point R, the root of T. Denote by $T_i$ the tree that consists of all the vertices and arcs of T connected to the vertex $\beta( e_i )$, where $\beta( e_i )$ denotes the initial point of the arc $e_i$. The tree $T_i$ has length at most L. Denote by $\sigma_i$ the state of $T_i$ that is the restriction of $\sigma$ to $T_i$.

Denote by $C_{(i)}$ the restriction of C to the tree $T_i$. It is easy to see that the function $C\#_{(i)}$ is a transformation of the states $S(T_i;R^n)$. Therefore, it follows by the inductive hypothesis that for each j>L,

$$\{C_{(i)}\cdot I( a )\}^j\sigma_i=\{C_{(i)}\cdot I( a )\}^{j+1}\sigma_i.$$

Let

$$[\{C_{(i)}\cdot I( a )\}^j\sigma]( \beta( e_i ) )=b_i.$$

Then

$$[\{C\cdot I( a )\}^{L+1}\sigma]( R )=$$

$$[(C\cdot I( a ))\{C\cdot I( a )\}^L\sigma]( R )=$$

$$C\#( R;b_1,\ldots,b_{r'} ).$$

Furthermore, for each j larger than L+1, the $C_{(j)}\cdot I( a )\sigma$ assigns the value $b_i$ to $\beta( e_i )$ $1\le i\le r'$ and therefore for each j larger than L+1, $\{C\cdot I( a )\}^j\sigma$ assigns to R the value

$$C\#( R;b_1,\ldots,b_{r'} ).$$

<u>Theorem C.1.</u> Let C be a (r,d)-network with a leaf-free input-output digraph G and let t be a positive integer. Suppose that G has a single output

vertex OV. There is an in-tree T(t) with root R and a mapping of digraphs Θ(t) from T(t) to G that satisfy the following conditions:

(i) Each walk from a leaf of T(t) to R has length t,

(ii) for each walk W of length t in G that connects a vertex w of G to the vertex OV, there is a leaf Λ(W) of T(t),

(iii) if W is a walk of length t in G from a vertex w to the vertex OV, then there is a unique directed path P(W) of T(t) of length t that connects Λ(W) to R,

(iv) Θ(t) carries R to OV and Θ(t) carries P(W) onto W .

(v) for each vertex v of T(t), in-degree(v)≤in-degree(Θ(t)v).

(vi) for each t≥1, T(t-1) is a subgraph of T(t) and the restriction of the map Θ(t) to the subgraph T(t-1) coincides with Θ(t-1) .

Proof. The proof is an induction on the lengths of the walks t. Suppose t=1. Because the digraph G is leaf-free, the vertex OV is the end point of at least one arc. Assume there are arcs $e_1,\ldots,e_{r*}$ that end in OV, where $r*\le r$ is the in-degree of OV. Denote the initial vertex of the arc $e_i$ by $v_i$, $1\le i\le r*$. The walks of G of length 1 must each have as an initial point one of the vertices $v_i$ and each vertex $v_i$ is the initial point of exactly one walk $W_i$, $1 \le i \le r^*$, of length 1 that connects $v_i$ to OV. The vertices $v_i$ are not necessarily distinct. Construct a tree T(1) and a map $\Theta(1)$ as follows. T(1) has root R, and leaves

$$\Lambda( W_1 )=V_1,\ldots,\Lambda( W_{r*} )=V_{r*}.$$

Each leaf $V_i$, $1 \le i \le r^*$, is connected to R by a single arc $E_i$ that has initial point $V_i$ and end point R. The map $\Theta(1)$ carries R to OV, $\Theta(1)$ carries $V_i$ to $v_i$, and $\Theta(1)$ carries the arc $E_i$ to the arc $e_i$. Because the leaves $V_i$ are all distinct, the underlying abstract mixed graph of T(1) is a tree. Furthermore, the number of arcs that end at R is precisely r*. Therefore

$$\text{in-degree}(R) \le \text{in-degree}(OV) = \text{in-degree}(\Theta(1)R).$$

Each walk W of length 1 in G starts at one of the $v_i$ and exactly one path, $(\Lambda(W),E_i,R)$, is mapped by $\Theta(1)$ to the walk W. Because the in-degree of a leaf is 0, this completes the argument for t=1.

Assume now that the value of t is L, and that the assertion of the lemma is true for all networks with a single output vertex and for all walks of lengths t<L.

Because of the inductive hypothesis, we can construct a directed tree T(L-1) and a map $\Theta(L-1)$ from T(L-1) to G such that each walk of length L-1 in G that ends at OV is the image under $\Theta(L-1)$ of a directed path of length L-1 and such that for each vertex v of T(L-1),

in-degree(v)≤in-degree(Θ(L-1)v).

Each leaf of T(L-1) is uniquely associated to a walk in G of length L-1 that ends in OV. These leaves are mapped by Θ(L-1) to vertices in G.

By inductive hypothesis (v), for each vertex v of T(L-1),

in-degree(v)≤in-degree(Θ(L-1)v).

If W is a walk in G of length L-1 that ends in OV, then there is, by inductive hypothesis (iii), a unique path P(W) in T(L-1) that starts from a leaf Λ( W ), ends at the vertex R, and (because of assumption (iv)) it is mapped by Θ(L-1) onto W. If $(B_0,A_1,...,A_n,B_n)$ is a walk with n≥1, then $B_1$ is the second vertex of W. Suppose now that $W_1,...,W_z$ are the walks in G of length L that end at OV. For 1≤i≤z, $W_i$ is a walk $(B_{i0},A_{i1},B_{i1},A_{i2},...,A_{iL},B_{iL})$, where each $B_{ij}$ is a vertex of G and each $A_{ij}$ is an arc of G. For each 1≤i≤z, the sequence of vertices and arcs $(B_{i1},A_{i2},...,A_{iL},B_{iL})$ is a walk $W_i$# in G of length L-1. (Note that even if $W_i$ and $W_j$ are distinct, the walks $W_i$# and $W_j$# may not be distinct walks.) Therefore, there is a unique path P($W_i$#) in T(L-1) that begins with the leaf Λ( $W_i$# ), that ends at R, and is such that the path P($W_i$#) is mapped by Θ(L-1) onto the walk

$W_i$#. Construct the tree T(L) and the map Θ(L) as follows. Form a new set of vertices $\Lambda(W_i)$, $1 \le i \le z$, that are to be leaves of the tree T(L). For each $1 \le i \le z$, connect the vertex $\Lambda( W_i )$ to the vertex $\Lambda( W_i\# )$ of T(L-1) by an arc $a_i$. Note that only one arc now connects $\Lambda(W_i)$ to a vertex of T(L-1), and that arc has as end point the vertex $\Lambda( W_i\# )$ of T(L-1). We define the map Θ(L) so that it coincides with Θ(L-1) on the vertices and arcs of T(L-1), while $\Theta(L)_V( \Lambda( W_i ) )=B_{i0}$ and $\Theta(L)_A( a_i )=A_{i1}$.

Suppose that W is a walk in T(L) from a leaf of T(L) to R. Such a walk must start at some leaf $\Lambda( W_i )$. Because L>1, the leaf $\Lambda( W_i )$ is not R. Therefore, for the walk W to end at R, the initial arc of W must be the initial arc of $W_i$. Such an arc has, by the construction carried out, an end point in T(L-1) and that endpoint is a leaf Λ( W* ) where W* is a walk in T(L-1). If W# is the walk that starts at the second vertex of W and otherwise coincides with the walk W, then the walk W# begins at a leaf of T(L-1) and ends at R. Therefore, by (i), the walk W# has length L-1. It follows that the walk W has length L, because W has one more vertex and arc than W#. This establishes condition (i). Condition (ii) is clearly satisfied, by construction.

We turn now to the proof of (iii). Suppose that W is a walk of length L in G from a vertex v in G to the vertex OV. Use the same notation as that in the previous paragraph. Denote by W# the walk that starts with the second vertex of W and otherwise coincides with W. The walk W# is a walk of length L-1 that is entirely a walk in T(L-1), because the only vertices of T(L) that are not in T(L-1) are leaves. The inductive hypothesis (iii) states that there is a unique directed path P(W#) in T(L-1) of length L-1 that connects Λ( W# ) to R, and hypothesis (iv) states that the map Θ(L-1) carries P(W#) to W#. The path P(W#) is a path from the leaf Λ( W# ) of T(L-1) to R. In T(L) the leaf Λ( W ) is connected by an arc A to the vertex Λ(W#) in T(L-1). By construction, the map Θ(L) carries A to the initial arc of W. The path P=(Λ( W ),A,P(W#)) is carried by Θ(L) onto W. To prove (iii) we need to establish that the path P is the unique directed path of T(t) that connects Λ( W ) to R. If P′ is a second such path, since it begins at Λ( W ) and ends at R, and since R≠Λ( W ), it follows that the first arc of P′ must be the same as the first arc of P because, by construction, there is only one arc A from Λ( W ). If the two paths P and P′ are different, then the two paths that start from the second vertices of P and P′ must be different. However, each of these new paths is a path in T(L-1) that maps from the second vertex of P to R, and (iii) for T(L-1) asserts that there is only one such path. Therefore (iii) is established for t=L. The tree T(L) has been constructed so that T(L-1) is a subgraph of the tree T(L) and the maps Θ(L) and Θ(L-1) coincide on T(L-1), therefore (vi) is satisfied.

Finally, we need to show that (v) is true. We need only establish this for those vertices of T(L-1) that are the leaves (in T(L-1)), $\Lambda( W_i\# )$, because they are the only vertices for which the in-degree is changed. For each $1 \le i \le z$, the in-degree of $\Lambda( W_i\# )$ is the number of the walks among the $\{W_j\}$ that have as second vertex the initial vertex of $W_i\#$. If $\Lambda( W_a )$ and $\Lambda( W_b )$ are to be connected to the same $\Lambda( W_a\# )$, the initial arcs of $W_a$ and $W_b$ must be different, because they must start from different leaves of T(L). On the other hand, these initial arcs must have the same end point which is the initial point of the walk $W_a\#$. Therefore, the number of arcs that end in $\Lambda( W_a\# )$ is at most the number of arcs that are connected to the initial point of $W_a\#$. However, $\Theta(L)( \Lambda( W_a\# ) )$ is the initial point of $W_a\#$. Therefore,

$$\text{in-degree}(\Lambda( W_a\# )) \le \text{in-degree}(\Theta(L)( \Lambda( W_a\# ) )).$$ ▒

The following corollary of Theorem C.1 allows us to conclude that if an (r,d)-network computes a function F in time t, then there is an (r,d)-network with input-output graph that is a tree of length t that also computes the function F in time t. The proof is easy and it is not included.

Corollary C.1. Suppose that C is an (r,d)-network with input-output digraph G that computes a function F( $x_1,\ldots,x_n$ ) in time t. There is an (r,d)-network C′ with input-output digraph a directed tree T of length t that also computes F in time t. Furthermore, if v is a vertex of the input-output digraph T, then the function C′#( v;. ) associated to that vertex is either one of the functions associated to a vertex of the input-output digraph of F, or the function is an identity function.

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