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THE AGGREGATE EXCESS DEMAND FUNCTION AND OTHER AGGREGATION PROCEDURES

by

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Abstract

Two theorems are given; the first extends the Sonnenschein-Mantel-Debreu theorem characterizing aggregate demand functions from the set of $n \geq 2$ commodities to all $2^n - (n+1)$ subsets of two or more commodities. The second theorem concerns spatial voting models for $k \geq 2$ candidates over a space of n issues. The relationships among the sincere election rankings of the candidates for all of the sets of $2^n - 1$ issues are given. Both theorems have the same kind of conclusion; anything can happen. By showing the mathematical reasons for these results and by recalling some recent results from statistics, voting, and economics, it is argued that this "anything can happen" conclusion is the type one must anticipate from aggregation procedures; particularly processes of the type commonly used in economic models where the procedure is responsive to changes in agents' preferences, changes in data, etc.

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THE AGGREGATE EXCESS DEMAND FUNCTION AND OTHER AGGREGATION PROCEDURES[1]

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In 1972, H. Sonnenschein asked whether the aggregate excess demand function possesses any general properties other than the classical assertions about continuity, homogeneity, and Walras' Law. The stunning conclusion, discovered by Sonnenschein [26, 27], Mantel [10], and Debreu [3], is that there are no others. (Also see A. Mas-Colell [11] and the nice exposition found in W. Shafer and H. Sonnenschein [25].) The SMD theory proves for $n \geq 2$ commodities that starting with any choice of a continuous, homogeneous (of degree zero) function $\xi(\mathbf{p})$ satisfying Walras' Law, an assignment of monotone, continuous, strictly convex preferences and initial endowments for n agents can be found so that $\xi(\mathbf{p})$ is the aggregate excess demand function for this economy over most values of the prices $\mathbf{p} = (p_1, \ldots, p_n)$.

A related problem concerns the kind of information revealed about an economy once $\xi(\mathbf{p})$ is specified. So, for a given $\xi(\mathbf{p})$ where $n \geq 3$, let $\mathbf{e}$ represent one of the associated economies guaranteed by the SMD theory. Instead of exchanging all n commodities, suppose these n agents exchange only the k commodities, $C = \{c_1, \ldots, c_k\}$, $2 \leq k < n$; the remaining $n-k$ commodities, $\{c_{k+1}, \ldots, c_n\}$, are withheld from the market for legal or other reasons. Represent the relevant prices and the aggregate excess demand function for this restricted market by $\mathbf{p}_C = (p_1, \ldots, p_k)$ and $\xi_C(\mathbf{p}_C)$.

An extension of Sonnenschein's question that is in the spirit of the revealed preference literature is to determine all relationships that exist between $\xi(\mathbf{p})$ and $\xi_C(\mathbf{p}_C)$. For instance, suppose for $k = 3, n > 3$ that $\xi(\mathbf{p})$ satisfies the Arrow-Hurwicz [1] condition of gross substitutes; a condition that ensures $\xi(\mathbf{p})$ defines a well behaved dynamic (gradient-like) with a unique stable equilibrium. How much of this regularity must $\xi_C(\mathbf{p}_C)$ inherit? Does this assumption on ξ preclude the possibility that ξ_C exhibits a limit cycle of the form created by Scarf [24], or that it prevents ξ_C from admitting an even more extreme structure with many different equilibria where each is locally unstable? (See Saari-Simon [21].) Conversely, what happens when new commodities are introduced into a market? As an illustration, suppose liberalized laws permit previously proscribed commodities to be exchanged in the market place. What properties of ξ_C impose restrictions on the behavior of ξ?

More generally, let $C^n = \{c_1, \ldots, c_n\}$ denote the set of all $n \geq 3$ commodities. The set C^n admits 2^n different subsets of commodities. One of these subsets is empty and n of them consist of a single commodity, so denote the remaining $2^n - (n+1)$ subsets

[1] I am pleased to acknowledge conversations with L. Hurwicz, J. Jordan, A. Mas-Colell, and C. Simon about earlier versions of this manuscript. Also, I benefited from comments made at meetings and seminars where these results were presented. In particular this includes a June, 1990, conference in Stockholm, Sweden. Some stylistic suggestions were made by Arvid and Padon Kalinen. This research was supported, in part, by NSF Grant IRI-8803505

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(with at least two commodities) by $D_1, \dots, D_{2^n-(n+1)}$. When only the goods listed in D_j are exchanged in this economy, represent the corresponding prices as $\mathbf{p}_{D_j}$ and the aggregate excess demand function as $\xi_{D_j}(\mathbf{p}_{D_j})$. The more general issue for an exchange economy is to determine all possible relationships among these $2^n-(n+1)$ aggregate excess demand functions. For instance, using the intuition behind the consumer surplus literature, one might wonder whether the excess demand functions for the restricted markets tell us anything about the excess demand function for the full market. As an extreme example, suppose for each of the $2^n-(n+2)$ proper subset of goods that the excess demand function, ξ_{D_j}, satisfies the gross substitutes property. Does this strong regularity assumption impose any restrictions on the excess demand for the total economy? What are the relationships among the $2^n-(n+1)$ excess demand functions?

The somewhat unexpected conclusion, as proved here, is that *for most prices there need not be any relationship whatsoever among the demand functions* $\{\xi_{D_j}\}_{j=1}^{2^n-(n+1)}$; *anything can happen!* Indeed, the formal statement has the exact same flavor as the SMD theory. As asserted in Theorem 1, for each D_j one can choose *any* continuous function $\xi_{D_j}(\mathbf{p}_{D_j})$ that is homogeneous of degree zero and satisfies Walras' law for $\mathbf{p}_{D_j}$. Theorem 1 ensures the existence of an economy $\mathbf{e}$ – an assignment of monotone, continuous, strictly convex preferences and initial endowments for n agents – so that whenever only the goods listed in D_j are exchanged, the chosen function $\xi_{D_j}(\mathbf{p}_{D_j})$ is the aggregate excess demand function for most values of the prices $\mathbf{p}_{D_j}; j = 1, \dots, 2^n-(n+1)$.

As it will be apparent from the proof, this conclusion that "anything can happen" extends to other price models. In fact these results extend, in part, even to individual excess demand functions. This last assertion is an immediate by-product of the nature of the proof of Theorem 1. Here, following the lead of Debreu, each aggregate excess demand function $\xi_{D_j}(\mathbf{p}_{D_j})$ is divided into an individual excess demand for each of the n agents. Then an individual preference relationship and initial endowment is constructed for each agent that generate the assigned demand functions. Because these individual excess demand functions for the different choices of sets D_j can be unrelated, it follows immediately that even for an individual, very few constraints can be imposed upon the excess demand functions defined by the different subsets of commodities.[2] This assertion, for example, suggests an immediate explanation for some of the well known difficulties in consumer surplus and other topics. However, rather than discussing the consequences and corollaries of these results, I choose to emphasize the nature of this conclusion which asserts the existence of chaotic outcomes from an economic model.

This claim that "anything can happen" is surprising. It appears to add further support for W. Hildenbrand's opinion that *"an exchange economy can no longer serve as an appropriate prototype example for an economy if one wants to go beyond the existence and optimality problem."* [3, pp26]. Perhaps; but perhaps there is another interpretation. Perhaps, as argued here, the deeper message is that this conclusion where anything can happen is the type of outcome we must anticipate for large classes of economic models. As I show, there is a growing body of evidence to indicate that this kind of outcome occurs with many of the aggregation processes – from statistics, choice, and allocation systems –

[2]The qualification arises because, in the proof, the assigned individual excess demand functions are of a particular form. However many of these restrictions can be eliminated.

that are widely used in economic modeling. This in itself suggests that unless proved otherwise one must expect that anything can happen with many different economic models. To support this assertion I offer another theorem with a very similar conclusion, I discuss some related results from the literature, and I indicate how some standard assumptions from economics and decision theory are responsible for this kind of outcome.

Before discussing the prevalance of this behavior, it is worth considering some immediate consequences. In particular this assertion means that, unless and until proven otherwise, one must not assume that conclusions based on economic models are robust. Without proof, we cannot assume that a "minor" change in assumptions or procedures will cause only slight changes in the conclusions. Instead one must suspect, until proven otherwise, that the outcome of an aggregation, decision, or allocation procedure can be highly sensitive to modifications of certain basic assumptions; e.g., changes in the set of agents, restrictions on the set of commodities, variations in the set of issues, preferences, etc. As shown here, "small" changes in these variables can lead to radical differences in the conclusions. Consequently one must wonder, for example, whether "simplifying assumptions" about such variables (e.g., "let all agents have preferences of the type . . .") serve merely to simplify the analysis, or whether they critically dictate the final outcome. One must wonder whether conclusions derived from such a model indicate universal truths about economics, or whether they reflect only what occurs in special cases. If it is the latter, then, as argued in (Saari and Williams [22]), different mechanisms and procedures are required for different economic settings.

The assertion that a chaotic[3] state of affairs should be anticipated with economic models is strongly supported by the behavior of the commonly used system of plurality voting; a system that admits strikingly similar conclusions. To motivate the discussion, suppose that the outcome of a departmental plurality election for the one tenure-track position is $c_1 \succ c_2 \succ \cdots \succ c_n$ and that the bottom ranked candidate, c_n, withdraws from consideration. Confronted with such a situation, it would not be uncommon for a department to act as though nothing important has changed; the tacit assumption would seem to be that the situation changed so slightly that c_1 remains the department's top-choice. Would she?

As above, from the $n \geq 3$ candidates $C^n = \{c_1, \ldots, c_n\}$, construct the $2^n - (n+1)$ sets of two or more candidates $\{D_j\}_{j=1}^{2^n-(n+1)}$. With voting, the outcome for each D_j is an election ranking rather than an excess demand function. So, for each D_j arbitrarily select a (complete, binary, transitive) ranking of the candidates, α_j. The conclusion (Saari [17]) is that there exists a profile of voters, $\mathbf{p}$, where the sincere plurality election outcome of $\mathbf{p}$ for the subset of candidates D_j is $\alpha_j; j = 1, \ldots, 2^n - (n+1)$. This conclusion asserting that "anything can happen" holds for almost all ways there are to tally ballots; the singular exception where relationships always exist among the election rankings for $n \geq 3$ is the Borda Count (Saari [17, 20]). (Recall, the Borda Count for n candidates is where a voter's ith ranked candidate is assigned $n - i$ points, $i = 1, \cdots, n$.)[4] In fact, even in a

[3] The word "chaos" is used both in its generic sense as well as to invoke comparisons with recent developments from dynamical systems. One such comparison is made above with the comment about the sensitivity of aggregation procedures with respect to small changes in certain assumptions. The connections with dynamical systems are much deeper; the technical approach used in this program was motivated by ideas from "chaos" and "symbolic dynamics." These connections are described in Saari [15, 16].

[4] In a conversation (May, 1990) L. Hurwicz wondered whether there are allocation procedures for exchange

more general social choice setting, one can argue (Saari [19]) that Arrow's Impossibility Theorem, Sen's pareto liberal paradox, etc. are further manifestations of this property that almost anything can happen with aggregation processes.

Similar assertions about chaotic outcomes hold for many other kinds of aggregation processes including, say, statistics. As just one example, consider the widely used, non-parametric Kruskal-Wallis procedure (see, for instance, Kruskal [8]). This method permits the alternatives from a given set to be ranked. As above, from the set of all n alternatives, $C^n = \{c_1, \ldots, c_n\}$, construct the sets $D_1, \ldots, D_{2^n-(n+1)}$ consisting of two or more alternatives. In her PhD dissertation, D. Haunsperger [4, 5] shows, in part, that the K-W test does admit relationships among the rankings of the various sets of candidates, and these relationships are the exact same kind admitted by the Borda Count from voting. As Haunsperger also shows, should only slight differences be made in K-W procedure, then there need not be any relationships whatsoever among the rankings of the different subsets of alternatives. This kind of conclusion for statistics is not restricted to non-parametric methods. A somewhat worrisome fact is that similar conclusions arise with Bayesian statistics and Bayesian decision analysis – procedures commonly used in game theory and other aspects of economic theory. (For a related type of problem see Saari [18] and Haunsperger and Saari [6].)

"Incentive Theory" is an economic topic specifically designed to avoid allowing everything to happen. After all, by admitting additional alternatives, one may be creating extra opportunities to manipulate the outcome. One incentive approach, the Bayesian-Nash equilibrium of games, has enjoyed great success in explaining a wide variety of economic behaviors. But, as J. Ledyard [9] noted, such research incorporates a standard set of simplifying assumptions that frequently include "risk-neutral agents with quasi-linear preferences and independently distributed private values." He goes on to wonder whether *"it is the assumption of Bayes equilibrium behavior or the assumptions of specific utility functions and beliefs which drive the results. If the former, then the assumptions are merely simplifying and the conclusions of research in this area can be widely applied; if the latter, then the assumptions are substantive and care must be taken not to attribute too much to any particular result."* Ledyard then proves it is the latter! When a wider class of utility functions and beliefs are admitted, anything can happen. By being free to select preferences, he essentially proves that *"any non-dominated behavior can be rationalized as Bayesian equilibrium behavior."* Ledyard's conclusion, then, supports the theme of this paper – we must anticipate that anything can happen. One must expect economic procedures to be preference specific and sensitive to basic assumptions. (See Section 3.)

The basic reason for the above assertions is that for each system, each agent's preferences (or the data) can vary over a wide selection of quite different rankings. Each of the above conclusions is a manifestation of the diversity of heterogeneous profiles and the sensitivity of

economies where, instead of Theorem 1, a Borda type of conclusion holds. There are. One is obtained by replacing each agent's individual excess demand function with the sum of the agent's excess demands for each pair of commodities. This procedure provides a Borda type of conclusion, but the resulting equilibrium need not be a Pareto point. Indeed, with only a slight modification of the proof of Theorem 1 one can show that the general properties among the sets of pareto points for the different sets of commodities have little to do with one another. This probably can be used to show there does not exist a procedure that always satisfies both Hurwicz's requirement and the pareto property.

the procedure to changes in these profiles. Thus the plethora of different possible outcomes follows from the richness of the space of profiles; a richness represented in Section 3 by the geometric dimension of a space. To illustrate these comments a second theorem (of independent interest) is given. One of the advantages of this theorem is that the associated geometry is sufficiently simple to illustrate the source of these assertions about chaotic outcomes. (As a by-product, the geometry also indicates why these outcomes are "robust" – they persist for open sets of preferences.)

To introduce this second theorem, note that in the above discussion about positional and plurality voting, the outcomes can vary randomly with changes in the sets of candidates. What happens should the set of candidates remains fixed, the voters remain the same, but, say, the sets of issues change? (This modeling involves spatial models for voting.) As asserted in Theorem 2, anything can happen. For each subset of issues, arbitrarily choose a ranking for the candidates. There exist assignment of voters' preferences so that the plurality ranking of the candidates for each subset of issues is the chosen one.

2. Excess Demand and Spatial Models for Voting

Let $\mathbf{p} \in R^n_+$. Recall that a function $f(\mathbf{p}) : R^n_+ \to R^n$ is homogeneous of degree zero iff for any positive scalar λ, $f(\mathbf{p})$ satisfies

$$f(\lambda\mathbf{p}) = f(\mathbf{p}). \tag{2.1}$$

Function $f(\mathbf{p})$ satisfies Walras' Law iff

$$< \mathbf{p}, f(\mathbf{p}) >= 0 \tag{2.2}$$

where $< -, - >$ is the standard Euclidean inner product in R^n. If $f(\mathbf{p})$ is homogeneous, we can restrict attention to

$$S^{n-1}_+ = \{\mathbf{p} \in R^n_+ | < \mathbf{p}, \mathbf{p} >= \sum_{j=1}^n p_j^2 = 1.\}$$

Thus S^{n-1}_+ is the portion of the unit sphere in the positive orthant R^n_+.

THEOREM (SMD). *Let $\xi(\mathbf{p})$ be a continuous function satisfying Eqs. 2.1, 2.2. For $\epsilon >$ 0, there exists an economy — an assignment of monotone, continuous, strictly convex preferences $\succeq_i$ and initial endowments for the agents — so that $\xi(\mathbf{p})$ is the aggregate excess demand function for all $\mathbf{p}$ in*

$$S^{n-1}_\epsilon = \{\mathbf{p} \in S^{n-1}_+ | p_j \geq \epsilon, j = 1, \ldots, n\}. \tag{2.3}$$

This assignment process can be accomplished with only n agents; there exist examples where it cannot be accomplished with fewer than n agents.

The culmination of the SMD theory for exchange economies, as expressed in the above strong, general assertion, is due to G. Debreu [3].

Using the notation from the previous section, we have that if a function is homogeneous of degree zero in $\mathbf{p}_{D_j}$, then attention can be restricted to $\mathbf{p}_{D_j} \in S^{|D_j|-1}_+$.

THEOREM 1. *Let $\epsilon > 0$ be given. Let $\{\xi_{D_j}(\mathbf{p}_{D_j})\}_{j=1}^{2^n-(n+1)}$ be a collection of continuous, homogeneous (of degree zero) functions where $\xi_{D_j}(\mathbf{p}_{D_j}) \in R^{|D_j|}$ satisfies Walras' law $< \xi_{D_j}(\mathbf{p}_{D_j}), \mathbf{p}_{D_j} >= 0$. There exists an exchange economy consisting of n agents, each with a fixed initial endowment and with monotone, strictly convex, continuous preferences, with the property that should these agents exchange only the commodities in set D_j and hold fixed the remaining commodities at their level of initial endowment, then the aggregate excess demand function is $\xi_{D_j}(\mathbf{p}_{D_j})$, $\mathbf{p}_{D_j} \in S_\epsilon^{|D_j|-1}$; $j = 1, \ldots, 2^n - (n+1)$.*

A similar theorem also asserting that anything can happen occurs in spatial models for voting[5]where the issues confronting the electorate can change. See, for instance, the papers by McKelvey [12] and Kramer [7] and the book by Ordeshook and Riker [13] and their references. In this setting the issues $C^n = \{c_1, \ldots, c_n\}$ are represented as points in an issue space R^n where the ith issue is assigned the coordinate direction $\mathbf{e}_i$; the value of a component of a point in R^n determines the degree of intensity in support of the corresponding issue – perhaps on a "liberal – conservative" scale. The ith voter is characterized by a particular point $\mathbf{q}_i \in R^n$ and, quite naturally, the closer another point is to $\mathbf{q}_i$, the more the ith voter prefers it. Thus this agent's utility function is $U_i(\mathbf{q}) = -||\mathbf{q}_i - \mathbf{q}||$ where $|| - ||$ is the Euclidean distance. A voter votes for the candidate whose views on the issues are closest to his; namely, voter i votes for the jth candidate characterized by her stand on the issues $\mathbf{a}_j$ iff $-||\mathbf{q}_i - \mathbf{a}_j|| \geq -||\mathbf{q}_i - \mathbf{a}_k||$ for all $k \neq j$. To become elected a candidate attempts to position herself in issue space so she receives the most votes.

The plurality method is a special case of a positional voting method. A positional voting method for k candidates is defined by a voting vector $\mathbf{W}^k = (w_1, \ldots, w_k)$ where the scalars w_i satisfy the inequalities $w_1 > w_k = 0, w_i \geq w_{i+1}, i = 1, \ldots, k-1$. In tallying a voter's ballot, w_i points are assigned to the voter's ith ranked candidate. The candidates are ranked according to their point totals where "more is better." Thus, plurality voting is defined by the voting vector $(1, 0, \ldots, 0)$ while the Borda Count is defined by $(k-1, k-2, \ldots, 1, 0)$.

Corresponding to each of the $2^n - 1$ subsets of one or more issues, $\{D_j\}_{j=1}^{2^n-1}$ from C^n, let $R^{|D_j|}$ be the appropriate coordinate plane (or axis) of R^n that represents the issue space for D_j. It is natural to wonder how a particular positioning of a candidate's views affects her chances with respect to different subsets of issues. It is obvious that if a candidate's views about abortion, foreign aid, or some other previously undiscussed issue suddenly become disclosed, this changes the set of issues and it can affect her standing. The interesting issue is to determine whether there exist relationships among the election rankings for the different sets of issues.

To pose this question in more specific terms, note that a basic assumption is that a candidate attempts to choose her positions on the issues to appeal to the largest number of voters. Presumably she does this by using surveys. So, suppose by use of totally reliable polling information a candidate chooses the winning position for each of the n issues. Is it possible for her to end up in a bottom-ranked position whenever sets of two or more issues

[5]This model has close connections to other areas such as location theory, statistics, etc. As such, Theorem 2 illustrates that this "anything can happen" conclusion extends to several other topics. Indeed, it is easy to modify Theorem 2 so that the conclusion applies to these areas.

are considered? As an even more dramatic example, suppose by using very sophisticated polling techniques a candidate skillfully positions herself so she has the winning position for each of the $2^n - (n+2)$ proper subsets of issues. Is it possible for her to end up being bottom-ranked for the set of all n issues? As asserted in the following theorem, anything is possible with a wide selection of choices of positional voting methods. This is because there need be no relationship whatsoever among the candidates' rankings for the different sets of issues. As true for the earlier statements about positional voting, the Borda Count provides some relief from all possible paradoxes. However, in this setting, the BC is not alone; there are other choices of voting vectors that avoid even more paradoxes.

The second part of this theorem depends on the following non-degeneracy condition imposed upon the positions taken by the candidates. It is a vector independence condition that requires each candidate to distinguish herself in that her position is not a simple linear combination of the positions assumed by certain other candidates.

DEFINITION. *Represent the candidates' stands on the issues by the vectors* $\{\mathbf{a}_i\}_{i=1}^k$. *The* k *candidates are* **distinguishing on the issues** *if for each* D_j *the following condition holds. For the set of issues* D_j, *let* $\gamma(D_j) = min(k, |D_j| + 1)$. *For each set of* $\gamma(D_j)$ *candidates' stands on the issues in* D_j, *the convex hull of the points has dimension equal to* $\gamma(D_j) - 1$.

THEOREM 2. *Assume there are* $n \geq 2$ *issues,* $k \geq 2$ *aspiring candidates defined by their position on the issues* $\{\mathbf{a}_j\}_{j=1}^k$.

a. *Assume that for each set of issues, the candidates' stands are distinct – no two are the same. There exists an open set of voting vectors* $\mathcal{W}_1^k$ *containing the plurality voting vector such that if* $\mathbf{W}^k \in \mathcal{W}_1^k$ *is used to tally the elections, then the following holds:*
Let α_j *be a ranking of the* k *candidates for* D_j, $j = 1, \ldots, 2^n - (n+1)$. *There exists a choice of the* $m \geq 2$ *voters characterized by their beliefs over the issues,* $\{\mathbf{q}_i\}_{i=1}^m$, *so that when these* m *voters vote on the candidates subject to the set of issues* D_j, *the outcome is* α_j, $j = 1, \ldots, 2^n - 1$.

b. *Assume the candidates' stand on the issues is distinguishing. For each set of issues* $D_j, |D_j| \geq k$, *choose a ranking* α_j *of the* k *candidates. For any choice of a voting vector* $\mathbf{W}^k$ *used to tally the elections, there exists a choice of* $m \geq 2$ *voters characterized by their beliefs over the issues,* $\{\mathbf{q}_i\}_{i=1}^m$, *so that when these* m *voters vote on the candidates subject to the set of issues* $D_j, |D_j| \geq k$, *the outcome is* α_j.

From this assertion, it follows that anything can happen in the election rankings (based on a wide choice of positional voting methods) when the sets of issues change. The conclusion of the theorem extends to where voter's preferences are modeled by use of any other choice of norms (where, say, the level sets are ellipsis or rectangles rather than circles), to where different sets of voters can use different kinds of norms (to reflect different weightings of the issues), and even to the setting where some voters' preferences are given by semi-norms used to model those situations where some voters' interests are restricted to single or a limited number of issues. Some extensions to other classes of utility functions can be made. In fact, the same conclusion holds even if the choice of the positional voting method changes with the set of issues. (This last extension is more of interest for statistical procedures than for election processes where the tallying method is designated in advance and it is independent of what issues may emerge in an election.)

To see the size of the $\mathcal{W}_1^k$ sets, it follows from the proof that $\mathcal{W}_1^2$ contains all methods (there is only one), and that

$$\mathcal{W}_1^3 = \{\mathbf{W}^3 = (w_1, w_2, 0) | w_1 > 2w_2\}. \tag{2.4}$$

As a consequence, about half of all positional voting methods available for three candidate elections are in $\mathcal{W}_1^3$. Moreover, the BC just avoids being in $\mathcal{W}_1^3$ by being on the boundary. The anti-plurality method $(1,1,0)$ is the extreme case of a procedure not in $\mathcal{W}_1^3$. To see why these last two voting vectors are not in $\mathcal{W}_1^3$, suppose for each set of issues that the positions of the three candidates form a straight line. This means that one candidate, say c_2, always is in the middle. In such a degenerate (but admissible) setting, for each D_j c_2 never can be bottom ranked by any voter. This restriction on the admissible profiles forces c_2 to be BC ranked in first or second place. Thus the worse electoral fate she could encounter is a complete tie with all three candidates. Should the anti-plurality method be used, c_2 can do no worse than be tied for first- place with one other candidate. (This is the only positional voting method $\mathbf{W}^3$ with this property.) So, because of the restrictions on the profiles, not all election rankings can occur with these procedures. In fact for all $s, k \geq 2$, the Borda Count, the voting vectors

$$(\overbrace{1, \ldots, 1}^{s \text{ times}}, 0, \ldots, 0),$$

and the convex combinations of these vectors are not in $\mathcal{W}_1^k$.

By imposing more realistic non-degeneracy conditions on $\{\mathbf{a}_j\}_{j=1}^k$, more types of profiles are admitted. The effect is that Theorem 2 holds for larger classes of positional voting methods. For instance, for integer δ, $1 \leq \delta \leq k-1$, and for each $D_j, |D_j| > \delta$, suppose that the convex hull defined by each subset of $\delta + 1$ positions of the candidates has dimension δ. This non-degeneracy condition admits more profiles, so there are fewer restrictions on the voting vectors leading to the above type of conclusion. As such, the conclusion of Theorem 2 holds for much larger classes of voting vectors $\mathcal{W}_\delta^k$ (but the election outcome over sets of issues fewer than δ may be restricted.) This conclusion is indicated in part b of the theorem for $\delta = k - 1$. When the candidates have distinguishing stands on the issues and an arbitrary choice of $\mathbf{W}^k$ is used, then there may be restrictions on the rankings when the election is based on a fewer than k issues, but the restrictions lift as the value of $|D_j|$ increases, and they disappear once $|D_j| \geq k$.

With the Euclidean norm, the flexibility in choosing the ith voter's preferences, characterized by $\mathbf{q}_i$, allows us to restrict the number of voters needed for the plurality vote; it appears that the conclusion holds for m that is an integer multiple of k such that $m \geq \binom{k+1}{2}$. To appreciate this value of m, note that $m = \binom{k+1}{2}$ voters are needed should each candidate receive at least one vote and should there be a one vote margin between each candidate's tally in the selected ranking. The condition that m is a multiple of k is needed only to accommodate a ranking of a complete tie vote among the candidates.

While Theorem 2 appears to be a new, different type of result for spatial models of voting, there are other statements in this large literature with this same "anything can happen" flavor. As a particularly well known example demonstrating how this flexibility can lead to problems of "manipulation," I point to McKelevey's important paper [12] where, with some mild assumptions on the locations of the $\mathbf{q}_i$'s, he proves that one can start with two *arbitrary* points $\mathbf{x}_0, \mathbf{y}_0 \in R^n$ and then build an agenda (a listing of points that are compared, in a specified order, with a majority vote) starting at point $\mathbf{x}_0$, passing through $\mathbf{y}_0$ and returning to $\mathbf{x}_0$. In other words, the set of issues and the voters' points, $\{\mathbf{q}_i\}_{i=1}^m$, points remain fixed, and the $\{\mathbf{a}_j\}_{j=1}^k$ points are chosen appropriately to create a cycle[6].

3. The Basic Idea and Other Applications

The purpose of this section is to indicate that this kind of conclusion results from assumptions that are most natural to many economic models. The mathematical approach is a modification of the main theme in (Saari [14, 19]). It starts with the fact that aggregation processes with m agents and n alternatives can be viewed as defining a mapping

$$F : \prod_{j=1}^{m} P_j \to \prod_{i=1}^{\beta} A_i \tag{3.1}$$

where P_j is the space of preferences for the jth agent, A_i is the outcome space characterizing the different outcomes or rankings for the alternatives associated with the subset D_i, and the ith component of F specifies the outcome of the profiles for D_i, $i = 1, \dots, \beta$. If F is onto, then it follows that any outcome admitted by A_j can be coupled with any outcome from A_i, $j \neq i$. This means that "anything can happen;" the outcome for one set D_j need not have anything whatsoever to do with the ranking for D_i, $j \neq i$.

Proving that a mapping is surjective concerns rank conditions for the Jacobian DF. Here the proof involves either a direct, or an indirect argument concerning how the value of F varies with changes of profiles. *These rank conditions, then, are identified with the responsiveness of the procedure to changes in agents' preferences.* As responsiveness is a common assumption in economics, one must expect, in general, that the rank conditions will be satisfied. Thus, what remains is to compare the dimension of the domain and range; if $dim(\prod_{j=1}^m P_j) \geq dim(\prod_{i=1}^{\beta} A_i)$, then we must question whether F is surjective; we must anticipate the possibility of chaotic lists of outcomes. Stated in slightly different terms, should a responsive model admit a rich selection of heterogeneous profiles, then one must expect "anything to occur."

Notice how these comments reflect the spirit of standard assumptions from social choice and game theory. For example, a "no dictator" assumption requires a choice procedure to

[6] With the methods developed in this paper, one can show that a similar theorem holds should positional voting methods be used instead of an agenda. Here values of β, s are specified where $\beta > s \geq 1$. For a arbitrarily selected $\mathbf{a}_1, \mathbf{a}_n$, a sequence of points $\{\mathbf{a}_1, \dots, \mathbf{a}_n, \dots, \mathbf{a}_k, \mathbf{a}_1\}$ are given (but chosen appropriately). The first β candidates are voted upon and then the s top-ranked candidates are advanced to be compared with the next $\beta - s$ listed candidates. The procedure is continued until the last set of candidates is selected, and then the top ranked candidate is selected. At one of the steps, $\mathbf{a}_n$ will be the top-ranked candidate; the final winner is $\mathbf{a}_1$. So, if $\beta = 2$ and $s = 1$, then this becomes the McKelevey Theorem. Should there be at least β issues, it can be shown that this conclusion holds for all positional voting methods and that it follows from the results in Saari [17]. Thus the technically interesting case is for fewer than β issues with the accompanying restrictions on profiles.

be responsive to the preferences of more than one agent, while other assumptions, such as independence of irrelevant alternatives, contribute to the rank considerations or serve to determine the dimensions of the domain and range. Therefore one might correctly suspect that the kind of ideas developed here can be used to explain, extend, and unify many of the results from social choice, including Arrow's Theorem – one just needs to create an discrete analog for the Jacobian. (Saari [19].)

One way to illustrate these ideas is with the simple geometric proof of Theorem 2 for $k = 2, n = 2$, where there are no tie votes for the selected $\{\alpha_j\}_{j=1}^{3}$. Let $\mathbf{a}_{i,D_j}$ denote the ith candidate's position on the issues in D_j. As each candidate's position on each of the coordinate axes of $R^n = R^2$ uniquely defines her position for the space of both issues, it follows that

$$\mathbf{a}_{i,\{1,2\}} = (\mathbf{a}_{i,\{1\}}, \mathbf{a}_{i,\{2\}}) \tag{3.2}$$

For the jth axis (representing the jth single issue), $j = 1, 2$, find the perpendicular bisector for the points $\mathbf{a}_{1,\{j\}}, \mathbf{a}_{1,\{j\}}$. These two lines, one horizontal and the other vertical, divide R^2 into four regions. (See Figure 1.) The ith voter's preferences based on the two single issues is determined by which quadrant contains $\mathbf{q}_i$. Thus each of these quadrants is characterized by one of the four pairs of ordinal rankings of the single issues. For instance, the rankings $\mathbf{a}_{1,\{1\}} \succ \mathbf{a}_{2,\{1\}}$, $\mathbf{a}_{2,\{2\}} \succ \mathbf{a}_{1,\{2\}}$ correspond to where $\mathbf{q}_i$ is in the quadrant on the $\mathbf{a}_{1,\{1\}}$ side of the vertical line and the $\mathbf{a}_{2,\{2\}}$ side of the horizontal line.

To determine the ith voter's preferences for the full set of issues $\{1, 2\}$, find the perpendicular bisector of the line segment connecting the two points $\mathbf{a}_{1,\{1,2\}}, \mathbf{a}_{2,\{1,2\}}$. For single issues each candidate has a distinct position, therefore this line segment is not parallel to either coordinate axis. With the assistance of simple trigonometry (using the fact that in similar triangles the ratios of similar sides is a fixed constant), it follows that this perpendicular bisector passes through the intersection of the horizontal and vertical lines. Moreover, this bisector passes through the two quadrants where a voter changes his ranking of the candidates depending on which single issue is being considered. What this bisector determines, then, is how such a voter ranks the candidates when both issues are being considered. Geometrically, these three bisectors divide R^2 into six regions, so there are six different "types of voters." Thus it is the geometry that increases the number of voter types when issues are added; the geometry of the coordinate axis determine the voter types for single issues with the geometry of the interior determine the voter types when both issues are involved.

This increase in voter types leads to the proof of the assertion. To illustrate the ideas, I use the special case represented in Figure 1. Let n_i be the number of voters in the ith region in this figure, let $x_i = \frac{n_i}{\sum_{j=1}^{6} n_j}$, $i = 1, \ldots, 6$, and let $\mathbf{x} = (x_1, \ldots, x_6)$. By definition, $\mathbf{x}$ is an element of the five dimensional unit simplex $Si(6) = \{\mathbf{y} \in R^6 \,|\, \sum_{j=1}^{6} y_j = 1,\, y_j \geq 0\}$. The combined election outcomes for all three sets of issues can be represented as the mapping $F : Si(6) \to R^3$ defined by

$$F = \begin{pmatrix} F_{\{1\}} \\ F_{\{2\}} \\ F_{\{1,2\}} \end{pmatrix} = \begin{pmatrix} (x_1 + x_2 + x_6) - (x_3 + x_4 + x_5) \\ (x_4 + x_5 + x_6) - (x_1 + x_2 + x_3) \\ (x_1 + x_5 + x_6) - (x_2 + x_3 + x_4) \end{pmatrix} \tag{3.3}$$

The sign of F_{D_j} determines the winner of the election for the set of issues D_j; a positive value means that a_1 wins, a negative value means that a_2 wins, and a zero value represents a tie vote. As $F(\mathbf{p}')) = 0 \in R^3$ for the interior point $\mathbf{p}' = (\frac{1}{6}, \dots, \frac{1}{6}) \in Si(6)$, and as the Jacobian DF has maximal rank, it follows that F maps an open neighborhood of $\mathbf{p}'$ to V, an open neighborhood of 0. Thus the image of F contains an open set about the origin in R^3. As such, V intersects all eight orthants of R^3, all coordinate planes, and all coordinate axes. Consequently any combination of the $3^3 = 27$ election rankings can occur.

To review the steps of this argument, note that the geometry of the issue space defines a domain for F with a larger dimension than that of the image space. The responsiveness properties of the election procedure ensures that the rank conditions are satisfied. Thus, the existence of the asserted chaotic outcomes is ensured.

It is worth indicating how the minimal value of $m = 3$ arises; it is based on the symmetry of the problem. Select rankings $\{\alpha_{D_j}\}$ where none is a tie vote. Assign a voter to each of the quadrants defined by the vertical and horizontal bisectors that contains the diagonal bisector. For example let voter 1 have the rankings $\mathbf{a}_{1,\{1\}} \succ \mathbf{a}_{2,\{1\}}, \mathbf{a}_{2,\{2\}} \succ \mathbf{a}_{1,\{2\}}$ and voter 2 has $\mathbf{a}_{2,\{1\}} \succ \mathbf{a}_{1,\{1\}}, \mathbf{a}_{1,\{2\}} \succ \mathbf{a}_{2,\{2\}}$. (Thus, using Figure 1, voter 1 is in either region 1 or 2 while voter 2 is in either region 4 or 5.) For each voter, choose his ranking over the issues $\{1, 2\}$ to coincide with the ranking $\alpha_{\{1,2\}}$. With this choice, $\alpha_{\{1,2\}}$ is the election outcome, and this outcome is independent of the third voter's preferences. On the other hand, for each of the single issue elections, these two voters split the vote, so the outcome is uniquely determined by the third voter. Thus, let the third voter's preferences for the issue $\{j\}, j = 1, 2$, be the selected ranking $\alpha_{\{j\}}$.

The above existence argument exploits the difference between the dimension of the domain and range; a differential that increases significantly with the number of issues. To see this, note that $n = 3$ issues creates seven subsets of issues, so R^7 is the image space of the map corresponding to F in Eq. 3.3. On the other hand, the geometry of the various perpendicular bisectors in R^3 leads to a situation where there are at least 29 different types of voters where no voter is indifferent between candidates for any set of issues. Thus the domain of F is the 28-dimensional $Si(29)$. Consequently, when $n = 2$, the difference in dimensions between the domain and range is two, and when $n = 3$ it is at least 21. In general, the difference between the dimension of the domain and the range grows exponentially with n. As such, all that is required in the proof is to find expeditious ways to verify the rank condition. Moreover, not only must we expect the conclusion of the theorem to hold, but, accompanying the increase in the differential between the dimensions, we must anticipate many other kinds of "counter- intuitive" conclusions to occur. (To see how to construct these other examples, see Section 2 of [16].)

As indicated above, the basic ideas used to verify the different sorts of conclusions are captured by the representation of Equation 3.1. If the dimension of the domain exceeds that of the image space, then, unless proven otherwise, one must anticipate the standard assumption of responsiveness to ensure that the relevant rank conditions are satisfied and that anything can happen. This comment can be further illustrated with Ledyard's nice paper discussed above. Posing his model in the framework of Eq. 3.1, the domain, which includes the space of all utility functions, becomes infinite dimensional. Thus, one must expect all sorts of different kinds of outcomes to be admitted should the rank conditions be satisfied. As the rank conditions reflect the responsiveness of the procedure to changes within the domain – in Ledyard's formulation this corresponds to changes in strategies – one must expect a conclusion of his type whenever this responsiveness occurs. This is Ledyard's conclusion that *"any non-dominated behavior can be rationalized as Bayesian equilibrium behavior."* As one can show, dominate strategies represent settings where DF has a lower rank.[7]

As a final comment to further suggest the wide-spread generality of this approach for economic theory, consider the much analyzed Saumuelson's transfer paradox. (This has an extensive literature; to get the flavor, see Samuelson [23] and Balasko [2].) This paradox is where in an exchange economy agent one surrenders some of his initial endowment to agent two. The unexpected conclusion is that agent one's Walrasian allocation can be strictly better than it would have been, and agent two's allocation can be strictly worse. With $c \geq 2$ commodities, the ith agent is representing by an element from $R^c_+ \times \mathcal{U}^c$ where $\mathcal{U}^c$ is the set of convex utility functions from R^c_+ to R. With $k \geq 2$ agents, the allocation mapping can be considered as

$$F : \prod_{i=1}^{k}(R^c_+ \times \mathcal{U}^c) \rightarrow \prod_{i=1}^{a} R^c_+. \tag{3.4}$$

This is a mapping from an *infinite dimensional domain to a finite dimensional one.* Therefore, only the rank condition – the responsiveness of the Walrasian allocation to changes in preferences — remains to be verified in order to prove and extend the various kinds of transfer paradoxes that appear in the literature.

As true with the spatial model for voting, an increase in the values of k, c add to the dimension differential between the domain and range. As such, one can correctly expect that the added flexibility is manifested by having the transfer paradox conclusion accompanied with many other conditions, such as requiring the Walrasian allocations to be unique for the allocations, etc.. For instance, it is possible to extend this theorem so that it has the flavor of the above two theorems. Namely, for each set of two or more commodities, D_j, select changes both in each agent's initial endowment $\mathbf{w}_{D_j} = (\mathbf{w}_{1,D_j}, \dots, \mathbf{w}_{a,D_j})$ and in the in final allocations $\mathbf{v}_{D_j} = (\mathbf{v}_{1,D_j}, \dots, \mathbf{v}_{a,D_j})$. Then there exists an economy where the ith agent is represented by her initial endowment ω_i and the utility function $U_i, i = 1, \dots, a$. For this economy, if initial endowments were $\{\omega_i + \mathbf{w}_{i,D_j}\}_{i=1}^{a}$, then the Walrasian allocation would be changed by $\{\mathbf{v}_{i,D_j}\}_{i=1}^{a}; j = 1, \dots, 2^n - (n+1)$. The minimal assumptions on the changes (which define the directional derivatives of the procedure) are easy to compute; they reflect the rank conditions as restricted by standard conditions such as the weak axiom of revealed preferences, etc.

4. Proofs

Two different approaches are given for the proof of Theorem 1. The first is a complete proof based on modifying Debreu's construction of a preference relationship. A second proof that follows the comments of Section 3 is outlined.

[7]Intuitively, this is because the same strategy remains optimal even with changes in other agents' strategies. Thus, a dominate strategy has the form of a critical point; a point with lower rank for the mapping.

Both approaches rely on the observation that the analysis concerning different sets of commodities can involve disjoint regions in commodity space. For instance, compare the region of R^n used to analyze the exchange of all n commodities and the region used when c_n is withheld from the market. The difference involves the plane in R^n passing through the initial endowment with normal vector $\mathbf{e}_n$. When all n commodities are exchanged, it can be that the demand vector is on one side of this plane, so the excess demand function is based on the properties of the utility function in this half space. On the other hand, when c_n is withheld from the market, the individual's excess demand is determined strictly by the properties of the utility function on the plane — the previously important properties of the utility function on the half space are irrelevant. Should these two regions be disjoint, then a "cut and paste" argument can be used on each of these regions to define a new utility function that has the appropriate properties on each of these separate portions of R^n. In general this approach succeeds because utility functions can be modified in infinite number of ways – this manifests the larger (infinite) dimension of the domain that is central to the argument described in Section 3. (This argument has the flavor of the "spatial model for voting" argument presented in Section 3; for different subsets of issues (commodities), the preferences are determined by the geometry of the different subspaces of the space.) What leads to the separation of the regions of R^n is the assumption that each price in $\mathbf{p}_{D_j}$ is bounded below by $\epsilon > 0$.

In both proofs D_j denotes both a specific subset of commodities and the indices of these commodities. This dual use should cause no confusion, but it significantly simplifies the notation.

Proof of Theorem 1 Assume given the functions, $\{\xi_{D_j}(\mathbf{p}_{D_j})\}_{j=1}^{2^n-(n+1)}$, with the properties specified in the statement of Theorem 1. In the indicated variable each function is homogeneous of degree zero and satisfies Walras' Law, so it follows that ξ_{D_j} can be viewed as a tangent vector field on $S_+^{|D_j|-1}$.

There will be n agents in this economy. List the indices $1, \dots, n$ on a circle in a clockwise fashion; so $i+1$ is immediately clockwise of i and 1 is immediately clockwise of n. For agent i consider the hyperspace passing through an initial endowment ω_i^* with the normal vector $\mathbf{e}_i$. In this hyperplane, consider a smooth, monotone, strictly convex utility function v_i^*. To simplify the proof, we assume that v_i^* satisfies the *expansion property* whereby all translates of each coordinate axis of $R^{|D_j|}$ meets each level set of v_i^*.

In $R_+^{|D_j|}$ let $T_{\mathbf{p}_{D_j}} S_+^{|D_j|-1}$ represent the tangent plane to $S_+^{|D_j|-1}$ at $\mathbf{p}_{D_j}$. For $i \in D_j$ let $\mathbf{b}_{i,D_j}(\mathbf{p}_{D_j})$ be the projection of the unit coordinate vector $\mathbf{e}_i$ on this plane. Thus the ith component of $\mathbf{b}_{i,D_j}(\mathbf{p}_{D_j})$ is $1-(p_{i,D_j})^2$ and the sth component, $s \in D_j, s \neq i$, is $-p_{i,D_j}p_{s,D_j}$. If $i \notin D_j$, then $\mathbf{b}_i(\mathbf{p}_{D_j})$ is not defined.

Let $\epsilon > 0$ be given. Following Debreu, for each D_j there exists a continuous, positive scalar function $b_{D_j}(\mathbf{p}_{D_j})$ so that each of the f_{i,D_j} components uniquely defined in the relationship

$$\sum_{\{i|c_i \in D_j\}} f_{i,D_j}\mathbf{e}_i = [\xi_{D_j}(\mathbf{p}_{D_j}) - g_{D_j}(\mathbf{p}_{D_j})] + b_{D_j}(\mathbf{p}_{D_j})\mathbf{p}_{D_j} \tag{4.1}$$

exceeds any specified positive bound for $\mathbf{p}_{D_j} \in S_\epsilon^{|D_j|-1}$. Choose $b_{D_j}(\mathbf{p}_{D_j})$ so that

$$f_{i,D_j}(\mathbf{p}_{D_j})\|\mathbf{b}_i(\mathbf{p}_{D_j})\| \geq 1. \tag{4.2}$$

For each D_j and for $i \in D_j$ assign agent i the individual excess demand function $\xi_{i,D_j} = f_{i,D_j}(\mathbf{p}_{D_j})\mathbf{b}_i(\mathbf{p}_{D_j})$. For $i \notin D_j$, let ξ_{i,D_j} be the excess demand function defined by the utility function v_i^*. By construction, it follows that for each D_j, the sum of the individual excess demand functions equals ξ_{D_j} for $\mathbf{p}_{D_j} \in S_\epsilon^{|D_j|-1}$.

Let $S^{n-1}_{-\epsilon/80} = \{\mathbf{p} \in S^{n-1} | p_i \geq -\epsilon/80\}$. (Thus, if $\mathbf{p} \in S^{n-1}_{-\epsilon/80}\}$ is viewed as a price vector, some of the prices may be zero or negative.) For each i, let $\xi_i(\mathbf{p})$ be a bounded, continuous extension of $\{\xi_{i,D_j}\}_{i\in D_j}$ from the various portions of S_+^{n-1} to $S^{n-1}_{-\epsilon/80}$. Moreover, for each D_j assume for the portion where $S_+^{n-1} \cap R^{|D_j|} = S_+^{|D_j|-1}$ that $\xi(\mathbf{p}_{D_j})$ is along the ray defined by $\mathbf{b}(\mathbf{p}_{D_j})$. Where extended, $\xi(\mathbf{p})$ need not admit the interpretation of being a demand function.

The ith agent's preference relation is constructed. To facilitate comparisons of the following construction with that of Debreu's proof, I use similar notation. We start by assuming that the ith agent's initial endowment is at the origin of R^n, and then, by standard translation arguments, the endowment will be translated to a more suitable position. For the ith agent there is a distinguished direction, $\mathbf{e}_i$; call that direction the *vertical direction*. The *horizontal plane* is the linear subspace passing through the origin with normal vector $\mathbf{e}_i$.

Let $\eta = -\epsilon/2$ and define $g : R \to R$ as

$$g(x) = \begin{cases} x^2 \text{ for } x < \eta < 0, \\ \dfrac{1}{10\eta}x^3 + \dfrac{4}{5}x^2 + \dfrac{1}{10}\eta x \text{ for } \eta \leq x \leq -\dfrac{\eta}{20} \\ ax + b \text{ for } 0 < -\eta < x \end{cases} \tag{4.3}$$

where a, b are chosen so that $g \in C^1$. (Thus $a = 83\eta/4000 < 0$ and $b = -241\eta^2/80,000$.) The convex function g is strictly convex in the region where it is not linear.

In the affine plane passing through $\mathbf{e}_i$ with normal vector $\mathbf{e}_i$, consider the level sets $\partial C_t = \{\mathbf{x} | \sum_{j\neq i} g(x_j) = t,\ \alpha^* = \epsilon/\sqrt{4-\epsilon^2} \leq t \leq (2-\epsilon)/\sqrt{4-(2-\epsilon)^2} = \omega^*\}$. Let C_t be the convex, closed region in this plane that contains $\mathbf{e}_i$ and that has ∂C_t as its boundary. Thus, if $t_1 < t$, then C_{t_1} is in the interior of C_t. The equation

$$t = \frac{\alpha}{\sqrt{1-\alpha^2}}$$

defines the connection in this construction between C_t and $p_i = \alpha$.

Let L_t^* be the cone defined with vertex at the origin and with rays passing through C_t. The boundary of this cone, ∂L_t^*, is defined by the rays passing through the set ∂C_t. Moreover, all n components of the interior normal for ∂L_t^* at a point on $\partial L_t^* \cap R^{|D_j|}, i \in D_j$, are positive.

Let $\mathcal{D} = \{\xi(\mathbf{p})|\mathbf{p} \in S_+^{n-1}\}$ and define $\mathcal{C}_t = \mathcal{D} \cap \partial L_t^*$ and $\mathcal{D}_t = \mathcal{D} \cap L_t^*$. By construction,

$$\mathcal{C}_{t,D_j} = \{\xi_{i,D_j}(\mathbf{p}_{D_j})|p_i = \alpha, \mathbf{p}_{D_j} \in S_\epsilon^{|D_j|-1}\} \subset \mathcal{C}_t \cap R^{|D_j|}.$$

Likewise, the set

$$\mathcal{D}_{t,D_j} = \{\xi_{i,D_j}(\mathbf{p}_{D_j})|p_i \geq \alpha, \mathbf{p}_{D_j} \in S_\epsilon^{|D_j|-1}\} \subset \mathcal{D}_t \cap R^{|D_j|}$$

and $\mathcal{C}_{t,D_j}$ is in the boundary of $\mathcal{D}_{t,D_j}$. Also, by construction, if $\mathbf{p}_{D_j} \in S_\epsilon^{|D_j|}$ where $p_i = \alpha$ then the corresponding hyperplane (in $R^{|D_j|}$) meets ∂L_t^* only along the ray $\mathbf{b}(\mathbf{p}_{D_j})$, thus it meets $\mathcal{C}_t$ at the single point $\xi_{D_j}(\mathbf{p}_{D_j})$.

With only slight modifications, the argument of the next four paragraphs follows Debreu's construction but applied to these more general sets $\mathcal{C}_t, \mathcal{D}_t$. In fact the only modification is to note that the constructed sets lie above the horizontal plane. Therefore, this argument is only outlined; more detail can be found in Debreu's paper.

To start, let $B(\mathbf{x}, r)$ be the closed ball with center $\mathbf{x}$ and radius r and let $d(\mathbf{x}, t)$ be the distance from $\mathbf{x} \in \mathcal{D}_t$ to ∂L_t^*. Define $\mathcal{E}_t = \cup_{\mathbf{x} \in D_t} B(\mathbf{x}, \frac{1}{2}d(\mathbf{x}, t))$. The set $\mathcal{E}_t \subset L_t^*$ is compact and $\mathcal{E}_t \cap \partial L_t^* = \mathcal{C}_t$. Because of the factor of $\frac{1}{2}$ in the radius of the balls, $\mathcal{E}_t$ is strictly above the horizontal plane. To obtain convexity, let $\mathcal{F}_t$ be the convex hull of $\mathcal{E}_t$ and define $G_t = \mathcal{F}_t + R_+^n$. The set $G_t \subset L_t^*$, defined to ensure monotony, is closed, convex, and satisfies $G_t \cap \partial L_t^* = \mathcal{C}_t$ as well as $G_{t'} \subset Int G_t$ for $t' < t$. It is clear that $t \to G_t$ is continuous in the topology of Hausdorff closed convergence. Also, G_t is strictly above the horizontal plane. In fact, with the lower bound assumption on ξ and the above construction, it follows that G_t is bounded below by the horizontal plane translated $\frac{1}{4}$ units upward along the vertical axis.

In general, the set G_t is not strictly convex as it can have segments of straight lines on the boundary. Let H be the hyperplane passing through the origin where the $s \neq i$ component of the normal vector has the value $\epsilon/4$ and the ith component has the value $\sqrt{1-(n-1)(\epsilon/4)^2}$. Thus, $H \cap L_t^* = 0$ for all admissible values of t. Let $\delta = (1, \ldots, 1) \in R^n$. For every $\mathbf{x} \in H$ let $\lambda(t, \mathbf{x}) = min(\{s|\mathbf{x}+s\delta \in G_t\})$ and $\gamma(t, \mathbf{x}) = min(\{s|\mathbf{x}+s\delta \in L_t^*\})$. Thus, $\gamma(t, \mathbf{x}) \leq \lambda(t, \mathbf{x})$ where equality holds iff $\mathbf{x} + \lambda(t, \mathbf{x})\delta \in \mathcal{C}_t$. Moreover, both functions are continuous functions of the two variables.

Let $\rho(x, y) = \frac{1}{2}\sqrt{(x^2+y^2)/2} + \frac{1}{2}[\sqrt{(y-x)^2+1} + x - 1]$ be defined on $\Delta = \{(x, y) \in R_+^2 | x \leq y\}$. As shown in Debreu, ρ is continuous, convex, strictly increasing in each variable and it satisfies $\rho(s, s) = s$ for $s \in R_+$. Moreover, if $x \neq y$, and/or $x' \neq y'$, the ρ is strictly convex on the segment $[(x, y), (x', y')]$. The function $\mu(t, \mathbf{x}) = \rho(\lambda(t, \mathbf{x}), \gamma(t, \mathbf{x}))$ is continuous, convex and satisfies $\lambda(t, \mathbf{x}) \leq \mu(t, \mathbf{x}) \leq \gamma(t, \mathbf{x})$ where one of the equalities occurs iff both occur iff $\mathbf{x} + \mu(t, \mathbf{x})\delta \in \mathcal{C}_t$. If $M_t = \{\mathbf{x} + s\delta | s \geq \mu(t, \mathbf{x})\}$ then M_t is strictly convex and $G_t \subset M_t \subset L_t^*$, $M_t \cap \partial L_t^* = \mathcal{C}_t$, and $t \to M_t$ is continuous. Moreover, Debreu's argument serves to show that the set M_t is strictly convex. While M_t is strictly above the horizontal plane, the distance from this plane approaches zero as $||\mathbf{x}|| \to \infty$ in directions where $x_i \to \infty$.

To select an initial endowment, choose any point $\omega_i \in R_+^n$ so that $\omega_i + \xi(\mathbf{p}_{D_j})$ is at least one unit from the boundaries of R_+^n for all D_j and all $\mathbf{p}_{D_j} \in S_\epsilon^{|D_j|-1}$. Let $Q_t =$

$\partial(M_t + \omega_i) \cap R^n_+$ be the boundary of the translated set M_t. Q_t is the level set of the preference relation for $\alpha^* \leq t \leq \omega^*$. For $t \geq \omega^*$, let $Q_t = \frac{t}{\omega^*} Q_{\omega^*}$.

We now construct the indifference sets below Q_{α^*}. They are based on the assigned utility function v_i^*. Translate v_i^* so that the initial endowment $\omega_i^* = \omega_i$. The level sets of v_i^* are defined only on the translated horizontal plane, so the definition of these sets needs to be extended to R^n. Toward this end, let $m^* = max_{\{D_j | i \notin D_j\}}(\{v_i^*(\xi_{D_j}(\mathbf{p}_{D_j}) | \mathbf{p}_{D_j} \in S^{|D_j|-1}_{\epsilon/2}\}$. Restrict attention to those level sets of v_i^* where $v_i^* \leq m^*$. According to the expansion requirement imposed on v_i^*, each of these level sets passes through all coordinate planes of R^n. Let d be the minimum distance between $\{\mathbf{x}$ on the translated horizontal plane $|v_i^*(\mathbf{x}) \leq m^*.\}$ and Q_{α^*}. By construction (where M_t is above the horizontal plane) and compactness, $d > 0$.

In the two dimensional plane P^2 spanned by $\mathbf{e}_i$ and $\delta - \mathbf{e}_i$, consider a smooth, strictly convex curve h with the parametric representation $h : R \to P^2, h(0) = 0$ and $h(-\|\omega_i\|) = \frac{d}{2}$, and the $\mathbf{e}_i$ component of h approaches $-\infty$ as $t \to \infty$. Let $v_{i,m} = \{\mathbf{x} + h(t) | t \in R, \mathbf{x}$ is in the translated horizontal plane and $v_i^*(\mathbf{x}) = m \leq m*\}$. Each level set $v_{i,m}$ is strictly convex and this set of indifference surfaces satisfies the monotone property.

It remains to fill the gap between v_{i,m^*} and Q_{α^*}. So, for each $\mathbf{x} \in v_{i,m^*}$ let $\delta(\mathbf{x})$ be the unique point $Q_{\alpha^*} \cap \{\mathbf{x} + s\delta | s \in R\}$. For $t \in [0, 1]$, define $V_t = \{(1-t)\mathbf{x} + t\delta(\mathbf{x}) | \mathbf{x} \in v_{i,m^*}\} \cap R^n_+$. Because of the expansion property on v_i^*, each $\mathbf{x} \in R^n_+$ is on a unique level set; either one of the $v_{i,m}$ sets, one of the V_t sets, or one of the Q_t sets. All that remains is to show that the V_t sets are strictly convex and monotone.

Select any two points, $\mathbf{y}_i = t\mathbf{x}_i + (1-t)\delta(\mathbf{x}_i) \in V_t$. To show that V_t is strictly convex, it must be shown that for any $\lambda \in (0, 1)$ that $\lambda \mathbf{y}_1 + (1-\lambda)\mathbf{y}_2$ is above the surface. However, by the strict concavity of v_{i,m^*} and Q_{α^*}, both $\lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2$ and $\lambda\delta(\mathbf{x}_1) + (1-\lambda)\delta(\mathbf{x}_2)$ lie strictly above their respective level sets. Thus, by simple vector algebra, the conclusion holds for the surface V_t. The monotone condition is proved is a similar manner. This completes the proof.

Outline of an alternative proof. Let $\mathcal{U}^n$ be the set of strictly convex, continuous, monotone functions on R^n. Let Ψ_{ϵ,D_j} be the set of continuous tangent vector fields on $S^{|D_j|-1}_{\epsilon}$ and let $\Psi_\epsilon = \prod_{D_j} \Psi_{\epsilon,D_j}$. Endow both spaces with the C^0 topology on compact sets. Let

$$F_m : E^m = \prod_{j=1}^{m} [\mathcal{U}^n \times R^n_+] \to \Psi_\epsilon$$

be the mapping where $F((u_1, \omega_1), \ldots, (u_m, \omega_m))$ is the listing of the $2^n - (n+1)$ aggregate excess demand functions defined by the economy $\mathbf{e} = (u_1, \omega_1), \ldots, (u_m, \omega_m)$. Theorem 1 asserts that F_n is surjective. This is true if it can be shown that

i. each $\mathbf{e} \in E^n$ has an open neighborhood where its F_n image contains an open neighborhood of $F_n(\mathbf{e})$, and
ii. the image of F_n is dense in Ψ_ϵ.

Outline of part (i.) This is done by showing that the tangent map $DF_{n,\mathbf{e}} : T_{\mathbf{e}}E^n \to T_{F_n(\mathbf{e})}\Psi_\epsilon$ is surjective. First we show that $T_{F_n(\mathbf{e})}\Psi_\epsilon$ can be identified with Ψ_ϵ. This follows from the definition of $DF_{n,\mathbf{e}}(\eta\mathbf{e}')$ which is the linear approximation for $F_n(\mathbf{e}+\eta\mathbf{e}') - F_n(\mathbf{p})$

for scalar η. At each point $\mathbf{p} = (\mathbf{p}_{D_1}, \ldots, \mathbf{p}_{D_{2^n-(n+1)}}) \in \Psi_\epsilon$, both terms of this difference define a tangent vector in each of the components of Ψ_ϵ, so the difference also is a tangent vector. As this holds for all $\mathbf{p}$, the conclusion follows after using a simple continuity argument.

As $F_n = \sum_{j=1}^n F_1^j$ where F_1^j is the individual excess demand mapping $F_1^j : E^1 \to \Psi_\epsilon$ for the jth agent, many of the properties of F_n are determined by the properties of F_1. We use this fact to characterize certain subspaces of $T_{\mathbf{e}}E^n$. For instance, $T_{\mathbf{e}}E^1$ includes *all* C^2 functions from R^n_+ to R^1; not just those that are monotone and strictly convex. This can be seen from the Taylor series expansion $\eta v(\mathbf{x} + \mathbf{h}) = \eta[v(\mathbf{h}) + (\nabla v(\mathbf{x}), \mathbf{h}) + \frac{1}{2}D^2 v(\mathbf{x})(\mathbf{h}, \mathbf{h}) + R(\mathbf{h})]$. It then follows for any C^1 function v that on any compact set and for sufficiently small values of η, the function $u + \eta v \in \mathcal{U}^n$. Similarly, by use of the strict concavity of functions in $\mathcal{U}^n$, it follows that the continuous functions are in the tangent space – it is important to note that v need not be monotone or convex.

Let $\mathbf{e} \in E^n$ be given. To show that $DF_{n,\mathbf{e}}$ is surjective, let $\xi = (\xi_{D_1}, \ldots, \xi_{C^n}) \in \Psi_\epsilon$ be given. Decompose this function into individual excess demand functions in same manner used in the proof of Theorem 1. According to Debreu, on each of the (translated) spaces $R^{|D_j|}$ passing through ω_i there is a continuous, monotone, strictly convex preference that gives rise to the indicated demand functions. These $2^n - (n+1)$ utility functions need not be compatible with one another. However, the part necessary to define the demand function is inside of a cone that, with the exception of the vertex, is totally in the interior of the orthant. Thus, a "cut and paste" argument is used where we keep only those relevant portions of each of the utility functions. Now, it is easy to paste these portions together – to extend the portions of the function that remain over the whole space. As this extended function is only in the tangent space, it only needs to be continuous; it need not satisfy any other properties of monotonicity or convexity. Thus this function v is defined so that $DF_{1,\mathbf{e}}(v)$ is the individual excess demand function. This is the basic step in showing that $DF_{n,\mathbf{e}}$ is surjective.

An outline of part (ii.). For a given ξ and a given open neighborhood of ξ, V, it must be shown that there is a utility function u so that $F(u) \in V$. Let C^n represent the set of all n commodities and start with ξ_{i,C^n}. Chose $\eta, 0 < \eta \leq \epsilon/2$. Corresponding to the set $D_j, |D_j| < n$, is the vector $\mathbf{n}_{D_j}$ that has the value η in the ith component iff $i \notin D_j$ where all other components are zero. The plane passing through $\omega_i + \mathbf{n}_{D_j}$ with normal vector $\mathbf{n}_{D_j}$ is parallel to the coordinate plane containing ξ_{i,D_j}. Continuously extend ξ_{i,C^n} so that on each such translated plane it has the form ξ_{i,D_j}. There exists $\delta > 0$ where the precise value depends upon the value of η and the maximum values of $||\xi_{D_j}||$ so that the extended ξ_{C^n} is a tangent vector for $\mathbf{p}_{C^n} \in S^{n-1}_\delta$. Let $U_{i,\eta}$ be the corresponding utility function given by Debreu's construction. This function give the appropriate individual excess demand function for the set of all n commodities, and the appropriate excess demand function for the set D_j if the initial endowment had been $\omega + \mathbf{n}_{D_j}$ rather than ω_i. Now, by a continuity argument using properties of the extension, one can show that for sufficiently small values of η, the utility function u_η has the desired properties.

Proof of Theorem 2. The utility function for an agent is defined in Section 2; points closer to $\mathbf{q}_i$ are accorded a higher utility. Assume that the k candidates' positions on the issues $\{\mathbf{a}_j\}_{j=1}^k$ are specified and that they satisfy the conditions of the theorem. For each set of issues D_j the location of $\mathbf{q}_i$ uniquely determines the ith voter's rankings of the k candidates. The *voter's type* is a listing of these rankings where the jth entry is the voter's rankings of the candidates subject to the issues in $D_j, j = 1, \ldots, 2^n - 1$.

Let $\gamma(n)$ represent the number of voter types. The value of $\gamma(n)$ depends not only on n but also on the candidates' positions $\{\mathbf{a}_j\}$. For instance, in the degenerate (but admissible) setting where for each set of issues the points all lie on a straight line, certain rankings never can occur. In particular, a candidate whose position is not an endpoint on the line, never is bottom ranked by any voter. At the other extreme, if for a set of issues the convex hull of the points $\{\mathbf{a}_j\}$ have dimension $k-1$ then all rankings of the candidates are admitted. Let

$$Si(\gamma(n)) = \{\mathbf{x} \in R^{\gamma(n)} | x_i \geq 0, \sum_{i=1}^{\gamma(n)} x_i = 1\} \tag{4.4}$$

be the unit simplex. A rational point $\mathbf{x} \in Si(\gamma(n)$ can be viewed as being a profile where component x_i determines the fraction of all voters that are of the ith type.

To discuss the image space, note that the voting vectors $\mathbf{W}^k$ and $\mathbf{W}^k - a(1, \ldots, 1)$, $a > 0$, are equivalent in the sense that the election rankings always are the same. The vectors are normalized by choosing a value of a so that the sum of the components of a voting vector is zero. In this manner, for instance, the normalized form of the plurality voting vector, $(1, 0, \ldots, 0)$, is $(1, 0, \ldots, 0) - (\frac{1}{k}, \ldots, \frac{1}{k}) = (1 - \frac{1}{k}, -\frac{1}{k}, \ldots, -\frac{1}{k})$. The tally for each candidate can be represented as a vector in R^k where the value of the jth component is the election tally for a_j. In this way, a voter voting for the jth candidate would cast an appropriate permutation of the coordinates of $\mathbf{W}^k$.

An election can be viewed as being the obvious mapping from $Si(\gamma(n)) \to R^k$. Because the voting vectors are normalized, the sum of the tallies for all candidates is equal to zero, so the vector for the tally is in the vector subspace $V(D_j)$ which passes through the origin 0 and has $(1, \ldots, 1)$ as a normal vector. In this manner, the election outcome can be viewed as being a mapping $F_{D_j} : Si(\gamma(n)) \to V(D_j)$.

The election outcome over all sets of issues is an element of $\prod_{j=1}^{2^n-1} V(D_j)$. The election over all subsets of issues can be represented as the mapping

$$F = (F_{D_1}, \cdots, F_{D_{2^n-1}}) : Si(\gamma(n)) \to \prod_{j=1}^{2^n-1} V(D_j). \tag{4.5}$$

By following the approach of [14], it follows that if the image of F contains an open set about the origin, then the proof is completed. This proof requires verifying the two steps described in Section 3.

1. First it must be shown that there is a profile $\mathbf{p}$ in the *interior* of $Si(\gamma(n)$ so that $F(\mathbf{p}) = 0$; namely, $\mathbf{p}$ is mapped to the origin.
2. The second step is to prove that the rank of the Jacobian of F is $(k-1)(2^n-1)$ – the dimension of $\prod_{j=1}^{2^n-1} V(D_j)$. This implies that an open neighborhood of $\mathbf{p}$ is mapped to an open neighborhood of 0. This open set meets all combinations of the rankings, so the conclusion holds.

Both steps involve algebraic, open conditions, so if they hold for a single voting method, then they must hold for an open set of voting methods. The strategy of the proof, then, is to show that each step holds for the plurality vote. The restriction on the voting vectors, $\mathcal{W}_1^k$, arises in the first step, and only when one considers the (admissible) degenerate situation where the vectors $\{\mathbf{a}_{i,D_j}\}_{i=1}^k$ form a straight line for each choice of D_j.

Proof of Step 1: First I show that a tie vote can occur with the plurality vote for each set of issues. Such a conclusion holds for the profile where for each D_j and each candidate c_i, $\frac{1}{k}$ of the voters have c_i top ranked. The first part of this proof proves that such a profile can be defined for any admissible configuration of the candidates' positions.

Consider a single issue set D_j. By hypothesis, the position points are distinct. The perpendicular bisectors between adjacent points $\{\mathbf{a}_{i,D_j}\}$ on the line divide this coordinate axis $R^{|D_j|}$ into k regions. Each region is identified with a particular candidate; if $\mathbf{q}_i$ is in the region identified with the sth candidate, then the ith voter has the sth candidate top-ranked. In the total issue space, R^n, the perpendicular bisectors have dimension $n-1$. These bisectors from the n coordinate axis intersect to create k^n rectangles in R^n. Similarly, for each set of issues $D_j, |D_j| \geq 2$, these bisectors divide the issue space (the coordinate plane) $R^{|D_j|}$ into $k^{|D_j|}$ rectangles. (These rectangles are based on the division of the coordinate axes for the individual issues defining D_j. The perpendicular bisectors defined by a set of issues disjoint from D_j form planes parallel to the coordinate plane $R^{|D_j|}$.)

For $D_s \subset D_j$ let $Pr_{D_j,D_s} : R^{|D_j|} \to R^{|D_j|}$ be the natural projection mapping. If $D_j = C^n$, denote the mapping by Pr_{n,D_s}. As the k^n rectangles in R^n are similar to a coordinate grid for R^n, it follows that for the ith candidate there is a unique rectangle $Rec(i;n) \subset R^n$ so that for any single issue D_j, $Pr_{n,D_j}(Rec(i;n))$ is the region associated with the ith candidate. Similarly, for each set of issues D_j there is a unique rectangle $Pr_{n,D_j}(Rec(i;n)) = Rec(i;D_j) \subset R^{|D_j|}$ with this projection property

$$Pr_{D_j,D_s}(Rec(i;D_j)) = Rec(i;D_s). \tag{4.6}$$

Construct a profile by assigning $\frac{1}{k}$ of the voters to $Rec(i;n), i = 1, \ldots, k$. This means that for any set of issues D_j, $\frac{1}{k}$ of the voters are assigned to $Rec(i;D_j)$. Next I show that this assignment leads to a plurality completely tied vote among the candidates for each set of issues. This assertion clearly holds for each set of a single issue. For a set of issues $D_j, |D_j| \geq 2$, the space $R^{|D_j|}$ is divided into k regions where if $\mathbf{q}_s$ is in the ith region then the sth voter has the ith candidate top-ranked with respect to the issues D_j. The region for the ith candidate is defined in the following manner: For each $s \neq i$, let $P_{i,s:D_j}$ be the perpendicular bisector of $\mathbf{a}_{i,D_j}$ and $\mathbf{a}_{s,D_j}$. The plane $P_{i,s:D_j}$ divides $R^{|D_j|}$ into two halfspaces; let $H(i,s;D_j)$ be the one containing $\mathbf{a}_{i,D_j}$. Let $T(i;D_j) = \cap_{s\neq i} H(i,s;D_j)$. It is clear from construction that if a voter's beliefs are in the convex set $T(i;D_j)$, then he has the ith candidate top-ranked. (See Figure 2 for the case $k = 3, n = 2$.) Clearly if $i \neq s$ then $T(i;D_j)$ and $T(s;D_j)$ are disjoint sets. It also follows for $|D_j| = 1$ that $T(i,D_j) = Rec(i;D_j)$. Because of the assumption that the points $\{\mathbf{a}_{i,D_j}$ are distinct, it follows that $T(i;D_j)$ contains an open set about $\mathbf{a}_{i,D_j}$, so $T(i;D_j)$ is non-empty. Indeed, it follows from the definition of a perpendicular bisector, the triangle inequality, standard facts from trigonometry (that the ratio of similar sides of similar triangles are constant), and the construction of the grid that

$$Rec(i; D_j) \subset T(i; D_j) \text{ for all } D_j. \tag{4.7}$$

Thus for each D_j and candidate i, the above constructed profile has $\frac{1}{k}$ of the voters in $T(i; D_j)$; for all sets of issues the election outcome is a completely tied plurality vote.

This profile does not suffice for our purposes because it does not define an interior point of $Si(\gamma(n))$. (An interior point must have a positive number of voters for each of the $\gamma(n)$ different voter types.) There are many different ways to construct such a profile using symmetry and the fact that these points form a convex set in $Si(\gamma(n)$, or by using an analytic argument involving the implicit function theorem and the above profile. Instead I use the following more elementary geometric argument that seems to be easier to follow, simpler to construct, and more intuitive. The ideas are described with $k = 2, n = 2$ and Figure 1. Here, the $\gamma(2) = 6$ regions are indicated in the figure. These regions can be labeled according to the voter type where, say, $(1, 2 : 2)$ corresponds to a voter preferring candidate one on the first issue and candidate two on the second issue and on both issues. Thus, $Rec(1; \{2\})$ and $Rec(2; \{2\})$ are, respectively, the rectangles with the labels $(1, 1; 1)$ and $(2, 2; 2)$. The above constructed profile places $\frac{1}{2}$ of the voters in each of these rectangles. To create a profile that is an interior point of $Si(6)$ and that results in a completely tied plurality vote for all sets of issues, some of the voters are moved from these to regions into the other four regions. However, this must be done so that the sum of the fractions on each side of each perpendicular bisector remains $\frac{1}{2}$, so for each move, a compensating move must be made. The sum for each voter and each set of issues is obtained in the following manner. A specified candidate and specified set of issues determines an entry in the triplet. (The first candidate on the second issue corresponds to all triplets $(-, 1;)$.) Thus, the sum is obtained by summing the fraction of voters over all regions with an entry of that form. (So, for the first candidate with respect to the second issue, the sum is obtained by summing the fraction of candidates in the three regions that have 1 for the second entry.)

If α voters are moved from $(1, 1; 1)$ to $(1, 2; 1)$, then the regions have, respectively, $\frac{1}{2} - \alpha, \alpha$ of the voters. By the summation process, this has not changed vote totals for the first issue or the set of two issues, but it has for an election with respect to the second issue. To compensate, a symmetric modification must be made with α of the voters moved from $(2, 2; 2)$ to $(2, 1; 2)$. With this compensating move elections over all sets of issues are plurality tied. The last change is to move β voters from $(1, 1; 1)$ to $(2, 1; 1)$ with the compensating symmetric change of β voters from $(2, 2; 2)$ to $(1, 2; 2)$. With the constraint $\alpha + \beta \le \frac{1}{2}$, all possible profiles leading to a complete tie vote are determined. A similar construction is used in the general case.

A general argument depends on the following facts.

1. *For each set D_j and $i \neq s$, the vector $\mathbf{a}_{i,D_j} - \mathbf{a}_{s,D_j} \in R^{|D_j|}$ has no zero components.* This conclusion follows because if the tth component of this vector difference were zero, then the ith and the sth candidates have identical stands on the tth issue — a direct contradiction to the basic assumption.

2. *If* $D_s \subset D_j$ *then* $Pr_{D_j,D_s}(P(i,t;D_j)) = R^{|D_s|}$ *for all* $i \neq t$. If this were not true, then the normal vector for $P(i,t;D_j)$ must have a zero component in some coordinate direction contained in $R^{|D_s|}$. As $\mathbf{a}_{i,D_j} - \mathbf{a}_{t,D_j}$ is a normal vector for this bisector, the assertion follows from the first statement.

3. *If* $D_s \cap D_j = \emptyset$, *then* $P(i,t;D_j)$ *and* $Pr^{-1}_{n,D_j}(T(i;D_j))$ *are parallel to the coordinate plane* $R^{|D_s|}$. This is immediate from the construction. Alternatively, the normal vector $\mathbf{a}_{i,D_j} - \mathbf{a}_{t,D_j}$ expressed as a vector in R^n has zero components in all component directions corresponding to issues not in D_j.

4. *A perpendicular bisector from one set of issues transversely intersects all perpendicular bisectors from all other sets of issues.* This is immediate from the above.

5. *If* $D_s \subset D_j$, *then for each* i, *the set* $Pr^{-1}_{D_j,D_s}(T(i;D_s))$ *meets at least two sets from* $\{T(r;D_j)\}_{r=1}^k$. If not, then $Pr^{-1}_{D_j,D_s}(T(i;D_s))$ never meets a perpendicular bisector $P(t,s;D_j)$. This means that the projection of this bisector does not cover all of $R^{|D_s|}$. This last statement is a contradiction to Assertion 2.

6. *If* $|D_j| < n$, *if* $Pr^{-1}_{n,D_j}(T(i;D_j)), Pr^{-1}_{n,D_j}(T(s;D_j))$ *have a common boundary and both meet* $T(t;n)$, *then both sets also meet at least one other set from* $\{T(r;n)\}_{r=1}^k$. As already shown, the common boundary must intersect the boundary of $T(t;n)$. There are two situations. The first is that there is a point in this intersection away from all other hyperplanes defining the boundary of $T(t;n)$. In this setting the conclusion follows from the fact that the intersection must be transverse. In the contrary case, the common boundary contains the intersection of several of the bounding hyperplanes for $T(t;n)$. (A situation where this occurs is when the configuration formed by the positions always form a straight line.) The portion of this boundary outside of $T(t;n)$ cannot coincide with a boundary plane of any other $T(s;n)$ by Assertions 2 & 3. Thus the boundary must meet the interior of at least one other $T(s;n)$. This competes the proof.

In R^n, the perpendicular bisectors from each set of issues is an $n-1$ dimensional plane. In the manner described above, their intersections form the regions defining the voter types. Thus, order the $2^n - 1$ sets of issues; the sets of single issues first, then the sets of two issues, etc. Assigned to each region formed by the perpendicular bisectors is a label given by a vector with $2^n - 1$ entries; the sth entry is the name of the candidate who is top-ranked for a voter in this region with respect to the sth listed set of issues. For instance, the rectangle $Rec(i;n)$ has the label with i as the entry for each component.

7. *If a region shares a common boundary (face) with* $Rec(i;n)$ *then its address differs from* $(i,i,\dots,i)$ *only in an entry for a single issue. Moreover, if* i *is changed to* s, *then there is a region adjacent to* $Rec(s;n)$ *where the only non-*s *entry is in the same component and it is* i. The first part follows from the fact that $Rec(i;n)$ is in the interior of $T(i;D_j)$; $|D_j| \geq 2$, and the above description of the boundaries of the various sets. The second statement follows from the fact that, for the indicated single issue, i and s must be adjacent candidates. (Otherwise the grid would not allow such an adjacent rectangle.) Thus, adjacent to $Rec(s;n)$ must be a rectangle of the indicated type.

8. *If two ranking regions share a common boundary, then their labels differ in only one entry. If this entry corresponds to a set of issues* $D_j, |D_j| < n$, *then there is at least one*

other pair of regions that differ only in the D_j entry with the same two names. For each coordinate corresponding to a set of issues $D_j, |D_j| < n$, and for each candidate, there is a pair of regions with common boundary where this candidate's name is one of the two entries. The first and third assertions are obvious. For the middle assertion, assume without loss of generality that the names of the candidates are 1 and 2. If the assertion were false, then the common boundary between these two issues in R^n would not intersect any other perpendicular bisector. This contradicts Assertion 2.

9. *Let positive integer $s < n$ and let $|D_t| > s$. Then $\cap_{|D_j|=s}T(i; D_j) \subset T(I; D_t)$.* This is fairly immediate from the construction and the argument showing the existence of $Rec(i; n)$.

We now are ready to start the construction of an interior profile by modifying the profile used above with $\frac{1}{k}$ of the voters in each $Rec(i; n)$. Start with the rectangle $Rec(1; n)$ and move a fraction of the voters into each of the adjacent rectangles that are in the same ranking region. By the fact that $Rec(1; n)$ is properly contained in $T(i; D_j)$ for all $|D_j| \geq 2$, it follows from the above assertions concerning the boundaries of these regions that this is true for all rectangles sharing a common boundary face with $Rec(1; n)$. For each such move, only one entry – an entry corresponding to a single issue D_j – changes from 1 to, say, s. Each such move keeps invariant the number of voters in each region with the exception of the set D_j; here candidate 1 loses votes and candidate s gains votes. To compensate for this, note that according to the above assertions there is a rectangle sharing a face with $Rec(s; n)$ where the D_j entry is 1. Thus, a compensating move is made so that the same fraction of voters are moved from $Rec(s; n)$ to this adjacent rectangle. With this compensating move, the fraction of voters supporting each candidate for each set of issues remains at $\frac{1}{k}$, but more types of voters are represented. Once all of the rectangles adjacent to $Rec(1; n)$ have voters, then the same process is continued for each of $\{Rec(s; n)\}_{s=2}^{k}$ where, for any adjacent rectangle to one of these distinguished regions that is not represented in the profile, a move of voters is made to have it represented. Notice that for certain configurations of $\{\mathbf{a}_i\}$, the associated compensating move may add voters from $Rec(i; n)$ to a rectangle already containing voters.

After all adjacent rectangles are represented, then a portion of the voters are moved from each of these regions to rectangles adjacent and in the same $T(i; D_j)$ regions for $|D_j| \geq 2$. The same argument used above, along with the assertions, shows that compensating moves can be made where the compensating move may require movement of voters between rectangles already containing voters. In this manner, all ranking regions with a fixed letter for all sets of more than one issue are represented in the profile.

The argument now moves to provide representation for sets of issues $D_j, |D_j| = 2$. Choose such a set of issues. In $R^{|D_j|}$, all regions are filled as $\{T(i; D_j)\}_{i=1}^{k}$ forms a partition. However, in R^n, the intersection of such regions based on sets of two issues forms another kind of grid in R^n, and only k of these grid regions are represented in the modified profile. In particular, for each candidate i there is a unique grid region that contains $Rec(i; n)$, and, because of the assumption on the candidates positions, it does not share a face with the grid region containing $Rec(s; n)$ for $s \neq i$. Consequently, such regions are not represented at this state in the construction of the modified profile. The boundary between a grid region with representation and one without must pass through rectangles (ranking regions) containing representation. Thus, crossing this boundary, but staying in the same rectangle, only effects the vote totals for a particular set of two issues. So, starting with the grid region containing $Rec(1;n)$, move a portion of the voters to an adjacent grid region by using these boundary rectangles. For the same geometric kinds of reasons used for changes in single issue directions, if a grid region share a face with a grid region representing candidate s over issues D_j, then adjacent to the grid region containing $Rec(s;n)$ in the D_j component label direction, is a grid region associated with candidate 1. Thus, the compensating move involves a rectangle, with voter representation (from stage 1), that includes the boundary between these grid regions. Once this is done for all boundaries, the same construction used in stage 1 is continued – voters are moved into adjacent rectangles with the appropriate compensating moves (which may involve rectangles already assigned voters.)

This same construction is continued over all sets of issues $D_j, |D_j| < n$. The important points about this construction is that each move is either within a rectangle, or between adjacent rectangles. When it is within a rectangle, it crosses the surface associated with the regions for a higher dimensional set of issues. As only one component is being changed and as that is for a set of issues $D_j; |D_j| < n$, there is a compensating move. Moves between adjacent rectangles are made to fill up grid regions defined by sets of higher number of issues – here the compensating move can involve rectangles already represented by voters. As long as the portion of voters chosen at each step is sufficiently smaller than the number at the previous step, there is no danger of a region being emptied of voters. In this manner, an interior point of $Si(\gamma(n)$ is constructed whereby for each set of issues the election leads to a completely tied outcome.

If for each pair of candidates and for each set of issues, the perpendicular bisectors are considered in R^n, then the regions defined by the intersections defines ranking regions where a representation is given for which candidates are ranked in second, third, etc. place. These ranking regions are a refinement of the ranking regions based only on who is top-ranked. In this space, the above modified profile can be extended; take an original region specifying which candidates are top-ranked over each set of issues and split the number of voters into each of the refined regions. This, in no manner, affects the plurality vote outcome. This is the final, modified profile.

To see why the above construction cannot be modified to hold for all voting vectors, consider the special case of $k = 3$ where $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ lie on a straight line with $\mathbf{a}_2$ in the middle. This configuration restricts the kinds of profiles that can be admitted. So, for the BC voting vector $(2, 1, 0)$, a completely tied vote over the first issue is possible only should there be no voters with candidate 2 as top-ranked. From this, it follows it is impossible to construct a profile – even for a single issue – that is an interior point of $Si(\gamma(n))$. From this it can be shown that the conclusion of Theorem 2 does not hold. However, it turns out that if, for each single issue, an interior profile can be constructed, then the above kinds of modifications produce an interior profile over all sets of issues.

Proof of Step 2: The linear mapping F can be expressed as a $k(2^n - 1) \times \gamma(n)$ matrix where the sth vector component of the jth column vector represents how the jth voter type votes on the issues D_s. (Recall, the sum of the entries for each voting vector component equals zero.) To show that the rank of the matrix is $(k-1)(2^n - 1)$, it suffices to show that these column vectors span a space of this dimension. To do this, take any set of issues, D_j and a candidate i. According to the above geometric assertions, $T(i; D_j)$ shares a boundary with at least one other $\{T(s; D_j)\}$. It also follows that there are two ranking regions where the only difference in the labels is that one has i for top-ranked candidate in D_j, while the other has s. (These are any two ranking regions created from a single ranking region (in the same rectangle) in R^n by the bounding hyperplane dividing $T(i; D_j)$ from $T(s; D_j)$.) All the entries of the column vectors representing these two voter types are the same except for the entry corresponding to the set of issues D_j. Here the vectors differ in that the one where candidate i is top ranked has $1 - \frac{1}{k}$ for the ith entry and $-\frac{1}{k}$ for all others, while the one where candidate s is top ranked has the $1 - \frac{1}{k}$ value for the sth entry. The difference between the two column vectors, then, is a zero vector for all entries corresponding to any set of issues not D_j. The vector value for the set of issues D_j is the scalar $1 - \frac{1}{k}$ times a vector with zero for all entries except the ith and the sth; one is unity and the other is (-1). This is true for all choices of $i = 1, \ldots, k$, and the resulting set of vectors spans a space of dimension $k - 1$. Applying the same argument to all $2^n - 1$ sets of issues creates set of vectors of dimension $k - 1$. Since these vectors are zero valued in all but one component space, the collection trivially spans a space of dimension $(k - 1)(2^n - 1)$. This completes the proof.

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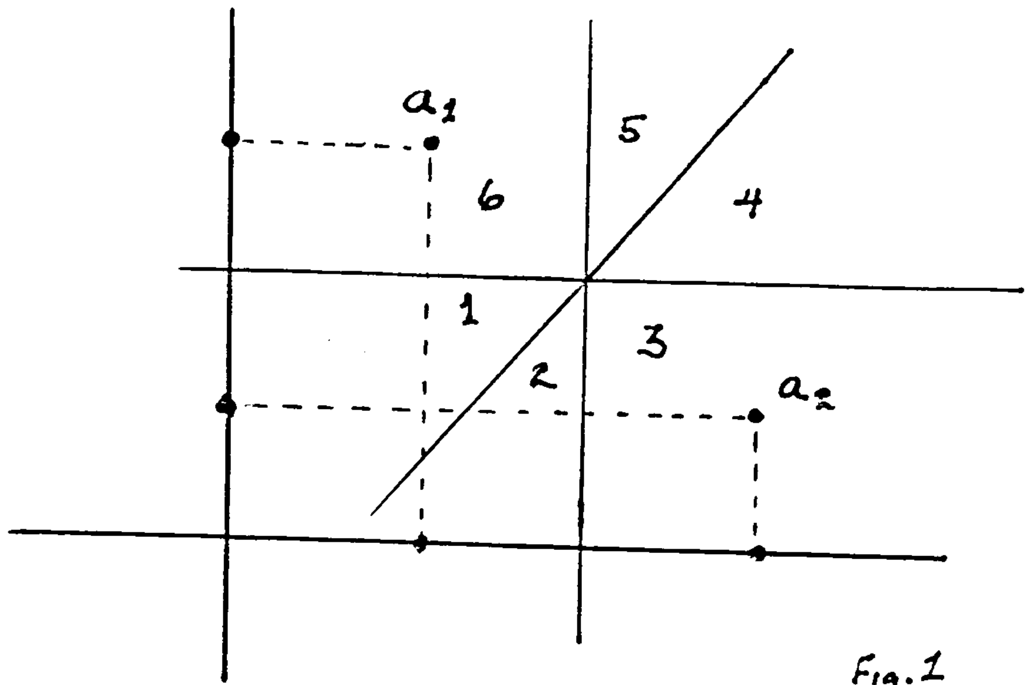

Fig. 1

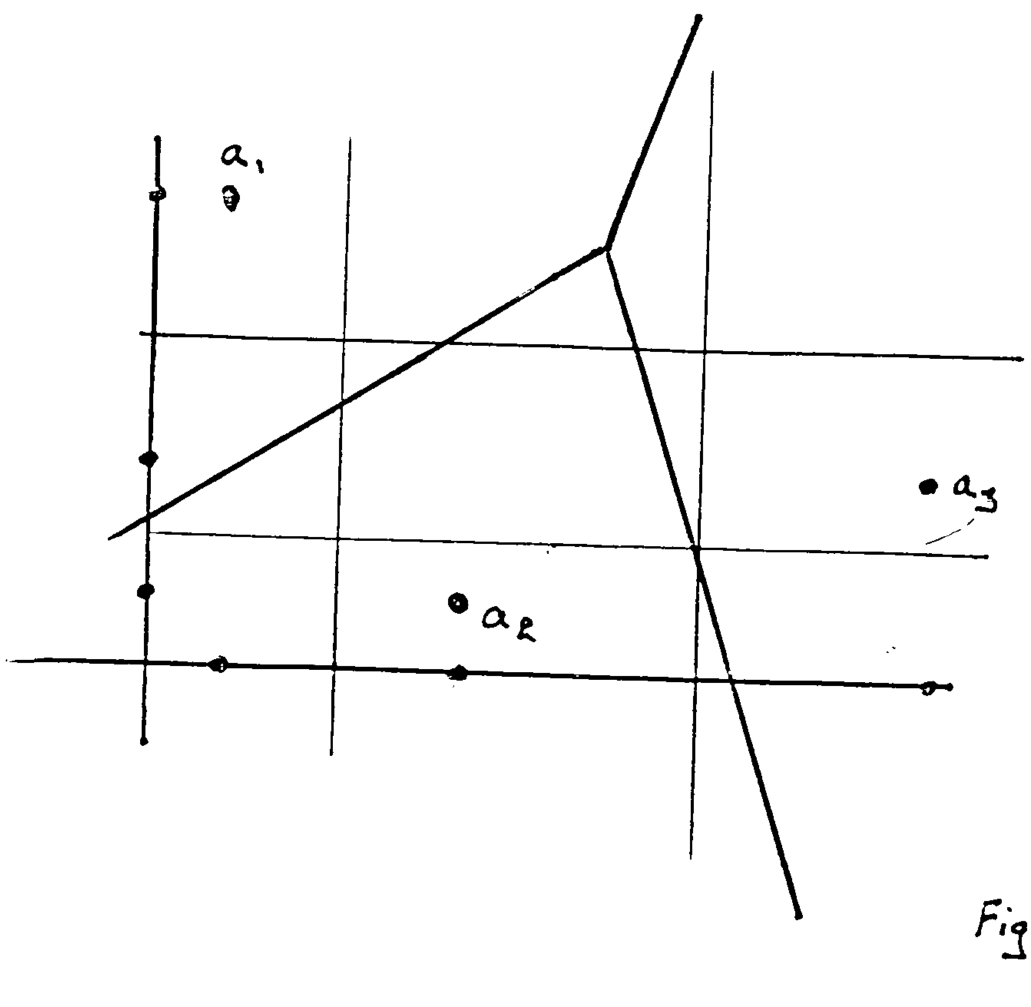

Fig 2