pricing", NBER Working Paper No. 3076.

Flam, H. and E. Helpman (1987), "Industrial Policy under Monopolistic Competition", *Journal of International Economics*, 22, 79-102.

Friedman, J. W. (1986), *Game Theory with Applications to Economics*, Oxford University Press.

Koenker, R. W. and M. K. Perry (1981), "Product Differentiation, Monopolistic Competition, and Public Policy", *The Bell Journal of Economics*, 12, 217-231.

Krugman, P. (1982), "Trade in Differentiated Products and the Political Economy of Trade Liberalisation", in *Import Competition and Response*, J. Bhagwati (ed.), NBER, University of Chicago Press.

Lawrence, C. and P. T. Spiller (1983), "Product Diversity, Economies of Scale, and International Trade", *Quarterly Journal of Economics*, 98, 63-84.

Lederer, P. J. and A. P. Hurter, Jr. (1986), "Competition of Firms: Discriminatory Pricing and Location", *Econometrica*, 54, 623-640.

McFadden, D. (1973), "Conditional Logit Analysis of Qualitative Choice Behavior", in P. Zarembka (ed.), *Frontiers in Econometrics*, Academic Press, New York.

McFadden, D. (1978), "Modelling the Choice of Residential Location", in *Spatial Interaction Theory and Residential Location*, A. Karlquist et al., eds., North Holland, Amsterdam, 75-96.

McFadden, D. (1989), "A Method of Simulated Moments for Estimation of Discrete Response Models without Numerical Integration", *Econometrica*, 57, 995-1026.

Perloff, J. M. and S. C. Salop (1985), "Equilibrium with Product Differentiation", *Review of Economic Studies*, 52, 107-120.

Raubitschek, R. (1987), "A Model of Product Proliferation with Multiproduct Firms", *The Journal of Industrial Economics*, 35, 269-279.

Sattinger, M. (1984), "Value of an Additional Firm in Monopolistic Competition", *Review of Economic Studies*, 51, 321-332.

Spence, A. M. (1976), "Product Selection, Fixed Costs, and Monopolistic Competition", *Review of Economic Studies*, 43, 217-235.

Train, K. (1986), *Qualitative Choice Analysis; Theory, Econometrics, and an Application to Automobile Demand Analysis*, MIT Press, Cambridge, Mass.

Train, K. E., D. L. McFadden and M. Ben-Akiva (1987), "The Demand for Local Telephone Service: A Fully Discrete Model of Residential Calling Patterns and Service Choices", *Rand Journal of Economics*, 18, 109-123.

Appendix 1. Equilibrium Existence for the General Model

The second order condition is

$$\frac{d^2\Pi_i}{dp_i^2} = 2\sum_{h=1}^{N}\frac{\mathbb{P}_{ih}(\mathbb{P}_{ih}-1)}{\mu} + \left(\frac{p_i - c_i}{\mu}\right)\sum_{h=1}^{N}(2\mathbb{P}_{ih}-1)\ \frac{\mathbb{P}_{ih}(\mathbb{P}_{ih}-1)}{\mu}$$

Evaluating this express at $d\Pi_i/dp_i = 0$ yields (using (III.7)):

$$\operatorname{sgn}\frac{d^2\Pi_i}{dp_i^2} = \operatorname{sgn}\Big(2\sum_{h=1}^{N}\mathbb{P}_{ih}(\mathbb{P}_{ih}-1)\cdot\sum_{k=1}^{N}\mathbb{P}_{ik}(1-\mathbb{P}_{ik}) + \sum_{k=1}^{N}\mathbb{P}_{ik}\cdot\sum_{h=1}^{N}(2\mathbb{P}_{ih}-1)(\mathbb{P}_{ih}^2-\mathbb{P}_{ih})\Big)$$

We therefore wish to show

$$-2\Big(\sum_h \mathbb{P}_{ih}^2\Big)^2 - 2\Big(\sum_h \mathbb{P}_{ih}\Big)^2 + 4\Big(\sum_h \mathbb{P}_{ih}^2\Big)\Big(\sum_h \mathbb{P}_{ih}\Big) + 2\Big(\sum_h \mathbb{P}_{ih}\Big)\Big(\sum_h \mathbb{P}_{ih}^3\Big) - 3\Big(\sum_h \mathbb{P}_{ih}\Big)\Big(\sum_h \mathbb{P}_{ih}^2\Big) + \Big(\sum_h \mathbb{P}_{ih}\Big)^2 < 0\ ;$$

or, rewriting:

$$-2\Big(\sum_h \mathbb{P}_{ih}^2\Big)^2 - 2\Big(\sum_h \mathbb{P}_{ih}\Big)\Big(\sum_h \mathbb{P}_{ih}^3\Big) + \Big(\sum_h \mathbb{P}_{ih}^2\Big)\Big(\sum_h \mathbb{P}_{ih}\Big) - \Big(\sum_h \mathbb{P}_{ih}\Big)^2 < 0. \quad (*)$$

To sign this expression, note that

a) $$\sum_h\sum_k(\mathbb{P}_{ih}-\mathbb{P}_{ik})^2\mathbb{P}_{ih}\mathbb{P}_{ik} = 2\Big(\sum_h \mathbb{P}_{ih}^3\Big)\Big(\sum_k \mathbb{P}_{ik}\Big) - 2\Big(\sum_h \mathbb{P}_{ih}^2\Big)\Big(\sum_k \mathbb{P}_{ik}^2\Big)$$

b) $$\sum_h \sum_k \mathbb{P}_{ih}\mathbb{P}_{ik}(\mathbb{P}_{ih} - 1) = (\sum_h \mathbb{P}^2_{ih})(\sum_k \mathbb{P}_{ik}) - (\sum_h \mathbb{P}_{ih})(\sum_k \mathbb{P}_{ik})$$

From (a) and (b), the desired condition (*) can be rewritten

$$\sum_h \sum_k \mathbb{P}_{ih}\mathbb{P}_{ik}\{(\mathbb{P}_{ik} - \mathbb{P}_{ih})^2 - (1 - \mathbb{P}_{ih})\} < 0$$

which is the condition given in the text. $\square$

If this condition holds at any extremum (when the first order condition is zero) then the profit function is necessarily strictly quasiconcave.

Appendix 2. Uniqueness of Equilibrium

Let us show uniqueness for the case when there is but one consumer class. The proof does though cover the cases where the a_i's and c_i's are not necessarily the same for all firms. From (III.6) the first-order conditions are now (given N statistically identical consumers):

$$\frac{d\Pi_i}{dp_i} = N\mathbb{P}_i[1 + (\frac{p_i - c_i}{\mu})(\mathbb{P}_i - 1)] = 0, \quad i = 1\ldots n$$

which is the implicit form of the best-reply function. From the analysis of Appendix 1 we know this function has a unique solution $p_i^{br}(p_1\ldots p_{i-1}, p_{i+1}\ldots p_n)$. Now let us show the best-reply function is a contraction, that is $p_i^{br}(\cdot)$ has the property

$$\sum_{\substack{j=1\\ j\neq i}}^{n} \left| \frac{\partial p_i^{br}}{\partial p_j} \right| < 1 \quad .$$

Once this is shown, uniqueness of equilibrium follows directly (see e.g. Friedman (1986)). Now, $\frac{\partial p_i^{br}}{\partial p_j} = - \left(\frac{\partial^2 \Pi_i/\partial p_i \partial p_j}{\partial^2 \Pi_i/\partial p_i^2}\right)$ by the implicit function theorem. For the logit model above we have (from the first order condition given at the beginning of this appendix):

$$\text{a)} \qquad \frac{1}{N}\frac{\partial^2 \Pi_i}{\partial p_i^2} = \frac{P_i(P_i - 1)}{\mu}\left[2 + \frac{p_i - c_i}{\mu}(2P_i - 1)\right]$$

$$= - P_i/\mu < 0 \quad ;$$

$$\text{b)} \qquad \frac{1}{N}\frac{\partial^2 \Pi_i}{\partial p_i \partial p_j} = \frac{P_i P_j}{\mu}\left[1 + \frac{p_i - c_i}{\mu}(2P_i - 1)\right], \; i \neq j, \; i,j = 1 \ldots n$$

$$= \frac{P_i^2 P_j}{(1 - P_i)\mu} \; ;$$

where the last expression in each case is given by substituting the first order condition, $\partial\Pi_i/\partial p_i = 0$, or $p_i - c_i = \mu/(1 - P_i)$ - that is, we are evaluating along the best reply function.

From the above, we have $\frac{\partial p_i^{br}}{\partial p_j} = \frac{P_i P_j}{(1 - P_i)} > 0$, so the desired contraction property becomes

$$\frac{P_i}{1 - P_i} \sum_{\substack{j=1 \\ j \neq i}}^{n} P_j = P_i \left[(1 - \frac{P_0}{1 - P_i} \right] < 1,$$

which is ensured to hold.

Let us now show that for $c_i = c_j = c$ and $a_i = a_j = a$, i, j = 1...n, the unique equilibrium is symmetric. This is true as there is a unique solution p^* to $\frac{p^* - c}{\mu} = \frac{1}{1 - P^*}$ (which is equation (III.7) when all consumers and firms are the same), with $P^* = [n + \exp((a_0 - p_0 - a + p^*)/\mu)]^{-1}$.

Appendix 3. An Example of Non-Existence of a Pure Strategy Price Equilibrium

We here find conditions for existence of pure price strategy Nash equilibrium for the demand system (III.11) where firms have zero production costs.

We first show there can be no equilibrium with $X_1 = N_A$. Firm 1's best price in this range would be $p_1 = p_2 + (a_{1A} - a_{2A}) - \delta$, with δ an arbitrarily small positive constant.[11] However, 2's best price in this range is $p_2 = p_1 + (a_{2B} - a_{1B}) - \delta$. Given $(a_{1A} - a_{2A}) - (a_{1B} - a_{2B}) = k$, where k is a positive constant, these two conditions are obviously inconsistent. Now, $X_1 = 0$ cannot be a possible equilibrium as 1 could undercut any positive price 2 could set. Likewise for $X_1 = N_A/2$.

The only other equilibrium involves $X_1 = N_A + N_B$, with $p_2 = 0$ (by an argument à la Bertrand) and $p_1 = a_{1B} - a_{2B} - \delta$ (1 setting the highest price that ensures the whole market). It must now be the case that 1 does not wish

[11] A more technical argument with the same principles and conclusions could be undertaken along the lines of Lederer and Hurter (1986).

to raise p_1 and serve only type A consumers (at best price $p_1 = a_{1A} - a_{2A} - \delta$). Comparing profits under these two strategies shows the equilibrium described above exists only if $(N_A + N_B)(a_{1B} - a_{2B}) > N_A(a_{1A} - a_{2A})$. Note that no equilibrium exists if the B's prefer good 2.[12]

Appendix 4. Existence of a Price Equilibrium with Multiproduct Firms for Given Product Ranges[13]

Here we show there is a unique price equilibrium for the subgame in which all firms but firm 1 sell m_2 products, whereas firm 1 sells m_1 products. For the nested logit formulation, firm i's profit is given by

$$\Pi_i = N \sum_{k=1}^{m_i} (p_{ik} - c)\mathbb{P}_{ik} - m_i F - K, \quad i=1\ldots n,$$

where $\mathbb{P}_{ik}$ is given by (IV.1) through (IV.3). The first order condition with respect to p_{ik} yields

$$p_{ik} - c = \mu_2 + \quad [1 - \frac{\mu_2}{\mu_1}(1 - \mathbb{P}_i)] \sum_{h=1}^{m_i} (p_{ih} - c)\ \mathbb{P}_{h|i}, \quad k=1\ldots m_i,\ i=1\ldots n,$$

so that the price is the same for all firm i's products (and is necessarily positive since $\mu_1 \geq \mu_2$). The first order conditions therefore reduce to

---

[12] Note also that when $a_{1A} = a_{2A} = a_{1B} = a_{2B}$, there is the zero price homogeneous good Bertrand equilibrium with $X_1 = X_2 = (N_A + N_B)/2$.

[13] We would like to thank Jacques Thisse for encouraging and helping us with this proof.

$$p_1 - c = \frac{\mu_1}{1-P_1} \text{ and } p_i - c = \frac{\mu_1}{1-P_i}, \quad i=2\ldots n, \text{ where}$$

$$P_j = \frac{m_j^{\mu_2/\mu_1} \exp[-p_j/\mu_1]}{D}, \quad j=1\ldots n, \text{ and}$$

$$D = m_2^{\mu_2/\mu_1} \sum_{j=2}^{n} \exp[-p_j/\mu_1] + m_1^{\mu_2/\mu_1} \exp[-p_1/\mu_1].$$

Hence $p_j = p_2$ for $j=2\ldots n$, and the first order conditions become

$$\frac{p_1 - c}{\mu_1} = 1 + \frac{M \exp[(p_2 - p_1)/\mu_1]}{n - 1}$$

and

$$\frac{p_2 - c}{\mu_1} = 1 + \frac{1}{n - 2 + M \exp[(p_2 - p_1)/\mu_1]}$$

where we have defined $M = \left(\frac{m_1}{m_2}\right)^{\mu_2/\mu_1}$.

Clearly, the first expression solves for a unique p_1 given any p_2; it can be shown that a similar property holds for all other firms. Subtracting the latter expression from the former yields

$$\frac{p_2 - p_1}{\mu_1} = \frac{1}{n - 2 + M \exp[(p_2 - p_1)/\mu_1]} - \frac{M \exp[(p_2 - p_1)/\mu_1]}{n - 1} \qquad \text{(A.4.1)}$$

As the left hand side of this expression is linearly increasing in $\frac{p_2 - p_1}{\mu_1}$ and the right hand side is decreasing, (A.4.1) has a unique solution in $\frac{p_2 - p_1}{\mu_1}$. Hence there is a unique solution (p_1^*, p_2^*) to the first order conditions.

Let $\underline{p}_1^*$ be the m_1-dimensional vector with all components equal to p_1^* and $\underline{p}_2^*$ the m_2-dimensional vector with all components equal to p_2^*. Now, we wish to show that $\underline{p}_1^*$ and $\underline{p}_2^* = \ldots \underline{p}_n^*$ is an equilibrium of the price subgame defined from (m_1, m_2).

Consider any firm i. The price derivative of Π_i with respect to p_{ik} is given by

$$\frac{\partial \Pi_i}{\partial p_{ik}} = N\mathbb{P}_{ik}\left[1 + \left(\frac{1}{\mu_2} - \frac{1-\mathbb{P}_i}{\mu_1}\right)\sum_{h=1}^{m_i}(p_{ih} - c)\mathbb{P}_{h|i} - \frac{p_{ik} - c}{\mu_2}\right] .$$

Since $\mu_1 \geq \mu_2$, it follows immediately that

(i) $\left.\frac{\partial \Pi_i}{\partial p_{ik}}\right|_{p_{ik}=c} > 0 \quad , \; k=1\ldots m_i \quad .$

Moreover, it can be shown that

(ii) there exists $\bar{p}$ such that $\left.\frac{\partial \Pi_i}{\partial p_{ik}}\right|_{p_{ik}=\bar{p}} < 0$ for all $p_{ih} \leq \bar{p}$, whatever $h \neq k$.

The argument is as follows. When $p_{ik} = \bar{p}$, we have

$$\left.\frac{\partial \Pi_i}{\partial p_{ik}}\right|_{p_{ik}=\bar{p}} = N\mathbb{P}_{ik}\left[1 + \left(\frac{1}{\mu_2} - \frac{1 - \mathbb{P}_i}{\mu_1}\right)\sum_{h=1}^{m_i}(p_{ih} - c)\mathbb{P}_{h|i} - \frac{\bar{p} - c}{\mu_2}\right]$$

$$\leq N\mathbb{P}_{ik}\left[1 + \left(\frac{1}{\mu_2} - \frac{1 - \mathbb{P}_i}{\mu_1}\right)(\bar{p} - c) - \frac{\bar{p} - c}{\mu_2}\right] \text{ since } \sum_{h=1}^{m_i} \mathbb{P}_{h|i} = 1$$

$$\leq N\mathbb{P}_{ik}\left[1 - \left(\frac{1 - \mathbb{P}_i}{\mu_1}\right)(\bar{p} - c)\right]$$

$$\leq N\mathbb{P}_{ik}\left[1 - \frac{1 - \kappa}{\mu_1}(\bar{p} - c)\right]$$

where $1 - \mathbb{P}_i \geq 1 - \sup_{p_i} \mathbb{P}_i = 1 - \mathbb{P}_i|_{p_i=0} = \kappa > 0$. Consequently, for $\bar{p}$ large enough the last expression is negative for all i=1...n, which completes the proof.

Clearly, $\pi_i(p_i, p^*_{-i}; m)$ has a global maximizer p_i^M in the compact set $[c,\bar{p}]^{m_i}$. If p_i^M belongs to the boundary of $[c,\bar{p}]^{m_i}$, at least one component of p_i^M is equal either to c or to $\bar{p}$. But then, property (i) or (ii) would imply that $\pi_i(p_i, p^*_{-i}; m)$ is higher at some point in $]c,\bar{p}[^{m_i}$, a contradiction. Hence p_i^M must belong to $]c,\bar{p}[^{m_i}$ and is, therefore, a solution to the first order conditions applied to π_i. As p_i^* is the only solution of these conditions, $p_i^* = p_i^M$ so that p_i^* maximizes $\pi_i(p_i, p^*_{-i}; m)$.

□

Appendix 5. Existence of a Symmetric Product Range Equilibrium with Multiproduct Firms

Here we prove the assertion that $m_1 = m^*$, where m^* is given by (V.13), is the number of products that maximizes firm one's profit, (V.16), given all other (n-1) firms choose m^* products. By construction the derivative $d\Pi_1/dm_1$ as given by (V.17) is zero when evaluated at $m_1 = m^*$, (and furthermore m^* is uniquely determined).

We wish to show that the profit function $\Pi_1(m_1, m^*)$ is strictly quasi-concave in m_1. We know from (V.16) that

$$\Pi_1 = \frac{N\mu_1}{n - 1}\Delta - m_1F - K \tag{A.5.1}$$

with

$$\Delta = \left(\frac{m_1}{m^*}\right)^{\frac{\mu_1}{\mu_2}} \exp\left[(p^* - p_1^*)/\mu_1\right] .$$

Clearly, we have

$$\frac{d\Delta}{dm_1} = \Delta\left[\frac{\mu_2}{\mu_1}\frac{1}{m_1} + \frac{d}{dm_1}\left(\frac{p^* - p_1^*}{\mu_1}\right)\right] . \tag{A.5.2}$$

Furthermore, (A.4.1) implies that

$$\frac{p^* - p_1^*}{\mu_1} = \frac{-\Delta^2 - (n - 2)\Delta + n - 1}{(\Delta + n - 2)(n - 1)}$$

rom which it follows that

$$\frac{d}{dm_1}\left(\frac{p^* - p_1^*}{\mu_1}\right) = -\left[\frac{1}{(\Delta + n - 2)^2} + \frac{1}{n - 1}\right]\frac{d\Delta}{dm_1}. \qquad (A.5.3)$$

Combining (A.5.2) and (A.5.3) then gives

$$\frac{d\Delta}{dm_1} = \frac{\mu_2}{\mu_1}\frac{\Delta}{m_1}\varphi^{-1} \qquad (A.5.4)$$

where

$$\varphi = \frac{\Delta}{(\Delta + n - 2)^2} + \frac{\Delta}{n - 1} + 1 > 1. \qquad (A.5.5)$$

Differentiating (A.5.3) with respect to m_1 and using (A.5.4) yields

$$\frac{\partial \Pi_1}{\partial m_1} = \frac{N\mu_1}{n - 1}\frac{d\Delta}{dm_1} - F = \frac{N}{n - 1}\frac{\mu_1}{m_1}\Delta\ \varphi^{-1} - F$$

while the second derivative is given by

$$\frac{\partial^2 \Pi_1}{\partial m_1^2} = -\left(\frac{1}{m_1}\frac{\partial \Pi_1}{\partial m_1} + F\right) + \frac{N}{n - 1}\frac{\mu_2}{m_1}\left(\varphi^{-1} - \Delta\ \varphi^{-2}\frac{d\varphi}{d\Delta}\right)\frac{d\Delta}{dm_1}. \qquad (A.5.6)$$

We can now evaluate (A.5.6) at any solution of $\frac{\partial \Pi_1}{\partial m_1} = 0$ (so that we have $\frac{d\Delta}{dm_1} = \frac{F(n - 1)}{N\mu_1}$):

$$\left.\frac{\partial^2 \Pi_1}{\partial m_i^2}\right|_{\frac{\partial \Pi_i}{\partial m_1}=0} = \frac{F}{m_1}\{-1 + \frac{\mu_2}{\mu_1}(\varphi^{-1} - \Delta\,\varphi^{-2}\,\frac{d\varphi}{d\Delta})\} . \tag{A.5.7}$$

Since $\mu_1 \geq \mu_2$, this expression is negative when

$$\varphi - \Delta\,\frac{d\varphi}{d\Delta} < \varphi^2 \tag{A.5.8}$$

where (from (A.5.5)

$$\frac{d\varphi}{d\Delta} = \frac{-\Delta + n - 2}{(\Delta + n - 2)^3} + \frac{1}{n - 1} . \tag{A.5.9}$$

Substituting (A.5.9) in (A.5.8) and using (A.5.5), we obtain

$$1 + \frac{2\Delta^2}{(\Delta + n - 2)^3} < \left[\frac{\Delta}{(\Delta + n - 2)^2} + \frac{\Delta}{n - 1} + 1\right]^2$$

which can be shown to hold (recalling $n \geq 2$) by expanding the right hand side of the inequality. Hence (A.5.7) is negative.

$\square$