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A MODEL OF RANDOM MATCHING*

by

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Abstract

This paper presents a model of random matching between individuals chosen from large populations. We assume that the populations and the set of encounters are infinite but countable and that the encounters are i.i.d. random variables. Furthermore, the probability distribution on individuals according to which they are chosen for each encounter is "uniform", which also implies that it is only finitely additive.

Although the probability measure which governs the whole matching process also fails to be (fully) sigma-additive, it still retains enough continuity properties to allow for the use of the law of large numbers. This, in turn, guarantees that the aggregate process will (almost surely) behave "nicely", i.e., that there will be no aggregate uncertainty.

1. Introduction

Economic theory often requires models of random matching among few individuals out of large populations, as in search models, evolutionary game theory and so forth. Partly in order to avoid the messy computations involved with large but finite populations, it is quite prevalent to assume that there are uncountably many agents of various types, each of which has no effect whatsoever on the aggregate behavior, thus eliminating strategic considerations which extend beyond a specific encounter.

The purpose of this paper is to formalize this notion of random matching. Our goal is to construct a model of random matching, say, for simplicity, between individuals chosen out of two large populations, with the following properties: (1) The encounters are i.i.d. random variables, each of which specifies an individual from each population and the time of their encounter; (2) All individuals of each population have the same distribution over the encounters they may be involved in (in terms of both the time of encounter and the individual with whom they are matched;) (3) For each individual, the probability of being matched with any given individual (of the other population) is zero, and so is the probability of being matched in any given encounter, though with probability one that individual will be matched at some point with someone; and (4) The (realization of the) distribution of encounters over time equals their (common) distribution as random variables with probability one.

This task may seem straightforward: each encounter may be thought of as a random variable, specifying which individuals are matched and possibly also when does the encounter occur, and then one may stipulate that these random variables be i.i.d. with a specific distribution according to the economic assumptions of the model.

However, even much simpler models with uncountably many random variables in general, and i.i.d. ones in particular, pose some fundamental mathematical problems. A special attention has been given in the literature to the law of large numbers, which is often informally invoked to guarantee that there will be no aggregate uncertainty in such a model, though each individual does face "private" uncertainty. Judd (1985) showed that in a model with uncountably many random variables, there is an extension of the probability measure such that the law of large numbers is guaranteed. Since there also exist other extensions for which the law is not valid, there is a disturbing freedom in the choice of the extension. Feldman and Gilles (1985) considered a model with countably many random variables endowed with finitely additive measure and showed that the law of large numbers is always guaranteed.

We choose to model both the populations and the encounters as countable sets endowed with a finitely additive measure. This measure is assumed to be "uniform" in the sense that it equals the limit density of every subset, if the latter exists. Our model differs from that of Feldman and Gilles due to the additional complication of the matching process. More specifically, we have to use a finitely additive probability space. Indeed, if we consider events of the type "Individual number 1 (from population 1) is matched at some point of time with an individual belonging to a set A (a subset of population 2)," we would like its probability to equal the measure of the set A. In particular, it should be zero for every singleton and one for the entire (countable) population. Note that Feldman and Gilles use a finitely additive probability space only for measuring the "number" of random variables and a countably additive one for the underlying distribution of the random variables.

The difficulty is that a finitely additive probability may not guarantee the law of large numbers. However, our model shows that all the requirement mentioned above are consistent. That is to say, not only can one construct a model satisfying the random matching requirements (properties (1)-(3)), one can also do so with strong enough continuity properties of the underlying probability measure to obtain the law of large numbers for some random variables of interest (i.e., property (4)).

Although we restrict our attention to the specific random matching model described above, which suffices for a certain class of economic applications, we believe that our method of proof may be extended to other matching processes as well. For instance, in our model each individual is matched exactly once with probability one. For some applications it may be required to have a more interesting distribution of the number of encounters an individual is about to be involved in (e.g., a Poisson distribution). Some of the proofs may be turn out to be more cumbersome, but the basic procedure is likely to yield the desired results.

The result presented in this paper does not only mean that one may construct a random matching model as described above and use it as a starting point for economic analysis; it may also be interpreted as an alternative, more formal underlying model for many existing economic theory results, implicitly or explicitly assuming a model of this type. It may therefore be viewed as extending the "good news", i.e., the possibility results of Judd, Feldman and Gilles, and Green (1988), to a model which formally describes a random matching process.

Our model and formal results appear in the next section, which is followed by the proofs of the results. The final one is devoted to a brief discussion of finitely additive measures and their relevance to economic theory.

2. Model

Let $(N, 2^N, \mu)$ be a finitely additive measure space where $N=\{1,2,\dots\}$ is the set of natural numbers, and μ is a finitely additive measure constructed as follows. Let Z be the class of subsets of N which have a limit frequency, i.e., $Z=\{A\subset N: \lim_{n\to\infty} \frac{1}{n}|\{i\in A: i\leq n\}| \text{ exists}\}$. Given $A\in Z$, let μ' be this limit frequency: $\mu'(A)=\lim_{n\to\infty} \frac{1}{n}|\{i\in A: i\leq n\}|$. Then for any A and B in Z with $A\cap B=\Phi$, $A\cup B$ is also in Z, and

$$\mu'(A\cup B)=\mu'(A)+\mu'(B)$$

holds. Let μ be an extension of μ'. The proof of the existence of such an μ is fairly standard (see e.g. Feldman & Gilles (1985)).

Now, we construct a finitely additive probability space (Ω, Ξ, ν) for a random matching device. It is constructed by introducing countably many random variables. Formally, let Ω be the set of all functions from N to $[0,1)\times N\times N$ where, for a state of the world $\omega\in\Omega$, a typical value $\omega(k)=(\omega_1(k),\omega_2(k),\omega_3(k))=(t,i,j)$ implies that the i-th individual of type 1 and the j-th individual of type 2 meet in the k-th encounter at time $t\in[0,1)$. We also write $\omega=(\omega_1,\omega_2,\omega_3)$ where $\omega_1:N\to[0,1)$, $\omega_2:N\to N$, and $\omega_3:N\to N$ with the obvious meaning. We define $\Xi=\sigma([\mathcal{B}([0,1))\times 2^N\times 2^N]^N)$, the σ-algebra generated by $[\mathcal{B}([0,1))\times 2^N\times 2^N]^N$ where $\mathcal{B}([0,1))$ is the set of Borel subsets of

$[0,1)$.

We are looking for a finitely additive measure ν on (Ω,Ξ) which satisfies the following conditions. First of all, we need σ-additivity on the sub-σ-algebra denoting only the time of encounters. To do this, let G be a distribution function on $[0,1)$, and let μ_G be a Lebesgue-Stieltjes measure induced by G. We introduce two auxiliary spaces: first, consider $(\Omega^1,\mathcal{E}',\lambda')$ where $\Omega^1=[0,1)^N$, $\mathcal{E}'$ is the σ-algebra generated by $[\mathcal{B}([0,1))]^N$, λ' is a (countably additive) measure on $\mathcal{E}'$ satisfying $\lambda'(E)=\Pi_{k\in K}\mu_G(T^k)$ if $E=\{\omega_1|\omega_1(k)\in T^k$ for all $k\in K\}$ with $|K|<\infty$. Next, consider $(\Omega^1,\mathcal{E},\lambda)$ where $\mathcal{E}=\sigma[\mathcal{E}'\cup\{E\subset\Omega^1|\ \exists F\colon E\subset F,\ \lambda'(F)=0\}]$, and λ on $\mathcal{E}$ is the completion of λ' on $\mathcal{E}'$. Notice that λ is uniquely determined by μ_G (see, e.g., Halmos, 1974). We would like to stipulate that ν be an "extension" of λ, a condition which is formally given below as a special case of P-1 (with $A_k=B_k=N$).

The second condition is that the random variables, i.e., the encounters are i.i.d. and, moreover, for each encounter k, $\omega_1(k)$, $\omega_2(k)$, and $\omega_3(k)$ are independent as well. More formally:

P-1. For all $K\subset N$ (a set of encounters) with $|K|<\infty$, and A_k, $B_k\subseteq N$ $(k\in K)$, and all $E\in\mathcal{E}$, we have

$$\nu(\{\omega\in\Omega|\ \omega_1\in E,\ \omega_2(k)\in A_k \text{ and } \omega_3(k)\in B_k \text{ for } k\in K\}) = \lambda(E)\Pi_{k\in K}\mu(A_k)\mu(B_k).$$

The third condition focuses on individuals. It states that for each individual the time of encounter is independent of the matched partner, and that these are i.i.d. across individuals. It also implies that with probability one, a person is matched once and only once.

P-2. For all $J \subset N$ (a set of individuals) with $|J|<\infty$, for all T^j in $\mathcal{B}([0,1))$ and all $C^j \subseteq N$ $(j \in J)$, we have

$$\nu(\cap_{j\in J}\cup_{k\in N}\{\omega\in\Omega|\ \omega(k)\in T^j\times\{j\}\times C^j,\ \omega_2(k')\neq j \text{ for } k'\neq k\}$$
$$= \nu(\cap_{j\in J}\cup_{k\in N}\{\omega\in\Omega|\ \omega(k)\in T^j\times C^j\times\{j\},\ \omega_2(k')\neq j \text{ for } k'\neq k\} = \Pi_{j\in J}\mu_G(T^j)\mu(C^j).$$

Our main results are:

Theorem: <u>There exists a finitely additive measure ν on (Ω,Ξ) satisfying P-1 and P-2.</u>

With this construction, we will also show that the distribution of encounters over time follows G almost surely.

Proposition. $\nu(\{\omega\in\Omega|\ \mu(\{k|\ \omega_1(k)\leq t\})=G(t) \text{ for all } t\in[0,1)\})=1.$

3. Proofs

(1) Proof of the theorem:

We begin with the following lemma.

Lemma 1: Let Ω be an arbitrary set, and let $\mathcal{B}$ be a subset of 2^Ω satisfying:

i) $\Phi,\ \Omega\in\mathcal{B}$,

ii) if A and B are in $\mathcal{B}$, then $A\cap B$ is also in $\mathcal{B}$,

iii) for any A in $\mathcal{B}$, there exist finitely many pairwise disjoint elements $A_1,\ldots,A_k$ in $\mathcal{B}$ such that $A^c=\cup_{i=1}^k A_i$.

Moreover, let $\nu:\mathcal{B}\to\mathbb{R}$ satisfy:

iv) $\nu(\Phi)=0$, $\nu(\Omega)=1$, and $\nu(A)\geq 0$ for all $A\in\mathcal{B}$,

v) $\nu(A) = \Sigma_{i=1}^{k}\nu(A_i)$ whenever $A = \cup_{i=1}^{k}A_i$ where A and $\{A_i\}$ are in $\mathcal{B}$, and $\{A_i\}$ are pairwise disjoint.

Then there exists an extension of ν to a finitely additive measure on the algebra generated by $\mathcal{B}$.

Proof. Assume that the conditions of the lemma are satisfied. Let $\mathcal{A}$ be the algebra generated by $\mathcal{B}$. We first observe that any set A in $\mathcal{A}$ can be written as a finite union of some pairwise disjoint sets in $\mathcal{B}$. Indeed, by (iii) it suffices to consider events of the type $A=\cap_{i=1}^{n}\cup_{j=1}^{m_i}A_{ij}$ where $\{A_{ij}\}$ are in $\mathcal{B}$. By the distributive law we have

$$A = \cup_{j_1}\cup\ldots\cup_{j_n}[A_{1j_1}\cap\ldots\cap A_{nj_n}] = \cup_{j_1}\cup\ldots\cup_{j_n}B_{(j_1,\ldots,j_n)}$$

where $B_{(j_1,\ldots,j_n)} = A_{1j_1}\cap\ldots\cap A_{nj_n} \in \mathcal{B}$. It follows by induction that any A in $\mathcal{A}$ is written as a finite union of some (not necessarily pairwise disjoint) sets in $\mathcal{B}$. To see that the sets may be assumed disjoint without loss of generality, suppose that $A=\cup_{i=1}^{k}A_i$ holds where A is in $\mathcal{A}$, and $\{A_i\}$ are in $\mathcal{B}$. For each i=1,...,k, consider a partition $\mathcal{P}_i=\{A_i, E_{i1},\ldots, E_{in_i}\}$ of Ω where $\{E_{ij}\}$ are in $\mathcal{B}$. Such a partition exists by virtue of (iii). Then let $\mathcal{P}$ be the join of $\mathcal{P}_1,\ldots,\mathcal{P}_k$. Each element in $\mathcal{P}$ is in $\mathcal{B}$ by virtue of (ii). Then A may be written as

$$A = \cup_{B\in\mathcal{P},\ B\subset A_1\cup\ldots\cup A_k}B.$$

Now, for any $A=\cup_{i=1}^{k}A_i$ with $\{A_i\}$ being pairwise disjoint sets in $\mathcal{B}$, we let

$$\nu(A) = \Sigma_{i=1}^{k}\nu(A_i).$$

First, we claim that $\nu(A)$ is well-defined. To see this, suppose that

$A=\cup_{i=1}^{\ell} B_i$ holds where $\{B_i\}$ are again pairwise disjoint sets in $\mathcal{B}$. Take the join of $\{A_1,\ldots,A_k,\ A^c\}$ and $\{B_1,\ldots,B_\ell,\ A^c\}$; denote it $\{C_1,\ldots,C_m,\ A^c\}$ with $\{C_i\}$ in $\mathcal{B}$. Then we have, by (v),

$$\Sigma_{i=1}^{k}\nu(A_i) = \Sigma_{i=1}^{m}\nu(C_i) = \Sigma_{i=1}^{\ell}\nu(B_i).$$

Next, note that $\nu(A)\geq 0$ for all $A\in\mathcal{A}$. Finally, for A_1, $A_2\in\mathcal{A}$ with $A_1\cap A_2=\Phi$, it immediately follows that $\nu(A_1)+\nu(A_2)=\nu(A_1\cup A_2)$, which completes the proof of the lemma. Q.E.D.

Next, we construct for our space $\Omega=[[0,1)\times N\times N]^N$ a collection of sets $\mathcal{B}$ and a function ν satisfying (i)-(v) of Lemma 1 as well as P-1 and P-2. Some additional notations will be needed. Let N_j' (resp. N_j'') be the set of states of the world such that the j-th player of type 1 (resp. type 2) is selected either not at all or more than once, i.e.,

$$N_j' = \{\omega\in\Omega \mid \text{ either } \omega_2(k)\neq j \text{ for all } k\in N, \text{ or } \omega_2(k)=\omega_2(k')=j \text{ for some } k,\ k'\in N \text{ with } k\neq k'\},$$

and

$$N_j'' = \{\omega\in\Omega \mid \text{ either } \omega_3(k)\neq j \text{ for all } k\in N, \text{ or } \omega_3(k)=\omega_3(k')=j \text{ for some } k,\ k'\in N \text{ with } k\neq k'\}.$$

Next, let $\mathcal{B}_1$ be the class of sets of the form:

$$M^1 = \{\omega\in\Omega \mid \omega_1\in E,\ \omega_2(k)\in A_k,\ \omega_3(k)\in B_k \text{ for } k\in K\},$$

for some $K\subset N$ with $|K|<\infty$, $E\in\mathcal{E}$, and $A_k,B_k\subseteq N$; let $\mathcal{B}_2$ be the class of the sets of the form:

$$M^2 = \cap_{j\in J}\cup_{k\in N}\{\omega\in\Omega|\ \omega(k)\in T_j\times\{j\}\times C_j,\ \omega_2(k')\neq j \text{ for } k'\neq k\},$$

for some $J\subset N$ with $|J|<\infty$, $T_j\in\mathcal{B}([0,1))$, and $C_j\subseteq N$; let $\mathcal{B}_3$ be the class of the sets of of the form:

$$M^3 = \cap_{j\in J}\cup_{k\in N}\{\omega\in\Omega|\ \omega(k)\in T_j\times C_j\times\{j\},\ \omega_3(k')\neq j \text{ for } k'\neq k\},$$

where J, T_j, and C_j are as above; and let $\mathcal{B}_4$ be the class of the sets of the form:

$$M^4 = [\cap_{j\in J'}N'_j] \cap [\cap_{j\in J''}N''_j].$$

Then let $\mathcal{B}$ be a family of sets of the form:

$$M^1\cap M^2\cap M^3\cap M^4$$

where $M^\ell\in\mathcal{B}_\ell$ ($\ell=1,2,3,4$). Note that each $\mathcal{B}_\ell$ is closed under intersection, hence so is $\mathcal{B}$. Let $\nu:\mathcal{B}\to\mathbb{R}$ satisfy $\nu(\cap_{\ell=1}^4 M^\ell)=\prod_{\ell=1}^4\nu(M^\ell)$ where $\nu(M^1)$, and $\nu(M^2)$ and $\nu(M^3)$ are defined by P-1, and P-2 respectively, and $\nu(M^4)=0$ if $M^4\neq\Omega$, i.e., if $J'\neq\Phi$ or $J''\neq\Phi$. Note that if $M=\cap_{\ell=1}^4 M^\ell=\cap_{\ell=1}^4 M^{\ell'}$ and $\nu(M^\ell)>0$ for all $\ell=1,2,3,4$, then $M\neq\Phi$ and $\nu(M^\ell\Delta M^{\ell'})=0$, whence M and ν are well defined. The proof is provided in the appendix.

Now, we have the following lemma.

Lemma 2. $\mathcal{B}$ and ν defined above satisfy i)-v) of Lemma 1.

Proof. Notice that (i), (ii), and (iv) are immediate. As for (iii), note that it suffices to show that for all $M^\ell \in \mathcal{B}_\ell$ ($\ell=1,2,3,4$) $(M^\ell)^c$ is the union of pairwise disjoint elements of $\mathcal{B}$. Moreover, we can restrict our attention to sets M^1 generated by $|K|=1$, M^2 and M^3--by $|J|=1$ and M^4--by $|J'|+|J''|=1$, and for these the claim is straightforward. All we have to prove is, therefore, that $\nu(A)=\Sigma_{i=1}^n \nu(A_i)$ holds whenever A and $\{A_i\}$ are in $\mathcal{B}$, and $\{A_i\}$ are pairwise disjoint. For each $i=1,\dots,n$, A_i can be written as the intersection of M_i and M_i' where

$$M_i = \{\omega\in\Omega|\ \omega_1\in E_i,\ \omega_2(k)\in A_{ik} \text{ and } \omega_3(k)\in B_{ik} \text{ for } k\in K_i\}$$

with $|K_i|<\infty$, and

$$M_i' = [\cap_{j\in I_i'}\cup_{k\in N}\{\omega\in\Omega|\ \omega(k)\in T_{ij}'\times\{j\}\times C_{ij}',\ \omega_2(k')\neq j \text{ for } k'\neq k\}\cap_{j\in J_i'}N_j']$$
$$\cap\ [\cap_{j\in I_i''}\cup_{k\in N}\{\omega\in\Omega|\ \omega(k)\in T_{ij}''\times C_{ij}''\times\{j\},\ \omega_3(k')\neq j \text{ for } k'\neq k\}\cap_{j\in J_i''}N_j'']$$

with $|I_i'|+|I_i''|+|J_i'|+|J_i''|<\infty$, and $M_i'=\Omega$ if $I_i'=I_i''=J_i'=J_i''=\Phi$. Then consider the partitions $\{E_i,E_i{}^c\}$ on $[0,1)^N$, $\{T_{ij}',T_{ij}'{}^c\}$ $(j\in I_i')$ and $\{T_{ij}'',T_{ij}''\}$ $(j\in I_i'')$ on $[0,1)$, and $\{A_{ik},A_{ik}{}^c\}$, $\{B_{ik},B_{ik}{}^c\}$ $(k\in K_i)$, $\{C_{ij}',C_{ij}'{}^c\}$ $(j\in I_i')$, and $\{C_{ij}'',C_{ij}''\}$ $(j\in I_i'')$ on N. Now, consider the join of $\{E_i,E_i{}^c\}$ $(i=1,\dots,n)$, henceforth denoted by Ψ, that of the partitions on $[0,1)$, to which we refer by T, and that of the partitions on N, denoted by Γ. Also consider $\Theta=\{\tilde{N},\tilde{M}\}$ with a measure δ such that $\delta(\tilde{N})=0$ and $\delta(\tilde{M})=1$. Since Ψ, T, Γ, and Θ are finite, $(\Psi, 2^\Psi, \lambda|_{2^\Psi})$, $(T, 2^T, \mu_G|_{2^T})$, $(\Gamma, 2^\Gamma, \mu|_{2^\Gamma})$, and $(\Theta, 2^\Theta, \delta)$ are measure spaces. Let $K=\cup_{i=1}^n K_i$, $I'=\cup_{i=1}^n I_i'$, $I''=\cup_{i=1}^n I_i''$, $J'=\cup_{i=1}^n(I_i'\cup J_i')$, and $J''=\cup_{i=1}^n(I_i''\cup J_i'')$. Consider a product measure space:

$$\tilde{\Omega} = (\Psi\times\Gamma^2)^{|K|}\times(T\times\Gamma)^{|I'|+|I''|}\times\Theta^{|J'|+|J''|}.$$

To each atom of $\tilde{\Omega}$, there corresponds a unique event in Ξ. Also, A and $\{A_i\}$ are associated with events X and $\{X_i\}$ in $2^{\tilde{\Omega}}$, which are rectangles in this product space, and $\nu(A)=\nu''(X)$ and $\nu(A_i)=\nu''(X_i)$ hold where ν'' is a product measure defined by

$$\nu'' = [(\lambda|_{2^\Psi} \times (\mu_G|_{2^T})^2]^{|K|} \times (\mu_G|_{2^T} \times \mu|_{2^\Gamma})^{|I'|+|I''|} \times \delta^{|J'|+|J''|}.$$

Therefore, v) follows, which completes the proof of the lemma. Q.E.D.

These lemmata imply that ν has an extension to a finitely additive measure on (Ω, Ξ) by the Hahn-Banach theorem, which completes the proof of the theorem.

Proof of the proposition. First of all, we have, for every $t \in [0,1)$,

$$\nu(\{\omega \in \Omega | \ \mu(\{k| \ \omega_1(k) \le t\}) = G(t)\}) = \lambda(\{\omega_1 \in [0,1)^N | \ \mu(\{k| \ \omega_1(k) \le t\}) = G(t)\}) = 1$$

where the second equality holds by virtue of the law of large numbers. Let Q denote the set of rational numbers between zero and unity and τ denote the set of discontinuity points of G, which is a countable set since G is monotone. Then

$$\begin{aligned}
&\nu(\{\omega \in \Omega | \ \mu(\{k| \ \omega_1(k) \le t\}) = G(t) \text{ for all } t \in Q \cup \tau\}) \\
&= \lambda(\cap_{t \in Q \cup \tau} \{\omega_1 \in [0,1)^N | \ \mu(\{k| \ \omega_1(k) \le t\}) = G(t)\}) \\
&= \lambda(\Omega) - \lambda(\cup_{t \in Q \cup \tau} \{\omega_1 \in [0,1)^N | \ \mu(\{k| \ \omega_1(k) \le t\}) \ne G(t)\}) \\
&\ge 1 - \Sigma_{t \in Q \cup \tau} \lambda(\{\omega_1 \in [0,1)^N | \ \mu(\{k| \ \omega_1(k) \le t\}) \ne G(t)\}) = 1
\end{aligned}$$

holds where the inequality holds by virtue of σ-additivity of λ. Note that at each $\omega \in \Omega$, both $\mu(\{k| \ \omega_1(k) \le t\})$ and $G(t)$ are monotone functions of t. Should they equal for all $t \in Q \cup \tau$, they would also equal for all $t \in [0,1)$. Hence we have

$$\nu(\{\omega \in \Omega | \ \mu(\{k| \ \omega_1(k) \le t\}) = G(t) \text{ for all } t \in [0,1)\}) = 1. \qquad \text{Q.E.D.}$$

4. Finitely Additive Measures -- A Discussion

Finitely additive measures were not introduced into economic theory for the sole purpose of having the benefits of the law of large numbers with many i.i.d. random variables. They were also used in other economic models, for instance, Armstrong and Richter (1984) and Weiss (1981). Yet one may wonder why should we deal with these somewhat unconventional mathematical tools, and are we not better off with a countably additive measure on uncountable sets. Without any claim to originality, we would like to draw the reader's attention to the following points:

a. A finitely additive measure is, of course, a weaker notion than a countably additive one. Hence, the question should rather be: How can one justify countable additivity? As is well known, countable additivity is equivalent to continuity of a measure (with respect to chains of inclusion,) and while continuity is a nice property to work with, in many cases it is simply a matter of mathematical convenience, rather than a fundamental axiom.

De Finetti (see, for instance, 1949, 1950) was a strong proponent of finitely additive probability measures, and Savage (1954) and Dubins and Savage (1965) followed in this vein. As a matter of fact, Savage, which is often considered to provide the most satisfactory axiomatic justification of Bayesian decision making, suggests axioms which only guarantee a finitely additive probability measure.

For the sake of honesty we should admit that one can relatively easily supplement Savage's axioms so that they be equivalent to a continuous measure. (See, for instance, Gilboa, 1985, 1989.) However, these continuity axioms fail to be as compelling as the more basic ones (such as the Sure Thing Principle.) Needless to say, almost all axiomatic models have some of these (such as an archimedian and/or a non-atomicity axioms;) however, the less technical restrictions one imposes, the happier should one be.

b. Though a finitely additive measure may be a somewhat unintuitive construct, it should be compared to its viable alternatives. We feel that it fares quite well, at least if the alternative is a continuum of agents, of encounters and so forth. One should recall that countable additivity is typically obtained at the cost of restricting the sigma-algebra of measurable events, whereas finitely additive measures allow for every subset to be measurable. Put differently, to avoid discontinuity problems we have become accustomed to say that the problematic entities do not exist, a solution which can hardly be suggested as the more intuitive.

Once we swallow the pill of finite additivity, there seems to be little point in modeling intuitively discrete entities -- such as agents -- by a continuum. As opposed to points of time, quantities or prices -- all of which are naturally modeled as continua -- agents, encounters etc. do not necessarily make one think of the real line.

c. It has been suggested quite often (as in some of the papers mentioned in the introduction) that an alternative to infinite populations, countable or not, is simply to consider finite large populations and study their asymptotic behavior. Many authors seem to believe that this would be the "right" thing to do, and only appallingly complicated mathematical computations, if anything, should allow us to use infinite, simpler models.

We would like to propose here a different viewpoint, according to which infinite models are sometimes conceptually "better" than finite ones. Of course, we are interested in finitely many agents, so that a finite model is a more accurate one <u>objectively</u>. However, when a very large population is considered, it may often be the case that individuals, being boundedly rational, perceive the situation as if the population were infinite, as if their behavior had no effect on aggregate variables and so forth. For these cases an infinite model would be a better subjective description of reality unless we succeed in incorporating bounded rationality in a finite model. Until then, the validity of the infinite model's results does not depend on the limit behavior of the finite model; should the latter differ from the former, it is the infinite model we should expect to better approximate reality.

APPENDIX

Claim. If $M=\cap_{\ell=1}^{4}M^{\ell}=\cap_{\ell=1}^{4}M^{\ell'}$, and if $\nu(M^{\ell})>0$ for all $\ell=1,2,3,4$, then

$$\nu(M^{\ell}\Delta M^{\ell'}) \equiv \nu((M^{\ell}\setminus M^{\ell'})\cup(M^{\ell'}\setminus M^{\ell})) = 0$$

holds for $\ell=1,2,3,4$.

Proof. Assume that the provisions of the claim hold. First of all, observe that $\nu(M)>0$ implies that $M^4=M^{4'}=\Omega$. Next, M^1 may be written as

$$M^1 = \{\omega\in\Omega \mid \omega_1\in E,\ \omega_2(k)\in A_k,\ \omega_3(k)\in B_k \text{ for } k\in K\},$$

where $E\in\mathcal{E}$, $A_k,B_k\subseteq N$, and $K\subset N$ with $|K|<\infty$. Likewise, M^2 and M^3 are written as

$$M^2 = \cap_{j\in I_a}\cup_{k\in N}\{\omega\in\Omega \mid \omega(k)\in T_j\times\{j\}\times C_j,\ \omega_2(k'')\neq j \text{ for } k''\neq k\},$$

where $I_a\subset N$ with $|I_a|<\infty$, $T_j\in\mathcal{B}([0,1))$, and $C_j\subseteq N$, and

$$M^3 = \cap_{j\in I_b}\cup_{k\in N}\{\omega\in\Omega \mid \omega(k)\in S_j\times D_j\times\{j\},\ \omega_3(k'')\neq j \text{ for } k''\neq k\},$$

where $I_b\subset N$ with $|I_b|<\infty$, $S_j\in\mathcal{B}([0,1))$, and $D_j\subseteq N$, respectively. $M^{1'}$, $M^{2'}$, and $M^{3'}$ are assumed to be represented in the same way with primed symbols. In the following, we shall prove $\lambda(E\Delta E')=0$ only. The proofs for other sets are similar.

Assume the contrary, i.e., that $\lambda(E\Delta E')>0$. We assume without loss of generality that $\lambda(E\setminus E')>0$ holds. We now construct ω^* which belongs to M^1, M^2, and M^3, but not $M^{1'}$. For $k\in K$, let $\omega_2^*(k)$ and $\omega_3^*(k)$ belong to $A_k\setminus I_a$ and $B_k\setminus I_b$ respectively where $A_k\setminus I_a$ and $B_k\setminus I_b$ are nonempty since $|I_a|+|I_b|<\infty$, while $|A_k|=\infty$ and $|B_k|=\infty$ hold as $\nu(M)>0$. Next, it is claimed that there exist ω_1^*, $\{k_j\}_{j\in I_a}$ and $\{k_j''\}_{j\in I_b}$ such that $\omega_1^*\in E\setminus E'$, $\omega_1^*(k_j)\in T_j$, and $\omega_1^*(k_j'')\in S_j$. Indeed, if not, for any $\omega_1\in E\setminus E'$, there exist either $ja=ja(\omega_1)\in I_a$ or $jb=jb(\omega_1)\in I_b$, and $\tilde{K}=\tilde{K}(\omega_1)$ with $|\tilde{K}|\leq|I_a|+|I_b|<\infty$ and $\tilde{K}\cap K=\phi$ such that for all $k''\notin\tilde{K}\cup K$ we have either $\omega_1(k'')\in T_{ja}{}^c$ or $\omega_1(k'')\in T_{jb}{}^c$. Let $E(ja,\tilde{K})$ and

$E(jb,\widetilde{K})$ denote the set of such ω_1's. Then we have

$$E\setminus E' \subset \cup_{|\widetilde{K}|\leq|I_a|+|I_b|}[\{\cup_{ja\in I_a}E(ja,\widetilde{K})\}\cup\{\cup_{jb\in I_b}E(jb,\widetilde{K})\}]$$

where the right hand side of the relation is a countable union of $E(\bullet)$'s. Observe that $\lambda(E(ja,\widetilde{K}))=0$ holds since $\{\omega_1(k)\colon k\in N\}$ are i.i.d., and $\mu_G(T_j{}^c)<1$ and $\mu_G(S_j{}^c)<1$ hold for all $j\in I_a$ and all $j\in I_b$, respectively. Thus, by countable subadditivity of λ, we get $\lambda(E\setminus E')=0$, which is a contradiction. Next, for each k_j $(j\in I_a)$, let $\omega^*(k_j)\in T_j\times\{j\}\times C_j$, and for each k_j'' $(j\in I_b)$, let $\omega^*(k_j'')\in S_j\times D_j\times\{j\}$. Finally, let $\omega_2^*(k)\notin I_a$ for $k\notin K\cup\{k_j\colon j\in I_a\}$, and let $\omega_3^*(k)\notin I_b$ for $k\notin K\cup\{k_j''\colon j\in I_b\}$. Then ω^* belongs to M^1, M^2, and M^3, but not $M^{1\prime}$, which is a contradiction.

Q.E.D.

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