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AN INFORMATIONALLY MORE EFFICIENT ENDOWMENT GAME

by

Lu Hong
University of Minnesota

Scott E. Page
Northwestern University*

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1. INDRODUCTION

An important question in the theory of implementation is whether or not a given social choice function (correspondence) is consistent with the self-interested behavior of agents in some institutional setting, i.e., whether or not a certain social choice function can be implemented by some informationally decentralized resource allocation mechanism. This question was systematically studied in Maskin (1977). The main result in that paper is the following: With the feasible set known, a necessary and sufficient condition for a social choice correspondence satisfying the "no veto power" to be Nash implementable is a "monotonicity" condition. Hurwicz, Maskin and Postlewaite (1986) (This will be refered to as HMP later. Earlier versions of the paper date back to 1980) later studied the feasible implementation problem of a social choice function when the feasible set is not known. With the assumption that preferences are known, they first constructed an endowment game that implements in Nash equilibria individually rational social choice functions (as functions of agents' initial endowments) for pure exchange economies with three or more agents, strictly monotone preferences and semi-positive endowments. Combining this with Maskin's result, they proved that for those economic environments just mentioned, any individually rational and monotone social choice function is Nash implementable.

In their endowment mechanism, a message from any agent is an endowment profile, i.e., n - tuples of endowment vectors, one is a claim about his own endowment, the rest are estimations about others' endowments. Every agent may conceal part of his endowment but is not allowed to exaggerate it. The dimension of the message space is thus $n^2\ell$, where n is the number of agents and ℓ is the number of goods. In this paper,we will present an alternative mechanism with $n(\ell+1)$-dimensional message space and with an outcome function no more complicated, if not simpler, than that in HMP. In this sense, our mechanism is informationally more efficient.

Our main ideas are as follows: First, in the HMP mechanism, the claim of agent i about his own endowment is compared to others' estimations about his endowment (component by component) in order to provide incentives for truthful revelation of endowments. Since he is not allowed to overstate his endowment, the following is true:

1) The "size" of his claimed endowment vector equals the "size" of his true endowment vector if and only if he tells the truth about the endowment.

2) The "size" of his claimed endowment vector is smaller than the "size" of his true endowment vector if and only if he conceals part of his endowment.

(The "size" of a vector is later defined as its norm.)

This is most easily understood through an example. Suppose that there are two goods in the economy: apple and bread. Suppose that we know Mr. Smith has a total of 12 units of goods. If Mr. Smith claims that he has 3 apples and 9 loaves of bread, then this has to be his true endowment vector. The reason is that if he is not telling the truth, it must be that he has either more apple or bread or both so that the total is not 12 units.

This suggests that it might be enough to have each agent estimate only the "size" of every other agent's endowment (not a whole vector) in addition to announcing his own endowment. Further, if we think of agents as being arranged in a circle, we may only need each agent's neighbor to provide him incentives to reveal his true endowment. This way, a message from any agent consists of a claim about his own endowment and his estimation of the "size" of his neighbor's endowment.

The rest of the paper is organized as follows: In Section 2, notation and assumptions are discussed. They are almost exactly the same as those in HMP. Section 3 contains the main contents of the paper: a mechanism with $n(\ell+1)$-dimensional message space is formally described and it is shown that it implements in Nash equilibria social choice functions under our consideration. Section 4 provides concluding remarks.

2. NOTATION AND ASSUMPTIONS

Let ℓ be a positive integer. Let $x=(x_1,\ldots,x_\ell)\in\mathbb{R}^\ell$ & $y=(y_1,\ldots,y_\ell)\in\mathbb{R}^\ell$.

We use the following conventions:

$x \leqq y$ iff $x_j \leqq y_j$ for all $j = 1,\dots,\ell$

$x \leq y$ iff $x \leqq y$ & $x \neq y$

$x < y$ iff $x_j < y_j$ for all $j = 1,\dots,\ell$

$\mathbb{R}^\ell_+ = \{ x \in \mathbb{R}^\ell \mid x \geqq 0 \}$

$\mathbb{R}^\ell_{++} = \{ x \in \mathbb{R}^\ell \mid x > 0 \}$

$\mathbb{R}^\ell_{+o} = \mathbb{R}^\ell_+ \setminus \{0\}$

$\mathbb{R}^{\ell n}_{+o} = \mathbb{R}^\ell_{+o} \times \dots \times \mathbb{R}^\ell_{+o}$ (n times)

2.1. The Economic Environment

$N = \{ 1,\dots,n \}$ = the set of agents; $n \geqq 3$.

$L = \{ 1,\dots,\ell \}$ = the set of commodities.

$\mathring{\omega}_i$ = the true initial endowment of agent i.

$\vec{\mathring{\omega}} = (\mathring{\omega}_1,\dots,\mathring{\omega}_n)$ = the true endowment profile.

$\mathbb{R}^\ell_+$ is the consumption set for every agent.

$\mathring{R}_i$ = the true preference relation of agent i defined on $\mathbb{R}^\ell_+$.

$\mathring{P}_i$ = the true strict preference of agent i.

<u>Assumption 1</u> For all $i \in N$, $\mathring{\omega}_i \in \mathbb{R}^\ell_{+o}$.

<u>Assumption 2</u> For all $i \in N$, $\mathring{R}_i$ is reflexive, transitive, total and strictly increasing.

2.2. The Social Choice Function

The social choice function under our consideration is a function $f = (f_1,\dots,f_n)$ where $f_i\colon \mathbb{R}^{\ell n}_{+o} \to \mathbb{R}^\ell$ for all $i \in N$ such that Assumptions 3, 4 & 5 are satisfied.

<u>Assumption 3</u> f is balanced, i.e. $\forall\, \vec{v} = (v_1,\dots,v_n) \in \mathbb{R}^{\ell n}_{+o}$, $\sum_{i \in N} f_i(\vec{v}) = 0$.

<u>Assumption 4</u> f is feasible, i.e. $\forall\, \vec{v} = (v_1,\dots,v_n) \in \mathbb{R}^{\ell n}_{+o}$, $f_i(\vec{v}) \geqq -v_i$ for all $i \in N$.

Assumption 5 f is individually rational, i.e. $\forall\ \underset{\rightarrow}{v} = (v_1,\dots,v_n)\in\mathbb{R}^{\ell n}_{+o}$, $(v_i + f_i(\underset{\rightarrow}{v}))\mathring{R}_i v_i$ for all $i\in N$.

3. THE MECHANISM

We construct an individually feasible and balanced mechanism (S,h) such that it implements the social choice function for true preference profiles, i.e. for any true endowment profile $\underset{\rightarrow}{\mathring{\omega}}$, a Nash equilibrium exists, and further, for any Nash equilibrium strategy s^*, $h(s^*)=f(\underset{\rightarrow}{\mathring{\omega}})$.

The message space of the mechanism is $S = S_1\times\dots\times S_n$, where $S_i= (0,\mathring{\omega}_i]\times\mathbb{R}^1_{++}$. A generic element of S_i is $s_i= (v_i,t_i^{i+1})$ where v_i is agent i's claim about his endowment and t_i^{i+1} is interpreted as agent i's estimation about the "size" of his neighbor's endowment. Since $0 \le v_i \lneqq \mathring{\omega}_i$, agent i is allowed to withhold part of his endowment, but not all of it. The outcome function $h(s) = (h_1(s),\dots,h_n(s))$ specifies each agent's net trade as follows:

$\forall\ s \in S$, define $M(s) = \{i \in N:\ \|v_i\| > t_{i-1}^i\}$

$$\beta_i(s) = \sum_{j\neq i,i+1} |\,\|v_j\| - t_{j-1}^j| \qquad \forall\ i \in N$$

$$\beta(s) = \sum_{i\in N} \beta_i(s)$$

Remark: In this paper, $\|\cdot\|$ can be either $\|\cdot\|_1$-norm or $\|\cdot\|_2$-norm where for any $x \in \mathbb{R}^\ell$, $\|x\|_1= (|x_1|+\dots+|x_\ell|)$ & $\|x\|_2= ((x_1)^2+\dots+(x_\ell)^2)^{1/2}$. In fact, it can be any positive function that is strictly increasing in each argument. Of particular interest is the value function (the inner product of the endowments and a strictly positive price vector). This allows us to make the following statement: "Given a strictly positive vector of world prices, the agents need only report their own endowment vector and an estimate of the total wealth of his neighbor." It is much easier to know the wealth of an agent than to know the precise vector of goods the agent owns. The concept of the "size" of a vector x in this paper is expressed in terms of $\|x\|$.

Case (1) (Norm-unanimity) If $\forall\ i\in N$, $\|v_i\| = t_{i-1}^i$, we set $h(s) = f(\underset{\rightarrow}{v})$.

Case (2) If $M(s)=\emptyset$ and there is no norm-unanimity, we set

$h_i(s)=[(\beta_i(s)/\beta(s))\sum_{j\in N} v_j] - v_i$ for all $i\in N$

(This is well-defined because in this case, $\beta(s)\neq 0$. In fact, $\beta(s)=(n-2)\sum_{j\in N}|\,\|v_j\|-t^j_{j-1}| > 0$ since there is no norm-unanimity.)

Case (3) If $M(s)\neq\emptyset$, then we set

$$h_i(s) = \begin{cases} (1/(\#M(s)))\sum_{j\in N} v_j - v_i & \text{for } i\in M(s) \\ -v_i & \text{for } i\notin M(s) \end{cases}$$

These three cases exhaust all the possibilities of $s\in S$, so that the outcome function is completely defined. It is also easily seen that the individual feasibility and the balance condition are satisfied.

As in HMP, we prove three claims which together imply that our mechanism implements the social choice function f.

Claim 1 Truthful norm-unanimity is a Nash equilibium, i.e. $\bar{s}=(\bar{s}_1,\ldots,\bar{s}_n)$, where $\bar{s}_i=(\bar{v}_i,\bar{t}_i^{i+1})=(\mathring{\omega}_i,\|\mathring{\omega}_{i+1}\|)$ for all $i\in N$, is a Nash equilibium.

Proof. Without loss of generality, we show that agent 1 has no incentive to unilaterally deviate from $\bar{s}_1$. $\forall s_1=(v_1,t_1^2)\in S_1$ such that $s_1\neq\bar{s}_1=(\mathring{\omega}_1,\|\mathring{\omega}_2\|)$, it has to be the case that $v_1\leqq\mathring{\omega}_1$, so $\|v_1\|\leqq\|\mathring{\omega}_1\|=\bar{t}^1_n$. Then $1\notin M(s_1,\bar{s}_{)1(})$. There are two possible cases. One is $M(s_1,\bar{s}_{)1(})=\emptyset$ in which case rule (2) applies and $\beta_1(s_1,\bar{s}_{)1(})=\beta_1(\bar{s})=0$ so that $h_1(s_1,\bar{s}_{)1(})=-v_1$. The other case is $M(s_1,\bar{s}_{)1(})\neq\emptyset$ so that rule (3) applies and since $1\notin M(s_1,\bar{s}_{)1(})$, $h_1(s_1,\bar{s}_{)1(})=-v_1$. By rule (1), $h_1(\bar{s})=f_1(\underset{\rightarrow}{\mathring{\omega}})\ \mathring{R}_1 0\ \mathring{P}_1(-v_1)$ since f is individually rational, $\mathring{R}_1$ is strictly increasing and v_1 is semi-positive.[1] So $\forall s_1\in S_1$ s.t. $s_1\neq\bar{s}_1$, $h_1(\bar{s})\ \mathring{P}_1 h_1(s_1,\bar{s}_{)1(})$. Therefore, agent 1 does not want to deviate from

1 Strictly speaking, it should be $\mathring{\omega}_1+h_1(\bar{s})=\mathring{\omega}_1+f_1(\underset{\rightarrow}{\mathring{\omega}})\mathring{R}_1\mathring{\omega}_1\mathring{P}_1\mathring{\omega}_1+h_1(s_1,\bar{s}_{)1(})$. We abuse the notation in this paper since it should be clear what it really means from the context.

Claim 2 False norm-unanimity is not a Nash equilibrium, i.e.

$s = (s_1, \ldots, s_n)$ where $s_i = (v_i, t_i^{i+1})$ is not a Nash equilibrium if $\|v_i\| = t_{i-1}^{i}$ $\forall\, i \in N$ and there exists a $k \in N$ s.t. $v_k \neq \mathring{\omega}_k$.

Proof. Since $v_k \neq \mathring{\omega}_k$, $v_k \leq \mathring{\omega}_k$. Let $\tilde{s}_k = (\tilde{v}_k, \tilde{t}_k^{k+1})$ where $\tilde{v}_k = \mathring{\omega}_k$ and $\tilde{t}_k^{k+1} = t_k^{k+1}$. Since $\|\tilde{v}_k\| = \|\mathring{\omega}_k\| > \|v_k\| = t_{k-1}^{k}$, $k \in M(\tilde{s}_k, s_{)k(})$. Others are not in $M(\tilde{s}_k, s_{)k(})$ because their strategies stay the same and $\tilde{t}_k^{k+1} = t_k^{k+1}$. So $M(\tilde{s}_k, s_{)k(}) = \{k\}$. By rule (3), $h_k(\tilde{s}_k, s_{)k(}) = \sum_{j \neq k} v_j$. Since $\forall\, i \in N$, $h_i(s) \geqq - v_i$ and since $\sum_{i \in N} h_i(s) = 0$, $h_k(s) \leqq \sum_{i \neq k} v_i$. Now we want to show $h_k(s) \neq \sum_{i \neq k} v_i$. Suppose the equality holds. Then $\forall\, i \neq k$, $h_i(s) = - v_i$. But by rule (1), $h_i(s) = f_i(\underset{\to}{v})$, so that $f_i(\underset{\to}{v}) = - v_i$. Since v_i is semi-positive and $\mathring{R}_i$ is strictly increasing, $0 \mathring{P}_i (-v_i)$, i.e., $0 \mathring{P}_i f_i(\underset{\to}{v})$ which contradicts the assumption that f is individually rational. Thus, $h_k(s) \leq \sum_{i \neq k} v_i$. Since $\mathring{R}_k$ is strictly increasing, $h_k(\tilde{s}_k, s_{)k(})\ \mathring{P}_k\ h_k(s_k, s_{)k(})$. Thus s is not a N.E..

Claim 3 No norm-unanimity is not a N.E., i.e. $s = (s_1, \ldots, s_n)$ where $s_i = (v_i, t_i^{i+1})$ is not a N.E. if there exists a $k \in N$ such that $\|v_k\| \neq t_{k-1}^{k}$.

Proof. 1) M(s)=N. In this case, $h_{k-1}(s) = (1/n) \sum_{j \in N} v_j - v_{k-1}$. Let $\tilde{s}_{k-1} = (\tilde{v}_{k-1}, \tilde{t}_{k-1}^{k})$ where $\tilde{v}_{k-1} = v_{k-1}$ and $\tilde{t}_{k-1}^{k} = \|v_k\|$. Then $\tilde{t}_{k-1}^{k} = \|v_k\|$ implies that $k \notin M(\tilde{s}_{k-1}, s_{)k-1(})$. For all $j \neq k$, $j \in M(s)$ implies $j \in M(\tilde{s}_{k-1}, s_{)k-1(})$. In particular $(k-1) \in (\tilde{s}_{k-1}, s_{)k-1(})$. So by rule (3), $h_{k-1}(\tilde{s}_{k-1}, s_{)k-1(}) = (1/(n-1)) \sum_{j \in N} v_j - v_{k-1} \geq h_{k-1}(s)$. Since $\mathring{R}_{k-1}$ is strictly increasing, $h_{k-1}(\tilde{s}_{k-1}, s_{)k-1(}) \mathring{P}_{k-1} h_{k-1}(s)$. Therefore, s is not a N.E..

2) $M(s) \neq \emptyset$ & $M(s) \neq N$. Suppose $i \notin M(s)$ & $j \in M(s)$. Then $h_i(s) = -v_i$. Let $\tilde{s}_i = (\tilde{v}_i, \tilde{t}_i^{i+1})$ where $\tilde{v}_i = (1/2) v_i$ and $\tilde{t}_i^{i+1} = t_i^{i+1}$. $\|\tilde{v}_i\| = \|(1/2) v_i\| < \|v_i\| \leqq t_{i-1}^{i}$ (since $i \notin M(s)$) implies $i \notin M(\tilde{s}_i, s_{)i(})$ and $j \in M(s)$ implies

$j \in M(\tilde{s}_i, s_{)i(})$, so $M(\tilde{s}_i, s_{)i(}) \neq \emptyset$. Then rule (3) applies and $h_i(\tilde{s}_i, s_{)i(}) = -\tilde{v}_i = -(1/2)v_i \geq -v_i = h_i(s)$. So, $h_i(\tilde{s}_i, s_{)i(}) \ \mathring{P}_i \ h_i(s)$. Thus s is not a N.E..

3) $M(s)=\emptyset$. Since this does not satisfy norm-unanimity, we assume that $\|v_k\| \neq t^k_{k-1}$. We analyze two subcases.

i) $\beta_{k-1}(s) = 0$. In this case, $h_{k-1}(s) = -v_{k-1}$. Let $\tilde{s}_{k-1} = (\tilde{v}_{k-1}, \tilde{t}^k_{k-1})$ where $\tilde{v}_{k-1} = (1/2)v_{k-1}$ and $\tilde{t}^k_{k-1} = t^k_{k-1}$. Then $(\tilde{s}_{k-1}, s_{)k-1(})$ is still not norm-unanimous because $\tilde{t}^k_{k-1} = t^k_{k-1} \neq \|v_k\|$. Since $\|\tilde{v}_{k-1}\| = \|(1/2)v_{k-1}\| < \|v_{k-1}\| \leq t^{k-1}_{k-2}$ (because $(k-1) \notin M(s)$), $(k-1) \notin M(\tilde{s}_{k-1}, s_{)k-1(})$. $\forall j \neq k-1$, $j \notin M(s)$ implies $j \notin M(\tilde{s}_{k-1}, s_{)k-1(})$. Thus $M(\tilde{s}_{k-1}, s_{)k-1(}) = \emptyset$ and there is no norm-unanimity. Apply rule (2), $h_{k-1}(\tilde{s}_{k-1}, s_{)k-1(}) = -\tilde{v}_{k-1} =$

$= -(1/2)v_{k-1} \geq -v_{k-1}$ because $\beta_{k-1}(\tilde{s}_{k-1}, s_{)k-1(}) = \beta_{k-1}(s) = 0$. Thus, $h_{k-1}(\tilde{s}_{k-1}, s_{)k-1(}) \mathring{P}_{k-1} h_{k-1}(s)$, so that s is not a Nash equilibrium.

ii) $\beta_{k-1}(s) > 0$. Consider $\tilde{s}_{k-1} = (\tilde{v}_{k-1}, \tilde{t}^k_{k-1})$ where $\tilde{v}_{k-1} = v_{k-1}$ and $\tilde{t}^k_{k-1} = \|v_k\|$. Then $M(\tilde{s}_{k-1}, s_{)k-1(}) = \emptyset$. Since $\beta_{k-1}(\tilde{s}_{k-1}, s_{)k-1(})$ does not depend on $\tilde{s}_{k-1}$, $\beta_{k-1}(\tilde{s}_{k-1}, s_{)k-1(}) = \beta_{k-1}(s) > 0$. Since $\beta_k(\tilde{s}_{k-1}, s_{)k-1(})$ depends on $\tilde{v}_{k-1}$ not on $\tilde{t}^k_{k-1}$, $\tilde{v}_{k-1} = v_{k-1}$ implies that $\beta_k(\tilde{s}_{k-1}, s_{)k-1(}) = \beta_k(s)$.

$\forall\ j \neq k-1\ \&\ k$, $\beta_j(\tilde{s}_{k-1}, s_{)k-1(}) = \sum_{i \neq j, j+1, k} |\|v_i\| - t^i_{i-1}| + |\|v_k\| - \tilde{t}^k_{k-1}|$

$= \sum_{i \neq j, j+1, k} |\|v_i\| - t^i_{i-1}| < \sum_{i \neq j, j+1, k} |\|v_i\| - t^i_{i-1}| + |\|v_k\| - t^k_{k-1}| = \beta_j(s)$, the inequality is because $\|v_k\| \neq t^k_{k-1}$. So $0 < \beta(\tilde{s}_{k-1}, s_{)k-1(}) < \beta(s)$. Apply rule (2), $h_{k-1}(\tilde{s}_{k-1}, s_{)k-1(}) = (\beta_{k-1}(\tilde{s}_{k-1}, s_{)k-1(}) / \beta(\tilde{s}_{k-1}, s_{)k-1(})) \sum_{j \in N} v_j - v_{k-1}$

$\geq (\beta_{k-1}(s)/\beta(s)) \sum_{j \in N} v_j - v_{k-1} = h_{k-1}(s)$. So $h_{k-1}(\tilde{s}_{k-1}, s_{)k-1(}) \mathring{P}_{k-1} h_{k-1}(s)$. Therefore, s is not a Nash equilibrium.

From the above three claims, we immediately have the following:

THEOREM The mechanism described in this paper has a message space of dimension $n(\ell+1)$ and it implements the social choice function.

4. CONCLUSION

The idea of having each agent propose his estimation of the "size" (not the whole vector) of his neighbor's (not everybody else's) endowment enables us to reduce the message space of the HMP mechanism a great deal without complicating the outcome function. Although our mechanism is informationally more efficient, we do not know if it is in fact informationally efficient. That is to say, we do not know if $n(\ell+1)$ is the minimum dimension of the message space required to carry out the implementation.

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