

point R , the root of T . Denote by T_i the tree that consists of all the vertices and arcs of T connected to the vertex $\beta(e_i)$, where $\beta(e_i)$ denotes the initial point of the arc e_i . The tree T_i has length at most L . Denote by σ_i the state of T_i that is the restriction of σ to T_i .

Denote by $C_{(i)}$ the restriction of C to the tree T_i . It is easy to see that the function $C\#_{(i)}$ is a transformation of the states $S(T_i; R^n)$. Therefore, it follows by the inductive hypothesis that for each $j > L$,

$$\{C_{(i)} \cdot I(a)\}^j \sigma_i = \{C_{(i)} \cdot I(a)\}^{j+1} \sigma_i.$$

Let

$$[\{C_{(i)} \cdot I(a)\}^j \sigma](\beta(e_i)) = b_i.$$

Then

$$\begin{aligned} & [\{C \cdot I(a)\}^{L+1} \sigma](R) = \\ & [(\{C \cdot I(a)\} \{C \cdot I(a)\}^L \sigma)](R) = \\ & C\#(R; b_1, \dots, b_{r'}). \end{aligned}$$

Furthermore, for each j larger than $L+1$, the $C_{(j)} \cdot I(a) \sigma$ assigns the value b_i to $\beta(e_i)$ $1 \leq i \leq r'$ and therefore for each j larger than $L+1$, $\{C \cdot I(a)\}^j \sigma$ assigns to R the value

$$C\#(R; b_1, \dots, b_{r'}).$$

Theorem C.1. Let C be a (r, d) -network with a leaf-free input-output digraph G and let t be a positive integer. Suppose that G has a single output

vertex OV . There is an in-tree $T(t)$ with root R and a mapping of digraphs $\theta(t)$ from $T(t)$ to G that satisfy the following conditions:

- (i) Each walk from a leaf of $T(t)$ to R has length t ,
- (ii) for each walk W of length t in G that connects a vertex w of G to the vertex OV , there is a leaf $\Lambda(W)$ of $T(t)$,
- (iii) if W is a walk of length t in G from a vertex w to the vertex OV , then there is a unique directed path $P(W)$ of $T(t)$ of length t that connects $\Lambda(W)$ to R ,
- (iv) $\theta(t)$ carries R to OV and $\theta(t)$ carries $P(W)$ onto W .
- (v) for each vertex v of $T(t)$,
 $\text{in-degree}(v) \leq \text{in-degree}(\theta(t)v)$.
- (vi) for each $t \geq 1$, $T(t-1)$ is a subgraph of $T(t)$ and the restriction of the map $\theta(t)$ to the subgraph $T(t-1)$ coincides with $\theta(t-1)$.

Proof. The proof is an induction on the lengths of the walks t . Suppose $t=1$. Because the digraph G is leaf-free, the vertex OV is the end point of at least one arc. Assume there are arcs e_1, \dots, e_{r^*} that end in OV , where $r^* \leq r$ is the in-degree of OV . Denote the initial vertex of the arc e_i by v_i , $1 \leq i \leq r^*$. The walks

of G of length 1 must each have as an initial point one of the vertices v_i and each vertex v_i is the initial point of exactly one walk W_i , $1 \leq i \leq r^*$, of length 1 that connects v_i to OV . The vertices v_i are not necessarily distinct. Construct a tree $T(1)$ and a map $\theta(1)$ as follows. $T(1)$ has root R , and leaves

$$\Lambda(W_1) = V_1, \dots, \Lambda(W_{r^*}) = V_{r^*}.$$

Each leaf V_i , $1 \leq i \leq r^*$, is connected to R by a single arc E_i that has initial point V_i and end point R . The map $\theta(1)$ carries R to OV , $\theta(1)$ carries V_i to v_i , and $\theta(1)$ carries the arc E_i to the arc e_i . Because the leaves V_i are all distinct, the underlying abstract mixed graph of $T(1)$ is a tree. Furthermore, the number of arcs that end at R is precisely r^* . Therefore

$$\text{in-degree}(R) \leq \text{in-degree}(OV) = \text{in-degree}(\theta(1)R).$$

Each walk W of length 1 in G starts at one of the v_i and exactly one path, $(\Lambda(W), E_i, R)$, is mapped by $\theta(1)$ to the walk W . Because the in-degree of a leaf is 0, this completes the argument for $t=1$.

Assume now that the value of t is L , and that the assertion of the lemma is true for all networks with a single output vertex and for all walks of lengths $t < L$.

Because of the inductive hypothesis, we can construct a directed tree $T(L-1)$ and a map $\theta(L-1)$ from $T(L-1)$ to G such that each walk of length $L-1$ in G that ends at OV is the image under $\theta(L-1)$ of a directed path

of length $L-1$ and such that for each vertex v of $T(L-1)$,

$$\text{in-degree}(v) \leq \text{in-degree}(\theta(L-1)v).$$

Each leaf of $T(L-1)$ is uniquely associated to a walk in G of length $L-1$ that ends in OV . These leaves are mapped by $\theta(L-1)$ to vertices in G .

By inductive hypothesis (v), for each vertex v of $T(L-1)$,

$$\text{in-degree}(v) \leq \text{in-degree}(\theta(L-1)v).$$

If W is a walk in G of length $L-1$ that ends in OV , then there is, by inductive hypothesis (iii), a unique path $P(W)$ in $T(L-1)$ that starts from a leaf $\Lambda(W)$, ends at the vertex R , and (because of assumption (iv)) it is mapped by $\theta(L-1)$ onto W . If $(B_0, A_1, \dots, A_n, B_n)$ is a walk with $n \geq 1$, then B_1 is the second vertex of W .

Suppose now that W_1, \dots, W_z are the walks in G of length L that end at OV . For $1 \leq i \leq z$, W_i is a walk

$(B_{i0}, A_{i1}, B_{i1}, A_{i2}, \dots, A_{iL}, B_{iL})$, where each B_{ij} is a vertex of G and each A_{ij} is an arc of G . For each

$1 \leq i \leq z$, the sequence of vertices and arcs

$(B_{i1}, A_{i2}, \dots, A_{iL}, B_{iL})$ is a walk $W_i\#$ in G of length $L-1$.

(Note that even if W_i and W_j are distinct, the walks $W_i\#$ and $W_j\#$ may not be distinct walks.) Therefore,

there is a unique path $P(W_i\#)$ in $T(L-1)$ that begins with the leaf $\Lambda(W_i\#)$, that ends at R , and is such

that the path $P(W_i\#)$ is mapped by $\theta(L-1)$ onto the walk

$W_i\#$. Construct the tree $T(L)$ and the map $\theta(L)$ as follows. Form a new set of vertices $\Lambda(W_i)$, $1 \leq i \leq z$, that are to be leaves of the tree $T(L)$. For each $1 \leq i \leq z$, connect the vertex $\Lambda(W_i)$ to the vertex $\Lambda(W_i\#)$ of $T(L-1)$ by an arc a_i . Note that only one arc now connects $\Lambda(W_i)$ to a vertex of $T(L-1)$, and that arc has as end point the vertex $\Lambda(W_i\#)$ of $T(L-1)$. We define the map $\theta(L)$ so that it coincides with $\theta(L-1)$ on the vertices and arcs of $T(L-1)$, while $\theta(L)_V(\Lambda(W_i)) = B_{i0}$ and $\theta(L)_A(a_i) = A_{i1}$.

Suppose that W is a walk in $T(L)$ from a leaf of $T(L)$ to R . Such a walk must start at some leaf $\Lambda(W_i)$. Because $L > 1$, the leaf $\Lambda(W_i)$ is not R . Therefore, for the walk W to end at R , the initial arc of W must be the initial arc of W_i . Such an arc has, by the construction carried out, an end point in $T(L-1)$ and that endpoint is a leaf $\Lambda(W^*)$ where W^* is a walk in $T(L-1)$. If $W\#$ is the walk that starts at the second vertex of W and otherwise coincides with the walk W , then the walk $W\#$ begins at a leaf of $T(L-1)$ and ends at R . Therefore, by (i), the walk $W\#$ has length $L-1$. It follows that the walk W has length L , because W has one more vertex and arc than $W\#$. This establishes condition (i). Condition (ii) is clearly satisfied, by construction.

We turn now to the proof of (iii). Suppose that W

is a walk of length L in G from a vertex v in G to the vertex OV . Use the same notation as that in the previous paragraph. Denote by $W\#$ the walk that starts with the second vertex of W and otherwise coincides with W . The walk $W\#$ is a walk of length $L-1$ that is entirely a walk in $T(L-1)$, because the only vertices of $T(L)$ that are not in $T(L-1)$ are leaves. The inductive hypothesis (iii) states that there is a unique directed path $P(W\#)$ in $T(L-1)$ of length $L-1$ that connects $\Lambda(W\#)$ to R , and hypothesis (iv) states that the map $\theta(L-1)$ carries $P(W\#)$ to $W\#$. The path $P(W\#)$ is a path from the leaf $\Lambda(W\#)$ of $T(L-1)$ to R . In $T(L)$ the leaf $\Lambda(W)$ is connected by an arc A to the vertex $\Lambda(W\#)$ in $T(L-1)$. By construction, the map $\theta(L)$ carries A to the initial arc of W . The path $P=(\Lambda(W), A, P(W\#))$ is carried by $\theta(L)$ onto W . To prove (iii) we need to establish that the path P is the unique directed path of $T(t)$ that connects $\Lambda(W)$ to R . If P' is a second such path, since it begins at $\Lambda(W)$ and ends at R , and since $R \neq \Lambda(W)$, it follows that the first arc of P' must be the same as the first arc of P because, by construction, there is only one arc A from $\Lambda(W)$. If the two paths P and P' are different, then the two paths that start from the second vertices of P and P' must be different. However, each of these new paths is a path in $T(L-1)$ that maps from the second vertex of P

to R, and (iii) for $T(L-1)$ asserts that there is only one such path. Therefore (iii) is established for $t=L$. The tree $T(L)$ has been constructed so that $T(L-1)$ is a subgraph of the tree $T(L)$ and the maps $\theta(L)$ and $\theta(L-1)$ coincide on $T(L-1)$, therefore (vi) is satisfied.

Finally, we need to show that (v) is true. We need only establish this for those vertices of $T(L-1)$ that are the leaves (in $T(L-1)$), $\Lambda(W_i\#)$, because they are the only vertices for which the in-degree is changed. For each $1 \leq i \leq z$, the in-degree of $\Lambda(W_i\#)$ is the number of the walks among the $\{W_j\}$ that have as second vertex the initial vertex of $W_i\#$. If $\Lambda(W_a)$ and $\Lambda(W_b)$ are to be connected to the same $\Lambda(W_a\#)$, the initial arcs of W_a and W_b must be different, because they must start from different leaves of $T(L)$. On the other hand, these initial arcs must have the same end point which is the initial point of the walk $W_a\#$. Therefore, the number of arcs that end in $\Lambda(W_a\#)$ is at most the number of arcs that are connected to the initial point of $W_a\#$. However, $\theta(L)(\Lambda(W_a\#))$ is the initial point of $W_a\#$. Therefore,

$$\text{in-degree}(\Lambda(W_a\#)) \leq \text{in-degree}(\theta(L)(\Lambda(W_a\#))). \quad \text{***}$$

The following corollary of Theorem C.1 allows us to conclude that if an (r,d) -network computes a

function F in time t , then there is an (r,d) -network with input-output graph that is a tree of length t that also computes the function F in time t . The proof is easy and it is not included.

Corollary C.1. Suppose that C is an (r,d) -network with input-output digraph G that computes a function $F(x_1, \dots, x_n)$ in time t . There is an (r,d) -network C' with input-output digraph a directed tree T of length t that also computes F in time t . Furthermore, if v is a vertex of the input-output digraph T , then the function $C'_{\#}(v; \cdot)$ associated to that vertex is either one of the functions associated to a vertex of the input-output digraph of F , or the function is an identity function.

BIBLIOGRAPHY

1. Abelson, H. Lower bounds on Information Transfer in Distributed Computations; JACM, Vol. 27, No. 2, April 1980, pp. 384-392.
2. Abhyankar, S.S., Local Analytic Geometry; Academic Press, New York 1964.
3. Arbib, M.A., Theories of Abstract Automata ; Prentice Hall, Inc. Englewood Cliff, New Jersey.
4. Buck, R. C., Advanced Calculus; McGraw -Hill Company, Inc., New York, 1956.
5. Burr, Stefan A,(Ed.), The Mathematics of Networks; Vol.26, Proceedings of Symposia in Applied Mathematics, American Mathematics Society, Providence, Rhode Island, 1982.
6. Fuita, C., The Complexity of Economic Decision Rules; Journal of Mathematical Economics, vol 4, pp.289-299, 1977.

7. Golubitsky, M. and V. Guillemin, Stable Mappings and Their Singularities; Graduate Texts in Mathematics No.14, Springer Verlag, New York, 1973.

8. Hilbert, D., Mathematische Probleme. Vortrag auf Internat. Math. Kongr. Paris. Nachr. Ges. Wiss. Gottingen, 253-297, 1900.

9. Hopcroft, John E. and Ullman, J.D., Introduction to Automata Theory, Languages and Computation; Addison and Wesley Publishing Co., Reading, Massachusetts, 1979.

10. Hurewicz, W. and Wallman, H.(1948), Dimension Theory; Princeton University Press Princeton, New Jersey, 1948.

11. Hurwicz, L., S. Reiter, and D. Saari, On Constructing an Informationally Decentralized Process Implementing a Given Performance Function; Mimeo, Presented and distributed at the Econometric Society World Congress, Aix-en-Provence, 1980.

12. Hurwicz, L., Optimality and Efficiency in Resource Allocation Mathematical Methods in the Social Sciences; K.J. Arrow, S. Karlin, and P. Suppes, Eds. Stanford University Press ,pp.27-48 Stanford, California, 1960.
13. Jordan, J.S., The Competitive Allocation Process is Informationally Efficient Uniquely; Journal of Economic Theory, Vol. 28, (1982), pp.1-18
14. Klein, Erwin and Thompson, Anthony C., Theory of Correspondences; John Wiley and Sons, New York, 1984.
15. Leontief, Wassily, A Note on the Interrelation of Subsets of Independent variables of a Continuous Function with Continuous First Derivatives; Bulletin of the AMS, vol. 53, (1947), pp.343-350
16. Lorentz, G. G., Approximation of Functions; Holt, Rinehart and Winston ,New York ,1966.
17. Lorentz, G. G., The 13-th Problem of Hilbert; Proceedings of the Symposia in Pure Mathematics, vol 28, (1976), pp. 419-430 .

18. McCulloch, W., and W. Pitts,
A Logical Calculus of the Ideas Immanent in
Nervous Activity; Bulletin of Mathematical
Biophysics, V,(1943) pp.115-133

19. Miller, George A., The Magic Number Seven, Plus or
Minus Two: Some Limits on Our Capacity for
Processing Information.; Psychological Review 63
(1956) p. 108.

20. Minsky, Marvin , The society of mind; Simon and
Schuster, New York, N.Y.,1986.

21. Mount, K. R., and S. Reiter, The Informational
Size of Message Spaces; Journal of Economic
Theory, vol. 8,(1974), pp.11-196

22. Mount, K.R., and S. Reiter, On the Existence of
a Locally Stable Dynamic Process with a
Statically Minimal Message Space; Discussion Paper
No. 550, The Center for Mathematical Studies in
Economics and Management Science, Northwestern
University ,Evanston, IL.,1983.

23. Oniki, H., The Cost of Communication in Economic Organization; Quarterly Journal of Economics, vol. 88,(1974), pp.529-550.
24. Pour-El, Marian Boykan, Abstract computability and its relation to the general purpose analog computer (Some connections between logic, differential equations and analog computers), Trans. Amer. Math Soc. 199,(1974), pp. 1-29.
25. Radner, Roy, The Organization of Decentralized Information Processing, AT &T Laboratories, Mimeo. (1990)
26. Reiter, S. ,There is no Adjustment Process with a 2-Dimensional Message Space for "Counter Examples"; Mimeo (1979).
27. Seidenberg, A. (1954), A New Decision Method for Elementary Algebra; Annals of Mathematics, vol 60,(1954), pp. 365-374.
28. Vituskin, A. G., Complexity of Tabulation Problems; translated from the Russian, Pergammon Press, Inc., New York, 1961.

29. Widder, D. V., Advanced Calculus; Prentice Hall,
New York, 1963.

Index

$(x _S Y)$	267
(G, C)	340
$(g_1, \dots, g_n, M_1 x \dots x M_n, h)$	109
(r, d) -module	43
(r, d) -modules	31
(r, d) -network	43, 340
(X/F)	58
(X_j/F)	62
$B(x)$	334
$M(x, z, x', z')$	125
$\Phi(x)$	334
Adequate C^k -revelation mechanism that realizes F	109
Adequate revelation	24
Adequate revelation mechanism realizing F	103
Adequate revelation message space	103
Alphabet	29, 43
Arbib and Spira	41
Arc	334
Arcs	334
Assumption 5.1.	92
Assumption 5.2	92
$BH(F: x_{i-1}, \dots, x_{i-d(i)}; x_{1-1}, \dots, x_{i-1-d(i-1)},$	
$x_{i+1-1}, \dots, x_{n-d(n)})$	97
$BH(F: x_i; x_{<-i>})$	97
$BH^*(F: x_i; x_{<-i>}) [x_i, p_{<-i>}]$	98

C(S)	267
Capacity	10
Closed trail	335
Computability	11
Computes an encoded version	38, 66
Connected	67, 335
Control Box	29
Coordinate correspondences	269
Corollary 6.5.1.	117
Corollary C.1	358
Cycle	335
$D(x_{1\alpha(1)} \cdots x_{N\alpha(N)}; F)$	96
Decentralized	15
Definition 4.1	61, 64
Definition 4.3	66
Definition 4.4	67
Definition 4.5	68
Definition 6.1.	103
Definition 6.2.	105
Definition 6.3.	106
Definition 6.4.	107
Definition 6.5	109
Definition 6.6.	109
Definition 6.7	114
Definition A1.1	269
Definition A1.2	273

Definition A2.1	279
Definition A2.2	283
Definition A2.3	285
Definition A3.1	290
Definition A4.1	297
Definition A5.1	306
Definition C.10	337
Definition C.11	339
Definition C.12	340
Definition C.13	341
Definition C.14	343
Definition C.2	334
Definition C.3	335
Definition C.4	335
Definition C.5	335
Definition C.6	336
Definition C.7	336
Definition C.8	336
Definition C.9	336
Delooping	342
Differentiably separable at (p_1, \dots, p_n)	107
Differentiably separable of rank (r_1, \dots, r_n)	106, 107
Digraph	44, 334
Dimension Based Lower Bound	68
Directed edge	333
Directed edges	334

Directed path	335
Directed tree	336
Directed walk	50
Edges	333
Efficient frontier	20, 93
Elementary loop vertex	336
Elementary vertex	336
Encoding	38
End point	334
Entropy	10
Essential revelation	24
Essential revelation mechanism	103
F factors through a product of manifolds	110
F-Equivalence	58
F-equivalent	62
Finite automaton	28
Finite state sequential machine	28
General abstract mixed graph	333
Gradient separability	24
Graph	334
$G \cdot f$	269
$H(F;x_i;x_{<-i>})$	97
$H^*(F;x_i;x_{<-i>})[x_i,p_{<-i>}]$	98
Head	28
Hilbert's Thirteenth Problem	51
H_j -separator set	40

$I(\cdot)$	339
I-separator set	68
I-separator set for the function F	64
Identity mapping	283
$I_G(a)$	339
In-degree	336
In-tree	336
Individual verifier functions	90
Information image	17
Informational image	93
Initial arc of the walk	334
Initial point	334
Initial vertex of the walk	334
Input vertices	339
Input-output digraph	339
Inputs	27
Interconnection function	43
Internal states	27
$INT[x]$	40
Isomorphic	265, 285
Isomorphism	159, 285
LE-i-separator set	68
Leaf	336
Lemma 2.1	40
Lemma 4.1	62
Lemma 4.2	82

Lemma 4.3	84
Lemma 6.1	99
Lemma 6.2.	107
Lemma 6.3	108
Lemma 8.1.	160
Lemma 8.2.	173
Lemma A1.1	270
Lemma A1.2	273
Lemma A2.1	280
Lemma A2.2	284
Lemma A2.3	285
Lemma A3.1	290
Lemma A3.2	293
Lemma A4.1	298
Lemma C.1	344
Lemma C.2	350
Length of the tree	50
Length of the walk	334
Leontief and Abelson	102
Line	67
$M: X \dashrightarrow Y$	268
$M(x)$	268
Map of digraphs	343
Mapping	279
Matrix of T	306
McCulloch and Pitts Networks	31

Message correspondence	88, 297
Message function of the adequate revelation mechanism	104
Message space	88, 269, 297
Modular network	27
Module	12, 341
Module f	43
Network C computes F in time t	341
Network input line	44
Network input lines	32
Network output lines	32
Next output function	28
Next state function	28
One step iteration	20
Out-degree	336
Out-tree	336
Outcome function	297
Outcome function	88
Output modules	44
Output vertex	44
Output vertices	339
Outputs	27
$P(x, z, x', z')$	133
Path	335
Points	333
Privacy preserving	88

Privacy preserving correspondence	257
$Q(x, z, x', z')$	124
$R(X)$	290
R^d state of G	337
R^d valued functions	337
Realization of F	297
Realize	15, 257
Realizes	88
Rectangle	273
Rectangle correspondence	290
Root	50, 336
R_X	290
$R_X(x)$	290
$S(G; R^d)$	337
Self loop	333
Separated	64
Separator set for an output line	21
Sequential machine	27
Standard basis	75
State	337
Strict mapping	279
Strong equivalence	114
Submersion	105
Superpositions	50
Tape	28
Theorem 2.1	37

Theorem 2.2	38, 41
Theorem 4.1	68
Theorem 6.1	100
Theorem 6.2	102
Theorem 6.3	104
Theorem 6.4	111
Theorem 6.5.	115
Theorem 7.1.	133
Theorem 8.1	167
Theorem A2.1	282
Theorem A3.1	291
Theorem A5.1	308
Theorem A5.2	310
Theorem C.1	351
Trail	335
Transformation of states	337
Tree	50, 335
Turing Machine	28
Type	336
$U \int_S V$	267
Underlying abstract mixed graph	335
Undirected edge	333
Verification function	90
Verification scenario	20, 90
Vertices	333
Walk	334

$X \times Y$	268
$X_{\langle -j \rangle}$	61
X_{ab}	174
X_S	267
$X \int_j^z$	61
\int	61
$\prod_{a \in A} X_a = \prod X_a$	267