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A GENERALIZED THEOREM OF THE MAXIMUM

by

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and

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This paper generalizes the Theorem of the Maximum (Berge, 1963) to allow for discontinuous changes in the domain and the objective function. It also provides a geometrical version of the (generalized) theorem.

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1. Introduction.

Almost every economic problem involves the study of an agent's optimal choice as a function of certain parameters or state variables. For example, demand theory is concerned with an agent's optimal consumption as a function of prices and income, while capital theory studies the optimal investment rule as a function of the existing capital stock.

A central tool in the analysis of such problems is Berge's (1963) Theorem of the Maximum. This theorem establishes the continuity of the optimal value and the upper hemicontinuity of the optimal choice (in the parameter or state variable). The former property is often employed in existence and characterization theorems for dynamic programming (see, e.g., Denardo, 1967); the latter property is used in applying fixed point theorems of the Kakutani variety (see, e.g., Debreu, 1959).

Berge's formulation of the maximum theorem requires that the objective function be jointly continuous in the parameter and the choice, and that the choice set be continuous in the parameter:

Theorem of the Maximum (Berge, 1963): Let X and Λ be topological spaces with X regular, let $f: X \times \Lambda \to \mathfrak{R}$ be a continuous function, and let $\gamma: \Lambda \to X$ be a continuous correspondence that is nonempty- and compact-valued. Then:

(i) The function $M: \Lambda \to \mathfrak{R}$ defined by $M(\lambda) = \max \{f(x,\lambda): x \in \gamma(\lambda)\}$ is continuous; and

(ii) the correspondence $m: \Lambda \to X$ defined by $m(\lambda) = \{x \in \gamma(\lambda): f(x,\lambda) = M(\lambda)\}$ is nonempty- and compact-valued and u.h.c.

Thus, Berge's theorem is applicable for objective functions situated as in Figure 1a. (For the present discussion of Figure 1, the choice $\gamma(\lambda)$ will be taken to be the closed interval [0,1] and will be suppressed.) Depicted in Figure 1a are $f(\bullet,\lambda_1)$ and $f(\bullet,\lambda_2)$, where λ_1 and λ_2 are nearby parameter values and each $f(\bullet,\lambda_i)$ is continuous in x. Berge's theorem establishes that the optimal values, $M(\lambda_1)$ and $M(\lambda_2)$, and the optimal choices, $m(\lambda_1)$ and $m(\lambda_2)$, are near each other.[1]

INSERT FIGURE 1 ABOUT HERE

Consider now objective functions situated as in Figure 1b. The functions $f(\bullet,\lambda_i)$ obviously are no longer continuous in x; furthermore, for fixed x, $f(x,\bullet)$ is no longer continuous in λ. Clearly, the hypotheses of Berge's theorem are not satisfied. Nevertheless, there is no fundamental reason why the Theorem of the Maximum should fail. Indeed, it should be observed that the optimal values, $M(\lambda_1)$ and $M(\lambda_2)$, and the optimal choices, $m(\lambda_1)$ and $m(\lambda_2)$, are still near each other.

In a nice paper, Leininger (1984) recognized that the upper hemicontinuity of the set of maximizers and the upper semicontinuity of the value function could be established under conditions weaker than a continuous objective function. For metric spaces, Leininger proved that if the choice set is continuous, the objective function is u.s.c., and an additional condition called "graph-continuity" is satisfied, then $m(\bullet)$ is u.h.c. and $M(\bullet)$ is u.s.c. The contribution of our paper is to provide a more complete generalization of the Theorem of the Maximum. First, we require the choice set merely to be u.h.c. and we weaken the condition of graph-continuity. Second, even under our weaker hypotheses, we conclude that the value function must in fact be continuous. Third, we prove the generalized result for the

[1]Since, in the examples of Figure 1, the optimal choice is single-valued, the upper hemicontinuity of the optimal choice **correspondence** implies the continuity of the optimal choice.

same general topological space setting as Berge's theorem. Finally, we provide a version of the result which has a geometrical interpretation close in spirit to the original Theorem of the Maximum.

Section 2 introduces the necessary concepts and proves the main theorems. Section 3 presents the geometrically more intuitive version of the result. Section 4 concludes with an example.

2. The Main Theorems.

Let (Y,T) be any topological space and let $P_0(Y)$ be the space of nonempty subsets of Y. We will utilize the (upper and lower) Vietoris topologies on $P_0(Y)$. The ***upper Vietoris topology*** on $P_0(Y)$, denoted by T_U, is defined as the coarsest topology with the property that, for every nonempty open subset G of Y, the set $[\bullet,G] \equiv \{U \in P_0(Y)\colon U \subset G\}$ is an open set. The ***lower Vietoris topology*** on $P_0(Y)$, denoted by T_L, is defined as the coarsest topology with the property that, for every closed subset F of Y, the set $[\bullet,F]$ is a closed set. The ***Vietoris topology*** on $P_0(Y)$, denoted by T_V, is defined as the coarsest topology that contains both the upper and lower Vietoris topologies. It is useful to observe that the collection $\{[\bullet,G]\colon G \in T\}$ is a basis for T_U. Meanwhile, let us define $I_G \equiv \{U \in P_0(Y)\colon U \cap G \neq \emptyset\}$, for every nonempty open subset G of Y. Then the collection $\{I_G\colon G \in T\}$ is a subbasis for T_L. The raison d'être for these topologies is the following. Suppose that Z is a topological space and g is a correspondence from Z to Y. Then g is upper hemicontinuous (lower hemicontinuous, continuous) if and only if, when viewed as a function from Z to $P_0(Y)$, g is continuous in T_U (T_L, T_V). For a detailed reference on these and further properties, see Klein and Thompson (1984, especially Section 1.3).

Our initial use of the Vietoris topologies will be to state necessary and sufficient conditions under which the value function is continuous. We need to define the following sets and functions. Let $\mathfrak{R}^{**}$ denote the extended real numbers $\mathfrak{R} \cup \{-\infty\} \cup \{+\infty\}$ with the topology generated by intervals of the form $[-\infty,a)$, (a,b) and $(b,+\infty]$, where $a,b \in \mathfrak{R}$. Let the "***subgraph***" $E(\lambda) = \{(x,y) \in X \times \mathfrak{R}: x \in \gamma(\lambda)$ and $y \leq f(x,\lambda)\}$ be the set of all points on or below the graph of $f(\bullet,\lambda)$, i.e., the flip side of the epigraph of $f(\bullet,\lambda)$. Let $\Pi(\lambda) = \{y \in \mathfrak{R}: (x,y) \in E(\lambda)$ for some $x \in X\}$ be the projection of $E(\lambda)$ onto the second coordinate and let $\overline{\Pi}(\lambda) = \text{cl}\, \Pi(\lambda)$. In the following theorem, we will view $\overline{\Pi}(\bullet)$ as a function from Λ to $P_0(\mathfrak{R})$.

Theorem 1: Let X and Λ be topological spaces, $f: X \times \Lambda \to \mathfrak{R}$ be a function, and $\gamma: \Lambda \to X$ be a correspondence. Define the function $M: \Lambda \to \mathfrak{R}^{**}$ by $M(\lambda) = \sup\{f(x,\lambda): x \in \gamma(\lambda)\}$. Then:

(a) $M(\bullet)$ is u.s.c. if and only if $\overline{\Pi}: \Lambda \to P_0(\mathfrak{R})$ is continuous in the upper Vietoris topology;

(b) $M(\bullet)$ is l.s.c. if and only if $\overline{\Pi}: \Lambda \to P_0(\mathfrak{R})$ is continuous in the lower Vietoris topology; and

(c) $M(\bullet)$ is continuous if and only if $\overline{\Pi}: \Lambda \to P_0(\mathfrak{R})$ is continuous in the Vietoris topology.

Proof: (a) For any $t \in \mathfrak{R}^{**}$, define $U_t = \{y \in \mathfrak{R}: y < t\}$. Observe that $\{\lambda \in \Lambda: M(\lambda) < t\} \equiv \{\lambda \in \Lambda: \overline{\Pi}(\lambda) \in [\bullet,U_t]\}$, since the definition of each set requires that $f(\bullet,\lambda) < t - \epsilon$, for some $\epsilon > 0$. Suppose $\overline{\Pi}(\bullet)$ is continuous. Since $[\bullet,U_t]$ is open in the upper Vietoris topology, the set $\{\lambda \in \Lambda: M(\lambda) < t\} = \overline{\Pi}^{-1}([\bullet,U_t])$ is open for all t, implying that $M(\bullet)$ is u.s.c. Conversely, suppose $M(\bullet)$ is u.s.c. For any open set $S \subset \mathfrak{R}$, define $t(S) = \sup\{t \in \mathfrak{R}^{**}: U_t \subset S\}$. Observe, since $\overline{\Pi}(\lambda)$ is of the form $\{y \in \mathfrak{R}: y \leq r\}$ for every λ (where $r \in \mathfrak{R}^{**}$), that $\overline{\Pi}^{-1}([\bullet,S]) = \overline{\Pi}^{-1}([\bullet,U_{t(S)}])$. The u.s.c. of $M(\bullet)$

implies that the latter set is open, establishing the continuity of $\overline{\Pi}(\bullet)$.

(b) For any $t \in \mathfrak{R}^{**}$, define $V_t = \{y \in \mathfrak{R}: y > t\}$. Observe that $\{\lambda \in \Lambda: M(\lambda) > t\} \equiv \{\lambda \in \Lambda: \overline{\Pi}(\lambda) \in I_{V_t}\}$. Suppose $\overline{\Pi}(\bullet)$ is continuous. Since I_{V_t} is open in the lower Vietoris topology, $\{\lambda \in \Lambda: M(\lambda) > t\} = \overline{\Pi}^{-1}(I_{V_t})$ is open for all t, implying that $M(\bullet)$ is l.s.c. Conversely, suppose $M(\bullet)$ is l.s.c. For any open set $S \subset \mathfrak{R}$, define $t'(S) = \inf\{t \in \mathfrak{R}: t \in S\}$. Observe that $\overline{\Pi}^{-1}(I_S) = \overline{\Pi}^{-1}(I_{V_{t'(S)}})$. The l.s.c. of $M(\bullet)$ implies that the latter set is open, establishing the continuity of $\overline{\Pi}(\bullet)$.

(c) Follows trivially from (a) and (b). Q.E.D.

Some remarks on Theorem 1 are in order. First, part (b) of the theorem remains true if $\overline{\Pi}(\bullet)$ is replaced by $\Pi(\bullet)$. However, the same substitution makes part (a) false. Second, suppose we view $\overline{\Pi}(\bullet)$ as a correspondence from Λ to $\mathfrak{R}$. Then part (a) may be restated: $M(\bullet)$ is u.s.c. if and only if $\overline{\Pi}(\bullet)$ is u.h.c. Parts (b) and (c) may be restated analogously.

Third, although Theorem 1 is quite straightforward, it implies the known results of continuity of the value (Berge, 1963). If $f(\bullet,\bullet)$ is continuous in (x,λ) and $\gamma(\bullet)$ is continuous in λ, then clearly $\overline{\Pi}(\bullet)$ is continuous in λ, establishing the continuity of $M(\bullet)$. Also, if $f(\bullet,\bullet)$ is merely u.s.c. and $\gamma(\bullet)$ is u.h.c., it easily follows that $M(\bullet)$ is u.s.c. Indeed, for every t, the inverse image $\overline{\Pi}^{-1}([\bullet,U_t])$ is open since its complement, $\{\lambda \in \Lambda: f(x,\lambda) \geq t$ for some $x \in \gamma(\lambda)\}$ is the projection of the closed set $\{(x,\lambda): f(x,\lambda) \geq t\} \cap \{(x,\lambda): x \in \gamma(\lambda)\}$ onto the second coordinate. It can also be shown that if $f(\bullet,\bullet)$ is l.s.c. and $\gamma(\bullet)$ is l.h.c., then $M(\bullet)$ is l.s.c.

We have so far only addressed the continuity of the value function. We now turn to a full statement of the maximum theorem, which also concerns the upper hemicontinuity of the maximizing correspondence $m(\bullet)$.

<u>Theorem 2</u>: Let X be a regular topological space, Λ be a topological space, and $\gamma\colon \Lambda \to X$ be a u.h.c. correspondence that is nonempty- and compact-valued. Suppose $f\colon X \times \Lambda \to \mathfrak{R}$ is a u.s.c. function and $\Pi\colon \Lambda \to P_0(\mathfrak{R})$ is continuous in the lower Vietoris topology. Then: (a) $M(\bullet)$ is a continuous function; and (b) $m(\bullet)$ is a u.h.c. correspondence.

<u>Proof</u>: The upper semicontinuity of $f(\bullet,\bullet)$ and upper hemicontinuity of $\gamma(\bullet)$ imply that $\Pi(\bullet)$ is continuous in the upper Vietoris topology. Theorem 1 then establishes that $M(\bullet)$ is continuous.

Let $(x_n,\lambda_n)_{n\in D}$ be any net converging to (x,λ), with $x_n \in \gamma(\lambda_n)$ and $f(x_n,\lambda_n) = M(\lambda_n)$. Since X is regular and $\gamma(\bullet)$ is u.h.c. and closed-valued, the correspondence $\gamma(\bullet)$ is closed (Klein and Thompson, 1984, Theorem 7.1.15), and therefore $x \in \gamma(\lambda)$. By the u.s.c. of $f(\bullet,\bullet)$, and the continuity of $M(\bullet)$, $f(x,\lambda) \geq \overline{\lim}\, f(x_n,\lambda_n) = M(\lambda)$. This proves that $x \in m(\lambda)$ and, hence, that $m(\bullet)$ is closed. Since $m(\lambda) = m(\lambda) \cap \gamma(\lambda)$ and $\gamma(\bullet)$ is u.h.c. and compact valued, we conclude (Klein and Thompson, 1984, Theorem 7.3.10(ii)) that $m(\bullet)$ is u.h.c.

Q.E.D.

It should be observed that Theorem 2 does not require $f(\bullet,\bullet)$ to be l.s.c. nor $\gamma(\bullet)$ to be l.h.c. Instead, we replace these hypotheses with the condition that $\Pi(\bullet)$ is continuous in the lower Vietoris topology. We do, however, maintain the assumption that $f(\bullet,\bullet)$ is u.s.c., assuring that the supremum of $f(\bullet,\lambda)$ is attained.

It is instructive to compare part (b) of Theorem 2 with Leininger's (1984) theorem. For metric spaces X and Λ, Leininger showed the upper hemicontinuity of $m(\bullet)$ when $f(\bullet,\bullet)$ is u.s.c., $\gamma(\bullet)$ is continuous, and a condition he called "graph-continuity" is satisfied. The latter condition may be restated: for every $\lambda \in \Lambda$, $x \in \gamma(\lambda)$ and $\epsilon > 0$, there exists a neighborhood O of λ with the property that $\lambda' \in O$ implies the existence of $x' \in \gamma(\lambda')$ such that $|f(x',\lambda') - f(x,\lambda)| < \epsilon$. In contrast, we establish the maximum theorem for topological spaces and only require that $\gamma(\bullet)$ be u.h.c. We replace "graph-continuity" with the hypothesis that $\Pi(\bullet)$ is continuous in the lower Vietoris topology, which may equivalently be stated: for every $\lambda \in \Lambda$, $x \in \gamma(\lambda)$ and $\epsilon > 0$, there exists a neighborhood O of λ with the property that $\lambda' \in O$ implies the existence of $x' \in \gamma(\lambda')$ such that $f(x',\lambda') > f(x,\lambda) - \epsilon$. Obviously, then, graph-continuity implies continuity of $\Pi(\bullet)$ in the lower Vietoris topology, but not the reverse.

3. A Geometrical Version of the Theorem.

The main hypothesis of the traditional Theorem of the Maximum, that $f(\bullet,\bullet)$ is jointly continuous, may be reinterpreted as requiring that $f(\bullet,\lambda)$ have a continuous graph and that the graph of $f(\bullet,\lambda)$ change continuously in λ. (See, for example, Figure 1a.) As we have seen in the previous section, it is unnecessary that $f(\bullet,\lambda)$ have a continuous graph for the conclusions of the maximum theorem to hold. In this section, we will establish a precise sense in which it is sufficient to show that the graph[2] of $f(\bullet,\lambda)$ changes continuously in λ. Theorem 3 will, for example, make it obvious on casual inspection that the objective function depicted in Figure 1b exhibits the desired continuity of the optimal value and upper hemicontinuity of the optimal choice.

Suppose that $f\colon X \times \Lambda \to \mathfrak{R}$ is u.s.c. in x at all $\lambda \in \Lambda$. Let $\mathfrak{R}^*$ denote the half-extended real numbers $\mathfrak{R} \cup \{-\infty\}$ with the topology generated by

[2] Or, more precisely, the closure of the graph, then convexified in the direction of the range. This is defined as $\tilde{G}(\bullet)$, below.

intervals of the form $[-\infty,a)$ and (a,b), where $a,b \in \mathfrak{R}$. Let $G(\lambda) = \{(x,f(x,\lambda)): x \in \gamma(\lambda)\}$ denote the graph of $f(\bullet,\lambda)$ restricted to $\gamma(\lambda)$. Let $\overline{G}(\lambda)$ be the closure of $G(\lambda)$ in the space $X \times \mathfrak{R}^*$ and let $\tilde{G}(\lambda) = \{(x,y):$ there exist $y',y'' \in \mathfrak{R}^*$ satisfying $y'' \le y \le y'$, $(x,y') \in \overline{G}(\lambda)$ and $(x,y'') \in \overline{G}(\lambda)\}$. Observe that, under the hypotheses of Theorem 3, $\tilde{G}(\lambda)$ is closed. Indeed, $\tilde{G}(\lambda)$ is the smallest u.h.c., convex-valued correspondence that contains $G(\lambda)$. Alternatively, $\tilde{G}(\lambda)$ can be interpreted as the (upper) boundary of the subgraph $E(\lambda)$. We may now state our result:

<u>Theorem 3</u>: Let X be a regular topological space, Λ be a topological space, and $\gamma: \Lambda \to X$ be a nonempty- and compact-valued correspondence. Suppose $f: X \times \Lambda \to \mathfrak{R}$ is a u.s.c. function of x, for every $\lambda \in \Lambda$, and suppose $\tilde{G}: \Lambda \to P_0(X \times \mathfrak{R}^*)$ is continuous in the Vietoris topology. Then:
(a) $M(\bullet)$ is a continuous function; and (b) $m(\bullet)$ is a u.h.c. correspondence.

<u>Proof</u>: We will demonstrate that the hypotheses of Theorem 2 are satisfied. Define the functions $\pi_1: X \times \mathfrak{R}^* \to X$ and $\pi_2: X \times \mathfrak{R}^* \to \mathfrak{R}^*$ to be the projections onto the first and second coordinates, respectively.

Since $\gamma(\bullet)$ is closed, observe that $\gamma = \pi_1 \circ \tilde{G}$. The composition of two continuous correspondences is continuous (Klein and Thompson, 1984, Theorem 7.3.11), implying that $\gamma(\bullet)$ is a continuous correspondence.

Similarly, $\beta = \pi_2 \circ \tilde{G}$ is continuous in the Vietoris topology of $P_0(\mathfrak{R}^*)$. Let $V_t = \{y \in \mathfrak{R}: y > t\}$, $I_{V_t} = \{U \in P_0(\mathfrak{R}): U \cap V_t \neq \emptyset\}$, and $I'_{V_t} = \{U \in P_0(\mathfrak{R}^*): U \cap V_t \neq \emptyset\}$. To demonstrate that $\overline{\Pi}(\bullet)$ is continuous in the lower Vietoris topology of $P_0(\mathfrak{R})$, it is sufficient to show (as in the proof of Theorem 1(b)) that inverse images of sets I_{V_t} are open. But observe that $\overline{\Pi}(\lambda) = \{y \in \mathfrak{R}: y \le y'$ for some $y' \in \beta(\lambda)\}$, implying that $\overline{\Pi}^{-1}(I_{V_t}) = \beta^{-1}(I'_{V_t})$, which is open from the continuity of β.

Finally, we will show that, on the graph of $\gamma(\bullet)$, $f(\bullet,\bullet)$ is u.s.c. jointly in x and λ. For every $\lambda \in \Lambda$, $x \in \gamma(\lambda)$, and $\epsilon > 0$, define $F_\epsilon = \{z \in \gamma(\lambda): f(z,\lambda) \geq f(x,\lambda) + \epsilon\}$. By the u.s.c. of $f(\bullet,\lambda)$ in x, F_ϵ is closed. Since $x \notin F_\epsilon$ and X is regular, there exist disjoint open sets O_ϵ and O'_ϵ such that $F_\epsilon \subset O_\epsilon$ and $x \in O'_\epsilon$. Define $W_\epsilon = \{y \in \Re^*: y \geq f(x,\lambda) + \epsilon\}$ and $K_\epsilon = (X\backslash O_\epsilon) \times W_\epsilon$. Note that K_ϵ is the product of two closed sets and hence is closed. Finally, define $L_\epsilon = (X \times \Re^*)\backslash K_\epsilon$. By construction, L_ϵ is open and contains $\tilde{G}(\lambda)$. Then $[\bullet,L_\epsilon]$ is a neighborhood of $\tilde{G}(\lambda)$ in the upper Vietoris topology. Let $(x_n,\lambda_n)_{n\in D}$ be any net converging to (x,λ). By the continuity of $\tilde{G}(\bullet)$, there exists $N^1_\epsilon \in D$ such that $\tilde{G}(\lambda_n) \in [\bullet,L_\epsilon]$ for every $n \geq N^1_\epsilon$. Since O'_ϵ is open, there exists $N^2_\epsilon \in D$ such that $x_n \in O'_\epsilon \subset X\backslash O_\epsilon$ for every $n \geq N^2_\epsilon$. Hence, $(x_n,y_n) \in \tilde{G}(\lambda_n)$ implies that $y_n \notin W_\epsilon$, whenever $n \geq N^1_\epsilon$ and $n \geq N^2_\epsilon$. This shows, for every $\epsilon > 0$, that $f(x_n,\lambda_n) < f(x,\lambda) + \epsilon$, and hence that $\overline{\lim}\, f(x_n,\lambda_n) \leq f(x,\lambda)$, establishing u.s.c. Q.E.D.

Suppose, additionally, that X is a metric space and $f(\bullet,\bullet)$ is uniformly bounded below. Since the familiar Haussdorff topology is now defined on $P_0(X \times \Re)$ and $\tilde{G}(\lambda)$ is now compact, continuity of $\tilde{G}(\bullet)$ in the Vietoris topology is equivalent to continuity of $\tilde{G}(\bullet)$ in the Haussdorff topology. This enables us to provide the most vivid interpretation of the hypothesis of Theorem 3. For any $S \in P_0(X \times \Re)$, define the ***ϵ-fattening*** of S as $S + \epsilon = \{z' \in X \times \Re: d(z,z') < \epsilon$ for some $z \in S\}$. Continuity of $\tilde{G}(\bullet)$ may now be restated: for every $\lambda \in \Lambda$ and $\epsilon > 0$, there exists a neighborhood O of λ such that $\lambda' \in O$ implies $\tilde{G}(\lambda') \subset \tilde{G}(\lambda) + \epsilon$ and $\tilde{G}(\lambda) \subset \tilde{G}(\lambda') + \epsilon$. In other words, every point on the closed, convexified graph of $f(\bullet,\lambda')$ lies within ϵ of a point on the closed, convexified graph of $f(\bullet,\lambda)$, and vice-versa. This is the precise sense in which the graph of $f(\bullet,\lambda)$ is required to vary continuously in the parameter.

It is useful to compare our notion of "nearness" of functions with that used in the traditional Theorem of the Maximum. Two functions $g(\bullet)$ and $h(\bullet)$ are close to one another in the uniform topology if the graph of $g(\bullet)$ can be carried onto the graph of $h(\bullet)$ by a (uniformly) small perturbation in the direction of the range. (Again see Figure 1a.) In contrast, we also permit a (uniformly) small deformation of the domain. More precisely, if X is a metric space,[3] f and g are close in the sense of Theorem 3 if $\sup_{x\in X} |f(x) - g(\tau(x))| < \epsilon$, where $\tau(\bullet)$ is a uniformly small perturbation of the identity on X, i.e., $\sup_{x\in X} \|\tau(x) - x\| < \epsilon$.

This is illustrated by the example from which Figure 1b was drawn. For $x,\lambda \in [0,1]$, define:

$$f_1(x,\lambda) = \begin{cases} x + \lambda, & \text{if } x \le \lambda \\ -x + \lambda, & \text{if } x > \lambda . \end{cases}$$

Then $\lim_{\lambda' \to \lambda} \sup_{x \in [0,1]} |f_1(x,\lambda') - f_1(x,\lambda)| = 2\lambda$, but the Haussdorff distance between $\bar{G}(\lambda')$ and $\bar{G}(\lambda)$ converges to zero as $\lambda' \to \lambda$. Since $f_1(\bullet,\lambda)$ is also u.s.c. in x, it satisfies the hypotheses of Theorem 3; indeed, $M(\lambda) = 2\lambda$ and $m(\lambda) = \lambda$ are continuous in the parameter.

In the previous example, we exploited the fact that if $\bar{G}(\bullet)$ is continuous in λ, then so is $\tilde{G}(\bullet)$. The reverse implication often does not hold. Indeed, suppose that $f(\bullet,\bullet)$ and $\gamma(\bullet)$ satisfy Theorem 3, $\lambda_n \to \lambda$, and

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[3]This same idea may be stated more generally for uniform spaces. For a general reference on uniform spaces, see Kelley (1955).

$\overline{G}(\lambda_n) \to G^*$. Then it may be the case that $G^* \neq \overline{G}(\lambda)$; in fact, G^* need not be the closure of the graph of ***any*** function. This is illustrated by the next example.

For $x,\lambda \in [-1,1]$, define:

$$f_2(x,\lambda) = \begin{cases} \max\{0,\ 1 - |x/\lambda|\}, & \text{if } \lambda \neq 0 \\ \chi_{\{0\}}(x), & \text{if } \lambda = 0, \end{cases}$$

where $\chi_{\{0\}}$ denotes the indicator function of the set $\{0\}$. Observe that G^* consists of all the points on the horizontal axis from -1 to $+1$ and all the points on the vertical axis from 0 to 1, while $\overline{G}(0)$ omits the interior points of the vertical segment. However, $\tilde{G}(\lambda) = \overline{G}(\lambda)$, for $\lambda \neq 0$, and $\tilde{G}(0) = G^*$. Moreover, for all λ, the original function may be recovered from $\tilde{G}(\bullet)$ by $f_2(x,\lambda) = \max\{y: (x,y) \in \tilde{G}(\lambda)\}$. It is easily checked that the latter property holds in generality.

4. Conclusion.

Let us conclude this paper with an application of the generalized Theorem of the Maximum. Consider the standard discounted dynamic programming problem:

$$W(x_0) = \sup_{\{x_t\}_{t=0}^{\infty}} \Sigma_{t=0}^{\infty}\ \delta^t V(x_t, x_{t+1}) \tag{1}$$

$$\text{subject to } x_{t+1} \in \gamma(x_t)\ (t = 0,1,2,\ldots)$$

$$x_0 \in X,\ \text{given},$$

where $\delta \in (0,1)$, X is a complete metric space, $\gamma: X \to X$ is a continuous correspondence that is nonempty and compact-valued, and V: graph(γ) $\to \mathfrak{R}$ is a real-valued function. If $V(\bullet,\bullet)$ is continuous and uniformly bounded above, then a standard application of Berge's theorem establishes that $W: X \to \mathfrak{R}$ is a continuous function. Furthermore, $W(\bullet)$ satisfies the Bellman equation:

$$W(x) = \max_{y \in \gamma(x)} \{V(x,y) + \delta W(y)\} \tag{2}$$

and the arg max correspondence in (2) is upper hemicontinuous.

In economic applications of the dynamic programming problem, it is often necessary to consider one-period payoff functions, $V(\bullet,\bullet)$, that are discontinuous. For example, in dynamic multi-person games, the payoff of each player depends on the future actions of his opponents. These actions may depend only upper hemicontinuously on the state, rendering the effective $V(\bullet,\bullet)$ a discontinuous function. One important context where this occurs is the two-player game of sequential bargaining with one-sided incomplete information.[4] If the seller's and buyer's actions are restricted to depend continuously on the state variable, an equilibrium often does not exist. However, if actions are permitted to be discontinuous in the state variable (and depend on the previous period's actions), then an equilibrium exists for every distribution function.[5] The existence theorem (Ausubel and Deneckere, 1989, Theorem 4.2) is proved using the techniques developed in this paper.

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[4]Or, equivalently, the problem of durable goods monopoly.

[5]In the standard terminology of the bargaining literature, a **strong-Markov** equilibrium need not exist, but a **weak-Markov** equilibrium exists for every distribution function.

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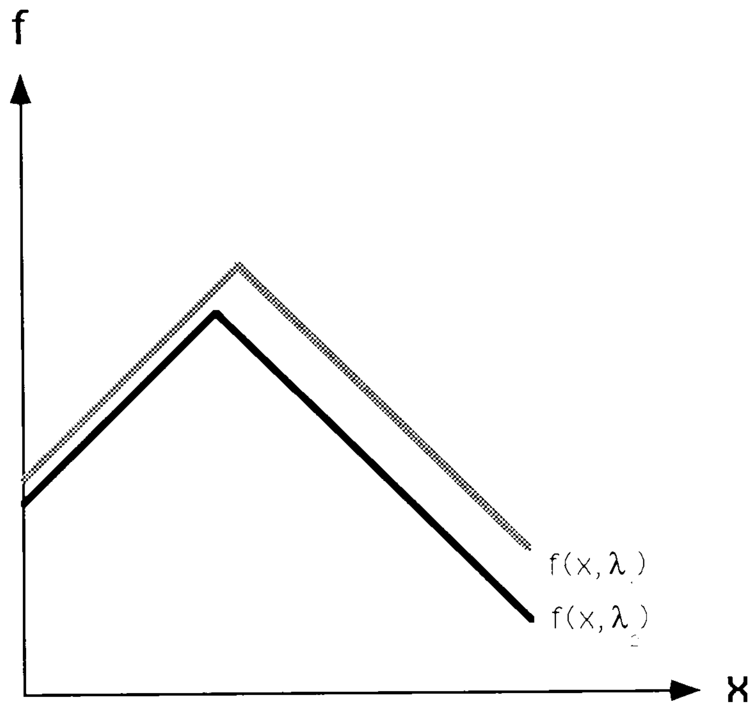

Figure 1a

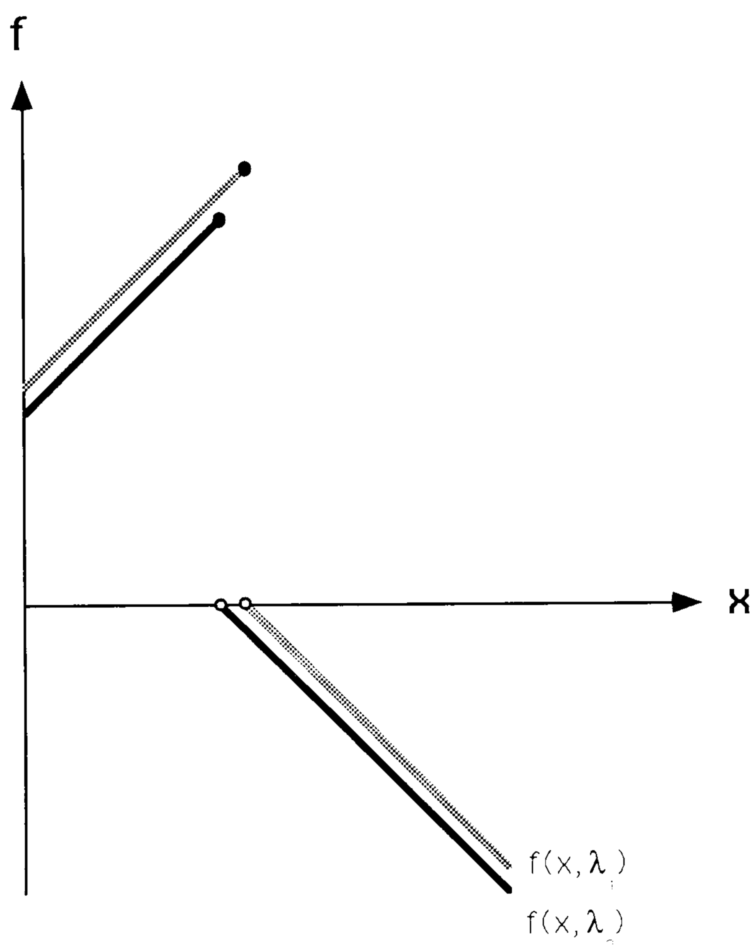

Figure 1b