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Sufficient Conditions for Insensitivity in Linear Models

by

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Abstract

Sufficient conditions are described such that small perturbations in the coefficients of a nonsingular system of linear equations have no effect on some components of the solution vector. Applications in linear programming and game theory are discussed.

For all values of t in some domain, let $B(t)$ be an $m \times m$ matrix of real numbers and let $b(t)$ be an m-vector of real numbers. Assume that B and b are both continuous functions of t. Suppose that for some t^o in an open subset of the domain, $B(t^o)$ is nonsingular. Then $B(t)$ is also nonsingular on some open neighborhood N containing t^o. In this paper, we shall be interested in discovering usable sufficient conditions such that certain components of $x(t) = B^{-1}(t)b(t)$ are constant on N. That is, we are interested in discovering conditions under which certain components of x are insensitive locally to continuous parametric variations in both B and b.

To provide some motivation for this problem, suppose that $B(t^o)$ is the optimal basis for some linear programming problem; and suppose that both the primal and dual problems are nondegenerate at $B(t^o)$. (See DANTZIG[2], for example, for a text on linear programming.) It is generally considered to be important that for small parametric variations in the linear program, the corresponding columns of $B(t)$ still constitute an optimal basis. This information often makes sensitivity analysis in applied linear programming problems relatively easy, especially when $B(t) = B(t^o)$. We seek conditions under which, even though $B(t)$ may not equal $B(t^o)$, computation of some of the components of $x(t)$ is unnecessary; since they are the same as the corresponding components of $x(t^o)$.

Two sufficient conditions will be established. The first, which is easy to check in specific problems, will be applied to dynamic Leontief-type models and to two models in noncooperative game theory. The second condition is somewhat more difficult to verify in specific problems; but an example will be presented to indicate its possible usefulness.

1. First Condition.

Suppose that B(t) and b(t) are partitioned as follows:

$B(t) = \begin{pmatrix} B_{11} & 0 \\ B_{21}(t) & B_{22}(t) \end{pmatrix}$, $b(t) = \begin{pmatrix} b_1 \\ b_2(t) \end{pmatrix}$; where B_{11} and $B_{22}(t^o)$ are nonsingular, 0 is a matrix of appropriate dimension each element of which is the constant zero, and B_{11} and b_1 are constant with respect to t. Then, whenever $B_{22}(t)$ is nonsingular, $x(t) = \begin{pmatrix} x_1 \\ x_2(t) \end{pmatrix} = \begin{pmatrix} B_{11}^{-1}b_1 \\ B_{22}(t)^{-1}[b_2(t) - B_{21}(t)B_{11}^{-1}b_1] \end{pmatrix}$

is the unique solution of $B(t)x(t) = b(t)$.

We now discuss three applications of this observation.

Application 1: Consider the linear programming problem:

$$\begin{aligned} &\text{minimize } c^T x \\ &\text{subject to } \quad Ax = b \\ &\qquad\qquad\qquad x \geq 0 \end{aligned}$$

where A may be partitioned in the form

$$\begin{bmatrix} A_{11} & & & \bigcirc \\ A_{21} & A_{22} & & \\ \vdots & & \ddots & \\ A_{n1} & \cdots & \cdots & A_{nn} \end{bmatrix},$$

A_{ii} is a Leontief matrix for $i = 1, \ldots, n$ and $A_{ij} \leq 0$ for $i > j$. Such programs have been widely used in economic modeling (see, for example, VEINOTT[6]). It is well-known that any optimal basis B for such a problem has the form

$$\begin{pmatrix} B_{11} & & & \bigcirc \\ B_{21} & B_{22} & & \\ \vdots & & \ddots & \\ B_{n1} & \cdots & \cdots & B_{nn} \end{pmatrix}$$ where each B_{ii} is nonsingular. Suppose further that the optimal basis is both primal and dual nondegenerate. Consider now sufficiently small continuous perturbations in the coefficients of c, A, and b. Partitioning x and b accordingly, if for some $k < n$ the perturbations do not affect B_{ij} and b_i for $i = 1,\ldots, k$, then it follows that the optimal values of $x_1,\ldots, x_k$ for the program are unaffected by the perturbations.

As an example, suppose that the problem is one of minimizing the cost of satisfying certain production requirements b in each of n time periods, when the A_{ij} are appropriate technological coefficients. The result then implies that changes in costs, future requirements, and future technological coefficients, small enough that primal and dual feasibility are maintained, do not affect optimal activity levels in earlier periods.

Application 2: It is known (see COTTLE AND DANTZIG[1] and LEMKE AND HOWSON[5]) that the problem of finding all Nash equilibrium points for a two-person non-zero-sum game may be solved by finding all solutions to a system

$$\left[\begin{array}{c|c|c} I & 0 & G \\ \hline & H^T & 0 \end{array}\right] \begin{bmatrix} w \\ x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ \vdots \\ -1 \end{bmatrix} \tag{1}$$

$$\langle w, (x, y) \rangle = 0 \tag{2}$$

$$w, x, y \geq 0 \tag{3}$$

where G and H are strictly negative matrices of the same dimensions, say $r \times s$, and I is the $(r+s) \times (r+s)$ identity matrix. A complementary basis for this problem is a nonsingular submatrix M composed of $(r+s)$ columns from the partitioned matrix with the property that

$$Mz = \begin{pmatrix} -1 \\ \vdots \\ -1 \end{pmatrix}$$

$$z \geq 0$$

has a solution and exactly one of the ith column of I and the ith column of $\begin{bmatrix} 0 & G \\ H^T & 0 \end{bmatrix}$ is present in M for i = 1,..., r+s. Under the usual nondegeneracy assumption, (x^o, y^o) is a Nash equilibrium for the game (G, H) if and only if $x^o = \frac{x^*}{e_r^T x^*}$, $y^o = \frac{y^*}{e_s^T y^*}$ and (w^*, x^*, y^*) is a solution of (1), (2), (3), the positive components of which correspond to columns of some complementary basis. (e_ℓ is an ℓ-vector of ones for ℓ = r, s.)

By rearranging columns, it is clear that any complementary basis can be written in the form $\begin{pmatrix} D & 0 \\ 0 & E \end{pmatrix}$ where D and E are nonsingular. All columns corresponding to positive y are in $\begin{pmatrix} D \\ 0 \end{pmatrix}$ and those corresponding to positive x are in $\begin{pmatrix} 0 \\ E \end{pmatrix}$. Suppose that the elements of the matrix H (i.e., the second player's payoff matrix) are perturbed a sufficiently small amount; then by the first sufficient condition, the same strategy y^o for player 2 is part of a Nash equilibrium in the perturbed game against a slightly different strategy for the first player. (Of course, we may rename the players and make the same statement about player 1.) That is, in a nondegenerate bimatrix game, small changes in one player's payoff matrix alter only the equilibrium strategies of his opponent.

Application 3: We now turn to a class of two-person, nonzero-sum games with the extensive-form structure pictured in Figure 1. (In words, chance makes one of K possible moves with probabilities $p_1, \ldots, p_K$, respectively, known to both players. Player 1 then makes one of r_k moves, having observed chance's move k. Player 2, in ignorance both of chance's move and 1's move

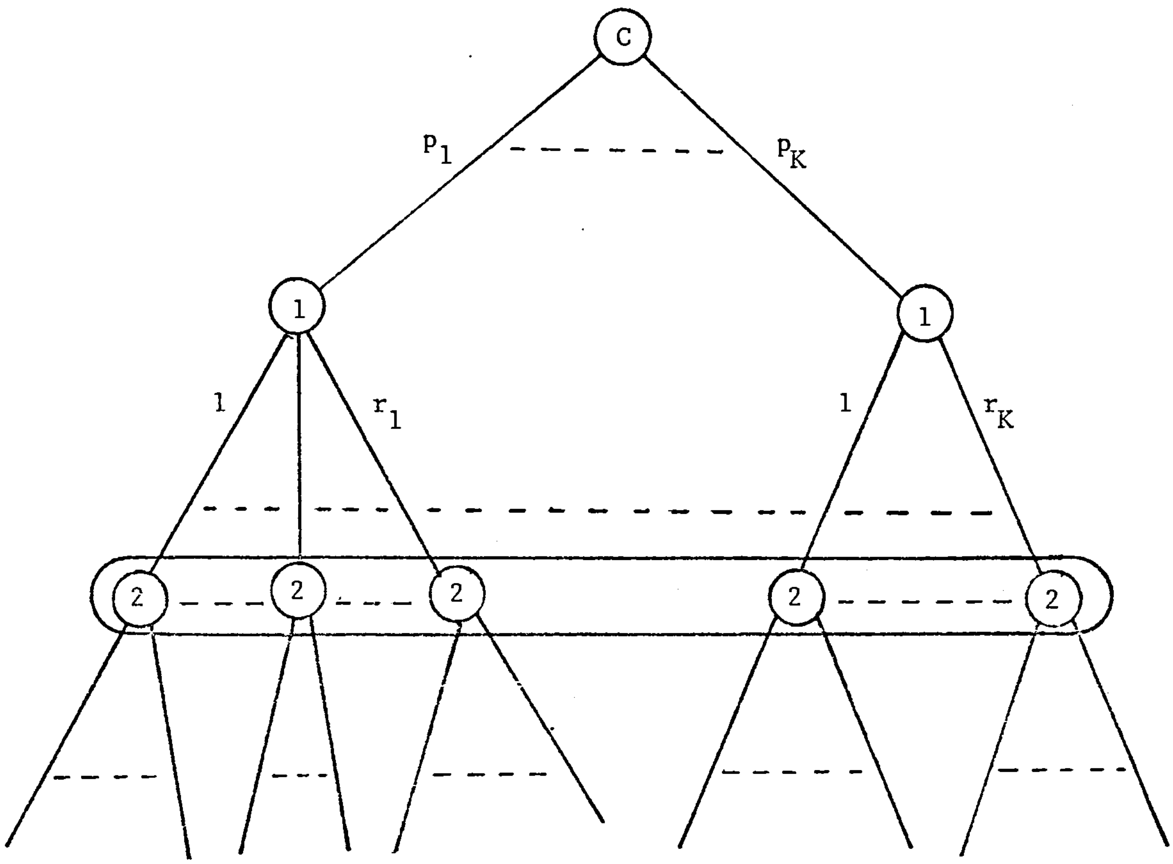

Figure 1

makes one of s moves. The payoffs for all possibilities are known.) As a special case of Theorem 1 in HOWSON AND ROSENTHAL[4] and the construction in HOWSON[3], it can be shown that Nash equilibria in behavioral strategies correspond to solutions of a linear complementarity problem with the same structure as in (1), (2), (3) except that for this case G is not a function of $(p_1,\ldots, p_K)$ while H is. Again applying the first condition, we conclude that in a nondegenerate game of the form represented in Figure 1, for sufficiently small changes in the chance-move probabilities, player 2's equilibrium strategies are unchanged.

Although applications 2 and 3 both concern two-person nonzero-sum games, they differ in that the perturbations of the probabilities in application 3 affect the payoff matrix of player 2 in the normal form of the game, but in such a way that his equilibrium strategies are locally constant. On the other hand, the results in both applications 2 and 3 can be understood intuitively in the following way. The equilibrium conditions require each player to make his opponent just indifferent between each of the opponent's basic strategies. For sufficiently small perturbations from a nondegenerate game the basic strategies do not change, and for the particular perturbations in these two cases neither does the opponent's expected payoff; hence, no change in the player's own equilibrium strategies. (I am indebted to Lloyd Shapley for this observation.)

2. Second Condition.

In this section, we derive a sufficient condition from considerations of duality.

Partition the original linear system as follows:

$$\begin{pmatrix} B_1(t) B_2(t) \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = b(t).$$

Since we seek conditions under which $x_1(t)$ is constant on a neighborhood of t^o, rewrite this as

$$B_2(t)x_2(t) = b(t) - B_1(t)x_1(t^o) .$$

From linear algebra, this system has a solution at t if and only if:

$$B_2^T(t)y = 0 \quad \text{implies} \quad \left(b(t) - B_1(t)x_1(t^o)\right)^T y = 0 . \tag{4}$$

Since it is generally difficult to ascertain whether or not (4) holds for all t in some neighborhood of t^o, this necessary and sufficient condition is useless for practical purposes. The following special case is, however, verifiable. Suppose that for all t in a neighborhood N about t^o,

$$S_1(t) = \{y : B_2^T(t)y = 0\} \text{ and } S_2(t) = \left\{y : \left(b(t) - B_1(t)x_1(t^o)\right)^T y = 0\right\}$$

are constant; then since (4) holds at t^o it must hold throughout N. In order that $S_1(t)$ be constant over N it is sufficient and necessary that the Hermite normal forms of all $B_2^T(t)$ are the same over N. In order that $S_2(t)$ be constant over N it is both sufficient and necessary that $b(t) - B_1(t)x_1(t^o)$ be a nonzero scalar multiple of $b(t^o) - B_1(t^o)x_1(t^o)$ throughout N. In many practical problems, both of these may be easy to verify when true.

Consider the following example.

$$\begin{bmatrix} -t & t & 1+3t & 0 \\ 0 & -(1-t^2) & 0 & (1+t)^2 \\ -3(1-t^2) & 3(1-t^2) & 3(1+3t) & 2(1+t)^2 \\ 2(1-t^2) & 2(1-t^2) & 5(1+3t) & 4(1+t)^2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} 1-t^2 \\ 2(1-t^2) \\ 9(1-t^2) \\ 21(1-t^2) \end{bmatrix}$$

At $t = 0$, $x(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 3 \end{pmatrix}$ is the unique solution. It is not immediately clear from inspection whether or not, for t near zero, the system remains solvable with $x_1(t) = 1$ and $x_2(t) = 1$. Checking the second sufficient condition,

$$B_2^T(t) = \begin{bmatrix} (1+3t) & 0 & 3(1+3t) & 5(1+3t) \\ 0 & (1+t)^2 & 2(1+t)^2 & 4(1+t)^2 \end{bmatrix}.$$

Clearly its Hermite normal form does not depend on t for $t > -\frac{1}{3}$. $b(t) - B_1(t)x_1(t^o) = (1 - t^2)(1, 3, 9, 17)$, a nonzero scalar multiple of $b(t^o) - B_1(t^o)x_1(t^o) = (1, 3, 9, 17)$ for $|t| < 1$. Hence, the system remains solvable with $x_1(t) = x_2(t) = 1$ for $-\frac{1}{3} < t < 1$.

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