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EVOLUTIONARY STABILITY IN GAMES WITH EQUIVALENT STRATEGIES, MIXED STRATEGY TYPES AND ASYMMETRIES*

by

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Abstract:

The first section briefly summarizes previous results in the literature. In the second section the concept of an Evolutionary Stable Strategy (ESS) is generalized for games with equivalent strategies. Dynamic stability results equivalent to the ones for the traditional definition of an ESS are proven. In the third section these results are applied to show that the assumption that types only use pure strategies can be relaxed to the case where types use finitely many different mixed strategies. In the fourth section the results are used to give conditions for dynamic stability of populations playing asymmetric games.

Introduction:

An Evolutionary Stable Strategy (ESS) is a strategy that meets a static condition in a symmetric 2 person game. This condition is closely related to the dynamic stability properties of a selection process in a population. A major result concerning this (proven by Taylor and Jonker [1978] and Zeeman [1980]) states roughly that given a specific population structure if the mean strategy of the population is an ESS then the population is locally asymptotically stable w.r.t. the dynamic process (see section 1).

This paper deals with games where equivalent strategies are present, i.e. strategies that cannot be distinguished w.r.t. their expected payoffs in the game.

The existence of equivalent strategies very much influences both the static ESS condition and the associated dynamic stability of populations:

- a strategy that is an ESS cannot be equivalent to another strategy
- the mean strategy of a population that is locally asymptotically stable cannot be equivalent to another strategy.

When do equivalent strategies appear?

Obviously when two different mixtures of pure strategies behave identically w.r.t. their payoffs in the game. This doesn't imply anything about their relation in the setup for which the game was developed. It is the nature of the game that makes them indistinguishable w.r.t. their payoffs. Equivalent strategies (often not pure strategies) that are associated to the same behavior in the original setup appear quite naturally

in the context of populations with mixed strategy types (section 3) and in asymmetric games (section 4). Of course equivalent strategies can also appear independently of some connection in the original setup.

How can equivalent strategies be avoided:

- eliminate purely equivalent strategies before analyzing the game: This measure is suggested in the literature (Samuelson [1989]). However it can't be done when the equivalent strategies are mixed. Besides how should the strategy left over be interpreted?
- perturb the payoffs slightly: this is quite radical and besides it doesn't help in mixed type populations or in asymmetric games.

The payoffs and the set of strategies for the game shouldn't be changed after the setup just because the solution concept doesn't work.

Why should equivalent strategies be treated as the same, independent of their relationship in the original setup?

The general procedure when modelling some process is to formalize the interactions that are believed to be relevant and then to derive some implications using e.g. game theory.

A first consequence of the model when equivalent strategies appear is that certain actions cannot be distinguished in the present structure imposed by the model on the process. So either the model has to be refined or the equivalent strategies have to be treated as the same.

Therefore in this paper (and often assumed in the literature (see Hines [1987])) equivalent strategies are treated as if they were the same. As a consequence the definitions of an ESS and of asymptotic stability should be consistent with this assumption.

However this isn't the fact for the traditional definitions of an ESS and of asymptotic stability and therefore these definitions have to be generalized.

In section 2 the ESS definition and the notion of asymptotically stable populations are generalized to make them consistent with equivalent strategies being treated as the same. In what way the new definitions relate to the traditional ones and to each other is analyzed. In the main theorem of the paper (theorem 2.1 (2.2)) an equivalent result to the classical one about the sufficiency of an ESS for the asymptotic stability of a population is stated and proven.

In addition this new theorem has some nice applications to other topics of the ESS theory:

The central theorem in ESS theory states the sufficiency of an ESS for the dynamic stability of a population consisting of types using pure strategies. In section 3 it is shown that the assumption that the types only use pure strategies can be weakened to assuming that there are finitely many types using mixed strategies in the population.

In section 4 it is shown that the results for symmetric games can be extended to asymmetric games.

I. Summary of results relevant for this paper

Notation: $S_N := \{e_1, \dots e_N\}$ set of pure strategies

$\Delta A :=$ set of all probability distributions on the finite set A (e.g. ΔS_N)

$C(x) := \{i \text{ s.t. } x(e_i) > 0\}$ support of x

$B(x) := \{i \text{ s.t. } e_i \text{ is best reply to } x\}$

$\text{conv}\{x^1, \dots x^n\} :=$ convex hull of ...

$\text{int}(K) :=$ relative interior of set K

$\partial K := K \setminus \text{int}(K)$

$[e_i]: S_N \to \{0,1\}$ s.t. $[e_i](e_i) := 1$, $[e_i](e_j) := 0 \ \forall j \neq i$

N = # of pure strategies in the game

$E(x,y)$ = expected payoff of playing strategy x against y , $x,y \in \Delta S_N$

$E(x,x) > 0 \quad \forall x \in \Delta S_N$ (need this for the dynamics later on)

DEF.: (Maynard Smith [1982])

$p \in \Delta S_N$ is an Evolutionary Stable Strategy (ESS) iff

$$\forall q \neq p \quad E(p,p) \geq E(q,p) \text{ and}$$

$$E(p,p) = E(q,p) \Rightarrow E(p,q) > E(q,q)$$

(This is also referred to as the ESS condition.)

Calling a strategy "evolutionary stable" becomes totally meaningless unless it can be related to the stable sets of some reasonable dynamic process. This process I will now introduce and then quote a theorem that shows its connection to the ESS condition.

Population dynamics associated when talking about evolutionary stability:

The population size is assumed to be infinite but with finitely many different types. Each member of the population (belonging to some type) is randomly matched against an opponent (also from the population) to play a one shot symmetric game. After the game the types (as a class of individuals) reproduce according to their relative fitness (= expected payoffs from game), breeding true (i.e. offspring are of same type) and then die leaving a new generation behind.

There are two ways of setting up the dynamics, discrete or continuous time changes:

discrete dynamics: $$dF^{t+1}(q) = \frac{E(q,m^t)}{E(m^t,m^t)}\, dF^t(q)$$

continuous dynamics: $$d\dot{F}^t(q) = (E(q,m^t) - E(m^t,m^t))\, dF^t(q)$$

where type q is a class of individuals using strategy q ($q \in \Delta S_N$), otherwise also referred to as pheno- or genotype.

$dF^t(q)$ frequency of types using strategy q at time t

There are only finitely many different types present, so $dF^t(q) > 0$ only for finitely many q

$$m^t = \sum_{q:\ dF^t(q)>0} dF^t(q) \cdot q \quad = \text{mean of population at timet}$$

$$d\dot{F}^t(q) = \partial/\partial t\ dF^t(q)$$

The analysis of the above dynamic adjustment process (also called the replicator model) depends on the set of types allowed and on whether local or global properties are being inferred.

Additionally when talking about a population being stable against mutation the type and rate of the mutation have to be specified.

Results regarding the connection between the ESS condition and the dynamic stability of the population:

(1) Monomorphic population:

The simplest setup for analyzing the stability of a population w.r.t. some mutation is the case of a monomorphic (i.e. only one type) population's stability against the one time invasion of one type of mutant. This is a very specific (therefore I call it "trivial") and hence quite unrealistic setup. The "good" property is that the ESS condition is both necessary and sufficient for the dynamic stability.

ASSUMPTION: discrete or continuous dynamics

THEOREM 1.0:

p is an ESS iff the monomorphic population consisting of type p is asymptotically stable w.r.t. the one time entrance of a mutant of arbitrary type q with sufficiently small frequency

$$
\begin{aligned}
\text{i.e. } \exists\, \varepsilon^* > 0 \ \text{ s.t. } & \forall q \in \Delta S_N \qquad \text{(type of mutant)} \\
& \forall \varepsilon \in (0, \varepsilon^*) \qquad \text{(frequency of mutant)} \\
& \text{setting } dF^0(p) := 1 - \varepsilon \ , \ dF^0(q) := \varepsilon \\
& \lim_{t \to \infty} dF^t(p) = 1, \ \ \lim_{t \to \infty} dF^t(q) = 0
\end{aligned}
$$

The proof is straightforward. Note that there is only one path leading from the population $dF(p) = 1 - \varepsilon$, $dF(q) = \varepsilon$ to the population $dF(p) = 1$, $dF(q) = 0$.

(2) Polymorphic population where types only use pure strategies

This setup allows for a more differentiated population. There are different types in the population, each type uses a pure strategy (the average strategy in the population will generally be mixed). The population stability is defined w.r.t. sufficiently small perturbations in the frequencies of the types.

ASSUMPTIONS:

Population structure: only types using pure strategies are present in the population

Note: The mean strategy of the population (short: population mean) uniquely determines the population distribution (i.e. the frequencies of the types), and vice versa.

Dynamics: continuous

THEOREM 1.1 (Taylor and Jonker [1978], Zeeman [1980]):

p ESS => population with mean p locally asymptotically stable

i.e. $\exists$ U(p) s.t. any population with mean in U(p) asymptotically converges to the population with mean p

Furthermore: U(p) can be chosen s.t. starting in U(p) the path of the dynamic process will stay in U(p).

For proof of "furthermore" see theorem 2.1 (section 2), set $Q(p) = \{p\}$: the U(Q(p)) constructed in the proof satisfies this property.

The following is a slight generalization of a theorem stated by Weissing [1989] for the case where $B(p) = \{1,\ldots N\}$:

COROLLARY 1.1:

p ESS => population with mean p globally asymptotically stable w.r.t. any population with mean q s.t. $C(p) \subseteq C(q) \subsetneq B(p)$

i.e. any population with mean q s.t. $C(p) \subseteq C(q) \subsetneq B(p)$ asymptotically converges to the population with mean p

PROOF: just define new simplex ΔS where $S := \{e_i \text{ s.t. } i \in B(p)\}$ and we get the case proven in Weissing [1989]

It seems that for the case of discrete dynamics the above results should also hold provided that the changes in fitness are sufficiently small i.e. the selection pressure is sufficiently weak. This rather intuitive result was proven by Weissing [1989]:

DEF.:

$\forall A_0 \in \mathfrak{R}^{N,N}$ s.t. $(A_0)_{ii} = 0$, $\forall a \in \mathfrak{R}^N$, $A(A_0,a) := A_0 + (a, \,..\, a)^T$
where $(a, \,..\, a)^T \in \mathfrak{R}^{N,N}$

NOTE:

If $A := (E(e_i,e_j))_{ij} \in \mathfrak{R}^{N,N}$ is the payoff matrix (i.e. $E(x,y) = x^TAy$)
=> $\exists$ unique $A_0 \in \mathfrak{R}^{N,N}$, $a \in \mathfrak{R}^N$ s.t. $A = A_0 + (a, \,..\, a)^T$
where $(a, \,..\, a)^T \in \mathfrak{R}^{N,N}$ and $(A_0)_{ii} = 0 \quad \forall i$

PROOF: $\forall i,j$: $(A_0)_{ij} := A_{ij} - A_{jj}$, $\forall j$: $a_j := A_{jj}$

NOTATION: p ESS of A <=> p ESS w.r.t. payoffs $E(p,q) = p^T Aq$

LEMMA:

p ESS of A_0 <=> $\forall a \in \mathbb{R}^N$ p ESS of $A(A_0, a)$

Proof: $E(q,p) = qTA_0 p + a^T p$

RESULT:

continuous dynamics independent of "a", i.e. independent of selection pressure

THEOREM 1.2 (Weissing [1989]):

$A_0 \in \mathbb{R}^{N,N}$ s.t. $(A_0)_{ii} = 0$, p ESS of A_0

=> $\exists\, a+ \in \mathbb{R}^N$ s.t. $\forall a \geq a+$ $(a \in \mathbb{R}^N)$ theorem 1.1 holds for discrete dynamics if the payoffs are $E(x,y) = x^T A(A_0, a) y$

PROOF: Weissing [1989] pp. 72-76

Again a slight generalization of Weissing [1989]:

COROLLARY 1.2:

$A_0 \in \mathfrak{R}^{N,N}$ s.t. $(A_0)_{ii} = 0$, p ESS of A_0

$\Rightarrow \exists\ a+ \in \mathfrak{R}^N$ s.t. $\forall a \geq a+$ $(a \in \mathfrak{R}^N)$ corollary 1.1 holds for discrete dynamics if the payoffs are $E(x,y)=x^T A(A_0,a)y$

NOTE:

Demanding that "a" is larger than "a+" means that the selection pressure in the population has to be small enough. Hence theorem and corollary 1.2. back up the intuition: if the selection pressure is small enough then the discrete case can be approximated by the continuous case.

Possible scenarios for the entrance of mutants in populations consisting of pure strategy types:

The above theorems give conditions for the dynamic stability of a population. What kind of stability of a population w.r.t. mutation is guaranteed if the population mean is an ESS?

The original scenario (Maynard Smith [1982]) relating to the ESS condition:

A population using pure strategy types is considered to be stable against mutation if after a one time entrance of a sufficiently small subpopulation of mutants using pure strategy types the population distribution converges back to the original distribution (i.e. the mutants die). Theorem 1.1 (1.2) gives a sufficient condition for stability of a population against mutants where "sufficiently small" is determined by the condition that the mean of the population including the mutants is in U(p).

Note that in the above scenario the population's stability is only tested against one arbitrary mutation. This can be relaxed somewhat by the following scenario of repeated entry of mutants:

The mutants can enter repeatedly, but each time "sufficiently few" s.t. the mean population isn't driven out of U(p) by the entrance. If no more mutants enter then the population distribution converges back to the original distribution. And again theorem 1.1 (1.2) gives us a sufficient condition for the stability of a population against the repeated entry of mutants.

2. The notion of evolutionary stable strategies (ESS) in games with equivalent strategies:

It is assumed from now on that equivalent strategies are the same strategy, just with different names. An immediate consequence is that the concept of an ESS and that of asymptotic dynamic stability have to be checked to whether they are compatible with this assumption. In the following it will be shown that the traditional definitions contradict this assumption. The appropriate new definitions for games with equivalent strategies will be stated and analyzed.

2.1 Equivalent strategies and the concept of an ESS

DEF.: x and y are equivalent iff $\forall z \in \Delta S_N$ $E(x,z) = E(y,z)$ and

$$E(z,x) = E(z,y)$$

DEF.: $e \in S_N$ is a pure equivalent strategy if $\exists\, y \in \Delta S_N$

s.t. e and y are equivalent

I define the notion of equivalence w.r.t. both components of E(,) since only then are the two strategies x and y indistinguishable in the symmetric setup.

EXAMPLE 2.1:

The simplest example for a symmetric game with equivalent strategies is the following:

	T	B
T	1	1
B	1	1

Notation: $a_{ij} := E(e_i, e_j)$ payoff to playing e_i against e_j

Although all strategies are equivalent there is no ESS in the above game.

The characterization of an ESS uses a strict inequality against best replies. Therefore a strategy that is an ESS cannot be equivalent to another strategy. Especially if the set of pure strategies is expanded by some strategies of which a mixture is equivalent to the ESS then this will destroy the existence of the ESS.

Elimination of equivalent strategies:

If one of the equivalent strategies is a pure strategy then we can just eliminate it justifying this by saying that we have two different names for the same strategy. But what happens if neither of the two is a pure strategy?

EXAMPLE 2.2:

	T	M	N	B
T	1	0	0	1
M	0	1	1	0
N	0	1	0	1
B	1	0	1	0

1/2 [T] + 1/2 [M] and 1/2 [N] + 1/2 [B] are equivalent strategies but there are no purely equivalent strategies in the game above.

ASIDE: this example coincides to a game originally w/o equivalent strategies but with mixed strategy types (see example 3.1)

To be consistent with the above assumption that equivalent strategies are the same we have to generalize the concept of an ESS.

2.1.1 Equivalent Evolutionary Stable Strategies

I will extend the notion of an ESS to allow for the ESS (now called Equivalent ESS) to be a set of equivalent strategies (not necessarily purely equivalent). Thereby a candidate for an ESS won't be ruled out just because there is another strategy in the game that "behaves" identically.

The definition of equivalent strategies allows us to define equivalence classes on ΔS_N, where Q(p) is the equivalence class containing p:

DEF.: $\forall p \in \Delta S_N$: $Q(p) := \{x \in \Delta S_N \text{ s.t. } x \text{ and } p \text{ are equivalent}\}$

NOTE: Q(p) is a subset of ΔS_N

$Q(p) = Q(x)$ if x and p are equivalent

Q(p) is the intersection of some hyperplane and ΔS_N

(easy to show)

DEF.: $p \in \Delta S_N$ is an Equivalent Evolutionary Stable Strategy (eESS) iff

$$\forall q \in \Delta S_N: \quad E(p,p) \geq E(q,p) \quad \text{and}$$
$$\forall q \text{ s.t. } q \notin Q(p): \quad E(p,p) = E(q,p) \quad \Rightarrow \quad E(p,q) > E(q,q)$$

NOTE: This definition is also referred to as the eESS condition.

NOTE: If p is an ESS then p is also an eESS so we get a broader definition.

NOTE: The concept of an eESS is really just evolutionary stability defined w.r.t. sets of equivalent strategies instead of w.r.t. separate strategies.

2.1.2 Stability

Since the definition is really w.r.t. equivalence classes, we get as an immediate consequence that an eESS is independent of the elimination or addition of purely equivalent strategies:

Original game: $\Gamma(\Delta S_N, E)$

Assume that e_N is a purely equivalent strategy

Define new game $\Gamma(\Delta T, E)$ where $\Delta T := \{x \in \Delta S_N \text{ s.t. } x(e_N) = 0\}$

RESULT 2.1:

p is an eESS of $\Gamma(\Delta S_N, E) \Rightarrow \forall p' \in Q(p) \cap \Delta T$: p' is an eESS of $\Gamma(\Delta T, E)$

$(Q(p) \cap \Delta T \neq \emptyset)$

i.e. eESS is independent of the elimination of purely equivalent strategies

PROOF: by definition of purely equivalent: $Q(p) \cap \Delta T \neq \emptyset$, o/w e_N couldn't have been eliminated; the rest is obvious since the structure of the payoffs wasn't changed.

The independence of adding a purely equivalent strategy can be shown just like above.

NOTE: In the case of equivalent strategies (in contrast to redundant strategies) there is no difference between sequential and simultaneous elimination of purely equivalent strategies.

2.1.3 Equivalent definitions of an eESS:

The following equivalent definitions of an eESS are the generalized versions of the ones known for an ESS.

DEF (equivalent to the original one):

p is an eESS iff $\forall q \in \Delta S_N \backslash Q(p) \quad \exists\, \varepsilon^* > 0$ s.t. $\forall \varepsilon \in (0,\varepsilon^*)$

$$E(p,(1-\varepsilon)p + \varepsilon q) > E(q,(1-\varepsilon)p + \varepsilon q)$$

This follows easily from the fact that E(,) is linear in both arguments.

DEF (equivalent to the original one):

p is an eESS iff $\exists\, \varepsilon^* > 0$ s.t. $\forall \varepsilon \in (0,\varepsilon^*)$, $\forall q \in \Delta S_N \backslash Q(p)$: E(...

i.e. ε^* can be chosen independently of q

PROOF: (compare to Van Damme [1987] p. 216)

"if" follows directly from last statement

"only if"

case #1: $\exists\, p' \in Q(p)$ s.t. $p' \in \mathrm{int}(\Delta S_N)$

$\Rightarrow B(p) = \{1,\ldots N\}$, set $\varepsilon^* := 1$

case #2: $\mathrm{int}(\Delta S_N) \cap Q(p) = \emptyset$:

$A := \partial\Delta S_N \backslash \{q \in \partial\Delta S_N$ s.t. for some $p' \in Q(p)$: $C(p') \subseteq C(q)\}$

$A \neq \emptyset$:

$Q(p)$ is a convex set and $\mathrm{int}(\Delta S_N) \cap Q(p) = \emptyset$

$\Rightarrow \exists\, e_i$ s.t. $\forall p' \in Q(p)$: $i \notin C(p') \Rightarrow A \neq \emptyset$

$\forall q \in A\ \exists$ a uniform ε^*:

A is closed and therefore compact

$\forall q \in A,\ \forall\ p' \in Q(p)$:

$\varepsilon(q,p') := \sup\ \{\varepsilon \in (0,1]$ s.t. $E(p',(1-\varepsilon)p' + \varepsilon q) > E(q,(1-\varepsilon)p' + \varepsilon q)\}$

$\varepsilon(q,p') = \varepsilon(q)$ independent of p' since $p' \in Q(p)$

$\varepsilon^* := \inf\ \{\varepsilon(q)$ s.t. $q \in A\}$

$\forall q \in A$: $\varepsilon(q) > 0$, $\varepsilon(.)$ continuous on compact $A \Rightarrow \varepsilon^* > 0$

This ε^* works $\forall q \in \Delta S_N \backslash Q(p)$:

$\forall q \in \Delta S_N \backslash Q(p)\ \exists\ p' \in Q(p),\ q_A \in A,\ \varepsilon_0 \in (0,1]$ s.t. $q = (1-\varepsilon_0)p' + \varepsilon_0 q_A$

$\forall \varepsilon < \varepsilon^*$: $(1-\varepsilon)p' + \varepsilon q = (1-\varepsilon\varepsilon_0)p' + \varepsilon\varepsilon_0 q_A$, $\varepsilon\varepsilon_0 < \varepsilon^*$

****

DEF (equivalent to the original one):

p is an eESS iff $\exists$ V open neighborhood of Q(p) w.r.t. ΔS_N s.t.

$$E(p,x) > E(x,x)\ \forall x \in V \backslash Q(p)$$

This follows directly from

$E(p',(1-\varepsilon)p' + \varepsilon q) > E((1-\varepsilon)p' + \varepsilon q,(1-\varepsilon)p' + \varepsilon q)\}$ so define

$V := \{(1-\varepsilon)p' + \varepsilon q$ s.t. $p' \in Q(p),\ q \in \Delta S_N,\ \varepsilon < \varepsilon^*\}$

NOTE: $\{x$ s.t. $C(x) \subseteq B(p)\} \subseteq V$

2.1.4 Static results (w/o proofs)

The familiar static results (see van Damme [1987]) can easily be extended and proven for the more general eESS condition:

RESULT 2.2: p eESS $\Rightarrow \forall p' \in Q(p)$ (p',p') is a symmetric proper equilibrium

RESULT 2.3: $\forall q \in \Delta S_N \backslash Q(p)$: $E(p,p) > E(q,p) \Rightarrow p$ eESS

This generalizes the result that if (p,p) is a strict Nash equilibrium then p is an ESS.

2.2 Asymptotic Stability and equivalent strategies

To be consistent with the assumption that equivalent strategies are the same, the definition of asymptotic stability has to formulated w.r.t. the set of strategies that are equivalent to p: Q(p).

DEF.:

Q(p) is locally asymptotically stable if there exists an open neighborhood of Q(p) U(Q(p)) s.t. any population with mean in U(Q(p)) converges to a population with mean in Q(p).

Global asymptotic stability defined in the same way

Does this stability w.r.t. the set Q(p) interfere with the stability of p? No!

THEOREM (Bomze [1986]):

If p is an asymptotically stable dynamic equilibrium then (p,p) is an isolated equilibrium, especially p isn't equivalent to another strategy (i.e. $Q(p) = \{p\}$)

So the mean strategy of a population that is asymptotically stable cannot be equivalent to another strategy. That means if p is asymptotically stable then so is Q(p). So the asymptotically stable strategies aren't lost if you only look for the asymptotically stable sets Q(p).

2.3 Dynamic results

In the last two sections we saw that equivalent strategies can destroy an ESS and the asymptotic stability of a population mean. How do the new concepts of Equivalent ESS and of asymptotic stability w.r.t. Q(p) relate to one another?

In the following I will analyze how these new definitions affect the known theorems (see section 1) on ESS and dynamic stability.

Theorem 1.0 (trivial process) can easily be generalized for the case of an eESS. More interestingly though we can also generalize the "Taylor and Jonker [1978], Zeeman [1980]" results concerning populations with pure strategy types:

ASSUMPTIONS:

Population structure: only types using pure strategies are present in the population

Dynamics: continuous

THEOREM 2.1:

p eESS => population with mean in Q(p) locally asymptotically stable

i.e. $\exists$ U(Q(p)) s.t. any population with mean in U(Q(p)) asymptotically converges to a population with mean in Q(p)

Furthermore: U(Q(p)) can be chosen s.t. starting in U(Q(p)) the path of the dynamic process will stay in U(Q(p)).

NOTE: If $Q(p) = \{p\}$ then theorem 2.1 is identical to theorem 1.1.

PROOF: $e_1, \ldots e_N$ pure strategies

x^t_i = frequency of pure strategy e_i in population at time t

continuous dynamics:

$$\dot{x}_i = (E(e_i,x) - E(x,x))\, x_i \qquad (*)$$

p eESS => define $V \subseteq \Delta S_N$ as in equivalent definition of eESS above

$\forall p' \in Q(p)$ consider the following construction:

$U_0(p') := V \cap \{x \in \Delta S_N \text{ s.t. } C(p') \subseteq C(x)\}$

Note that $U_0(p')$ open neighborhood of p'

$Z(x,p') := \prod_i x_i^{p'_i}$ (defined on U)

$x = p'$ is unique max of Z on ΔS_N (see van Damme [1987] p. 225)

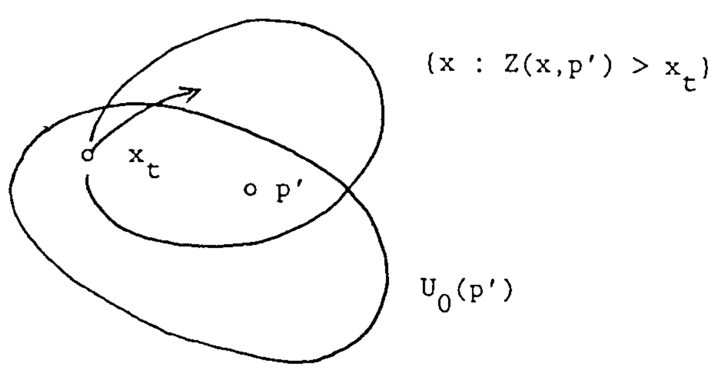
$\forall x \in U_0(p')$: $Z(x,p') > 0$ since "$x \in U_0$ and $p'_i > 0$" => "$x_i > 0$"

$\dot{Z}(x,p') = Z(x,p')\,(E(p',x) - E(x,x))$ (see van Damme [1987])

$\forall x \in U_0(p') \setminus Q(p)$: $\dot{Z}(x,p') > 0$

$\dot{Z}(x,p') = 0$ => x is a fixed point of (*)

Starting in $U_0(p')$ we cannot be sure that we stay in $U_0(p')$.



So take $K \subseteq U_0(p')$ s.t. K compact, $p' \in int(K)$

Z continuous on $\Delta S_N \Rightarrow \exists\, x^m$ s.t. $Z(x^m) = \max\{Z(x) : x \in \partial K\}$

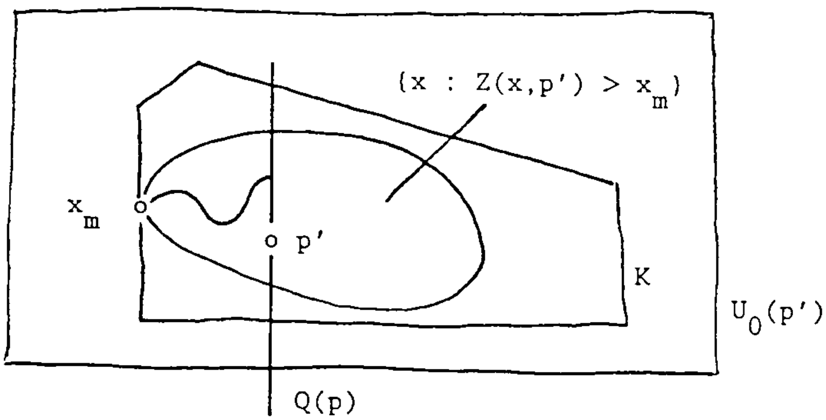
$U_1(p') := \{x \in U_0(p') : Z(x) > Z(x^m)\}$

$p' \in U_1$ since it's the unique max of Z and $p' \neq x^m$

U_1 is open

$\Rightarrow U_1(p') \neq \emptyset$

Now starting in $U_1(p')$ we will stay in $U_1(p')$.



Z will increase as long as $x \notin Q(p)$ so the dynamics once started in $U_1(p')$ will converge to an x in $U_1(p')$ s.t. $x \in Q(p)$.

Note that it isn't necessarily true that the dynamics converge to p'. The proof only shows that starting in $G := \bigcap_{p' \in P} U_1(p')$ for some $P \subseteq Q(p)$ the path will stay in G and eventually converge to an $x \in P$.

So far for every p' in Q(p) we constructed a $U_1(p')$. Now define

$U(Q(p)) := \bigcup_{p'} U_1(p')$

$p' \in U_1(p') \Rightarrow Q(p) \subseteq U(Q(p))$, $U_1(p')$ open $\Rightarrow U(Q(p))$ open

and any population starting with mean in U(Q(p)) will asymptotically converge to population with mean in Q(p) and the path will never leave U(Q(p)).

*****This ends the proof.

With the same assumptions as in theorem 2.1 we also get the generalized global result:

COROLLARY 2.1:

p eESS $\Rightarrow$ population with mean in $Q(p)$ globally asymptotically stable w.r.t. any population with mean q s.t. $C(q) \subseteq B(p)$ and $\exists\, p' \in Q(p)$ s.t. $C(p') \subseteq C(q)$

NOTE: If $Q(p) = \{p\}$ then corollary 2.1 is identical to corollary 1.1.

NOTE: the assumption "$\exists\, p' \in Q(p)$ s.t. $C(p') \subseteq C(q)$" is a necessary assumption for a population of mean q to converge to some population with mean $p' \in Q(p)$.

SKETCH of the proof: The proof follows closely the one above, there are just a few changes

$\forall p' \in Q(p)$: $U_0(p') := V \cap \{x \in \Delta S_N \text{ s.t. } C(p') \subseteq C(x)\}$

$x(0) \in U_0(p') \Rightarrow \forall t \geq 0$: $x(t) \in U_0(p')$ because:

$\Delta\{e_i \text{ s.t. } i \in B(p)\} \subseteq V$ so starting in U_0 we will stay in V.

The only way to leave $U_0(p')$ is to leave

$W(p') := \{x \in \Delta S_N \text{ s.t. } C(p') \subseteq C(x)\}$.

Assume that the path $x(t)$ doesn't stay in $W(p')$:

$x(t)$ is continuous in t

$\Rightarrow \exists\, t_1 > 0$ s.t. $x(t_1) \in \partial W(p')$

i.e. $\exists\ i \in C(p')$ s.t. $x(t_1)_i = 0$

$\Rightarrow Z(x(t_1),p') = 0$

but $x \in W(p') \Rightarrow \dot{Z}(x,p') > 0$ and $Z(.,p')$ continuous

$\Rightarrow Z(x(t_1),p') \geq Z(x(0),p') > 0 \quad (x(0) \in U_0(p'))$

and hence we get a contradiction

$$U(Q(p)) := \bigcup_{p' \in Q(p)} U_0(p') = \{q \in V \text{ s.t. } \exists\ p' \in Q(p) \text{ s.t. } C(p') \subseteq C(q)\}$$

This ends the proof.

****

Now to the results assuming discrete dynamics:

ASSUMPTIONS:

Population structure: only types using pure strategies are present in the population

Dynamics: discrete

THEOREM / COROLLARY 2.2:

$A_0 \in \mathbb{R}^{N,N}$ s.t. $(A_0)_{ii} = 0$, p eESS of A_0

$\Rightarrow \exists\ a+ \in \mathbb{R}^N$ s.t. $\forall a \geq a+$ $(a \in \mathbb{R}^N)$ theorem and corollary 2.1 hold for discrete dynamics if the payoffs are $E(x,y) = x^T A(A_0,a) y$

PROOF: Weissing [1989] pp. 72-76 can easily be generalized for Q(p) instead of p

3. ESS and dynamic stability in populations where types use mixed strategies:

We now want to drop the assumption that types only use pure strategies. Consider a population where finitely many different types (not necessarily using pure strategies) are present.

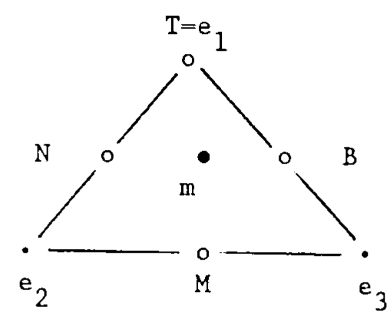
Taylor and Jonker [1978] and Zeeman [1980] proved that every ESS is an asymptotically stable dynamic equilibrium given that types only use pure strategies. The proof can't be directly applied to the case of (finitely many) mixed strategy types since it relies on the fact that the population mean uniquely determines the frequencies of the separate types. This generally doesn't hold for mixed strategy types:

EXAMPLE 3.1:

Game:

	e_1	e_2	e_3
e_1	1	-1	1
e_2	-1	1	3
e_3	1	3	-3

Population structure:



(o:= type)

types T,M,N,B in the population where

$T = e_1$, $\qquad M = 1/2\ e_2 + 1/2\ e_3$,

$N = 1/2\ e_1 + 1/2\ e_2$, $B = 1/2\ e_1 + 1/2\ e_3$

$$m := 1/2\ [e_1] + 1/4\ [e_2] + 1/4\ [e_3] = 1/2\ [N] + 1/2\ [B]$$
$$= 1/2\ [T] + 1/2\ [M]$$

So the mean of the population being at m doesn't determine the frequencies of T,M,N,B uniquely. If we define a new game where each original mixed strategy type coincides with a pure strategy type of the new game and define the payoffs respectively then we will generally get a game with equivalent strategies.

EXAMPLE 3.1.(continued):

Payoffs in above game when using T, M, N, and B as pure strategies are shown in example 2.1.

NOTE:

The equivalent strategies that appear when rewriting the game in the normal form w.r.t. the pure strategies T, M, N, B have to be considered the same since they are randomizing over the same strategies e_1, .. e_N.

If we use the results developed in section 2 on games with equivalent strategies, Taylor/Jonker's [1978] and Zeeman's [1980] results on dynamic stability can easily be extended to the more general case of finitely many different mixed strategy types being present in the population.

Just like in the former setup the population will be defined as stable w.r.t. changes in the frequencies of the fixed types, not w.r.t. arbitrary types.

The same scenarios for possible stories about the entrance of mutants hold as long as the total number of different types used by mutants or present in the initial population is finite and fixed.

As a special case it will follow that a monomorphic population is stable w.r.t. the repeated entrance of mutants belonging to a finite set of types of sufficiently small frequency. This generalizes the very specific setup of theorem 1.0 (trivial process) in the way that mutants can enter repeatedly. As a restriction the possible mutants have to belong to a specified finite set.

DEF.:

p is an eESS w.r.t. X ($X \subseteq \Delta S_N$) iff

$p \in X$ and $\forall q \in X \backslash Q(p)$ $\exists\ \varepsilon^* > 0$ s.t. $\forall \varepsilon \in (0, \varepsilon^*)$

$E(p,(1-\varepsilon)p + \varepsilon q) > E(q,(1-\varepsilon)p + \varepsilon q)$

ASSUMPTIONS:

N = # of pure strategies

Population structure: a finite number of different types $q^1, \ldots q^n$ are present in the population

Dynamics: continuous

THEOREM 3.1:

p eESS w.r.t. $\mathrm{conv}(q^1, \ldots q^n)$ => population with mean in Q(p) locally asymptotically stable

i.e. $\exists$ U(Q(p)) s.t. any population with mean in U(Q(p)) and distribution on $\{q^1, \ldots q^n\}$ converges to a population with mean in Q(p)

NOTE: the distribution of a population with mean p is unique iff the types q^i, $i=1,\ldots n$ are independent w.r.t. p, meaning:

$$\exists \text{ unique } {}_1, \ldots {}_n \text{ s.t. } p = \Sigma_i \; q^i$$

Furthermore: $U(Q(p))$ can be chosen s.t. starting in $U(Q(p))$ the path of the dynamic process will stay in $U(Q(p))$.

NOTE: If p is an ESS (i.e. $Q(p) = \{p\}$) then theorem 3.1. shows that in central theorem of ESS theory proven by Taylor/Jonker and Zeeman the assumption of types using pure strategies was unnecessary!

IDEA of the proof:

The space where the population dynamics take place $\mathrm{conv}\{q^i, i=1, \ldots n\} \subseteq \Delta S_N$ will be transformed into a simplex of dimension n. Each pure strategy e_i in the new simplex will correspond to the mixed strategy q^i in the original simplex. Next it will be shown that the eESS p corresponds to an eESS in the new simplex ΔS_n. The neighborhood $U'(Q())$ will be constructed following theorem 2.1 and finally transformed back into the original simplex to get $U(Q(p))$.

PROOF: N = # of pure strategies

types present in population $q^1, \ldots q^n$

frequencies at time t: $dF^t(q^1), \ldots dF^t(q^n)$ $(\Sigma_i \, dF^t(q^i) = 1)$

continuous dynamics:

$$d\dot{F}^t(q^i) = (E(q^i,m^t) - E(m^t,m^t))\, dF^t(q^i) \quad , i=1, \ldots n \qquad (*)$$

$$m^t = \sum_i q^i \, dF^t(q^i) \quad \text{mean of population at time } t$$

Define new simplex ΔS_n and payoff function $\tilde{E}(,)$ s.t. the pure strategy e_i in ΔS_n corresponds to the mixed strategy q^i in ΔS_N

Define $T: \Delta S_n \to \operatorname{conv}(q^1, \ldots q^n)$ s.t. $T(x) := \sum x_i q^i$

T is linear, surjective and $T(e_i) = q^i \quad i=1, \ldots n$

Define $\tilde{E}: \Delta S_n \times \Delta S_n \to \mathfrak{R}$ s.t. $\tilde{E}(x,y) := E(T(x),T(y))$

With $x^t := \sum_i dF^t(q^i) \, e_i \in \Delta S_n$ we get

$$E(q^i,m^t) = E(T(e_i),T(x^t)) = \tilde{E}(e_i,x^t) \quad \text{and} \quad E(m^t,m^t) = \tilde{E}(x^t,x^t)$$

(*) turns into $\dot{x}^t_i = (\tilde{E}(e_i,x^t) - \tilde{E}(x^t,x^t)) \, x^t_i \quad , i=1, \ldots n$

Lemma: p eESS w.r.t. $\operatorname{conv}(q^1, \ldots q^n)$ $\Leftrightarrow$

$\exists \, \pi \in \Delta S_n$ s.t. $T(\pi) \in Q(p)$ and π eESS w.r.t. ΔS_n

Proof: p eESS w.r.t. $\operatorname{conv}(q^1, \ldots q^n)$, T surjective $\Rightarrow \exists \, \pi \in \Delta S_n$ s.t. $T(\pi)=p$

The rest follows from: $\tilde{E}(x,y) := E(T(x),T(y))$ and $T(Q(x)) = Q(T(x))$

***(end of proof of lemma)

Now theorem 2.1 can be applied to the dynamics in ΔS_n since by definition all types are using pure strategies w.r.t. ΔS_n.

Theorem 2.1 $\Rightarrow Q(\pi)$ locally asymptotically stable

i.e. $\exists \, U'(Q(\pi)) \subseteq \Delta S_n$ s.t. ...

Using this $U'(Q(\pi))$ we will finally show that a $U(Q(p)) \subseteq \Delta S_N$ exists:

Need $U(Q(p)) \subseteq \Delta S_N$ s.t.

$$x \in \Delta S_n,\ \Sigma x_i\, q^i \in U(Q(p)) \quad \Rightarrow \quad x \in U'(Q(\pi)$$

Assume that U(Q(p)) doesn't exist:

$$\exists\ p' \in Q(p)\ \exists\ x^k \in \Delta S_n\ ,\ k=1,2,\ldots \quad \text{s.t.} \quad \lim T(x^k) = p'$$

$$\text{but} \quad \forall k : x^k \notin U'(Q(\pi))$$

x^k, k=1,2,.. bounded sequence $\Rightarrow$ $\exists$ convergent subsequence

so w.l.o.g. $\exists\ x^* \in \Delta S_n$ s.t. $x^* = \lim x^k$

T continuous so $T(x^*) = p'$ and since $T^{-1}(p') \subseteq Q(\pi)$: $x^* \in Q(\pi) \subseteq U'(Q(\pi))$

Since $U'(Q(\pi))$ open we get a contradiction to "$x^k \notin U'(Q(\pi))\ \forall k$"

So U(Q(p)) exists and we are done. Note that U(Q(p)) is not necessarily equal to $T(U'(Q(\pi)))$.

****

Again we get the global result assuming the same assumptions as in theorem 3.1:

COROLLARY 3.1:

p eESS $\Rightarrow$ population with mean in Q(p) globally asymptotically stable w.r.t. the set of populations s.t.

type q^i with frequency $dF(q^i)$, i=1, ... n , $\Sigma\, dF(q^i) = 1$

$C(q) \subseteq B(p)$ where $q := \Sigma\, dF(q^i)\, q^i$ is the population mean

$\exists\ p' \in Q(p)$ s.t. $p' \in \text{conv}\{q^i \text{ s.t. } dF(q^i) > 0\}$

PROOF: It follows directly from the one above.

Now to the results assuming discrete dynamics:

ASSUMPTIONS:

Population structure: only types using pure strategies are present in the population

Dynamics: discrete

THEOREM / COROLLARY 3.2:

$A_0 \in \mathbb{R}^{N,N}$ s.t. $(A_0)_{ii} = 0$, p eESS of A_0

$\Rightarrow \exists\ a+ \in \mathbb{R}^N$ s.t. $\forall a \geq a+$ $(a \in \mathbb{R}^N)$ theorem and corollary 3.1 hold for discrete dynamics if the payoffs are $E(x,y)=x^T A(A_0,a)y$

PROOF:

Just change (*) into the following and quote theorem / corollary 2.2.

$$\text{dynamics: } dF^{t+1}(q^i) = \frac{E(q^i,m^t)}{E(m^t,m^t)}\, dF^t(q^i) \quad , i=1, \ldots n \qquad (*)$$

$$(*) \text{ turns into } \quad x^{t+1}{}_i = -p\frac{\tilde{E}(e_i,x^t)}{\tilde{E}(x^t,x^t)}\, x^t{}_i \quad , i=1, \ldots n$$

*****

4. Dynamic stability of asymmetric games:

The assumption that the types in the population play a symmetric game (i.e. cannot distinguish between different states they're in) is in many cases counter intuitive. Therefore Selten [1980] extended the definition of an ESS to asymmetric games.

The ESS condition was generalized but either the dynamics were ignored or it was just assumed that the same results hold without proving them.

Of course if we are talking about a monomorphic population with everyone using the ESS strategy then this population will be stable against the one time entrance of an arbitrary type of sufficiently small frequency (see "trivial" process in section 1).

But what about the case of different types in the population or different types entering?

Theorem 1.1 generally cannot be applied. It applies to games in normal form. If an asymmetric game is written up in its symmetric normal form then equivalent strategies appear and these destroy the existence of an ESS if the ESS is not a pure strategy. However these equivalent strategies that appear when going to the normal form representation coincide to the same behavioral strategy and therefore should be regarded as the same. Therefore the theorems from section 2 can readily be applied and we get the same results for asymmetric games as for symmetric games.

ASYMMETRIC GAME $(U, (C_u)_{u \in U}, (w_{uv}, E_{uv})_{u,v \in U}, T)$:

U - finite set of information states

C_u - finite set of pure strategies in state u $(u \in U)$

ΔC_u - set of probability distributions on C_u

$a_u \in \Delta C_u$ local strategy in state u

$E_{uv}(a_u, b_v)$ = expected payoff to strategy a_u against strategy b_v,

$u,v \in U$, $a_u \in \Delta C_u$, $b_v \in \Delta C_v$

$B := \{p: U \to \cup \Delta C_u \text{ s.t. } \forall u \in U: p(u) \in \Delta C_u\}$ set of all behavioral strategies

$T \subseteq B$ finite set of types in the population

e.g. $T := \{f: U \to \cup C_u \text{ s.t. } \forall u \in U: f(u) \in C_u\}$ the set of all pure strategies of the normal form

Notation: $p_u := p(u)$

w_{uv} = probability of state u being matched with state v

$\forall u,v \in U: w_{uv} = w_{vu}$

$E(p,q) := \sum_{u,v} w_{uv} E_{uv}(p_u, q_v)$ expected payoff to strategy p against strategy q, $p,q \in B$

To illustrate the problems arising in asymmetric games consider the following example:

EXAMPLE 4.1:

There are two states of nature "rain" and "shine". Both players are in the same state.

$U := \{r,s\}$, $w_{rr} = w_{ss} = 1/2$, $C_r = \{t,b\}$, $C_s = \{T,B\}$

```
          /          \
   1/2   /            \  1/2
        /              \
```

	t	b
t	0	1
b	1	0

rain

	T	B
T	0	1
B	1	0

shine

types present in population (t,T), (t,B), (b,T), (b,B)

$p := (1/2\ [t] + 1/2\ [b], 1/2\ [T] + 1/2\ [B]) \in B$ is an ESS

To apply theorem 1.1 we have to rewrite the game into its symmetric normal form. The only relevant payoffs are the ones between the different types.

SYMMETRIC NORMAL FORM of an asymmetric game w.r.t. the types T present:

ΔT = set of probability distributions on T

$\tilde{E}(p,q) := \sum_{f,f' \in T} p(f)\, q(f')\, E(f,f')$ expected payoff to $p \in \Delta T$ against $q \in \Delta T$

EXAMPLE 4.1 (continuation):

The symmetric normal form w.r.t. the types (t,T), (t,B), (b,T), (b,B) is:

	(t,T)	(t,B)	(b,T)	(b,B)
(t,T)	0	1/2	1/2	1
(t,B)	1/2	0	1	1/2
(b,T)	1/2	1	0	1/2
(b,B)	1	1/2	1/2	0

We see right away that there are many equivalent strategies (none of which can be eliminated) in the symmetric normal form game although there aren't any in the asymmetric game in this example.

e.g. 1/2 (t,T) + 1/2 (b,B) equivalent to 1/2 (t,B) + 1/2 (b,T)

Note that this is independent of the payoffs.

As noted in section 2 the equivalent strategies will mostly destroy the existence of an ESS. This problem was noticed by Selten [1980] and hence he defined the concept of an ESS for asymmetric games in behavioral strategies.

With the framework built up in section 2 equivalent strategies don't pose a problem.

NOTATION:

In the following I will write beh. ESS (beh. eESS) if the ESS (eESS) is defined w.r.t. the set of behavioral strategies B.

NOTE:

Since only the types in T are allowed in the population, the population mean can only lie in conv(T) ($T \subseteq B$). However conv(T) and ΔT should not be confused, they lie in different spaces.

ΔT = set of probability distributions on the set of types

conv(T) = relevant set of behavioral strategies

In the following we will say that a behavioral strategy $p \in B$ and a strategy of the symmetric normal form $q \in \Delta T$ are behaviorally equivalent if $\forall u \in U$ they assign the same probabilities to the pure strategies in C_u. Formally this means:

DEF.: $p \in B$ and q in ΔT are behaviorally equivalent iff

$$\forall u \in U \ \forall e \in C_u: \quad p_u(e) = \sum_{f \in T \text{ s.t. } f(u)=e} q(f)$$

NOTE:

From the above definition it can be easily seen that every strategy of the normal form game has a unique behaviorally equivalent strategy in the extensive form. However the converse is not true, i.e. generally one strategy in conv(T) will correspond to a continuum of behaviorally equivalent strategies in ΔT (see example 4.1). And by definition strategies in B\conv(T) won't correspond to any strategies in ΔT.

We can define the map F that maps any strategy of the normal form into the behaviorally equivalent strategy in conv(T):

DEF.: $F: \Delta T \rightarrow conv(T)$ s.t. $\forall q \in \Delta T$

$$\forall u \in U\ \forall e \in C_u: \quad F(q)(u)(e) = \sum_{f \in T} q(f)\ f(u)(e)$$

NOTE: F is linear and $F(f) = f\ \forall e \in T$

CLAIM:

F is surjective, i.e. given any strategy in conv(T) $\exists$ a behaviorally equivalent strategy to it in ΔT

PROOF follows by definition

CLAIM: $\tilde{E}(p,q) = E(F(p),F(q))$

PROOF: (here just shown for $p = g \in T$)

$$\tilde{E}(g,q) := \sum_{f \in T} q(f)\ E(g,f) = \sum_{f \in T} q(f) \sum_{u,v \in U} w_{uv}\ E_{uv}(g(u),f(u)) =$$

$$= \sum_{u,v \in U} w_{uv} \sum_{e \in C_v} \sum_{f \in T} q(f)\ E_{uv}(g(u),e) =$$

$$= \sum_{u,v \in U} w_{uv} \sum_{e \in C_v} F(q)(v)(e)\ E_{uv}(g(u),e) =$$

$$= \sum_{u,v \in U} w_{uv}\ E_{uv}(g(u),F(q)(v)) = E(g,F(q))$$

*****

Let Eq(.) be the set of all equivalent classes,

i.e. $Eq(conv(T)) := \{Q(p) \text{ s.t. } p \in conv(T)\}$, $Eq(\Delta T) := \{Q(q) \text{ s.t. } q \in \Delta T\}$

CLAIM: $F: Eq(\Delta T) \xrightarrow{1\text{-}1} Eq(conv(T))$, $F(Q(p)) = Q(F(p))$

PROOF: $q' \in Q(q) \Rightarrow F(q') \in Q(F(q))$ so $F(Q(q)) \subseteq Q(F(q))$

$p \in Q(F(q))$, $\forall q'$ s.t. $F(q') = p \Rightarrow q' \in Q(q)$

so $Q(F(q)) \subseteq F(Q(q))$

$F^{-1}(Q(p)) = Q(q)$ where $F(q) = p$

THEOREM 4.1.:

p eESS in conv(T) <=> q eESS in ΔT (where $F(q)=p$)

i.e. the concept of an eESS doesn't depend on whether the game is represented in the behavioral or in the symmetric normal form.

NOTE: as a special case of the above theorem we get

p ESS w.r.t. conv(T) => q eESS w.r.t. ΔT (where $F(q) = p$)

PROOF of the above theorem: this follows readily

$p' \notin Q(p) \iff q' \notin Q(q)$ where $F(q') = p'$ (by previous claim)

$\exists\ \varepsilon^* > 0$ s.t. $\forall \varepsilon \in (0, \varepsilon^*)$

$\forall p' \notin Q(p) \iff \forall q' \notin Q(q)$ where $F(q') = p'$

$$\tilde{E}(p, (1-\varepsilon)p + \varepsilon p') > E(p', (1-\varepsilon)p + \varepsilon p')$$

$$\iff E(F(q), (1-\varepsilon)F(q) + \varepsilon F(q')) > E(F(q'), (1-\varepsilon)F(q) + \varepsilon F(q'))$$

$$\iff \tilde{E}(q, (1-\varepsilon)q + \varepsilon q') > E(q', (1-\varepsilon)q + \varepsilon q')$$

******

Now to the dynamic stability results:

In the asymmetric setup a type is characterized by the strategies it uses in each information state. Since in the symmetric case it is assumed that types breed true it is natural to assume that the same thing holds for the asymmetric case: the strategy profile is passed on to the next generation unchanged and the dynamics only influence the frequency of the type.

ASSUMPTIONS:

Asymmetric game $(U, (C_u)_{u \in U}, (w_{uv}, E_{uv})_{u,v \in U})$

Population structure: a finite number of different types $q^1, \dots q^n$, are present in the population ($q^i \in T \subseteq B$, so q^i is a behavioral strategy)

Dynamics: continuous

THEOREM 4.2:

p eESS w.r.t. $\mathrm{conv}\{q^1, \dots q^n\}$ of asymmetric game => population with mean in Q(p) locally asymptotically stable

i.e. $\exists$ U(Q(p)) s.t. any population with mean in U(Q(p)) and distribution on $\{q^1, \dots q^n\}$ converges to population with mean in Q(p)

PROOF: follows directly from theorem 3.1

And again we get the global result w.r.t. the same assumptions:

COROLLARY 4.2:

p eESS w.r.t. $\mathrm{conv}(q^1, \ldots q^n)$ of asymmetric game => population with mean in $Q(p)$ globally asymptotically stable w.r.t. the set of populations s.t.

type q^i with frequency $dF(q^i)$, $i=1, \ldots n$

$C(q) \subseteq B(p)$ where q is the population mean and

$\exists\, p' \in Q(p)$ s.t. $p' \in \mathrm{conv}\{q^i$ s.t. $dF(q^i) > 0\}$

Now to the results assuming discrete dynamics:

ASSUMPTIONS:

Asymmetric game $(U, (C_u)_{u \in U}, (w_{uv}, E_{uv})_{u,v \in U})$

Population structure: a finite number of different types $q^1, \ldots q^n$,

are present in the population ($q^i \in T$)

Dynamics: discrete

THEOREM / COROLLARY 4.3:

$A_0 \in \mathfrak{R}^{N,N}$ s.t. $(A_0)_{ii} = 0$, p eESS of A_0

=> $\exists\, a+ \in \mathfrak{R}^N$ s.t. $\forall a \geq a+$ ($a \in \mathfrak{R}^N$) theorem and corollary 4.2 hold for discrete dynamics if the payoffs are $E(x,y)=x^T A(A_0,a)y$

4.1 Truly asymmetric games:

In the following I would like to discuss the properties of the eESS in truly asymmetric games, i.e. players that are matched to play a game always belong to different information states.

DEF.: An asymmetric game $(U, (C_u)_{u \in U}, (w_{uv}, E_{uv})_{u,v \in U})$ is called truly asymmetric iff $\forall u \in U$: $w_{uu} = 0$

Selten [1980] showed that in truly asymmetric games the concept of an ESS is equivalent to that of strict Nash equilibria. Thus the ESS condition is very restrictive for these kind of games. Since an eESS is essentially an ESS condition on equivalent classes we get the same "bad" result for eESS:

CLAIM: p eESS $\iff$ $\forall q \in \mathrm{conv}(T)\backslash Q(p)$: $E(p,p) > E(q,p)$

$\iff$ $\forall u \in U\ \forall i \in \mathrm{conv}(T)(p_u)$: $(p\backslash p_u) \cup e_i \in Q(p)$

where $[(p\backslash p_u)\cup e_i](u) := e_i$, $\forall v \neq u$: $[(p\backslash p_u)\cup e_i](v) := p_v$

Selten [1983] developed the weaker concept of a limit ESS to resolve this problem. However it lacks asymptotic stability and therefore destroys this important characteristic:

EXAMPLE 4.2

	T	M	B
T	1	1	1
M	1	1	0
B	0	0	0

T is a limit ESS

but Q(T)={T} and {T} isn't asymptotically stable (just let M enter)

Discussion:

The concept of an Evolutionary Stable Strategy (ESS) developed by Maynard Smith [1982] presents a very intuitive and handy condition for a strategy to "survive" in a certain selection process. Taylor/Jonker [1978] and Zeeman [1980] were able to back up this intuition by showing that the mean strategy being an ESS is sufficient for the stability of a certain polymorphic population. This connection makes an ESS a very attractive equilibrium concept.

However when it comes to equivalent strategies the concept of an ESS turns out to be inconsistent: treating equivalent strategies as the same is quite a usual assumption (Samuelson [1989], Hines [1987]) and in many applications (see section 3 and 4) this becomes quite necessary. But the ESS concept doesn't include equivalent strategies being treated as the same. And getting rid of equivalent strategies by elimination doesn't work when they aren't pure strategies.

The concept of asymptotic stability of a strategy is inconsistent as well when equivalent strategies are being treated as the same: the stability must rather be defined w.r.t. the sets of strategies that can't be distinguished.

Due to these inconsistencies the traditional concepts don't capture all the stability properties they should.

In this paper these inconsistencies are resolved by introducing the concept of an Equivalent Evolutionary Stable Strategy (eESS) as a weaker notion of an ESS and considering the notion of asymptotically stable sets, the sets being sets of strategies that are equivalent.

The intuition that for these weaker concepts the same connections should hold between the static eESS condition and the stability of a population is confirmed by the theorems and proofs in section 2.

In fact the ESS are a subset of the eESS and the asymptotically stable strategies are also asymptotically stable sets of equivalent strategies.

Of course in practical examples finding an Equivalent ESS in a game with equivalent strategies can be much more difficult than finding an ESS. This is mainly due to the nature of games with equivalent strategies, they are harder to handle w.r.t. any equilibrium condition. So whenever possible purely equivalent strategies should be eliminated. This can be done without any problems since the elimination of purely equivalent strategies simplifies the computation but doesn't change the model when the eESS concept is used. Remember that this isn't the case for the more stringent concept of an ESS!

Summarizing the above: the concepts of an eESS and of asymptotically stable sets of equivalent strategies are consistent with equivalent strategies being treated the same and extend the traditional concepts and results of ESS theory.

In addition to these results on games with equivalent strategies there are some very nice applications to open questions of the ESS literature.

The first application (section 3) concerns populations where mixed strategy types are allowed:

In section 3 the assumption that the types use pure strategies is weakened to allow for types using mixed strategies and we still get the same results as in section 2.

The central theorem of the ESS theory is the result proven by Taylor/Jonker [1978] and Zeeman [1980] on the link between the ESS condition and the dynamic stability of a population. In retrospect we now see that their condition that types only use pure strategies was too stringent and can be generalized to the case of finitely many different types using mixed strategies.

Hines [1982] analyzed this case previously using covariance matrices. He proved the asymptotic stability of an ESS provided that the covariance matrix is non singular and can be assumed to be constant over time. Here these extra assumptions aren't needed and the theorem is more general since non singularity implies that the ESS must be interior.

The second application (section 4) concerns asymmetric games:

Selten [1980] extended the definition of an ESS to asymmetric games without analyzing the connection to the dynamics. However without this connection the stability of an ESS is only intuitively justified. This is made up for in section 4: using the concept of an eESS it is shown that the same dynamic stability results that hold for symmetric games also hold for asymmetric games. This gives more meaning to the name "evolutionary stable" when looking at asymmetric games.

Selten defined the concept of an ESS in asymmetric games on behavioral strategies since he ran into problems with equivalent strategies when looking at the symmetric normal form. It turns out that the concept of an eESS has a nice property in this respect: the concept of an eESS is independent of whether it is applied in the symmetric normal form or whether it is applied to behavioral strategies in the asymmetric game.

A different question is the "bad" properties (equivalent to strict Nash equilibria) of (e)ESS in truly asymmetric games . The concept of an eESS was developed to resolve inconsistencies in the ESS condition when equivalent strategies are considered to be the same. Otherwise ESS and eESS are based on the same ideas and hence subject to the same criticisms.

However a point should be made: whenever the concept of an ESS is changed to improve structural and existence properties and the dynamics are disregarded, in my opinion the central idea and strength of the (e)ESS theory is destroyed (e.g. limit ESS developed by Selten [1983]).

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