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CORE CONVERGENCE WITHOUT MONOTONE PREFERENCES

OR FREE DISPOSAL

by

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Abstract

We prove some core convergence theorems for finite economies without monotone preferences or free disposal of commodities. When these assumptions are relaxed, the relationship between the continuum and the large finite economies is lost. Extra conditions are needed, in general, to obtain convergence results.

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I. INTRODUCTION:

The purpose of this essay is to prove some core convergence results when preferences are not monotone (M) and there is no free disposal of commodities (FD). In an exchange economy, the two assumptions are technically related. More precisely, when preferences are transitive, assuming local non-satiation (LNS) and FD is equivalent to assuming M. Throughout the text we will use FD and M interchangeably.

Although the FD hypothesis is not crucial to the existence of competitive equilibria, little attention has been paid to relaxing that hypothesis in core convergence theorems.[2] Several examples in Manelli (1989), however, show that many of the existing core convergence results for large finite economies do not hold when M is weakened.[3] Some of these non-convergence examples consist of well-behaved sequences of economies, purely competitive in Hildenbrand's sense, with strictly convex and almost monotone preferences. Although equivalence of the core and the set of competitive equilibria may hold for the limit economy, there are no prices that can approximately decentralize a given core allocation of the finite economies along the sequence. The examples mentioned also demonstrate that without monotonicity, the convergence results which had been obtained for general sequences of economies with nonconvex preferences may not even hold in the replica case.

In this paper we prove some core equivalence relations for large finite economies. We do not assume monotonicity or free disposal. Preferences in our setting must be such that changes in consumption that belong to a given cone

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[2]McKenzie (1955, 1959, 1961) first established that FD is not required to prove the existence of a competitive equilibrium. This result was further generalized by Debreu (1962). Bergstrom (1973, 1976), Hart and Kuhn (1975) and Rader (1972) provided alternative proofs of somewhat similar results.

[3]For instance, Anderson (1978, 1981, 1985), Brown and Robinson (1974), Cheng (1983), Dierker (1975), Theorem 3 in Hildenbrand (1974) and Theorems 5 and 9 in Hildenbrand (1982).

in the positive orthant make consumers better off.

We show that for any core allocation there is a price system such that, on average, individuals' core bundles lie close to their budget sets and there is no bundle preferred to the core assignment which is far below the budget line. Theorem 1 finds a bound on the sum of a measure of non-competitiveness of a core allocation for a fixed economy. The bound depends on the size of the nonconvexities, the size of the endowment of the largest consumer, the total endowment and the number of agents with similar preferences, but it does not depend, essentially, on the size of the economy. Theorem 2 asserts that when the size of the economy increases, the average measure of non-competitiveness goes to zero. For this we require that first, as the economy becomes large, no consumer's preferences become arbitrarily different from those of most other consumers, and second, nonconvexities must not increase too rapidly. The non-convergence examples mentioned show that, in general, it is not possible to dispose of these requirements. Thus, core equivalence relations will hold without M or FD, but under more stringent conditions.

An example from Anderson and Mas-Colell shows that when preferences are nonconvex the stronger core convergence theorems may not hold.[4] Theorem 3 proves that when preferences are strictly convex in a uniform sense over all agents, the distance between core allocations and demand sets tends to zero in measure.

Our work is closely related to that of Anderson (1978), Arrow and Hahn (1971) and Dierker (1975). These authors obtain a bound on the sum of a measure of non-competitiveness of core allocations which depends on the endowment of the largest consumer but is independent of the number of agents

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[4] This example appeared as an appendix in Anderson (1985).

in the economy. Anderson assumes monotonicity and free disposal but makes almost no further restrictions on preferences,and in particular assumes no convexity, transitivity or continuity. Arrow and Hahn assume monotone preferences and place restrictions on endowments, core allocations and the size of nonconvexities of preferences.

Anderson (1981) imposes strict convexity of preferences uniformly over all agents in the economy and proves that the deviation between core allocations and demand sets tends to zero in measure. If in addition the endowments are uniformly integrable, he obtains convergence in mean. When preferences are not monotone, however, uniform integrability of the endowments does not imply uniform integrability of the core allocations and, therefore, convergence in mean may not occur.

Debreu and Scarf's (1963) equivalence result for replica sequences of economies is obtained under LNS and strict convexity. Aumann (1964) assumes monotone preferences in an economy with a continuum of agents, but M plays a minor role in his proof.[5] Vind (1965), without monotonicity or convexity, obtains a bound on the number of agents that violate a certain competitiveness condition.

Gabszewicz and Mertens (1971), Shitovitz (1973) and Khan (1976) are concerned with the presence of large traders in the economy. The non-convergence examples mentioned do not have significant traders either in the sense of having large endowments or in the sense of having large weights as in Shitovitz and Khan, but they do have significant traders, at least in the convex case, in the sense that their preferences are very different from the rest of the agents in the economy. It is this peculiarity, obliterated when preferences are monotone, that must be ruled out in order to obtain our

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[5] See Hildenbrand (1982).

results.

II. NOTATION:

For any x in $\mathbb{R}^k$, $\|x\| = \text{Max}\ \{|x^i| : 1 \le i \le k\}$, $\|x\|_1 = \sum_{i=1}^{k} |x^i|$, where x^i is the i^{th} component of x. For any subset D of $\mathbb{R}^k$, con(D) is the convex hull of the set D.

Let $\mathfrak{P}$ be the set of binary relations $\mathfrak{p}$ on $\mathbb{R}^k_+$ which are irreflexive. We write x $\mathfrak{p}'$ y for y is not preferred to x. A preference relation $\mathfrak{p}$ is convex if for any two commodity bundles x and x' such that x' $\mathfrak{p}$ x, the bundle $(\alpha x' + (1 - \alpha)x)\ \mathfrak{p}\ x$ for all α in (0,1).

An exchange economy is a map $\mathcal{E} : A \longrightarrow \mathfrak{P} \times \mathbb{R}^k_+$ that assigns to each agent a $\in$ A, a preference relation $\mathfrak{p}_a$ and an endowment e(a) in the consumption set $\mathbb{R}^k_+$. An allocation g is a consumption assignment that precisely exhausts the total endowments. The core of an economy $\mathcal{E}$, $\mathcal{C}(\mathcal{E})$, is the set of allocations that cannot be blocked by any coalition. A coalition B $\subseteq$ A blocks an allocation f if there is an assignment g, such that $\sum_B g(a) = \sum_B e(a)$ and $g(a)\ \mathfrak{p}_a\ f(a)\ \forall a \in B$.

Prices belong to $U = \{p \in \mathbb{R}^k : \|p\| = 1\}$. The demand set of an agent "a" at price p is $D_a(p) = \{x : x\ \mathfrak{p}'_a y,\ \forall y \text{ with } p\cdot y \le p\cdot e(a)\}$.

III. DEFINITIONS AND THEOREMS:

Given any open convex cone $V \subseteq \mathbb{R}^k_+$, let $\mathfrak{P}(V)$ be the set of preferences $\mathfrak{p}$ which are complete, transitive, continuous ($\{(x,y) \in \mathbb{R}^{2k} : x\ \mathfrak{p}\ y\}$ is an open set) and proper; that is $\forall x \in \mathbb{R}^k_+$, $(x + V)\ \mathfrak{p}\ x$.[6] When V is the strictly positive orthant, preferences are monotone.

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[6] We believe that the notion of proper preferences or cone monotonicity was first used by Grodal, Trockel and Weber (1984). Also see Mas-Colell (1986).

We employ the Hausdorff distance $\delta(\cdot,\cdot)$ between sets (Hildenbrand (1974) p. 16-18) both in measuring the size of nonconvexities and in the notion of similar preferences. We will discuss its use after stating our results.

Let $C(y) = \{z : z \succ_a y\}$ be the upper-contour set relative to bundle y. The nonconvexities of the preferences of consumer "a" are measure by

$$c(a) = \text{Sup}\ \{\delta(C(y),\ \text{con}[C(y)]) : y \in \mathbb{R}^k_+\}.$$

This asserts that for any bundle x in the convex hull of any given upper-contour set, there is a consumption bundle z (in the upper-contour set) which is at most c(a) away from x. The measure $c(\cdot)$ is less restrictive than the measure of nonconvexities used by Arrow and Hahn (1971) among others, which is based on the inner radius of upper-contour sets. For a given set of agents, the maximum individual nonconvexities in preferences are given by:

$$Q(A) = \underset{a\in A}{\text{Max}}\ c(a).$$

For any agent $a' \in A$ and any real number $S > 0$, let $B(a',S) = \{a \in A: \delta(\succ_a, \succ_{a'}) < S\}$. Note that $a' \in B(a',S)$. Thus,

$$I(A,S) = \underset{a\in A}{\text{Min}}\ |B(a,S)|,$$

is an index of how uncommon or "peculiar" preferences are in the given economy.

For a given economy with agents in A, we define respectively measures of the maximum individual endowment and the size of the average endowment:

$$E(A) = \underset{a\in A}{\text{Max}}\ \|e(a)\|,$$

$$M(A) = \|\textstyle\sum_A e(a)\|/|A|.$$

Finally, we employ the following notion of approximation which measures the extent to which an agent's commodity bundle x looks like a demand relative to a given price p.

$$\psi(x,a,p) = |p \cdot [x - e(a)]| + |\inf\{p \cdot [y - e(a)]: y \succ_a x\}|.^{7}$$

The first term is a measure of the budget deviation and the second one of the excess expenditure, both incurred when consumer "a" purchases bundle x at price p. Note that when p >> 0, $\psi(x,a,p) = 0$ implies that x belongs to the demand set.

Theorem 1 asserts that given an economy and a core allocation f(•) for that economy, there is a price system p that approximately decentralizes the core allocation, in the sense that there exists a bound on $(\sum \psi(f(a),a,p))$ the sum of the budget deviations and on the sum of the excess expenditures (incurred when purchasing the core bundle at the given price.) The bound depends on the total endowment, the size of the nonconvexities, the size of the largest endowment and the number of agents with similar preferences, but it does not depend, essentially, on the size of the economy. Theorem 2 provides conditions for the average measure of non-competitiveness to go to zero as the size of the economy increases. After stating Theorems 1 and 2, we discuss our assumptions, state Theorem 3 and end with the proofs.

Theorem 1: Let $\mathcal{E}$ be an exchange economy with preferences in $\mathfrak{P}(V)$. Let $f \in \mathcal{C}(\mathcal{E})$. Choose v so that $B(v,1) \subseteq V$ and define $L(A) = \dfrac{6k^2\|\sum_A e(a)\|}{I(A,S)} + 4kQ(A) + 6k$ $E(A) + 2kS$. Then, $\exists p \in U$ such that $\sum_A \psi(f(a),a,p) \leq 4kL(A)\|v\|_1$.

Theorem 2: Let $\mathcal{E}^n: A^n \longrightarrow \mathfrak{P}(V) \times \mathbb{R}^k_+$ be a sequence of economies. Let $f^n \in \mathcal{C}(\mathcal{E}^n)$. If:

(i) (NIN) Negligible Individual Nonconvexities of Preferences: $\lim\limits_{n \to \infty} \dfrac{Q(A^n)}{|A^n|} = 0$

[7] See Hildenbrand (1974), Khan (1974), Dierker (1975) and Anderson (1978) among others.

(ii) (NIE) Negligible Individual Endowment: $\displaystyle\lim_{n \to \infty} \frac{E(A^n)}{|A^n|} = 0$

(iii) (BAE) Bounded Average Endowment: $\displaystyle\sup_n M(A^n) < \infty$

(iv) (NPI) No Peculiar Individuals: $\exists S > 0$ such that $I(A^n, S) \to \infty$ as $n \to \infty$.

then $\exists \{p^n\} \in U$, such that

$$\lim_{n \to \infty} \frac{1}{|A^n|} \sum_{A^n} \psi\left[f^n(a),\ a,\ p^n\right] = 0.$$

Proof of Theorem 2: Divide both sides of the inequality in Theorem 1 by $|A^n|$. As the number of agents goes to infinity, $L(A^n)/\ |A^n| \to 0$.

Corollary 1: Let $\mathcal{E}^n: A^n \to \mathcal{P}(V) \times \mathbb{R}^k_+$ be a sequence of economies. Let $f^n \in \mathcal{C}(\mathcal{E}^n)$. If (i) (NIN) $\displaystyle\lim_{n \to \infty} \frac{Q(A^n)}{|A^n|} = 0$ and (ii) $\displaystyle\lim_{n \to \infty} E(A^n)/I(A^n, S) = 0$ then

$$\lim_{n \to \infty} \frac{1}{|A^n|} \sum_{A^n} \psi\left[f^n(a),\ a,\ p^n\right] = 0.$$

Corollary 2: Theorem 2 still holds if assumptions (iii) and (iv) are replaced by (and therefore weakened to) $\displaystyle\lim_{n \to \infty} M(A^n)/I(A^n, S) = 0$.

Remark: If all agents in the economy are drawn from a compact set of agents' characteristics, then Theorem 2 holds provided that nonconvexities do not increase too rapidly as the the economy becomes large. To see this, let T be a compact set of agents' characteristics, where the Hausdorff distance is the metric on T. Then there exists S such that $L(A) \le L(T)$ and $|A| = I(A,S)$ for all A with $\mathcal{E}(A) \subseteq T$. Corollary 1 justifies our claim.

NIN prevents nonconvexities from increasing to rapidly as the economy becomes large. If NIN does not hold, a counter-example to Theorem 2 may be found.[8] NIE requires that the endowment of each individual become negligible in relation to the size of the economy. A group of agents of increasing size, however, may progressively possess a larger fraction of the endowments. BAE states that the average endowment of all economies in the sequence should not go to infinity. This is a common assumption, implied by Hildenbrand's definition of purely competitive sequences of economies. The assumption is not required, however, to obtain the results in Theorem 2 which still hold if the average endowment goes to infinity with the size of the economy, provided the number of agents that are not too different $I(A^n,S)$ increases rapidly enough. This is stated in Corollary 2.

NPI is a weak assumption as well. Intuitively it requires that the economies should have no consumer with preferences that become arbitrarily different from those of most other individuals. A somewhat related assumption is the no "isolated" individuals (NII) condition:

$$\forall S \quad \operatorname*{Inf}_n \; I(A^n,S)/|A^n| > 0.$$

NPI is considerably weaker than NII in three aspects: First, NII requires that as the size of the economy increases there are many agents that are "arbitrarily close" to each other; the definition must hold for all $S > 0$. NPI requires instead that no agents are "arbitrarily different" from most others; the definition requires the existence of some $S > 0$. Second, NPI allows the fraction of agents endowed with similar preferences to go zero as the size of the economy increases. Thus, there may be preferences that are present throughout the sequence and that disappear in the limit. Third, NII has generally been stated in terms of agents' characteristics, thus requiring no isolated endowments or preferences.

[8] See Manelli (1989) for the example.

Finally note that if preferences are drawn from a compact set then NPI is satisfied, although this is not the case with NII. Furthermore, weak convergence of the distribution of preferences to a limit distribution is not enough to imply NII. The additional restriction of convergence of the supports must be placed.[9]

In order to define either NPI or NII, a notion of similar preferences is necessary. We consider preferences to be similar if they are close to each other in the Hausdorff distance sense. This notion of proximity is stronger than the one implied by the Hausdorff distance applied to the one-point compactification of the commodity space (i.e. closed convergence topology.)[10] The latter views points that are large as being close to each other, that is, no weight is assigned to occurrences at infinity.

Example 1 in Manelli (1989) introduces a sequence of economies with non-monotonic preferences. In the example there is a large coalition for which the joint core assignment is strictly less than its joint endowment. The excess goods are distributed among a few "peculiar" individuals who value those commodities. Thus, there is an individual whose core assignment of a certain commodity goes to infinity with the size of the economy.[11] Any price that approximately decentralizes the core allocation must assign to that commodity a price close to zero, so that our peculiar agent is able to purchase a large quantity of it. At zero price, however, most agents in the economy have a consumption bundle which is preferred to the core assignment and, on average, considerably less expensive than the endowments. By assuming

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[9]See Hildenbrand (1974) page 192 and Mas-Colell (1985) page 284.

[10]See Hildenbrand (1974), pages 16-18.

[11]See Wooders (1990) and the references there for a discussion of the competitive properties of economies where large coalitions are inessential.

that there are no peculiar individuals, we are able to eliminate this type of example. Some coalition (including agents relatively similar to our peculiar individual) will eventually be able to block the allocation proposed in the example. This is the role of assumption (ii) in Corollary 1 (or NPI in general.) The endowment of any individual (or coalition) potentially may be divided among many agents ($I(A^n,S)$) who value it.

With monotone preferences, Bewley (1973) first showed that bounded endowments imply that the core allocations are bounded. The one-point compactification topology is therefore sufficient in the monotone case because the core allocation is well-behaved (no ocurrences at infinity.) As soon as monotonicity and free disposal are abandoned, the non-convergence examples suggest that a stronger notion of similarity is necessary.

Theorem 3 asserts that if a form of strict convexity is imposed uniformly over agents, the distance between demands and core allocations tends to zero in measure. When preferences are not monotone, prices may be zero or negative. So budget sets need not be compact. Therefore non-emptiness of the demand set is not guaranteed by the standard theorems. A consequence of Theorem 3 is that, at the prices found in Theorem 1, the demand set is non-empty for most agents.

We reproduce the definition of equi-convexity of a set of preferences, notion introduced by Anderson (1981).[12] For x in $\mathbb{R}^k_+$, let $B(x,\delta) = \{y \in \mathbb{R}^k_+ : \|y - x\| < \delta\}$. For $x \neq y \in \mathbb{R}^k_+$, let

$$\sigma(x,y,\succ) = \sup\left\{ \delta : B\left[\frac{x+y}{2}, \delta\right] \succ B(x,\delta) \text{ or } B\left[\frac{x+y}{2}, \delta\right] \succ B(y,\delta)\right\}$$

If $P \subseteq \mathfrak{P}(V)$, $\sigma(x,y,P) = \inf\{\sigma(x,y,\succ) : \succ \in P\}$. We say that a subset P of $\mathfrak{P}(V)$

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[12]A related idea may be found in Grodal (1976).

is equi-convex if $\sigma(x,y,P) > 0$ for all $x \neq y \in \mathbb{R}^k_+$.

Note that $\sigma(x,y,P)$ is a continuous function of x and y for all P.

Theorem 3: Let $\mathcal{E}^n$: $A^n \longrightarrow \mathfrak{P}(V)$ be a sequence of economies which satisfies the hypothesis of Theorem 2. In addition, suppose $\mathfrak{P}(V)$ is equi-convex and $\exists\varepsilon > 0$ such that $e^n(a) \gg (\varepsilon,\varepsilon, \ldots,\varepsilon)$ $\forall a \in A^n$, $\forall n$. If $f^n \in \mathcal{C}(\mathcal{E}^n)$, then $\exists$ p^n such that $\forall\varepsilon > 0$

$$\lim_{n \longrightarrow \infty} \frac{|\{a \in A^n\text{: } \|f^n(a) - D_a(p^n)\| > \varepsilon\}|}{|A^n|} = 0$$

The proof of Theorem 1 uses, for convenience, three lemmas. It roughly follows these lines. For a given f in $\mathcal{C}(\mathcal{E})$ and for any agent a in A, define the set of "net preferred trades" by

$$\phi(a) = \{x - e(a)\text{: } x \in X(a),\ x\ p_a\ f(a)\} \cup \{0\},$$

and let $\Phi = \sum_A \phi(a)$. By definition of a core allocation, $\Phi \cap (-V) = \emptyset$. Then, if $\mathrm{con}(\Phi)$ does not go too far into $(-V)$, a separating hyperplane argument provides the p needed in the thesis of the theorem. Basically, these steps are in Anderson (1978) for the case where V is the positive orthant. To show that $\mathrm{con}(\Phi)$ does not go to far into $(-V)$ we need our extra assumptions: For any element w in $\mathrm{con}(\Phi) \cap (-V)$, there exists (by the Shapley-Folkman theorem) a coalition that could almost block the core allocation except for at most k troublesome members that may oppose the blocking. These k agents are not satisfied with their proposed bundles at the blocking allocation. It is possible, however, to find k new individuals (using NPI and NIN) who would actually accept (Lemmas 2 and 3) bundles "similar" to those refused by the k troublesome agents. Replacing these k agents with the new individuals we have a new coalition. Since both coalitions are very similar, if w is far inside

$(-V)$ then we can associate w' [in $\Phi \cap (-V)$] with the new coalition and therefore the core allocation could be blocked.

Lemma 1: Let a be any consumer and $g(a) \in \text{con}(\phi(a))$. Then there exists $h(a) \in \phi(a)$, $y(a) \in \mathbb{R}^k$, $\|y(a)\| < c(a)$ and $\lambda \in [0,1]$ so that $g(a)$ can be expressed as $g(a) = \lambda\, h(a) + y(a)$.

Proof of Lemma 1: First, we show that for each a, there is $h(a) \in \phi(a)$ and $y(a) \in \mathbb{R}^k$, $\|y(a)\| < c(a)$ so that $g(a)$ can be expressed as $g(a) = \lambda\, h(a) + y(a)$, with $\lambda \in [0,1]$: If $g(a) = 0$ then by taking $h(a^i) = 0$ the claim is established. If $g(a^i) \neq 0$, there is a set $\{x_j \in \phi(a)\colon 1 \leq j \leq k+2,\ x_1 = 0\}$ and non-negative weights λ_j so that $\sum_{j=1}^{k+2} \lambda_j = 1$, $\lambda_1 > 0$ and

$g(a) = \sum_{j=1}^{k+2} \lambda_j x_j = \lambda_1 0 + (1-\lambda_1) \sum_{j=2}^{k+2} \frac{\lambda_j x_j}{(1-\lambda_1)}$. Since $\sum_{j=2}^{k+2} \frac{\lambda_j}{(1-\lambda_1)} = 1$, $\sum_{j=2}^{k+2} \frac{\lambda_j x_j}{(1-\lambda_1)} \in \text{con}(\phi(a)\backslash\{0\})$. By definition of $c(a)$, there is $h(a) \in \phi(a)$ so that $\left\| \sum_{j=2}^{k+2} \frac{\lambda_j x_j}{(1-\lambda_1)} - h(a) \right\| < c(a)$. Finally, define

$y(a) = (1-\lambda_1)\left[\sum_{j=2}^{k+2} \frac{\lambda_j x_j}{(1-\lambda_1)} - h(a)\right]$. Taking $\lambda = (1-\lambda_1)$ we obtain $g(a) = \lambda\, h(a) + y(a)$.

Lemma 2: Given any agent a, $g(a) \in \text{con}(\phi(a))$ and any consumer b with $f(b)\ \succ'_a\ f(a)$, then there exists a net trade $g'(a)$ in $\phi(a)$ so that

$$\|g'(a) - g(a)\| \leq 2Q(A) + 3E(A) + \|f(b)\| \tag{1}$$

Proof of Lemma 2: By Lemma 1, given $g(a)$ there is $h(a) \in \phi(a)$, $y(a) \in \mathbb{R}^k$, $\|y(a)\| < c(a)$ and $\lambda \in [0,1]$ so that $g(a) = \lambda\, h(a) + y(a)$. Since $f(b)\ \succ'_a\ f(a)$, then $(\lambda\,[h(a) + e(a)] + (1-\lambda)\, f(b)) \in \text{con}(\{x : x\ \succ'_a\ f(a)\})$ and, by definition of $c(a)$, $\exists\ g'(a) \in \phi(a)$ and $y(a)$, $\|y'(a)\| < c(a)$ so that $g'(a) = \lambda [h(a) + e(a)] + (1 - \lambda) f(b) + y'(a) - e(a)$. Then $\|g'(a) - g(a)\| = \|\lambda [h(a) + e(a)] + (1 - \lambda) f(b) + y'(a) - e(a) - \lambda h(a) - y(a)\| \le \|h(a) - h(a)\| + \|y'(a) - y(a)\| + \|f(b) - e(a)\| \le 2c(a) + \|f(b) - e(b)\| + \|e(b) - e(a)\| \le 2c(a) + \|f(b)\| + 2\|e(b)\| + \|e(a)\|$.

Lemma 3: Given any consumer a, $g(a) \in con(\phi(a))$ and any consumer b with $f(a)\ \mathrm{p}_a\ f(b)$ and $d(\mathrm{p}_a, \mathrm{p}_b) < S$, then there exists $g'(b) \in \phi(b)$ so that

$$\|g'(b) - g(a)\| \le d(\mathrm{p}_a, \mathrm{p}_b) + 3E(A) + 2Q(A) + \|f(b)\| \tag{2}$$

Proof of Lemma 3: Let $h(a)$ be the net trade identified in Lemma 1. Let b be as in the hypothesis. Then $[h(a) + e(a)]\ \mathrm{p}_a f(b)$ by transitivity. Therefore there is a bundle x with $\|x - (h(a) + e(a))\| < d(\mathrm{p}_a, \mathrm{p}_b)$ and $x\ \mathrm{p}_{b^i} f(b^i)$. Hence, $\|x - e(b) - h(a)\| \le \|x - h(a) - e(a)\| + \|e(a) - e(b)]\| \le d(\mathrm{p}_a, \mathrm{p}_b) + \|e(a) - e(b)\|$. By construction, the net trade $[\lambda\ x + (1 - \lambda) f(b) - e(b)]$ belongs to $con(\phi(b))$. By definition of $c(b)$, there exists $g'(b) \in \phi(b)$, $y(b)$ with $\|y(b)\| < c(b)$ so that $g'(b) = \lambda\ x + (1 - \lambda) f(b) + y(b) - e(b)$. Hence, $\|g'(b) - g(a)\| = \|\lambda\ x + (1 - \lambda) f(b) + y(b) - e(b) - \lambda h(a^i) - y(a)\| \le \|\lambda [x - e(b)] + (1 - \lambda) [f(b) - e(b)] + y(b) - \lambda h(a) - y(a)\| \le \|x - e(b) - h(a)\| + \|y(b) - y(a)\| + \|f(b) - e(b)\| \le d(\mathrm{p}_a, \mathrm{p}_b) + \|e(a) - e(b)]\| + c(a) + c(b) + \|f(b) - e(b)\| \le d(\mathrm{p}_a, \mathrm{p}_b) + \|e(a)\| + 2\|e(b)]\| + c(a) + c(b) + \|f(b)\|$.

Proof of Theorem 1:

Define $N(a') = \{a \in B(a', S): \|f(a)\| < 2kM(A)|A|/ I(A,S)\}$. Then

$$|N(a')| \quad > \quad I(A,S)/2.$$

To see this, note that f is an allocation and let $\varepsilon = 2kM(A)/ I(A,S)$. Then, $|\{a \in B(a',S): \|f(a)\| \ge \varepsilon|A|\}| \le k\|\sum_A e(a)\|/\varepsilon|A| = kM(A)/\varepsilon$. Then $|N(a')| >$

$|B(a',S)| - kM(A)/\varepsilon = |B(a',S)| - I(A,S)/2 \geq I(A,S) - I(A,S)/2 = I(A,S)/2.$

Let $\Phi = \sum_A \phi(a)$ and $w \in \text{con}(\Phi)$. By the Shapley-Folkman theorem, there is a set $K = \{a^i\colon 1 \leq i \leq k\}$ so that $w = \sum_{A\setminus K} g(a) + \sum_K g(a^i)$ where $g(a) \in \phi(a)$ $\forall a \in A\setminus K$ and $g(a^i) \in \text{con}(\phi(a^i)$ $\forall i$.[13]

Let $C = \{a \in A\setminus K\colon g(a) \neq 0\}$ and let $w << 0$. Suppose that for some $a^j \in K$, $a^j \notin N(j)$ and $N(j) \subseteq C$. For simplicity, denote $N(a^j)$ by $N(j)$. Then w can be expressed as

$$w = \sum_{C\setminus N(j)} g(a) + \sum_{N(j)} g(a) + \sum_K g(a^i). \tag{3}$$

Since for all $g(a) \in \text{con}(\phi(a))$, $g(a) \geq -e(a)$ (and $w \ll 0$) then

$$-\sum_{N(j)} e(a) \leq \sum_{N(j)} g(a) = w - \sum_{C\setminus N(j)} g(a) - \sum_K g(a^i) << \sum_{C\setminus N(j)} e(a) + \sum_K e(a^i)$$

$$0 \leq \sum_{N(j)} [g(a) + e(a)] << \sum_{C\setminus N(j)} e(a) + \sum_{N(j)} e(a) + \sum_K e(a^i)$$

Thus, $\left\| \sum_{N(j)} [g(a) + e(a)] \right\| \leq \left\| \sum_C e(a) + \sum_K e(a^i) \right\|$ and therefore

$$\frac{|N(j)|}{k} \underset{a\in N(j)}{\text{Min}} \left\{ \|g(a) + e(a)\| \right\} \leq \left\| \sum_C e(a) + \sum_K e(a^i) \right\| \text{ and}$$

$$\underset{a\in N(j)}{\text{Min}} \left\{ \|g(a) + e(a)\| \right\} \leq \frac{k}{|N(j)|} \left\| \sum_C e(a) + \sum_K e(a^i) \right\| \tag{4}$$

Suppose that for all j, $1 \leq j \leq t$ with $t \leq k$, $a^j \notin N(j)$ and $N(j) \subseteq C$. Let d^j be the agent for which the minimum is attained in (4). Define

$$w'' = w - \sum_{j=1}^{t} g(d^j).$$

From (4), $\|w - w''\| = \left\| \sum_{j=1}^{t} g(d^j) \right\| \leq \sum_{j=1}^{t} \frac{k}{|N(j)|} \left\| \sum_C e(a) + \sum_K e(a^i) \right\| \leq$ $\frac{k^2}{\underset{1\leq j\leq t}{\text{Min}}\{|N(j)|\}} \left\| \sum_C e(a) + \sum_K e(a^i) \right\| \leq \frac{2k^2}{I(A,S)} \left\| \sum_A e(a) \right\| \leq \frac{2k^2 M(A)|A|}{I(A,S)}$. Thus,

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[13]For the Shapley-Folkman Theorem see Starr (1968).

$$\|w - w''\| \le \frac{2k^2M(A)|A|}{I(A,S)} \tag{5}$$

Note that w", by construction, can be written as

$$w'' = \sum_{C'} g(c) + \sum_{K} g(a^i), \tag{6}$$

where $C' = C \setminus \{d^j\colon d^j$ minimizes (4) for any j with $a^j \notin N(j)$ and $N(j) \subseteq C\}$.

We separate the agents in K into two disjoint subsets K' and K": Let K' = $\{i\colon a^i \in K$ and $\exists b^i \in N(a^i),\ f(b^i)\ p_{a^i}'\ f(a^i)\}$. Therefore, for any agent a^i whose index i is in $K'' = \{i\colon a^i \in K$ and $i \notin K\}$, it is the case that $f(a^i)\ p_{a^i}\ f(b^i)\ \forall b \in N(a^i)$. Note that if $i \in K''$ then $N(a^i)$ is not included in C'.

Given any a^i with i in K' and $g(a^i)$, there exists (by Lemma 2) a net trade $g'(a^i)$ so that (1) holds. Similarly given any a^i with i in K" and $g(a^i)$, for every agent b^i in $N(a^i)$ there exists (by Lemma 3) a net trade $g'(b^i)$ so that (2) holds. For each i in K", choose b^i in $N(a^i)$ so that b^i is not in C' and define

$$w' = \sum_{C'} g(a) + \sum_{i \in K'} g'(a^i) + \sum_{i \in K''} g'(b^i)$$

Using (1), (2) and (6),

$\|w'' - w'\| \le \|\sum_{K} g(a^i) - \sum_{i \in K'} g'(a^i) - \sum_{i \in K''} g'(b^i)\| \le \|\sum_{i \in K'} (g(a^i) - g'(a^i))\| + \|\sum_{i \in K''} (g(a^i) - g'(b^i))\| \le k\,(2Q(A) + 3E(A) + \frac{2kM(A)|A|}{I(A,S)}) + k\,(2Q(A) + S + 3E(A) + \frac{2kM(A)|A|}{I(A,S)})$. Thus,

$$\|w'' - w'\| \le 2k\,(2Q(A) + S + 3E(A) + \frac{2kM(A)|A|}{I(A,S)}).$$

Given that $\|w' - w\| \le \|w' - w''\| + \|w'' - w\|$, and using (5) we have

$$\|w' - w\| \le \frac{6k^2M(A)|A|}{I(A,S)} + 4kQ(A) + 6kE(A) + 2kS$$

By construction $w' \in \Phi$ and therefore $w' = \sum_A x(a)$, $x(a) \in \phi(a)$. If $w' \in (-V)$, define $x'(a) = x(a) + e(a) - \frac{G}{|B|}$, where $B = \{a : x(a) \neq 0\}$. Hence $\sum_B x'(a) = \sum_B x(a) + \sum_B e(a) - w' = \sum_B e(a)$. Since $(-w'/|B|) \in V$, and $x'(a)\ p_a$

$(x(a) + e(a))\ \succ_a\ f(a)$, which by transitivity implies $x'(a)\ \succ_a\ f(a)$. Thus $w' \notin (-V)^c$.

Let $L(A) = \dfrac{6k^2M(A)|A|}{I(A,S)} + 4kQ(A) + 6kE(A) + 2kS$. Let $r = kL(A)v$. Since $B(v,1) \subseteq V$ by hypothesis, $B(r, kL(A)) + V \subseteq V$. Therefore $w \notin -(r + V)$. We conclude that $\mathrm{con}(\Phi) \cap -(r + V) = \emptyset$.

By Minkowski's Theorem, there is $p \in U$, $p \neq 0$ such that $\mathrm{Inf}\ \{p \cdot \Phi\} \geq \sup \{p \cdot w\colon\ w \in -(r + V)\} = -p \cdot r = -kL(A)(p \cdot v) \geq -kL(A)\|v\|_1$. By continuity $p \cdot (f(a) - e(a)) \geq \mathrm{Inf}\ p \cdot \phi(a)$. Let $A' = \{a\colon\ p \cdot (f(a) - e(a)) < 0\}$. Then,

$$0 \geq \sum_{A'} p \cdot (f(a) - e(a)) \geq \sum_{A'} \mathrm{Inf}\ \{p \cdot \phi(a)\} \geq -kL(A)\|v\|_1.$$

Since f is an allocation, $\sum_A p \cdot (f(a) - e(a)) = 0$ and therefore

a) $\sum_A |p \cdot (f(a) - e(a))| = 2 \sum_{A'} |p \cdot (f(a) - e(a))| \leq 2\ kL(A)\|v\|_1$ and

b) $\sum_A |\mathrm{Inf}\ \{p \cdot (x - e(a))\colon\ x \in X(a),\ x\ \succ_a f(a)\}| \leq$

$$\leq \sum_{A'} |\mathrm{Inf}\ \{p \cdot \phi(a)\}| + \sum_{A\setminus A'} |p \cdot (f(a) - e(a))| \leq$$

$$\leq kL(A)\|v\|_1 + kL(A)\|v\|_1 \leq 2kL(A)\|v\|_1$$

Proof of Theorem 3:

Let a be any consumer in A^n, $\{p^n\}$ a sequence of prices, $\{f^n(a)\}$ a sequence of consumption bundles. Also let $T > 0$ and $\{\beta^n\}$, $\beta^n > 0$ with

$$\lim_{n \to \infty} \beta^n = 0, \quad \|f^n(a)\| < T \quad \text{and} \quad \|e^n(a)\| < T\ \forall n, \tag{7}$$

$$|p^n \cdot (f^n(a) - e^n(a))| < \beta^n \tag{8}$$

$$|\mathrm{Inf}\{p^n \cdot (x - e^n(a))\colon\ x\ \succ_a f^n(a)\}| < \beta^n. \tag{9}$$

Consider the set $\{x \in \mathbb{R}^k_+\colon\ p^n \cdot x \leq p^n \cdot e^n(a),\ \|x\| \leq 2T\}$ which is compact and therefore has a maximal element x^n for a's preferences. That is

$$x^n\ \succ_a'\ y,\ \forall y \text{ with } p^n \cdot y \leq p^n \cdot e^n(a),\ \|y\| \leq 2T \tag{10}$$

We now prove in three steps that if (7), (8) and (9) hold then

$$\lim_{n \to \infty} \|x^n - f^n(a)\| = 0. \tag{11}$$

Suppose (11) does not hold. Then there is γ, $0 < \gamma < T/2$, such that (in a subsequence) $\|x^n - f^n(a)\| > 2\gamma$ $\forall n$. Let $r^n = (x^n + f^n(a))/2$. Then $\|r^n\| \le 3T/2$.

First, we show that

$$|p^n \cdot (r^n - e^n(a))| < \beta^n \tag{12}$$

To prove (12) it is enough to show that $p^n \cdot (x^n - e^n(a)) > -\beta^n$. If $x^n \ \mathfrak{p}'_a \ f^n(a)$ then by (9), the assertion holds. If $f^n(a) \ \mathfrak{p}_a \ x^n$, then any convex combination of both bundles is, by strict convexity of preferences, preferred to x^n and, therefore, x^n is not maximal (expression (10) is violated). This proves (12).

Second, let $\xi = \text{Min}\{\sigma(x, y, \mathfrak{p}(V)) \colon x \le T,\ y \le 2T,\ \|x - y\| \ge 2\gamma\}$. Since $\sigma(\cdot, \cdot, \mathfrak{p}(V))$ is a continuous and strictly positive function defined on a compact set, ξ is greater than zero. Let $\delta = \text{Min}\ \{\xi,\ T/2\}$. Therefore, by equi-convexity, for all n

$$B(r^n, \delta) \ \mathfrak{p}_a \ x^n \quad \text{or} \quad B(r^n, \delta) \ \mathfrak{p}_a \ f^n(a).$$

Third, it may be assumed by passing to a subsequence that $p^n \longrightarrow p$. Suppose there is one commodity, say commodity 1, with price $p_1 < 0$. Note that for n large enough $p^n_1 < (p_1/2) < -\beta^n$. Defining $t^n = r^n + (\delta,\ 0,\ \ldots, 0)$ and using (12) we obtain that for large n, $p^n \cdot t^n \le p^n \cdot r^n + \delta p^n_1 \le p^n \cdot e^n(a) + \beta^n + (\delta p_1/2)$. Since for large n, $\beta^n + (\delta p_1/2) < -\beta^n$

$$p^n \cdot (t^n - e^n(a)) \le -\beta^n. \tag{13}$$

If $B(r^n, \delta) \ \mathfrak{p}_a \ f^n(a)$, then $t^n \ \mathfrak{p}_a \ f^n(a)$ and (13) contradicts (9). We must therefore consider the alternative $B(r^n,\delta)\ \succ_a\ x^n$, in which case $t^n \succ_a x^n$. Note, however that $\|t^n\| \le \|r^n\| + \delta \le 3T/2 + T/2 \le 2T$. Thus, given (13) x^n is not maximal and therefore (10) is contradicted.

Hence, it must be the case that all commodities have non-negative prices ($p > 0$). Given our assumptions on endowments, there is $\varepsilon > 0$ such that, for large enough n, $p^n \cdot e^n(a) > 2k\varepsilon$. For large n, $\beta^n < k\varepsilon$ and, given (12), $p^n \cdot r^n > k\varepsilon$. Therefore, there must be a commodity, say commodity 1, with $(p_1^n\ r_1^n) > \varepsilon$. Since $r_1^n \le 2T$ and $p_1^n \le 1$, then $p_1^n > (\varepsilon/(2T))$ and $r_1^n > \varepsilon$ for large n. Let $\delta' = \mathrm{Min}\ \{\varepsilon,\ \delta\}$ and let $t^n = r^n + (-\delta', 0, \ldots, 0)$. For n large $p^n \cdot t^n \le p^n \cdot e^n(a) + \beta^n - p_1^n\ \delta'$ and $\beta^n - p_1^n\ \delta' < -\beta^n$. Therefore we obtain (13): $p^n \cdot (t^n - e^n(a)) \le -\beta^n$. Similarly to the previous case this generates a contradiction. We conclude $\displaystyle\lim_{n \to \infty} \|f^n(a)-x^n\| = 0$, thus proving (11).

Since $\|f^n(a)\| \le T$, for n large enough $\|x^n\|$ is strictly less than 2T. Hence there is no consumption y with $p^n \cdot y \le p^n \cdot e^n(a)$ and $y \succ_a x^n$. This implies that $x^n = D_a(p^n)$ for large n. Hence

$$\lim_{n \to \infty} \|f^n(a) - D_a(p^n)\| = 0.$$

Since the average endowment and therefore the average core allocation is bounded (Assumption (ii), Theorem 2), given any $\alpha > 0$, $\exists T > 0$ and a set $E^n \subseteq A^n$ such that $\forall a \in E^n$ $\|f^n(a)\| \le T$, $\|e^n(a)\| \le T$ and

$$\frac{|A^n \setminus E^n|}{|A^n|} < \alpha \quad \forall n.$$

For any core allocation f^n of the economy $\mathcal{E}^n$ and for the price system p^n found in Theorem 2, there is $\exists\ \{\beta^n\}$, $\beta^n > 0$, $\beta^n \to 0$ so that

$$|\{a \in A^n:\ |p^n \cdot (f^n(a) - e^n(a))| < \beta^n\}/|A^n| > 1 - \beta^n$$

$$|\{a \in A^n:\ |\mathrm{Inf}\ \{p^n \cdot (x^n - e^n(a)):\ x \succ_a f^n(a)\}| < \beta^n\}|/|A^n| > 1 - \beta^n.$$

Thus, $\displaystyle\lim_{n \to \infty} |\{a \in E^n:\ (7),\ (8) \text{ and } (9) \text{ hold}\}|/|A^n| = 1$. But (7)-(9)

imply (11). Therefore, for any $\varepsilon > 0$,

$$\lim_{n \to \infty} |\{a \in E^n: \|f^n(a) - D_a(p^n)\| < \varepsilon\}|/|A^n| = 1.$$

Finally, for any $\varepsilon > 0$

$$\lim_{n \to \infty} \frac{|\{a \in A^n: \|f^n(a) - D_a(p^n)\| > \varepsilon\}|}{|A^n|} <$$

$$\lim_{n \to \infty} \frac{|\{a \in E^n: \|f^n(a) - D_a(p^n)\| > \varepsilon\}|}{|A^n|} + \frac{|A^n \setminus E^n|}{|A^n|} \leq 0 + \alpha.$$

REFERENCES

Anderson, Robert M., 1978, An elementary core equivalence theorem, *Econometrica* 46, 1483-1487.

Anderson, Robert M., 1981, Core theory with strongly convex preferences, *Econometrica* 49, 1457-1468.

Anderson, Robert M., 1985, Strong core theorems with nonconvex preferences, *Econometrica* 53, 1283-1293.

Anderson, Robert M., 1986, Notions of core convergence, chapter 2 in *Contributions in Mathematical Economics, In honor of Gerard Debreu*, W. Hildenbrand and A. Mas-Colell editors, North -Holland, New York.

Anderson, Robert M., 1988, The second welfare theorem with nonconvex preferences, *Econometrica* 56, 361-382.

Arrow, Kenneth J. and F.H. Hahn, 1971, *General Competitive Analysis*, North-Holland, New York

Aumann, Robert J., 1964, Markets with a continuum of traders, *Econometrica* 32, 39-50.

Bergstrom, Theodore, 1973, Competitive equilibrium without transitivity, monotonicity or local non-satiation, Department of Economics, Washington University.

Bergstrom, Theodore, 1976, How to discard free disposability at no cost, *Journal of Mathematical Economics* 3, 131-134.

Bewley, T., 1973, Edgeworth's conjecture, *Econometrica* 41, 415-452.

Brown, Donald J. and Abraham Robinson, 1974, The cores of large standard exchange economies, *Journal of Economic Theory* 9, 245-254.

Cheng, Hsueh-Cheng, 1983, A uniform core convergence result for nonconvex economies, *Journal of Economic Theory* 31, 269-282.

Debreu, Gérard, 1962, New concepts and techniques for equilibrium analysis, *International Economic Review* 4, 235-246.

Debreu, Gérard and Herbert Scarf, 1963, A limit theorem on the core of an economy, *International Economic Review* 4, 236-246.

Dierker, Egbert, 1975, Gains and losses at core allocations, *Journal of Mathematical Economics* 2, 119-128.

Gabszewicz, Jean and J. F. Mertens, 1971, An equivalence theorem for the core of an economy whose atoms are not too big, *Econometrica,* 39, 713-721.

Grodal B., W. Trockel and S. Weber, 1984, On approximate cores of non-convex economies, *Economics Letters,* 15, 197-202.

Hart, Oliver and Harold Kuhn, 1975, A proof of the existence of equilibrium without the free disposal assumption, *Journal of Mathematical Economics* 2, 335-343.

Hildenbrand, Werner, 1974, *Core and Equilibria of a Large Economy*, Princeton University Press, Princeton, NJ.

Hildenbrand, Werner, 1982, Core of an economy, Chapter 18 in *Handbook of Mathematical Economics,* edited by K. Arrow and M.D. Intrilligator, North-Holland, New York.

Khan, M. Ali, 1974, Some equivalence theorems, *Review of Economic Studies* 41, 549-565.

Khan, M. Ali, 1976, Oligopoly in markets with a continuum of traders: An asymptotic approach, *Journal of Economic Theory* 12, 273-297.

Manelli, Alejandro M., 1988, *Essays in Mathematical Economics,* Ph.D. Dissertation, University of California, Berkeley.

Manelli, Alejandro M., 1989, Monotonic preferences and core equivalence, forthcoming in *Econometrica*.

Mas-Colell, Andreu, 1985, *The Theory of General Economic Equilibrium: A Differentiable Approach*, Cambridge University Press, Cambridge.

McKenzie, Lionel, 1955, Competitive equilibrium with dependent consumer preferences, National Bureau of Standards and Department of the Air Force, The Second Symposium on Linear Programming, 277-294.

McKenzie, Lionel, 1959, On the existence of general equilibrium for a competitive market, *Econometrica* 27, 54-71.

McKenzie, Lionel, 1961, On the existence of general equilibrium: Some corrections, *Econometrica* 29, 247-248.

Rader, T., 1972, *Theory of General Economic Equilibrium,* Academic Press, New York and London.

Shitovitz, Benyamin, 1973, Oligopoly in markets with a continuum of traders, *Econometrica,* 41, 467-501.

Starr, R., 1969, Quasi-equilibria in markets with nonconvex preferences, *Econometrica* 37, 25-38.

Trockel, Walter, 1976, A limit theorem on the core, *Journal of Mathematical Economics,* 3, 247-264.

Vind, Karl, 1965, A theorem on the core of an economy, *Review of Economic Studies* 32, 47-48.

Wooders, Myrna H., 1990, Notes on inessentiality of large coalitions and boundedness of average and marginal contributions to coalitions, Department of Economics, University of Toronto.