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CARDINAL UTILITY AND THE PROBLEM OF SOCIAL CHOICE

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1. INTRODUCTION

In this paper we will argue that it is necessary to consider cardinal utility indexes, i.e., utility indexes which are constant up to positive linear transformations, if we want to develop a useful theory of social choice.

The literature on social choice usually considers a finite group of individuals, say n, each of them possessing a preference ordering defined on a finite set A of m alternative actions or social states. Then it tries to construct (Arrow [1]) a social welfare function, i.e. a function that assigns a social ordering to each n-tuple of individual preference orderings, satisfying certain desirable conditions; or it tries to construct (Sen [7]) a social decision function, i.e. a function that assigns to each n-tuple of individual preference orderings a social preference relation which generates a choice function, satisfying also certain desirable conditions.

During the last two decades the main efforts to solving the problem of social choice have been concentrated on these two approaches with the main results expressed in the form of impossibility theorems which state: in Arrow's approach, the no existence of a social welfare function satisfying the required conditions; in Sen's approach, the no existence of a social decision function satisfying also certain desirable conditions.[1]

It seems to this author that when we think of a social decision mechanism operating in a society, we do not think of it as deciding once and for all the action or actions that society is going to take. Rather, we view the decision mechanism as a kind of social strategy or rule that determines under each possible set of circumstances what action society should take when each particular set of circumstances occurs. Certainly this is the case of any social decision mechanism that we can think of in the real world. But if this is the case, then the virtues or defects of the mechanisms should be determined by judging its overall performance, i.e. by judging the desirability or no desirability of the sequence of actions that it generates when any possible sequence of "different sets of circumstances" comes along, and not by judging in isolation the action that the mechanism assigns under each different set of circumstances.

It appears to us that when we consider the unanimity rule as a bad mechanism because it immobilizes the system, we are not thinking of an isolate instance in which a single individual could block a movement; after all if an individual blocks a movement under the unanimity rule it is because he prefers the status quo to the proposed change. We are rather thinking of the possibility that, under different sets of circumstances, different individuals might block the proposed movements and force society to remain forever without undertaking any change at all, and that this long-run performance could be less desirable from the point of view of every individual in the society than the sequence of actions generated by other mechanisms, say a majority rule.

Thus, when we consider as unacceptable a mechanism as the unanimity rule, it seems that we are doing that more on the basis of its lack of long-run or overall efficiency or Pareto optimality, as defined by us later, than on the basis of fairness considerations as appears to be the case when we reject a social mechanism which is a dictatorship.[2]

Both the social welfare function and the social decision function approaches mentioned above are not appropriate for this type of overall analysis of the social choice problem because , although they allow in their models for changes in the preferences of the individuals with regard to the possible actions, they do not specify the overall preferences of the individuals. Thus, if in a situation a person prefers, say, action a_1 to action a_2 ($a_1 > a_2$) and then his preferences change and in a new situation he prefers a_2 to a_1 ($a_2 > a_1$), we should also know, in order to make an overall analysis of the social mechanism, whether he prefers the sequence "a_1 in the first situation and a_1 in the second" to the sequence "a_2 in the first situation and a_2 in the second" or if he is indifferent between them, etc.... And this information is not specified in either of the social welfare or social decision functions models.

In a previous paper this author [2] developed a model for analyzing the overall performance of social decision mechanisms. We developed there a rather attractive set of axioms regarding the preferences of the individuals that lead to the existence of a kind of cardinal utility indexes. Then by using these utility indexes we constructed a class of social decision mechanisms, all of whose members are Pareto-optimal over the class of societies considered in the paper in the sense that any <u>finite</u> sequence of actions generated by any mechanism of the class is such that no other finite sequence (of the same length, of course) of actions could have been generated that had made at least one individual better off and no one else worse off. We proved also there (Theorem 5) that no social decision mechanism exists which is Pareto-optimal in the sense described above over the class of societies considered by us, that is based only on the local orderings of actions by the individuals, and that satisfies a rather mild anonymity condition.

In this paper we will prove a theorem which is stronger than Theorem 5 presented in [2]. And on the light of this new theorem we will argue that it is necessary to consider cardinal utility indexes to develop a useful theory of social choice.

Any acceptable argument defending the use of cardinal utility has to be, it seems to us, convincing on two grounds. First, it must be shown that the set of axioms with regard to the preference orderings of the individuals that lead to the existence of cardinal utility indexes is reasonable enough in the sense that they (the axioms) do not impose unrealistic restrictions on the possible preference orderings of the individuals. Second, that there are definite advantages in adopting the use of cardinal utility indexes.

The axioms presented in [2], and which we will reproduce in this paper, and their economic interpretation appear to us rather appealing and we think that they will indeed pass the first test required by our argument. We should note here that our axioms do not involve directly the use of utility differences as in Frisch [3], and Suppes and Winet [8], or the ordering of uncertain prospects as in the expected utility theory of Morgenstern and von Neumann [5]. They involve the use of notions such as "rate of substitution," "independence," etc., that seem more natural from an economic point of view.

We will present our argument defending the advantages of using cardinal utility indexes in constructing and operating social decision mechanisms in the last section of this article. We want to mention here however that our argument is similar to that used in the debate on centralization versus decentralization by those favoring a decentralized approach in the organization of the economic activity. The construction and operation of social decision mechanisms which are Pareto-optimal, in a sense that we will make precise below, and that satisfy the rather mild requirement of anonymity, can be accomplished in a very simple way by using our cardinal utility indexes. To accomplish the same goal by using an ordinalist approach becomes extremely complicated.

This paper is organized as follows: In Section 2 the axioms on the preferences of the individuals that lead to the existence of cardinal utility indexes are presented; a society S is precisely described and the class $\mathcal{S}$ of societies to be considered by us is defined. In Section 3 a class $\mathcal{M}$ of social decision mechanisms which are Pareto-optimal is determined and the generalization of Theorem 5 of [2] proved. In Section 4, by using the conclusions of the theorem proved in Section 3, we will discuss the advantages of utilizing our cardinal utility indexes in the construction and operation of Pareto-optimal social decision mechanisms in comparison with the use of a pure ordinalist approach to accomplish the same goal.

2. DESCRIPTION OF A SOCIETY

A society is a group of a finite number of n individuals or members and two types of variables: one type over which the members of the society do have control and that we will call actions; another type over which the individuals do not have control, and that constitutes what we will call the environment.[3] Thus in our model the individuals of a society do not control the environment, they react to the environment by taking actions. And we assume that the different environments that may occur will affect, in general, the preferences of the individuals with regard to the possible actions that can be taken.

A society will then be precisely defined if we describe: the class of all the environments that may occur; the actions that can be taken when the different environments occur; the members of the society and how their preferences are affected when the different environments occur.

2.1. The Space of Environments

We represent by E the collection of all possible environments and by e a generic element of it. We assume that the occurrences of the different environments follow a probability law. Thus, the space of environments is defined by the triple $(E, \mathcal{S}, P)$ where $\mathcal{S}$ is a σ-field on E and P a probability measure on $\mathcal{S}$.

Remark. We assume that the space of environments $(E, \mathcal{S}, P)$ is rich enough for any application that we need to make in this paper. We want to note also that, although it helps our intuition to think of the environment as something concrete like weather conditions, etc., which may affect the preferences of the individuals, and we used illustrations of this type in [2] for that purpose, there is no need to do that. As it will become clearer later, we can look at the environment only as an auxiliary variable which is convenient in describing changes in the preferences of the individuals. These changes in preferences may be due to changes in things as concrete and visible as weather conditions, or they may be due to changes in less visible things as the blood pressure of the individuals, or even to changes as subtle as the state of mind of the individuals. What is important for our model is the assumption that each member of society has certain perception of the occurrence of these environments and of how these occurrences will affect his preferences.

2.2. Actions

There is a set of m actions $A = \{a_1,\ldots,a_m\}$ from which society can choose when different environments occur.

2.3. Members of a Society

Each society has a finite number n of members. A member i of a society is well defined for our purposes if we describe how the different environments affect his preferences with regard to the different actions that can be taken and the properties of these preferences. This we will do now.

For each member i $(i=1,\ldots,n)$, let λ_i be a function from E onto the finite set $K_i = \{1,\ldots,k_i\}$, where k_i is a natural number greater than or equal to 1, such that the inverse images $\lambda_i^{-1}(1) = E_i^1,\ldots,\lambda_i^{-1}(k_i) = E_i^{k_i}$ are all members of $\mathcal{J}$. Let $\mathcal{E}_i = \{E_i^1,\ldots,E_i^{k_i}\}$.

<u>Preference Orderings</u>. Consider the two sets of infinite sequences

$$A^{\infty} = \{(a_{\varphi_1},\ldots,a_{\varphi_h},\ldots): a_{\varphi_h} \in A \text{ for } h=1,2,\ldots,\} \text{ and}$$

$$\mathcal{E}_i^{\infty} = \{(E_i^{j_1},\ldots,E_i^{j_h},\ldots): E_i^{j_h} \in \mathcal{E}_i \text{ for } h=1,2,\ldots,\}.$$

Let a^{∞} represent a generic element of the set A^{∞} and E_i^{∞}, a generic element of $\mathcal{E}_i^{\infty}$.

We assume that to each member of $\mathcal{E}_i^{\infty}$, $(E_i^{j_1},\ldots,E_i^{j_h},\ldots)$, corresponds an ordering relation (i.e. a relation that is transitive, reflexive and connected), $Q_i(E_i^{j_1},\ldots,E_i^{j_h},\ldots) = Q_i(E_i^{\infty})$, of the elements of A^{∞}.

We will write $a^{\infty}|E_i^{\infty} \geqq_i \overline{a}^{\infty}|E_i^{\infty}$ to mean $a^{\infty}\ Q_i(E_i^{\infty})\ \overline{a}^{\infty}$. If we want to be more explicit we may also write

$$a_{\varphi_1}|E_i^{j_1},\ldots,a_{\varphi_h}|E_i^{j_h},\ldots, \geqq_i \overline{a}_{\varphi_1}|E_i^{j_h},\ldots,\overline{a}_{\varphi_h}|E_i^{j_h},\ldots \quad .$$

$$a^{\infty}|E_i^{\infty} >_i \overline{a}^{\infty}|E_i^{\infty} \text{ means } a^{\infty}|E_i^{\infty} \geqq_i \overline{a}^{\infty}|E_i^{\infty} \text{ and not } \overline{a}^{\infty}|E_i^{\infty} \geqq_i a^{\infty}|E_i^{\infty} \ ;$$

$$a^{\infty}|E_i^{\infty} \sim_i \overline{a}^{\infty}|E_i^{\infty} \text{ means } a^{\infty}|E_i^{\infty} \geqq_i \overline{a}^{\infty}|E_i^{\infty} \text{ and } \overline{a}^{\infty}|E_i^{\infty} \geqq_i a^{\infty}|E_i^{\infty} \ .$$

In the language of the social sciences $Q_i(E_i^{\infty})$ is the relation "is at least as good as" and $a^{\infty}Q_i(E_i^{\infty})\ \overline{a}^{\infty}$ means: that the member i of the society prefers society to take action a_{φ_1} when the event $E_i^{j_1}$ occurs,..., action a_{φ_h} when $E_i^{j_h}$ occurs, etc.,..., than to take action $\overline{a}_{\varphi_1}$ when $E_i^{j_1}$ happens,..., $\overline{a}_{\varphi_h}$ when $E_i^{j_h}$ happens, etc.,..., ; or that he is indifferent between the two sequences of actions.

Let $\mathcal{Q}_i = \{Q_i(E_i^\infty) : E_i^\infty \in \mathcal{E}_i^\infty\}$, i.e. $\mathcal{Q}_i$ is the collection of all preference orderings of the elements of the set A^∞ that correspond to infinite sequences of events of the outside world.

Properties of the Class of Orderings $\mathcal{Q}_i$. We assume that the class of orderings $\mathcal{Q}_i$ satisfies the following axioms:

Permutation Axiom.[4] Let ν be any natural number greater than 0. Let π_ν be any one to one function from the set of natural numbers $\{1,2,\ldots\}$ onto itself such that $\pi_\nu(h) = h$ for all $h > \nu$. Then we have: for any finite natural number ν and any π_ν,

$$(a_{\varphi_1},\ldots,a_{\varphi_h},\ldots)\ Q_i(E_i^{j_1},\ldots,E_i^{j_h},\ldots)(\bar{a}_{\varphi_1},\ldots,\bar{a}_{\varphi_h},\ldots) \iff$$

$$\iff (a_{\varphi_{\pi_\nu(1)}},\ldots,a_{\varphi_{\pi_\nu(h)}},\ldots)\ Q_i(E_i^{j_{\pi_\nu(1)}},\ldots,E_i^{j_{\pi_\nu(h)}},\ldots,)$$

$$(\bar{a}_{\varphi_{\pi_\nu(1)}},\ldots,\bar{a}_{\varphi_{\pi_\nu(h)}},\ldots).$$

Before we present in a formal way the independence axiom we need to develop some notation. Consider the infinite sequence

[1] $a_{\varphi_1}|E_i^{j_1},\ldots,a_{\varphi_h}|E_i^{j_h},\ldots, = (a^\infty,E_i^\infty)$ and take from it the terms

$$a_{\varphi_{h_1}}|E_i^{j_{h_1}},\ldots,a_{\varphi_{h_r}}|E_i^{j_{h_r}}\ .$$

Call this finite subsequence $F(h_1,\ldots,h_r;a^r;E_i^r)$ and the remainder infinite subsequence $C(h_1,\ldots,h_r;a^{\infty-r};E_i^{\infty-r})$. We will represent the infinite sequence [1] by

$$F(h_1,\ldots,h_r;a^r;E_i^r)\ C(h_1,\ldots,h_r;a^{\infty-r};E_i^{\infty-r}).$$

Suppose we obtain new sequences from the sequence [1] by changing in the finite part F of it: some, all or none of the a's; some, all or none of the E's; some, all or none of the a's and some, all or none of the E's. The new resulting sequences will be represented, respectively, by $F(h_1,\ldots,h_r;\bar{a}^r;E_1^r)\ C(h_1,\ldots,h_r;a^{\infty-r};E_1^{\infty-r})$;

$$F(h_1,\ldots,h_r;a^r;\bar{E}_1^r)\ C(h_1,\ldots,h_r;a^{\infty-r};E_1^{\infty-r});$$

$$F(h_1,\ldots,h_r;\bar{a}^r;\bar{E}_1^r)\ C(h_1,\ldots,h_r;a^{\infty-r};E_1^{\infty-r}).$$

The corresponding changes in the part C of sequence [1] will be represented in a similar way.

<u>Independence Axiom</u>. $F(h_1,\ldots,h_r;a^r;E_1^r)\ C(h_1,\ldots,h_r;a^{\infty-r};E_1^{\infty-r}) \gtrless_1$

$\gtrless_1 F(h_1,\ldots,h_r;\bar{a}^r;E_1^r)\ C(h_1,\ldots,h_r;a^{\infty-r};E_1^{\infty-r}) \iff$

$\iff F(h_1,\ldots,h_r;a^r;E_1^r)\ C(h_1,\ldots,h_r;\bar{a}^{\infty-r};\bar{E}_1^{\infty-r}) \gtrless_1$

$\gtrless_1 F(h_1,\ldots,h_r;\bar{a}^r;E_1^r)\ C(h_1,\ldots,h_r;\bar{a}^{\infty-r};\bar{E}_1^{\infty-r})$.

<u>Remark</u>. The class of preference orderings $\mathcal{Q}_1$ satisfying the Permutation and Independence axioms defines a class of preference orderings, which satisfies also the corresponding Permutation and Independence axioms, on the Cartesian product $A^r = A\times\ldots\times A$ (r times; r being any finite natural number) as follows:

$$a_{\varphi_1}|E_1^{j_1},\ldots,a_{\varphi_r}|E_1^{j_r} = F(1,\ldots,r;a^r;E_1^r) \geqq_1$$

$\geqq_1 \bar{a}_{\varphi_1}|E_1^{j_1},\ldots,\bar{a}_{\varphi_r}|E_1^{j_r} = F(1,\ldots,r;\bar{a}^r;E_1^r)$ if and only if there is a $C(1,\ldots,r;a^{\infty-r};E_1^{\infty-r})$ such that

$$F(1,\ldots,r;a^r;E_1^r)\ C(1,\ldots,r;a^{\infty-r};E_1^{\infty-r}) \geqq_1 F(1,\ldots,r;\bar{a}^r;E_1^r)\ C(1,\ldots,r;a^{\infty-r};E_1^{\infty-r}).$$

We represent these preference orderings by $Q_i(E_i^{j_1},\ldots,E_i^{j_r})$. In particular we have the preference orderings $Q_i(E_i^1),\ldots,Q_i(E_i^{k_i})$.

Rate of Substitution Axiom. If $a_{\varphi_p}|E_i^k \geq_i a_{\varphi_q}|E_i^k$ and $a_{\varphi_{\bar{p}}}|E_i^{\bar{k}} >_i$ $>_i a_{\varphi_{\bar{q}}}|E_i^{\bar{k}}$, then there exists a real and non-negative number (that depends on i, k, $\bar{k}$, φ_p, φ_q, $\varphi_{\bar{p}}$, $\varphi_{\bar{q}}$), $R_i(k, \bar{k}, \varphi_p, \varphi_q, \varphi_{\bar{p}}, \varphi_{\bar{q}})$ such that the following is true:

(a) If in a sequence $a_{\varphi_1}|E_i^{j_1},\ldots,a_{\varphi_h}|E_i^{j_h},\ldots$, we substitute a_{φ_q} for a_{φ_p} r times ($r > 0$) when the event E_i^k occurs and $a_{\varphi_{\bar{p}}}$ for $a_{\varphi_{\bar{q}}}$ s times ($s \geq 0$) when the event $E_i^{\bar{k}}$ occurs, then the resulting sequence is $>_i$, $<_i$, or $\sim_i$, with regard to the original one if and only if $\frac{s}{r} > R_i(k, \bar{k}, \varphi_p, \varphi_q, \varphi_{\bar{p}}. \varphi_{\bar{q}})$, $\frac{s}{r} < R_i(k, \bar{k}, \varphi_p, \varphi_q, \varphi_{\bar{p}}, \varphi_{\bar{q}})$, or $\frac{s}{r} = R_i(k, \bar{k}, \varphi_p, \varphi_q, \varphi_{\bar{p}}, \varphi_{\bar{q}})$, respectively.

(b) If in a sequence $a_{\varphi_1}|E_i^{j_1},\ldots,a_{\varphi_h}|E_i^{j_h},\ldots$, we substitute a_{φ_p} for a_{φ_q} r times when the event E_i^k occurs and $a_{\varphi_{\bar{q}}}$ for $a_{\varphi_{\bar{p}}}$ s times when the event $E_i^{\bar{k}}$ occurs, then the resulting sequence is $>_i$, $<_i$, or $\sim_i$, with regard to the original one if and only if

$\frac{s}{r} < R_i(k, \bar{k}, \varphi_p, \varphi_q, \varphi_{\bar{p}}, \varphi_{\bar{q}})$, $\frac{s}{r} > R_i(k, \bar{k}, \varphi_p, \varphi_q, \varphi_{\bar{p}}, \varphi_{\bar{q}})$, or $\frac{s}{r} = R_i(k, \bar{k}, \varphi_p, \varphi_q, \varphi_{\bar{p}}, \varphi_{\bar{q}})$, respectively.

We will represent sometimes a finite sequence containing c_1 times the term $a_{\varphi_1}|E_i^{j_1},\ldots,c_p$ times the term $a_{\varphi_p}|E_i^{j_p}$, where the c_h's (h=1,...,p) are integer and positive numbers, by $c_1 \times a_{\varphi_1}|E_i^{j_1},\ldots, c_p \times a_{\varphi_p}|E_i^{j_p}$.

We now turn to our last axiom, the

Repetition Axiom. For any integer and positive numbers p and c,

$$a_{\varphi_1}|E_i^{j_1},\ldots,a_{\varphi_p}|E_i^{j_p} \gtrsim_i a_{\varphi_1^*}|E_i^{j_1},\ldots,a_{\varphi_p^*}|E_i^{j_p} \Longleftrightarrow$$

$$\Longleftrightarrow\quad c \times a_{\varphi_1}|E_i^{j_1},\ldots,c \times a_{\varphi_p}|E_i^{j_p} \gtrsim_i c \times a_{\varphi_1^*}|E_i^{j_1},\ldots,c \times a_{\varphi_p^*}|E_i^{j_p}.$$

Utility Indexes. We try to determine now, if they exist, utility indexes for the different actions when the different events occur,

$$u^i(a_r|E_i^j) = u^i(r,j) \quad (r=1,\ldots,m;\ j=1,\ldots,k_i),$$

that satisfy the following condition:

$$[\alpha]\quad a_{\varphi_1}|E_i^{j_1},\ldots,a_{\varphi_p}|E_i^{j_p} \gtrsim_i a_{\varphi_1^*}|E_i^{j_1},\ldots,a_{\varphi_p^*}|E_i^{j_p} \Longleftrightarrow$$

$$\Longleftrightarrow\quad \sum_{h=1}^{p} u^i(\varphi_h,j_h) \geqq \sum_{h=1}^{p} u^i(\varphi_h^*,j_h),$$ where p is any finite natural number greater than 0.

Write $u^i = \begin{bmatrix} u^i(1,1),\ldots,u^i(1,k_i) \\ \vdots \qquad\qquad \vdots \\ u^i(m,1),\ldots,u^i(m,k_i) \end{bmatrix}$.

We will call the matrices u^i, utility matrices.

Theorem 1.

(i) There exists a class U^i of utility matrices u^i whose entries satisfy condition $[\alpha]$. Two matrices

$$\bar{u}^i = \begin{bmatrix} \bar{u}^i(1,1),\ldots,\bar{u}^i(1,k_i) \\ \vdots \qquad\qquad \vdots \\ \bar{u}^i(m,1),\ldots,\bar{u}^i(m,k_i) \end{bmatrix}, \quad \bar{\bar{u}}^i = \begin{bmatrix} \bar{\bar{u}}^i(1,1),\ldots,\bar{\bar{u}}^i(1,k_i) \\ \vdots \qquad\qquad \vdots \\ \bar{\bar{u}}^i(m,1),\ldots,\bar{\bar{u}}^i(m,k_i) \end{bmatrix}$$

belong to the class U^i if and only if $\bar{u}^i \in U^i$ and $\bar{\bar{u}}^i(r,j) = c\bar{u}^i(r,j) + b_j$, $(r=1,\ldots,m;\ j=1,\ldots,k_i)$, where c is a positive constant; each of the b_j's is a positive, null or negative constant.

(ii) If the entries of a matrix u^i satisfy condition $[\alpha]$, then $u^i \in U^i$.

We will omit the proof of Theorem 1. The reader interested in finding out how this theorem can be proved may look at [2].

We can now define precisely a society S, its different states s, and the class $\mathcal{S}$ of societies that we are going to consider in this paper.

<u>A Society</u>. Let $\lambda = (\lambda_1,\ldots,\lambda_i,\ldots,\lambda_n)$, $\mathcal{Q} = (\mathcal{Q}_1,\ldots,\mathcal{Q}_i,\ldots,\mathcal{Q}_n)$, $W = (E,\mathcal{A},P)$. We define S by the quadruple W, A, λ, $\mathcal{Q}$ and write

$$S \equiv (W,\ A,\ \lambda,\ \mathcal{Q}).$$

<u>States of a Society S</u>. Each of the functions $\lambda_i (i=1,\ldots,n)$ determines a partition $\mathcal{E}_i = \{E_i^1,\ldots,E_i^{k_i}\}$ on E. Let $\mathcal{E} = \bigcap_{i=1}^{n} \mathcal{E}_i$. The equivalence classes of the partition $\mathcal{E}$ are the states s of S. Thus, each state is the intersection of n events, and society S has $k_1 \times\ldots\times k_n$ different states. Write $s_{j_1,\ldots,j_n} = E_1^{j_1} \cap\ldots\cap E_n^{j_n}$, $j_i \in K_i$ for $i=1,\ldots,n$; and represent by $(s_{j_1,\ldots,j_n})$ the vector of $k_1 \times\ldots\times k_n$ different states of S.

<u>The class of societies $\mathcal{S}$</u>. We are interested in this article in the class of all societies S that have: a finite set of actions, a finite set of possible different states, and a finite number of members whose preferences satisfy our set of axioms and can, consequently, be represented by matrices of utility indexes as those of Theorem 1 above. We will designate this class of societies by $\mathcal{S}$.

3. SOCIAL DECISION FUNCTIONS

Definition 1. A social decision function or social decision mechanism M, is a function that assigns to each state s of each society S a subset of the set of actions A available to the society.

To indicate that the set of actions A corresponds to the society S, we will write A(S). By $\mathcal{A}(S)$ we will represent the power set of A(S), excluding the empty set.

Let $\mathcal{A} = \bigcup_{S \in \mathcal{S}} \mathcal{A}(S)$, and $\mathcal{B} = \{(S,s) : s \text{ is a state of } S \text{ and } S \in \mathcal{S}\}$. With this notation we can define a social decision mechanism M as a function from $\mathcal{B}$ into $\mathcal{A}$ such that for any $(S,s) \in \mathcal{B}$, $M(S,s) \in \mathcal{A}(S)$.

Definition 2. A finite sequence of actions $a_{r_1},\ldots,a_{r_p}$ is said to be Pareto-optimal for a society S when the states $s_1,\ldots,s_p$ of S prevail if

(i) $a_{r_1} \in A(S),\ldots,a_{r_p} \in A(S)$; and

(ii) there is no other finite sequence $\bar{a}_{r_1},\ldots,\bar{a}_{r_p}$ satisfying (i) and such that

$$\bar{a}_{r_1}|s_1,\ldots,\bar{a}_{r_p}|s_p \geqq_i a_{r_1}|s_1,\ldots,a_{r_p}|s_p \text{ for all } i=1,\ldots,n,$$

and

$$\bar{a}_{r_1}|s_1,\ldots,\bar{a}_{r_p}|s_p >_h a_{r_1}|s_1,\ldots,a_{r_p}|s_p \text{ for some } i=h.$$

Definition 3. A social decision mechanism M is Pareto-optimal over the class $\mathcal{S}$ of societies if for each $S \in \mathcal{S}$ and every finite sequence $s_1,\ldots,s_p$ of states of S, we have

$$a_{r_1} \in M(S,s_1),\ldots,a_{r_p} \in M(S,s_p) \Rightarrow a_{r_1}|s_1,\ldots,a_{r_p}|s_p$$

is Pareto-optimal for S when the states $s_1,\ldots,s_p$ prevail.

Pareto-Optimal Mechanisms. We will try to determine now a class of social decision mechanisms $\mathcal{M}$ whose members M are Pareto-optimal over the class of societies $\mathcal{S}$ in the sense of Definition 3.

Since for each state s of S we have $e \in s$ and $e' \in s \Rightarrow \Rightarrow \lambda_i(e) = \lambda_i(e')$, for $i=1,\ldots,n$, we can write $\lambda_i(s) = \lambda_i(e)$ for some $e \in s$. Let $U^i(S)$ be the class of matrices of utility indexes corresponding to the member i of S whose existence we stated in Theorem 1.

We can now determine a social decision mechanism M of the class $\mathcal{M}$ as follows: For each society $S \in \mathcal{S}$ and every member i of S pick a matrix of utility indexes u^i from the class $U^i(S)$ and calculate

$$\sigma(a_r) = \sum_{i=1}^{n} u^i \; [r, \lambda_i(s)] \; .$$

M(S,s) is then defined by the following condition:

$$a_r \in M(S,s) \Longleftrightarrow a_r \text{ maximizes } \sigma(a_r).$$

By picking for each $S \in \mathcal{S}$ and every member i of S matrices u^i from the class $U^i(S)$ in all the possible ways, we obtain the class $\mathcal{M}$ of social decision mechanisms that we are trying to define.

Theorem 2. Every $M \in \mathcal{M}$ is Pareto-optimal over $\mathcal{S}$ in the sense of Definition 3.

Proof. Very easy, taking into account condition [α] that the entries of each matrix of utility indexes u^i of the class $U^i(S)$ must satisfy.

As we have claimed above, every social decision mechanism of the class $\mathcal{M}$ is Pareto-optimal over $\mathcal{S}$ in the sense of Definition 3. To select a particular M from the class $\mathcal{M}$ some ethical principle can be added. To illustrate how this can be done, we now choose a particular mechanism that we will call $\overline{M}$, from the class $\mathfrak{M}$, by picking for each society S and every member i of it a matrix of utility indexes $\overline{u}^i$ from the class $U^i(S)$ in such a way that: if S were to adopt for each of its states s a most preferred action from the point of view of the i^{th} member, then his expected utility would be 1, if S were to adopt for each of its states a least preferred action from the point of view of i, and there were no states in S for which the i^{th} member is indifferent with regard to the actions to be taken, then his expected utility would be 0. More explicit:

Let

$$\begin{bmatrix} u^i(1,1),\ldots,u^i(1,k_i) \\ \vdots \qquad\qquad \vdots \\ u^i(m,1),\ldots,u^i(m,k_i) \end{bmatrix} = u^i$$

be any given matrix of the class $U^i(S)$.

Let $u^i(r^j,j)$ and $u^i(r_j,j)$ be a greatest and a smallest, respectively, utility indexes of the j^{th} column of the matrix u^i.

The matrix $\left(\overline{u}^i(r,j)\right) = \left(cu^i(r,j) + b_j\right)$ to be used in the mechanism $\overline{M}$ is then uniquely determined by the following relations:

(i) If for some column j, $u^i(r^j,j) = u^i(r_j,j)$, take $\overline{u}(r^j,j) = cu^i(r^j,j) + b_j = 1.$

(ii) If $u^i(r^j,j) > u^i(r_j,j)$, take $\overline{u}^i(r_j,j) = cu^i(r_j,j) + b_j = 0.$

(iii) $$\sum_{j=1}^{k_i} \left[cu^i(r^j,j) + b_j\right] P(E_i^j) = 1.$$

<u>Remark</u>. It is interesting to note that the only information with regard to a society S and the state s of it that prevails, that

the mechanisms of the class $\mathcal{M}$ use to determine the action to be adopted is given by the utility indexes of each member i of S that correspond to the different actions when s prevails.

We will show that, if for each point (S,s) of the domain of the social decision mechanism each individual i reveals <u>only</u> (i) his event $\lambda_i^{-1}(s)$ that corresponds to the state s and (ii) how he orders all the possible corresponding sequences of actions given a sequence of events in which each different event $E_i^1,\ldots,E_i^{k_i}$ repeats itself a finite number of times h, no matter how big h is, this information is <u>insufficient</u> (in a sense that we will make precise in Theorem 3 below) to allow society to select actions that are Pareto-optimal in the sense of Definition 3.[5]

Let us now introduce the notion of impersonality that will be used in Theorem 3.

<u>Impersonality</u>. The intuitive idea of <u>impersonality</u> or <u>anonymity</u> or <u>neutrality</u> of a social decision mechanism is that the action that it assigns to society(in each situation) is independent of who is who. That is, the action assigned by the social decision mechanism depends only on the information conveyed by the individuals of the society without consideration to who conveyed what information. Thus, if in a society composed of two individuals, Mr. 1 and Mr. 2, in one situation Mr. 1 conveys a certain information, say Y_1, and Mr. 2 conveys Y_2, and the decision mechanism assigns an action a, the same action must be assigned by the mechanism if a situation comes where the person that we call Mr. 1 in this new situation conveys Y_2 and the person designated as Mr. 2 conveys Y_1.

To formalize this notion we need to develop new notation.

Let $Q_i(h_i^1,\ldots,h_i^{k_i})$ be the preference ordering of sequences of actions, that correspond to the sequence of events $E_i^1,\ldots,E_i^1,\ldots;$ $;\ldots;E_i^{k_i},\ldots,E_i^{k_i}$ containing h_i^1 times the event $E_i^1,\ldots,h_i^{k_i}$ times the event $E_i^{k_i}$.

Write $M[Q_1(h_1^1,\ldots,h_1^{k_1}),\ E_1^{j_1};\ldots;\ Q_n(h_n^1,\ldots,h_n^{k_n}),\ E_n^{j_n}]$ to represent a social decision mechanism in which each individual i (i=1,...,n) reveals only his preference ordering $Q_i(h_i^1,\ldots,h_i^{k_i})$ and the event $E_i^{j_i}$ that obtains for him. We can state now

<u>Definition 4</u>. A social decision mechanism $M[Q_1(h_1^1,\ldots,h_1^{k_1}),$ $E_1^{j_1};\ldots;Q_n(h_n^1,\ldots,h_n^{k_n}),E_n^{j_n}]$ is said to be <u>impersonal</u> if,

(a) $M[Q_1(h_1^1,\ldots,h_1^{k_1}),E_1^{j_1};\ldots;Q_i(h_i^1;\ldots;h_i^{k_i}),E_i^{j_i};\ldots;Q_\ell(h_\ell^1,\ldots,h_\ell^{k_\ell}),$
$E_\ell^{j_\ell};\ldots;Q_n(h_n^1,\ldots,h_n^{k_n}),E_n^{j_n}] =$

$= M[Q_{\pi(1)}(h_{\pi(1)}^1,\ldots,h_{\pi(1)}^{k_{\pi(1)}})E_{\pi(1)}^{j_{\pi(1)}};\ldots;Q_{\pi(i)}(h_{\pi(i)}^1,\ldots,h_{\pi(i)}^{k_{\pi(i)}})$
$E_{\pi(i)}^{j_{\pi(i)}};\ldots;Q_{\pi(\ell)}(h_{\pi(\ell)}^1,\ldots,h_{\pi(\ell)}^{k_{\pi(\ell)}}),E_{\pi(\ell)}^{j_{\pi(\ell)}};\ldots;Q_{\pi(n)}(h_{\pi(n)}^1,\ldots,$
$,\ldots,h_{\pi(n)}^{k_{\pi(n)}}),E_{\pi(n)}^{j_{\pi(n)}}]$

where π is any one to one function from the finite set $\{1,\ldots,i,\ldots,\ell,$ $,\ldots,n\}$ onto itself, i.e., $\pi(1),\ldots,\pi(i),\ldots,\pi(\ell),\ldots,\pi(n)$ is any permutation of the ordered set $\{1,\ldots,i,\ldots,\ell,\ldots,n\}$; and

(b) $M[Q_1(h_1^1,\ldots,h_1^{k_1}),\ E_1^{j_1};\ldots;Q_i(h_i^1,\ldots,h_i^{k_i}),\ E_i^{j_i};\ldots;$
$;\ldots;Q_\ell(h_\ell^1,\ldots,h_\ell^{k_\ell}),\ E_\ell^{\varphi(j_\ell)};\ldots;Q_n(h_n^1,\ldots,h_n^{k_n}),E_n^{j_n}] =$

$= M[Q_1(h_1^1,\ldots,h_1^{k_1}),E_1^{j_1};\ldots;Q_i(h_i^1,\ldots,h_i^{k_i}),E_i^{j_\ell};\ldots;$
$;\ldots;Q_\ell(h_\ell^1,\ldots,h_\ell^{k_\ell}),\ E_\ell^{\varphi(j_i)};\ldots;Q_n(h_n^1,\ldots,h_n^{k_n}),\ E_n^{j_n}].$

whenever,

(i) $k_i = k_\ell$,

(ii) there is a one-to-one function φ from the set $\{1,\ldots,k_\ell\}$ onto itself such that $h_\ell^{\varphi(1)} = h_i^1,\ldots,h_\ell^{\varphi(k_\ell)} = h_i^{k_i}$,

(iii) $Q_i(h_i^1,\ldots,h_i^{k_i}) = Q_\ell(h_\ell^{\varphi(1)},\ldots,h_\ell^{\varphi(k_\ell)})$.[6]

Part (a) of the definition of impersonality does not need additional explanation. Part (b) is somehow more difficult to interpret. It captures the idea that if two individuals labeled, say, i and ℓ in a society reveal "similar" preferences, in the sense that conditions (i), (ii) and (iii) of (b) are satisfied, then if two situations occur which are the same from the point of view of the rest of the individuals and Mr. i is in situation 2 (event $E_i^{j_\ell}$ obtains for him) in an "equivalent" position to that at which Mr. ℓ was in situation 1 (event $E_\ell^{\varphi(j_\ell)}$ obtained for him) and vice versa, society should adopt the same action in both cases.

Part (b) of our definition of impersonality could be seen as a formalization of a type of interpersonal comparison which according to Arrow[7] it is exemplified by the following inscription supposedly found in an English graveyard.

"Here lies Martin Engelbrodde,
Ha'e mercy on my soul, Lord God,
As I would do were I Lord God,
And Thou wert Martin Engelbrodde."[8]

We are now ready to present the generalization of Theorem 5 of [2] that we announced in the introduction.

For any natural number h greater than 0, let $\mathcal{M}(h)$ be the class of social decision mechanisms $M[Q_1(h_1^1,\ldots,h_1^{k_1}), E_1^{j_1};\ldots; Q_n(h_n^1,\ldots,h_n^{k_n}), E_n^{k_n}]$ which are impersonal in the sense of definition 4, and where $h_1^1 = \ldots = h_1^{k_1} = \ldots = h_n^1 = \ldots = h_n^{k_n} = h$. Thus for the mechanisms of the class $\mathcal{M}(h)$ each individual i reveals the event that obtains for him together with his preference ordering, of sequences of actions, that corresponds to the sequence of events containing h times each of the different events $E_i^1,\ldots,E_i^{k_i}$ that can obtain for him.

Theorem 3. For any fixed h, no matter how big, there is no social decision mechanism in the class $\mathcal{M}(h)$ which is Pareto-optimal in the sense of Definition 3.

Proof. It will suffice to exhibit two societies S and S′ of the class $\mathcal{S}$ and show that there is no social decision mechanism in $\mathcal{M}(h)$ that is Pareto-optimal over S and S′.

Both societies S and S′ have two members, 1 and 2; and the set of actions from where they can choose is the same $\{a_1,a_2\}$.

In society S two different events E_1^1, E_1^2 can obtain for individual 1,and two E_2^1, E_2^2 for individual 2; similarly in S′, $E_1'^1$, $E_1'^2$ can obtain for individual 1 and $E_2'^1$, $E_2'^2$ for individual 2. Thus, in society S we have the following states:

$$s_{11} = E_1^1 \cap E_2^1,\ s_{12} = E_1^1 \cap E_2^2,\ s_{21} = E_1^2 \cap E_2^1,\ s_{22} = E_1^2 \cap E_2^2 .$$

In society S′:

$$s'_{11} = E_1'^1 \cap E_2'^1,\ s'_{12} = E_1'^1 \cap E_2'^2,\ s'_{21} = E_1'^2 \cap E_2'^1,\ s'_{22} = E_1'^2 \cap E_2'^2.$$

It remains to specify now $Q_1(h,h)$ and $Q_2(h,h)$ in S; $Q_1'(h,h)$ and $Q_2'(h,h)$ in S'. This is done below. Let:

(i) for individual 1 of S, $a_1|E_1^1 >_1 a_2|E_1^1$, $a_2|E_1^2 >_1 a_1|E_1^2$;

(ii) for individual 2 of S, $a_1|E_2^1 >_2 a_2|E_2^1$, $a_2|E_2^2 >_2 a_1|E_2^2$;

(i′) for individual 1 of S′, $a_1|E_1'^1 >_1 a_2|E_1'^1$, $a_2|E_1'^2 >_1 a_1|E_1'^2$;

(ii′) for individual 2 of S′, $a_1|E_2'^1 >_2 a_2|E_2'^1$, $a_2|E_2'^2 >_2 a_1|E_2'^2$.

In view of the definition of the Rate of Substitution $R_i(k,\bar{k},\varphi_p,\varphi_q,\varphi_{\bar{p}},\varphi_{\bar{q}})$, we clearly have that: $R_1(1,2,1,2,2,1)$ and (i) determines $Q_1(h,h)$; $R_2(1,2,1,2,2,1)$ and (ii) determines $Q_2(h,h)$; $R_1'(1,2,1,2,2,1)$ and (i′) determines $Q_1'(h,h)$; $R_2'(1,2,1,2,2,1)$ and (ii′) determines $Q_2'(h,h)$.

Remark. Observe however that the knowledge of, say, $Q_1(h,h)$ does not suffice, in general, to determine $R_1(1,2,1,2,2,1)$.

Take $R_1(1,2,1,2,2,1)$ and $R_2(1,2,1,2,2,1)$ in such a way that: $R_1(1,2,1,2,2,1) < R_2(1,2,1,2,2,1)$ and, for any two natural numbers s and r, $R_1(1,2,1,2,2,1) < \frac{s}{r} < R_2(1,2,1,2,2,1)$ only if both s and r are greater than h.

Now take $R_1'(1,2,1,2,2,1) = R_2(1,2,1,2,2,1)$ and $R_2'(1,2,1,2,2,1) = R_1(1,2,1,2,2,1)$. Thus, $R_2'(1,2,1,2,2,1) < R_1'(1,2,1,2,2,1)$ and $R_2'(1,2,1,2,2,1) < \frac{s}{r} < R_1'(1,2,1,2,2,1)$ only if both s and r are greater than h. It is easy to see, taking into account the Rate of Substitution Axiom, that, given the values assigned to $R_1(1,2,1,2,2,1)$, $R_2(1,2,1,2,2,1)$, $R_1'(1,2,1,2,2,1)$ and $R_2'(1,2,1,2,2,1)$, $Q_1(h,h) = Q_2(h,h) = Q_1'(h,h) = Q_2'(h,h)$.

The possible values of a social decision mechanism $M \in \mathcal{M}(h)$ at the point $[Q_1(h,h),E_1^1;\ Q_2(h,h),E_2^2]$ are: $\{a_1\}$, $\{a_2\}$, $\{a_1,a_2\}$. Assume that $M[Q_1(h,h),E_1^1;\ Q_2(h,h),E_2^2] = \{a_1\}$. Then, since M is impersonal in the sense of Definition 4 above, we have:

$$M[Q_1(h,h),E_1^1;\ Q_2(h,h),E_2^2] = M[Q_1(h,h),E_1^2;\ Q_2(h,h),E_2^1] =$$
$$= M[Q_1'(h,h),E_1'^2;\ Q_2'(h,h),E_2'^1] = M[Q_1'(h,h),E_1'^1;\ Q_2'(h,h),E_2'^2] = \{a_1\}.$$

Now suppose that a sequence of states comes along in society S containing r times the state s_{12}, and s times the state s_{21}:

$$\overbrace{s_{12},\ldots\ldots,s_{12}}^{r\text{ times}};\ \overbrace{s_{21},\ldots\ldots,s_{21}}^{s\text{ times}}.$$

The corresponding sequence of actions generated by M is

$$\overbrace{a_1,\ldots\ldots,a_1}^{r\text{ times}};\ \overbrace{a_1,\ldots\ldots,a_1}^{s\text{ times}}.$$

Thus we obtain the sequence

$$(1)\quad \overbrace{a_1|s_{12},\ldots\ldots,a_1|s_{12}}^{r\text{ times}};\ \overbrace{a_1|s_{21},\ldots\ldots,a_1|s_{21}}^{s\text{ times}}.$$

Sequence (1) is viewed from Mr. 1's point of view as the sequence

$$(1_1)\quad \overbrace{a_1|E_1^1,\ldots\ldots,a_1|E_1^1}^{r\text{ times}};\ \overbrace{a_1|E_1^2,\ldots\ldots,a_1|E_1^2}^{s\text{ times}}.$$

From the point of view of Mr. 2 as the sequence:

$$(1_2)\quad \overbrace{a_1|E_2^2,\ldots\ldots,a_1|E_2^2}^{r\text{ times}};\ \overbrace{a_1|E_2^1\ \ldots\ldots,a_1|E_2^1}^{s\text{ times}}.$$

Now consider the sequence

$$(1^*) \quad \overbrace{a_2|s_{12}, \ldots\ldots, a_2|s_{12}}^{r \text{ times}};\ \overbrace{a_2|s_{21}, \ldots\ldots, a_2|s_{21}}^{s \text{ times}},$$

which from Mr. 1's view becomes,

$$(1_1^*) \quad \overbrace{a_2|E_1^1, \ldots\ldots, a_2|E_1^1}^{r \text{ times}};\ \overbrace{a_2|E_1^2, \ldots\ldots, a_2|E_1^2}^{s \text{ times}}$$

and from Mr. 2's view is

$$(1_2^*) \quad \overbrace{a_2|E_2^2, \ldots\ldots, a_2|E_2^2}^{r \text{ times}};\ \overbrace{a_2|E_2^1, \ldots\ldots, a_2|E_2^1}^{s \text{ times}}.$$

Since $R_1(1,2,1,2,2,1) < \frac{s}{r} < R_2(1,2,1,2,2,1)$, we have:

(i) $(1_1^*) >_1 (1_1)$, by part (a) of the Rate of Substitution Axiom;

(ii) $(1_2^*) >_2 (1_2)$, by part (b) of the Rate of Substitution Axiom.

From (i) and (ii) it follows that M is not Pareto-optimal, in the sense of Definition 3, on S.

If we take $M[Q_1(h,h),E_1^1;\ Q_2(h,h),E_2^2] = \{a_2\}$, we can show in a similar way that M is not Pareto-optimal on S$'$. Finally, if we take $M[Q_1(h,h),E_1^1;\ Q_2(h,h),E_2^2] = \{a_1,a_2\}$, we can see that the mechanism M is not Pareto-optimal either on S or on S$'$. Q.E.D.

4. CARDINALITY VERSUS ORDINALITY IN THE PROBLEM OF SOCIAL CHOICE

As we mentioned in the introduction to this article, any satisfactory argument defending the use of cardinal utility has to be convincing on two grounds. First, the set of axioms on the preference ordering of the individuals must be plausible in the sense that they (the axioms) do not impose unrealistic restrictions on the individual preference orderings. Second, it must be shown that there are definite advantages in using cardinal utility indexes.

We believe that the axioms presented in this article, and that were extensively illustrated in [2], do not impose unrealistic restrictions on the preference orderings of the individuals. We will not discuss this matter further here. We will concentrate rather on showing the advantages that exist in using our cardinal utility indexes to construct and operate Pareto-optimal decision mechanisms that satisfy also the mild requirement of impersonality.

Consider the class of societies $\mathcal{S}$ and let us try to describe how the mechanisms of the class $\mathcal{M}$ can be put into operation to determine the action that any society of $\mathcal{S}$ should take whenever any of its possible states comes along. Pick the mechanism $\overline{M}$ of $\mathcal{M}$, obtained in Section 3, which is impersonal. To operate $\overline{M}$ on any society S of $\mathcal{S}$ we need only ask each member i of S to calculate his matrix of utility indexes $\overline{u^i}$, and then to reveal, when each state of society comes along, the utility indexes that correspond to the different actions when that state of society prevails. For instance, each individual might be instructed to feed to a computer these utility indexes. Then the computer can be programmed to add up, for each action, the utility indexes of the different individuals of the society and to select one action with the highest sum. Now suppose that after society S, having gone through a finite number of states, changes to a new society, say S′ (some people of S may disappear and some new people may join the remaining members of S). The only thing we need to do is to instruct the new members (the old ones already know how to proceed) to feed to our computer, when each state of S′ comes along, their utility indexes that correspond to the different actions. This mechanism, since it is a member of $\mathcal{M}$, is Pareto-optimal in the sense of Definition 3. Consequently, after societies S and S′ have gone through any finite number of states, we can be sure that the actions generated by $\overline{M}$ are such that no other actions could have been generated that had made any individual of S or S′ better off without having hurt somebody else at the same time.

In Section 3 we proved (Theorem 3) that if for each point (S,s) of the domain of the social decision mechanism each individual i reveals only (i) his event $\lambda_i^{-1}(s)$ that corresponds to the state s and (ii) his preference ordering, of sequences of actions, that corresponds to a sequence of events containing a finite number of times h, no matter how big h is, each of his different events $E_i^1,\ldots,E_i^{k_i}$, this information is insufficient in the sense that among the class of mechanisms $\mathcal{M}(h)$ that are based only on this information, and that are impersonal, there is no one which is Pareto-optimal over $\mathcal{S}$. Thus, if we want decision mechanisms which are impersonal and Pareto-optimal over $\mathcal{S}$, and we do not want to use the cardinal utility indexes derived in this paper, we have to require that for each (S,s) of the domain of the mechanism each individual i reveal, together with the event $\lambda_i^{-1}(s)$ that obtains for him, how he orders infinite sequences of actions, given an infinite sequence of events containing infinite many of each of his possible different events. We consider that this is operationally infeasible and hence our claim regarding the inadequacy of the ordinalist approach in finding solutions to the problem of social choice.

It is worth noting the similarity between the argument presented here and that developed by the advocates of decentralization during the controversy regarding centralization and decentralization as alternative ways of organizing the economic activity. In that controversy, it was argued by the advocates of decentralization that: you can achieve an allocation of resources that is Pareto-optimal (in the static sense, of course) through the, operationally rather simple, decentralized price mechanism; or alternatively, in a centralized and computationally very complicated way, by asking consumers, producers, and resource owners, to transmit to a central agency, respectively, their preference orderings, their production functions, and the amounts of resources owned, and have then the central agency calculate a Pareto-optimal allocation of resources. In our case we have argued similarly that you can achieve Pareto-optimality (in the sense of our Definition 3) in a very simple way by using our cardinal utility indexes; or alternatively, by going through the rather complicated procedure of requiring that each individual communicate to a kind of central agency, together with the event that prevails for him, how he orders infinite sequences of actions, given an infinite sequence of events containing infinite many of each of the different events that can obtain for him.

Several authors have argued (e.g. Frisch [4], Shapley [6]), in different contexts, for the need to consider cardinal utility. The argument presented in this article is new to the best of our knowledge.

Footnotes

1. For a nice summary of the main results of these two approaches, see the paper by Andreu Mass-Colell and Hugo Sonnenschein, "General Possibility Theorems for Group Decisions," The Review of Economic Studies, Vol. XXXIX (2), No. 118, (April 1972), pp.185-92.

2. This seems to be the spirit of the argument contained in the paper by Duncan Black, "On Arrow's Impossibility Theorem," the Journal of Law and Economics, Vol. XII (2) (October, 1969), pp. 227-48.

3. What we call in this paper "the environment," was designated in [2] as "state of the outside world."

4. For illustrations regarding the meaning of the axioms presented in this paper and other clarifying examples, see [2].

5. We should note here that in Theorem 5 of [2], for each point (S,s) each individual i was required to reveal only how he ordered the different actions given the event $\lambda_i^{-1}(s)$.

6. $Q_\ell(h_\ell^{\varphi(1)},\ldots,h_\ell^{\varphi(k_\ell)}$ represented obviously the preference ordering of Mr. ℓ, of sequences of actions, that corresponds to the sequence of events $E_\ell^{\varphi(1)},\ldots,E_\ell^{\varphi(1)};\ldots;E_\ell^{\varphi(k_\ell)},\ldots,E_\ell^{\varphi(k_\ell)}$ containing $h_\ell^{\varphi(1)}$ times the event $E_\ell^{\varphi(1)},\ldots,h_\ell^{\varphi(k_\ell)}$ times the event $E_\ell^{\varphi(k_\ell)}$.

7. See Arrow [1], p. 114.

8. The difficulty with this type of interpersonal comparison is that to be applicable, as formalized in part (b) of our definition of impersonality, the individuals need to have "similar" preferences in the sense stated by conditions (i), (ii) and (iii) of (b). This author wonders if the troubles of certain societies in the real world in reaching social decisions do not

lie on the fact that different groups of these societies have so different preferences that the type of interpersonal comparison formalized in part (b) of Definition 4 cannot be applied. They simply cannot understand each other!

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