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VINTAGE CAPITAL, INVESTMENT AND GROWTH *

by

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Abstract

We study the dynamics of growth and investment in a continuous time model with vintage capital. Vintage capital models may be characterized by non-exponential rates of depreciation and technical change and can incorporate "gestation lags" as well as "learning by doing". We investigate the effect of such features on the dynamics of investment and growth and show how they can contribute to explain the volatile nature of investment time-series.

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1. Introduction.

In models of vintage capital, equipments of different vintage may differ in their productivity due to technical progress or due to the effects of variable depreciation rates. In particular, such models allow for the possibility of non-uniform or non-exponential rates of depreciation and technical progress. Indeed, in certain certain plausible situations, equipment may, because of "learning by doing", become progressively more productive during an initial phase of its lifetime before it depreciates later on. Alternatively, there may be cases involving "gestation lags", where initially, new equipment is totally unproductive. In this paper we analyze the implications of non-exponential depreciation, and of "learning by doing", on the dynamics of investment in an optimal growth model with vintage capital. Essentially, in such a model one must keep track of equipments of different vintages to describe the investment dynamics. As we show below, non-exponential depreciation structures, with or without "learning by doing", may help explain the highly volatile nature of investment time-series. (For an empirical investigation in a discrete-time model, see also Benhabib and Rustichini [1989].) To introduce the discussion, it may be helpful to begin with the classical view of capital.

If we denote with $K(t)$ the capital stock at time t, and by $k(t)$ the investment (according to a notation which will be consistently used in the

rest of this paper) then the standard model of exponential depreciation.

$$\dot{K}(t) = k(t) - \gamma K(t) \tag{1.1}$$

has a solution (provided $K(t)e^{-\gamma t} \to 0$ as $t \to \infty$) given by:

$$K(t) = \int_{-\infty}^{t} k(s)e^{-\gamma(t-s)}ds, \text{ for } t \geq 0. \tag{1.2}$$

Note that in this formulation the efficiency of an investment good of vintage t has its efficiency reduced by a factor $e^{-\gamma t}$.

The dynamics of the optimal path in the exponential depreciation case are well known. We can take the case of a linear utility as an easy reference. It is known that if the initial capital stock is lower than the steady state value, then investment grows (with the optimal level of consumption equal to zero) until the steady state is reached. After that, both the capital stock and investment are constant. One of the points that we shall argue in this paper is that the assumption of exponential depreciation suffers by the virtue of its own simplicity (that is, by dramatically reducing the possible dynamics that an optimal growth model can describe).

In section 2 we provide a single discrete-time example to illustrate some of our main points. Section 3 sets out some notation and definitions. Section 4 describes our model of vintage capital and some properties of its solution. Section 5 contains a few technical results that characterize the solutions to the problems posed in section 4. Section 6, which contains the main results, describes the different investment dynamics that emerge under various assumptions on depreciation schemes, including those with "learning by doing" and "gestation lags". The standard "exponential depreciation" case turns out to be a particularly special case. Finally, section 7 shows how persistent and robust oscillations in investment can obtain in a model with "learning by doing" and with a strictly concave utility function.

2. A Simple Example.

The analysis of a simple example can probably be useful to clarify some of the issues we are going to discuss. Consider an economy where each investment good, once produced, lasts for only two periods, and then becomes completely useless. We also allow the relative efficiency of the investment good to change over its life span. The productive technology is given by a neoclassical production function where the quantity produced depends on a linear combination of the investment goods which are active at the time. The evaluation function is linear.

Formally, we are considering the problem:

$$\sup_{\{k_t\}_{t\geq 1}} \sum_{t=1}^{+\infty} \delta^t[f(ak_{t-2}+bk_{t-1}) - k_t] \tag{2.1}$$

$$\text{subject to } f(ak_{t-2}+bk_{t-1}) - k_t \geq 0 \quad k_t \geq 0 \quad t=1,2,\ldots,$$

where k_{-1}, k_0 are given. Here, $a,b \geq 0$. For concreteness, we shall set $f(x) = x^\alpha$, for $\alpha \in (0,1)$. To a given point (k_{t-1}, k_t), we associate the quantity $K_t \equiv ak_{t-1} + bk_t$, which we call the "capital stock".

Let us first consider the case where $a = b = 1$. The Euler equation for an interior solution is

$$-1 + \delta\alpha K_t^{\alpha-1} + \delta^2\alpha K_{t+1}^{\alpha-1}=0 \qquad t=1,\ 2,\ \ldots. \tag{2.2}$$

which gives the steady state value for the capital stock, K, as

$$\hat{K} = [\alpha\delta(1+\delta)]^{\frac{1}{1-\alpha}} \quad . \tag{2.3}$$

Consider now the curves: $\{(k_t, k_{t+1}):\ k_{t+1} + (k_t+k_{t+1})^\alpha = \hat{K}\} \equiv C$ and $\{(k_t, k_{t+1}): k_t + k_{t+1} = \hat{K}\} \equiv \tilde{J}$ in the state space (k_t, k_{t+1}). Their relative position is as in Figure 2.1A, if $\alpha < [\delta(1+\delta)]^{-1}$; and as in Figure 2.1B otherwise.

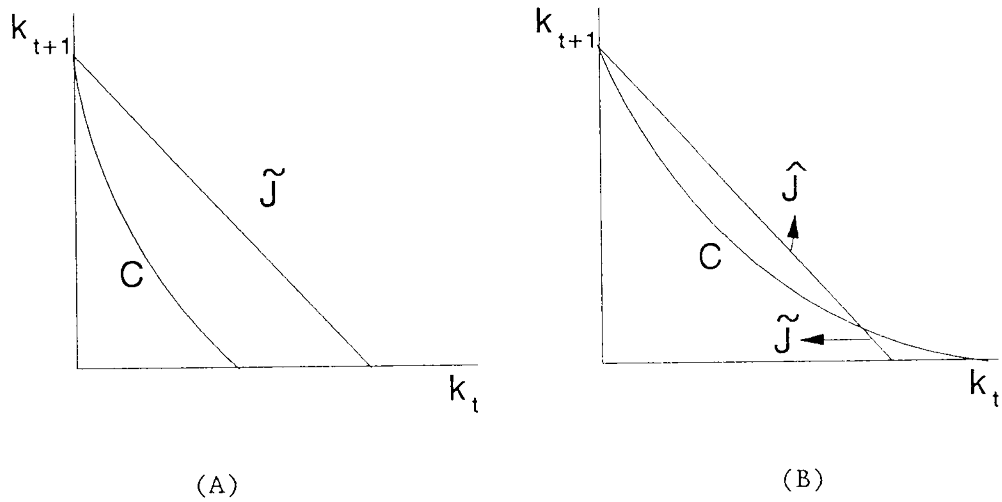

Figure 2.1

Let us consider case A first, and define the region

$$\{(k_t,\ k_{t+1}):\ k_t,\ k_{t+1} \geq 0,\ \ k_{t+1} + K_{t+1}^{\alpha} \geq \hat{K}\} \equiv F.$$

Proposition 1: For $(k_t,\ k_{t+1}) \in F$, the value function of problem (2.1) is given by

$$V(k_t,\ k_{t+1}) = (1+\delta)^{-1}k_{t+1} + K_{t+1}^{\alpha} + D \tag{2.4}$$

with $D \equiv \delta(1-\delta)^{-1}\hat{K}^{\alpha} - (1-\delta^2)^{-1}\hat{K}$.

The optimal policy is given by

$$\hat{c}(k_t, k_{t+1}) = \begin{cases} 0 & \text{if } (k_t, k_{t+1}) \in F^c \bigcup \tilde{J} \\ K_{t+1}^{\alpha} + k_{t+1} - \hat{K} & \text{otherwise.} \end{cases} \tag{2.5}$$

where F^c is the complement of F.

The proof is a standard computation.

The dynamics of the capital string (k_t, k_{t+1}) along the optimal path are easily described. Any point (k_t, k_{t+1}) in the region $\mathbb{R}_+^2 \setminus F$ is mapped into $(k_{t+1}, (k_t + k_{t+1})^{\alpha})$; any point in the region F is mapped into $(k_{t+1}, \hat{K} - k_{t+1}) \in \tilde{J}$. Notice that the set $\tilde{J}$ is an attracting invariant set. Once on $\tilde{J}$, the dynamics are purely oscillatory. So, in this simple case, the dynamics can be decomposed into convergence to $\tilde{J}$ (when out of J), and then oscillations inside $\tilde{J}$.

The situation is more complicated in case B. In this case only the component $\hat{J} \equiv \{(k_t, K_{t+1}) \in \tilde{J} : k_{t+1} + K_{t+1}^{\alpha} \geq \hat{K}\}$ is an attracting and invariant set. The component $\tilde{J} \setminus \hat{J}$ is unstable: capital strings in it cannot sustain the capital stock $\hat{K}$ (the "old" capital stock is relatively too high).

We may now consider the case where a and b are not equal to 1. Now $\hat{J} \equiv \{ (k_t, k_{t+1}): k_t, k_{t+1} \geq 0, \; ak_t + bk_{t+1} = \hat{K} \}$ where $\hat{K} = [\alpha\delta(b+a\delta)]^{\frac{1}{1-\alpha}}$; the analogues of Figure 1 for cases A and B hold if $\alpha \gtrless [\delta(b+a\delta)]^{-1}$ respectively. The set F is defined in an analogous way to the case where a = b = 1.

Now two distinct possibilities arise. We consider here case A for simplicity. If 0 < a < b, then the optimal consumption policy is given by

$$\hat{c}(k_t, k_{t+1}) = \begin{cases} 0 & \text{if } (k_t, k_{t+1}) \in F^c \bigcup \tilde{J} \\ K_{t+1}^{\alpha} + \frac{a}{b}k_{t+1} - \hat{K} & \text{otherwise.} \end{cases} \tag{2.6}$$

The set $\tilde{J}$ is an attractor and invariant set. The dynamics on $\tilde{J}$ are now different, and are described by

$$k_{t+1} = \hat{K}b^{-1} - ab^{-1}k_t \tag{2.7}$$

so that $\lim_{t \to +\infty} k_t = k^* = \hat{K}(a+b)^{-1}$ along the optimal path.

Let us now consider the case where 0 < b < a. This corresponds to a situation in which the investment goods, once installed, become more efficient before the final decay. This may be considered a form of learning by doing. Here the set $\tilde{J}$ is no longer an attractor. It is also not invariant, except for the point (k^*, k^*), where $k^* = \hat{K}(a+b)^{-1}$ as before. In fact, the difference equation $k_{t+1} = \hat{K}b^{-1} - ab^{-1}k_t$ is unstable.

Note that in this case it is no longer an optimal policy to set consumption equal to zero in order to reach the steady state value $\hat{K}$ of capital stock. Indeed, the policy of reaching $\hat{K}$ as soon as possible would lead to overinvestment. As the old capital becomes more efficient the effective capital stock K_t would overshoot $\hat{K}$, making necessary a negative investment in the following period in order to restore the value $\hat{K}$. But in the model that we are considering once a factory is built, it cannot be used for consumption, and negative investment is impossible.

One final remark: it is easy to show, by analyzing the Kuhn–Tucker conditions, that in case A the optimal policy for a state in the region below the curve C, is to set consumption in each period equal to zero, until the first time in which the capital profile exits such region.

There are further cases that may be analyzed in the simple context of the above example. The case of a one–period gestation lag corresponds to setting $0 = b < a$. Maintenance costs during the gestation period may be included by allowing negative values for b. Finally, the example may be modified to incorporate secondary markets for old capital goods. Old capital could either be sold on the secondary market at some price, or it could be consumed directly. Such a setup would require an additional decision variable to determine the quantity of the old capital to be carried over. A class of forestry models which allow old capital to be consumed has been studied by Mitra and Wan [1985, 1986].

3. **Notation and Definitions.**

Let $\mathbb{R}_+$ ($\mathbb{R}_-$, respectively) denote $[0, +\infty)$ ($(-\infty, 0]$) respectively. We let D be the family of positive finite measures on $(\mathbb{R}_-, B(\mathbb{R}_-))$; for $\mu, \gamma \in D$ we say $\mu \geq \gamma$ if $\mu(A) \geq \gamma(A)$ for $A \in B(\mathbb{R}_-)$.

The cone $D^* \subset D$ of measure of increasing depreciation is the set of measures which satisfy the condition that for every $\delta > 0$, the function $t \mapsto \mu([t, t+\delta))$, defined on $(-\infty, -\delta]$ is increasing. To any $\mu \in D$, we associate the measure $\mu^* \in D^*$ defined by $\mu^*([a,b)) \equiv \sup_{t \leq a} \mu([t, t+b-a))$.

M defines the space of (Lebesgue) measurable functions defined on $(-\infty, 0]$; $\overline{M}$ the analogous space defined over $\mathbb{R}$. M^+ and $\overline{M}^+$ denote the positive cones of these spaces.

For a function $k: (-\infty, t] \to \mathbb{R}$, with $t > 0$, with $t > 0$, we denote with $k_t(s) = k(t+s)$ the function $k_t: (-\infty, 0] \to \mathbb{R}$. Also, for any interval $[a, b]$ the function $k_{[a,b]}$ denotes the restriction to $[a, b]$.

To every $k \in M$, we associate the quantity $K(t) \equiv \int_{-\infty}^{t} k(s)\, d\mu(s-t)$.

Finally, χ_A denotes the characteristic function of the set A: $\chi_A(t) \equiv \{1 \text{ if } t \in A, \ \ 0 \text{ if } t \notin A\}$.

4. An Optimal Growth Model with Vintages.

The technology in our economy is given by a neoclassical production function of the capital stock K(t) and labor (which is normalized to 1). There is a exogenous deterministic technical progress of the labor augmenting type, with rate of growth $\lambda > 0$.

Let $F: \mathbb{R}^2_- \to \mathbb{R}$ be a neoclassical production function, i.e.,

$$\begin{aligned} &\text{(i)} \quad F \text{ is concave, } C^2 \text{ in the interior of } \mathbb{R}^2_+; \\ &\text{(ii)} \quad F \text{ is homogeneous of degree one;} \\ &\text{(iii)} \lim_{x \to +\infty} \frac{\partial F}{\partial x}(x, y) = 0 \quad \text{for every } y. \end{aligned} \tag{4.1}$$

We denote $f(x) = F(x, 1)$.

To a given $k \in M$ (the initial capital profile), we associate the set of admissible capital and consumption paths A(k), namely the set of pairs (c, k) $\in M^+ \times \overline{M}^+$ which satisfy ($\overline{c}$ is some positive constant):

(i) $k_0 = k$,

(4.2) (ii) $F\left(\int_{-\infty}^{t} k(s)\, d\mu(s-t),\ e^{\lambda t} \right) - c(t) = k(t)$ (Lebesque)-a.e.t

(iii) $\bar{c} \geq c(t) \geq 0$, $k(t) \geq 0$.

The optimal growth problem is then defined by:

$$\sup_{(c,k)\in A} \int_0^{+\infty} e^{-rt}\, U(c(t))\, dt \;. \tag{G}$$

In the following we shall consider a special form of the above problem. More precisely, if (4.2.ii) is replaced by

$$f\left(\int_{-\infty}^{t} k(s)\, d\mu(s-t) \right) - c(t) = k(t) \quad \text{Lebesque-a.e.t} \tag{4.2.ii'}$$

then the optimal growth problem G is said to be in the standard form. We can now show that this does not imply a loss of generality.

If $U(c) = c^\alpha$, $\alpha \in (0, 1]$, and $\lambda\alpha - \gamma < 0$, then the optimal growth problem can always be reduced to the standard form. This follows immediately from the homogeneity of F; it is enough to define the new quantities:

$\tilde{k}(t) \equiv e^{-\lambda t} k(t)$; $d\tilde{\mu}(t) = e^{\lambda t} d\mu(t)$.

Before discussing the characterization of the optimal path, we establish its existence in Lemmas 4.1 and 4.2. The conditions on F give, for every $\epsilon >$

0, a real number A_ϵ such that $f(x) \leq A_\epsilon + \epsilon x$. Then we have

<u>LEMMA 4.1</u> The path of maximal accumulation, defined by:

$$k(t) = f\left(\int_{-\infty}^{t} k(s)\, d\mu(s-t) \right)$$

satisfies the inequality:

$$\| k_{[t,t+1]} \|_\infty \leq C \left(\frac{A_\epsilon}{1-t} t + \| k_{[-1,0]} \| \right) \left(\frac{\epsilon}{1-\epsilon} \right)^t$$

where C denotes a positive constant.

<u>Proof</u>: The existence of such path is given by a standard contraction argument. By replacing μ by μ^* if necessary we may assume that $\mu \in D^*$. Then for any $h \in [0, 1]$, and $t \geq 0$:

$$k(t+h) \leq A_\epsilon + \epsilon k(t) + C\epsilon \|k_{[t,t+h]}\|_\infty \leq A_\epsilon + \epsilon k(t) + C\epsilon \|k_{[t,t+1]}\|_\infty,$$

and therefore

$$\|k_{[t,t+1]}\|_\infty \leq \frac{A_\epsilon}{1-\epsilon} + \frac{C\epsilon}{1-\epsilon} \|k_{[t-1,t]}\|_\infty$$

which implies the result. ∎

Now it is easy to prove:

LEMMA 4.2 The optimal growth problem has a solution for any $k_0 \in M_+$.

Proof Since the problem is concave, this follows immediately, by standard arguments, from the estimate in Lemma 4.1. ∎

We can now proceed to characterize the optimal path. We first need some conditions on the measure μ. More specifically, the following assumption (A1) will be standing for the remainder of the paper:

$$\begin{aligned} &\text{(i)} \quad d\mu(s) = m(s)\,d(s), \quad m \in L^1(\mathbb{R}_-; [0, 1]) \\ &\text{(ii)} \quad m(0) \neq 0 . \end{aligned} \tag{A1}$$

We shall see that optimal interior paths satisfy an integral equation which we immediately introduce:

$$\int_{-\infty}^{0} k(t+s)\,d\mu(s) = C \quad , \quad k_0 = \phi \tag{I}$$

where the constant, dependent only on μ and f, is given by:

$$C = C(\mu, f) \equiv f'^{-1}\left\{\left[\int_{-\infty}^{0} e^{rs} d\mu(s)\right]^{-1}\right\} \tag{4.3}$$

We say that a measurable function $k: \mathbb{R} \to \mathbb{R}$ which satisfies (4.2) for every $t \geq 0$ is a <u>solution</u> of the integral equation.

If, in addition:

$$\bar{c} \geq f\left(\int_{-\infty}^{0} k(t+s)\, d\mu(s)\right) - k(t) \geq 0 \tag{4.4}$$

and $k(t) \geq 0$, then the solution is said to be <u>feasible</u>; it is said to be <u>interior</u> if the inequalities hold strictly. We now discuss the relationship between the solutions of the optimal growth problem and the solutions of the integral equation.

<u>Theorem 4.1</u> Let A1 be satisifed. Then an interior optimal solution to the growth problem (G) satisifies the integral equation (I).

<u>Proof</u> See Appendix.

<u>Remark</u> In the case where the utility function is not linear we can derive the first order (Euler) equation:

$$\int_t^{+\infty} e^{-rs}U'(c(s))f'(K(s))m(t-s)\,ds - U'(c(t))e^{-rt} = 0 \quad \text{for every } t \geq 0,$$

which we shall use later.

5. The Dynamics of Stocks and Flows.

Theorem 4.1 showed the connection between interior optimal paths and solutions to the integral equation (I). In this section, we will establish conditions under which a unique solution to the integral equation (I) exists and provide a characterization of the solution (see Lemma 5.3 below). The characterization of the solution of the integral equation will allow us in section 6 to study the dynamics of the optimal investment path under different assumptions of schedules for depreciation and under "learning by doing". For the most part the results of this section are technical and may be skipped.

Definition $$\tilde{J} \equiv \left\{ \phi \in M^+ : \int_{-\infty}^{0} \phi(s)\,d\mu(s) = C \right\}.$$

We now define the family of operators $T_t: M \to M$ for every $t \geq 0$, and $S_t: \tilde{J} \to \tilde{J}$:

$T_t: M \to M$ for every $t \geq 0$ is the optimal solution operator;

$S_t: \tilde{J} \to \tilde{J}$ for every $t \geq 0$ is the integral equation operator.

We assume for the moment that $\{S_t\}$ is well defined, i.e., for every $\phi \in \tilde{J}$ there exists a $k \in \overline{M}$ such that $\int_{-\infty}^{0} k_t(s)\, d\mu(s) = 0$. We shall discuss later the conditions which insure that $\{S_t\}$ is well defined for every t. Note that T_t is well defined because a solution to the optimization problem exists and is unique if f is strictly concave (which we assume).

Theorem 4.1 established that an interior optimal solution solves the integral equation (I). Lemma 5.1 will establish that a solution to the integral equation (I) along which consumption and (gross) investment remains non-negative is an optimal solution.

<u>Lemma 5.1</u> If $S_t\phi$ satisfies $(S_t\phi)(0) \geq 0$ and $f\left(\int_{-\infty}^{0} S_t\phi(s)\, d\mu(s)\right) - S_t\phi(0) \geq 0$ for $t \geq 0$, then $T_t = S_t$ for every $t \geq 0$.

<u>Proof</u> Immediate from Theorem 4.1. □

If k is the optimal path with initial condition ϕ, then we define $T_t\phi \equiv k_t$. Note that since the problem is stationary, T is a semigroup, that is: $T_s(T_t\psi) = T_{s+t}\psi$ for every t, s > 0. Also note that T is continuous in the

L_p norm. Analogous statements hold for S_t. We now introduce a subset of $\tilde{J}$:

<u>Definition</u> $$\hat{J} \equiv \{ \psi \in \tilde{J} : \; S_t\psi = T_t\psi \quad \forall\, t \geq 0 \}$$

Note that $\hat{J}$ is non-empty because the constant function C (rescaled by a scalar) is in it. Note $\tilde{J} \supsetneq \hat{J}$ (indeed $\hat{J} = \{C\}$ is a possibility, as we shall see in the "learning by doing" example 6 of section 6. In the discrete time formulation, we have already seen this in the example of section 2, when $0 < b < a$).

The existence of (continuous) solutions to the integral equation is easy to derive if we assume:

$$m(s) \text{ is an absolutely continuous function.} \tag{A2}$$

Define now the problem D by

$$k(t) = m(0)^{-1} \int_{-\infty}^{0} k(t+s)m'(s)ds \quad , \; t \geq 0 \tag{D1}$$

$$k_0 = \phi \in M . \tag{D2}$$

Note that the function $k|_{[0,+\infty)}$ which satisfies the condition D1 is automatically continuous.

Lemma 5.2 Assume A1 and A2. Then a function $k \in \overline{M}$ which solves D with $\int_{-\infty}^{0} \phi(s)\,d\mu(s) = C$, also solves the integral equation (I).

Proof See Appendix.

The system D in fact characterizes the solution of the integral equation, when m′ is bounded. In fact:

Lemma 5.3 Let $k \in \overline{M}$, let $k_t \in L^\infty(\mathbb{R}_-)$ satisfy I for every t, and let $m' \in L^\infty(\mathbb{R}_-)$. Then for every $t \geq 0$, D1 and D2 hold. Also, if $\phi(s) = 0$ for $s \leq -T$, $T < +\infty$ then k is continuous at any $t \geq 0$.

Proof From the integral equation, for any $t \geq 0$ and $h > 0$, we have:

$$\int_{t}^{t+h} k(s)m(s-t-h)ds - \int_{-\infty}^{0} k(s)[m(s-t) - m(s-t-h)]ds = 0.$$

The first statement now follows from the Lebesque differentiation theorem and the dominated convergence theorem. The continuity of k now follows from the fact that k satisfies D, and continuity in L^p. ∎

<u>Lemma 5.4</u> Assume A1, A2, $m' \in L^\infty(\mathbb{R}_-)$. Then $S_t : \tilde{J} \to \tilde{J}$ is well defined, i.e., for any $\phi \in M$ there exists a solution of the integral equation. Furthermore, $S_t\phi|_{[0,t]}$ is a continuous function.

<u>Proof</u> From the equivalence condition of the Lemma, it suffices to prove that there is a unique solution, continuous on $[0, +\infty)$ to (D). The proof is now a standard application of a contraction mapping argument on the space of continuous functions on $\mathbb{R}_+$.

In the rest of the paper we shall sometimes be interested in examples where the condition A2 is not satisfied. It is interesting therefore to record results similar to Lemmata 5.2 and 5.3 above in a somewhat weaker situation.

Define the (A2′) condition on M as:

m is piecewise absolutely continuous, i.e.

$$m = \sum m_j \chi_{[t_{j+1}, t_j)}, \quad t_{j+1} < t_j \ldots < t_0 = 0, \text{ where} \qquad \text{(A2′)}$$

m_j is absolutely continuous on $[t_{j+1}, t_j)$.

Then define:

$$k(t) = m(0)^{-1}\left\{\sum_{j}^{\infty} k(t+t_{j+1}[m_{j+1}(t_{j+1}-)-m_j(t_{j+1}+)] + \right. \\ \left. - k(t+t_n)m_n(t_n+) - \int_{-\infty}^{0} k(t+s)m'(s)ds\right\} \qquad (D'1)$$

where the arguments t+ (t−) denote as usual limit from the right (left), and

$$\int_{-\infty}^{0} \phi(s)d\mu(s) = C. \qquad (D'2)$$

The analogues of Lemmata 5.2 and 5.3 hold:

<u>Lemma 5.2'</u> Let A1, A2' hold. Then a solution of D' is a solution of I.

<u>Lemma 5.3'</u> A solution of D' exists for an initial condition $\phi \in C((-\infty,0], \mathbb{R})$.

<u>Remark</u> A solution of the integral equation is not continuous, even for positive times, as the example $m(s) = \chi_{[-1,0]}(s)$, $k(s) = \{s-n: \text{ for } s \in [n,n+1)\}$ shows.

It may be of interest to note that, due to the linearity of the problem, the sets $\tilde{J}$ and $\hat{J}$ have a very simple form. We can describe it in the following lemma.

<u>Lemma 5.5</u> 1. $\tilde{J}$, $\hat{J}$ are convex sets;

2. The restrictions of T_t to $\hat{J}$ satisfies:

$$T_t\{\theta\psi_1 + (1-\theta)\psi_2\} = \theta T_t\psi_1 + (1-\theta)T_t\psi_2$$

(and therefore so does S_t, by the lemma.)

<u>Proof</u> If $\psi_i \in \hat{J}$, i = 1, 2, then by definition of $\hat{J}$,

$$f\left(\int_{-\infty}^{0} T_t\psi_i(s)d\mu(s)\right) - T_t\psi_i(0) \geq 0, \quad i = 1, 2, \quad t\geq 0, \text{ and so}$$

$\theta T_t\psi_1 - (1-\theta)T_t\psi_2$ also satisfies the above inequality for every t. It follows that $\theta T_t\psi_1 + (1-\theta)T_t\psi_2 \in \hat{J}$. By the uniqueness of the solution of the integral equation I, the claim follows. □

6. The Dynamic Implications of the Depreciation Profile.

In view of section 4 and 5, we can now use the integral equation I to describe the solutions to the optimal investment problem (G) in section 4. We associate with the integral equation (I) the <u>characteristic equation</u>

$$\int_{-\infty}^{0} e^{zs} d\mu(s) = 0 . \qquad (6.1)$$

Note that the left hand side of the equation is the Laplace transform of the function $s \mapsto m(-s)$, denoted $m(-\cdot)(z)$. When m has compact support, $m(-\cdot)$ is an analytic function defined on $\mathbf{C}$, otherwise it is defined on a proper subset of $\mathbf{C}$.

The <u>spectrum</u> associated with the characteristic equation (6.1) is the set

$$S \equiv \{z \in \mathbf{C}: m(-\cdot)(z) = 0\} \bigcup \{0\} \qquad (6.2)$$

Any linear combination of the set of eigenvalues of the form:

$$k(t) \equiv \sum_j A_j e^{z_j t}, \text{ with } z_0 = 0,\ A_0 = C,\ A_j \in \mathbf{C}$$

satisfies the integral equation (I).

The analysis of the asymptotic distribution of the elements of the spectrum can be reduced to the analysis of the distribution diagram of the pairs (p_j, m_j) of the exponential polynomial of the form

$$p(z) = \sum_{j=0}^{n} p_j z^{m_j} e^{\beta_j z}(1+\epsilon(z)) \qquad (6.3)$$

where $0 = p_0 < p_1 < \ldots < p_n$, and the m_j's are non-negative integers and ϵ is a continuous function of z such that $\epsilon(z) \to 0$ as $z \to \infty$. We refer the reader to Bellman and Cooke [1963], section 12.3, for an analysis of the distribution diagram.

As is well known, the set S prescribes the asymptotic behavior of the solutions of the integral equation. For those solutions which are feasible, they also describe the dynamics, as t tends to $+\infty$, in the manifold $\hat{J}$.

The examples below describe the investment dynamics for a variety of depreciation and learning by doing schedules. Note that Example 3, a special case, corresponds to the standard case of smooth exponential depreciation. Also note that for all examples other than Example 3, the spectrum S has complex elements and therefore that the dynamics of investment are necessarily oscillatory.

Example 1. Let

$$m(s) \equiv \max\{1+\frac{s}{T},\ 0\}, \quad \text{where } T > 0. \qquad (6.4)$$

This case corresponds to a linear depreciation schedule, where the lifetime of a machine is T, as illustrated in the Figure 6.1 below.

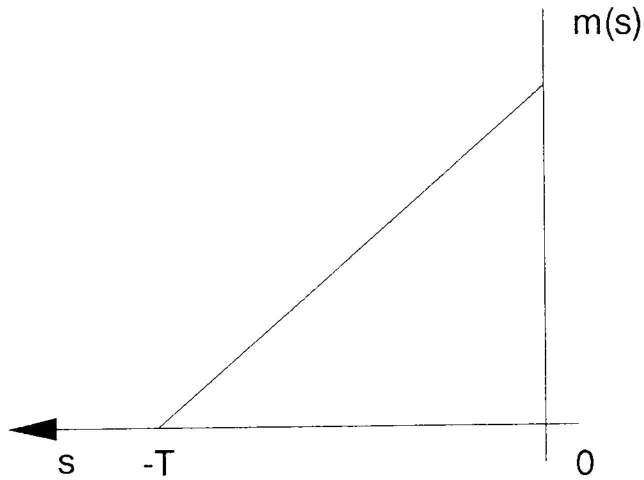

Figure 6.1

The characteristic equation is

$$zT + e^{-zT} - 1 = 0 \tag{6.5}$$

<u>Proposition 6.1</u> Any element in the spectrum associated with m in (6.4) has non-positive real parts. The only element with zero real part is $z_0 = 0$. Consequently for any initial capital profile ϕ, the solution $k^{\phi}: \mathbb{R} \to \mathbb{R}$ of the integral equation satisfies $\lim_{t \to +\infty} k^{\phi}(t) = C$.

<u>Proof</u> From the characteristic equation (6.5) and Hayes theorem (Bellman and Cooke [1963], p. 444), if $z \in S$, then $\text{Re } z \leq 0$.

A similar situation arises in the following case with an exponential depreciation schedule where a machine still lasts T periods. Note, however, that there is a discontinuity at time T, as illustrated in Figure 6.2 below.

Example 2 Let

$$m(s) = e^{\gamma s}\chi_{[-T,0]}, \quad \gamma > 0, \quad T > 0. \tag{6.7}$$

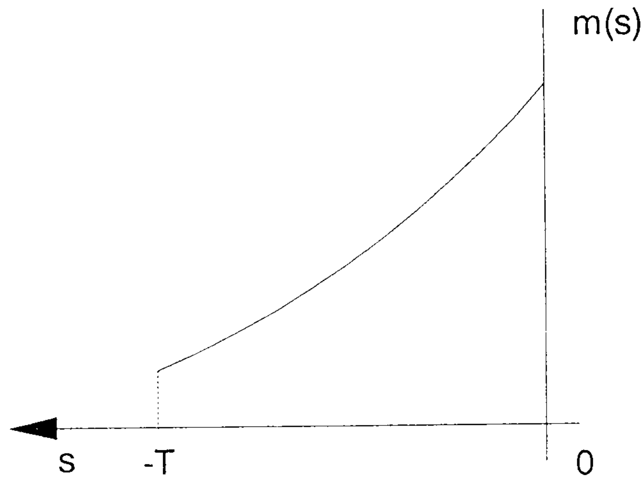

Figure 6.2

The characteristic equation is

$$1 - e^{-(\gamma+z)T} = 0, \tag{6.8}$$

and therefore, $S = \{0\} \bigcup \{-\gamma + (2\pi i)kT^{-1} \mid k = 0, \pm 1, \ldots \}$. The non-zero elements of the spectrum have negative real parts and the eigenfunctions have arbitrarily small periods. The solution from any initial condition converges to the constant solution at exponential rate γ.

This last fact is also immediate from the fact that any solution of the integral equation satisfies $k(t) = \gamma C + k(t-T)e^{-rt}$, and therefore

$$k(t+nT) = \gamma C \sum_{j}^{n} e^{-j\gamma T} + \phi(t-T)e^{-n\gamma T}, \quad t \in [0,T) \qquad (6.9)$$

and so $k(t) \to \frac{\gamma C}{1-e^{-rt}}$ uniformly as $t \to +\infty$.

As $T \to +\infty$ in Example 2, we get to the classical case of exponential decay, which we discuss next.

<u>Example 3</u> Let

$$m(s) = e^{\gamma s}, \quad s \in \mathbb{R}_- . \qquad (6.10)$$

Indeed this cooresponds to the classical case: if we define an aggregative jelly capital as $K(t) = \int_{-\infty}^{0} k(s+t)e^{\gamma s}\, ds$, differentiating $K(t)$ we obtain

$\dot{K}(t) = \int_{-\infty}^{0} \dot{k}(s+t)e^{\gamma s}\, ds$. Integrating the latter by parts we obtain the

standard accumulation equation for aggregate capital, given by

$\dot{K}(t) = k(t) - \gamma K(t)$. Figure 6.3 below illustrates the standard case.

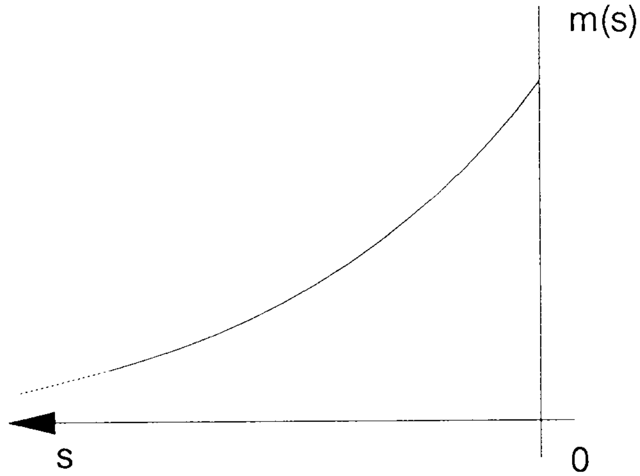

Figure 6.3

Now $m(-\cdot)$ is only defined on $\{z: \mathrm{Re}\, z > -\gamma\}$, and in this region the characteristic equation is $(z+\gamma)^{-1} = 0$, so that $S = \{0\}$.

Any solution of the integral equation is characterized as the solution of $k(t) - \gamma \int_{-\infty}^{0} k(t+s)d\mu(s) = 0$, so $k(t) = \gamma C$. In other words, as soon as the set $\tilde{J}$ is reached the investment becomes a constant. This of course is as expected, since it is the standard situation corresponding to an optimal growth model with a linear utility function. The following proposition sums this up:

Proposition 6.2   Any optimal path k^ϕ satisfies the equality

$k^\phi(t) \equiv \gamma C$ for any $t \geq 0$ such that $k_t^\phi \in \tilde{J}$.

In the examples we have seen so far, the dynamics of the optimal path are of the form described in Figure 6.4.

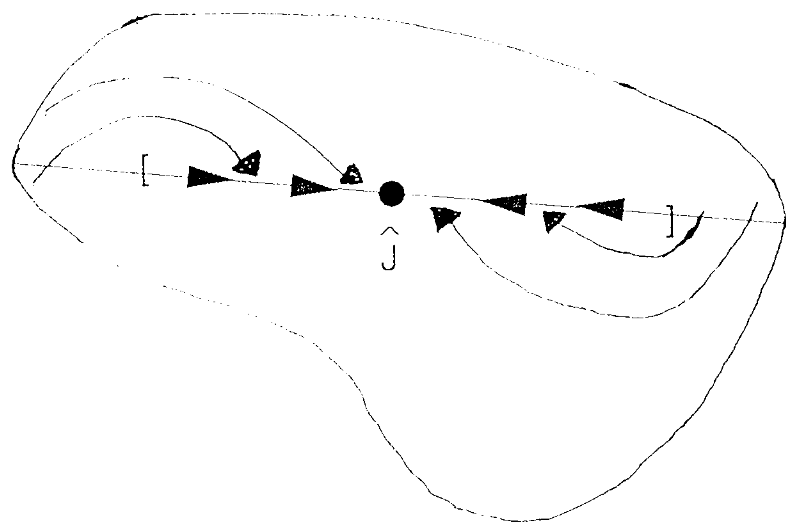

Figure 6.4

In particular the dynamics on $\hat{J}$ result in convergence to the unique constant steady state function with constant value γC.

We also know that the convergence to the steady state value is exponentially fast: indeed the convergence is estimated by the eigenvalue with the largest (negative) real part. This is the picture of a classical turnpike theorem. As we shall see in a moment, however, this is not the only possible case.

Example 4. (one-hoss shay) Let

$$m(s) = \chi_{[-T,0]}(s) , \quad T > 0 . \tag{6.11}$$

As illustrated in Fgure 6.5 below, in this case a machine does not depreciate but has a lifetime of T.

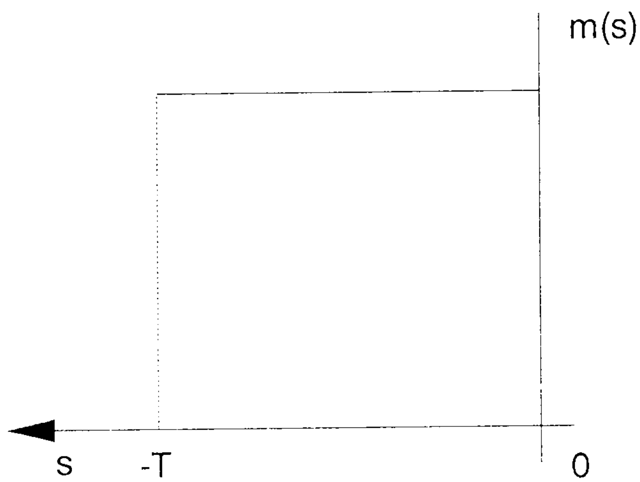

Figure 6.5

The characteristic equation is $1-e^{-zT} = 0$, and therefore $S = \{0\} \bigcup \{2\pi ik : k = \pm 1, \pm 2, \ldots\}$. All the eigenfunctions are purely periodic: there is no dampening of an initial perturbation. In fact, any solution of the integral equation satisfies $k(t) = k(t-T)$; so it is periodic with period T, and not necessarily continuous.

<u>Proposition 6.3</u> Any feasible solution of the integral equation, with $k_t^\phi \in \hat{J}$ satisfies $k_t^\phi(s) = k_t^\phi(s+T)$ for any $s \geq 0$.

The above case of one-hoss shay is an extreme example. To get a better understanding of its dynamic behavior it may be useful to consider it as a limit situation. In the following example, depreciation takes place at a more regular pace, and has the one-hoss shay case as its limit, as illustrated in figure 6.6.

<u>Example 5</u> Let

$$\begin{aligned} m(s) &= m_1(1-\theta) + m_2(\theta+s) \qquad && s \in [-\theta, 0] \\ m(s) &= m_1(1+s) && s \in [-1, -\theta]. \end{aligned} \tag{6.12}$$

where $\theta \in [0,1]$, $m_1(1-\theta) + m_2\theta = 1$. We are mostly interested in the case $m_1 > m_2 \geq 0$.

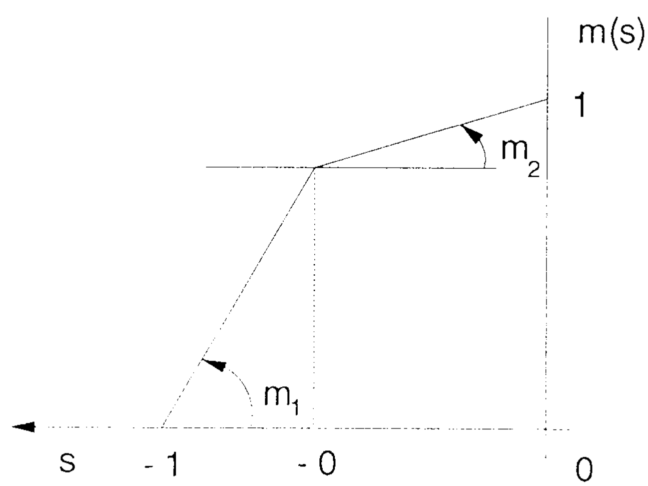

Figure 6.6

The characteristic equation is:

$$-ze^{z} + m_2e^{z} - m_1 + (m_1 - m_2)e^{z(1-\theta)} = 0. \tag{6.13}$$

Proposition 6.4 The spectrum associated with (6.13) is asymptotically distributed in the strip $\{t: |Re(z+\log z)| < C_1\}$ for some positive constant C_1; so the real part of the eigenvalues is eventually negative. As $m_2 \to 0$, $m_1 \to +\infty$, $\theta \to 1$, the spectrum tends pointwise on compact subsets of the complex plane to the spectrum associated with the case of "one-hoss shay" (6.11).

Proof This is a consequence of theorem 12.9, Bellman and Cooke [1963].

Note that setting m_1 and m_2 equal to 1, (6.13) reduces to (6.5), the case of Example 1.

We also remark that the characteristic equation given by (6.13) for $m_2 < 0$ corresponds to the case for which the efficiency of the investment goods increases over an initial period. Let us now consider the possibility that new investment goods do not reach the peak of their efficiency when they are introduced, but actually see their efficiency increase with time, at least for some time corresponds to "learning by doing". We shall first consider a simple example.

Example 6 Let

$$m(s) = (a+bs)\chi_{[-1,0]} \quad ; \quad a > 0, \ b < a \ .$$

m(s) is illustrated in Figure 6.7 for different values of b.

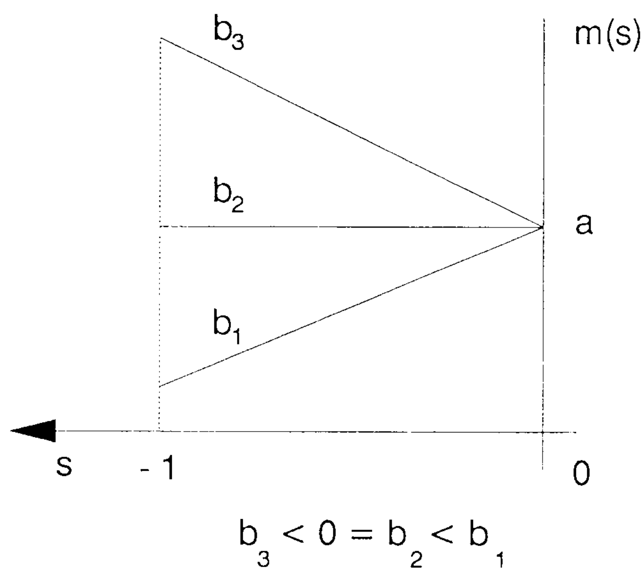

Figure 6.7

Then the characteristic equation is

$$\frac{a}{z} + (b-a)\frac{e^{-z}}{z} - \frac{b}{z^2} + \frac{be^{-z}}{z^2} = 0 \ .$$

By theorem 12.3 of Bellman and Cooke [1963], the spectrum associated with this characteristic equation is that of an equation of neutral type, that is with a spectrum asymptotically distributed on a strip in the complex plane of the form $\{z : | \mathrm{Re} z | \le C_1\}$ for some constant C_1. Indeed, integration by parts gives that differentiable solutions of the integral equation are solutions of the equation of the neutral type:

$$a\dot{k}(t) = - b\big(k(t) - k(t-1)\big) + (a-b)\dot{k}(t-1) \ . \tag{6.14}$$

The zeros of the characteristic equation are asymptotically distributed like the zeros of the equation

$$e^{z} - \left(1 - \frac{b}{a}\right) = 0 \tag{6.15}$$

so that asymptotically the real parts of the roots of (6.14) have $\mathrm{Re}\, z = \log\left(1 - \frac{b}{a}\right)$ and so have positive (or negative) real parts if $b < 0$ ($b > 0$, respectively). In other words, if the relative efficiency of investment increases with time, the steady state $\tilde{J}$ becomes unstable. Intuitively, current investment decreases the incentive to invest in the nearby future not only because of dimishing returns, but also becasuse the stock of aggregate capital tends to "increase" simply with the passage of time.

We also note that the sudden depreciation to zero for $s \leq -1$ is not the cause of the instability. If investment goods are fully efficient for a period after the time they reach their peak, as in the case of the following depreciation scheme:

$$m(s) = (a-b)\chi_{[-2,-1]} + (a+bs)\chi_{[-1,0]}, \tag{6.16}$$

the result does not change. In fact the characteristic equation is

$$(b-a)\frac{e^{-2z}}{z} + \frac{a}{z} - \frac{b}{z^2} + \frac{be^{-z}}{z^2} = 0 \tag{6.17}$$

which is again of neutral type; the zeros are asymptotically distributed like the zeros of $e^{2z} - \left(1 - \frac{b}{a}\right) = 0$, and the above analysis is unchanged.

For a final example, we turn to the case of pure "gestation lags".

Example 7.

We consider the case in which an investment becomes active only after a getation period T, after which it decays at an exponential rate.

Formally, we define:

$$m(s) = e^{\gamma(s+T)}\chi_{\{s \le -T\}} \tag{6.18}$$

Figure 6.8 below illustrates this case.

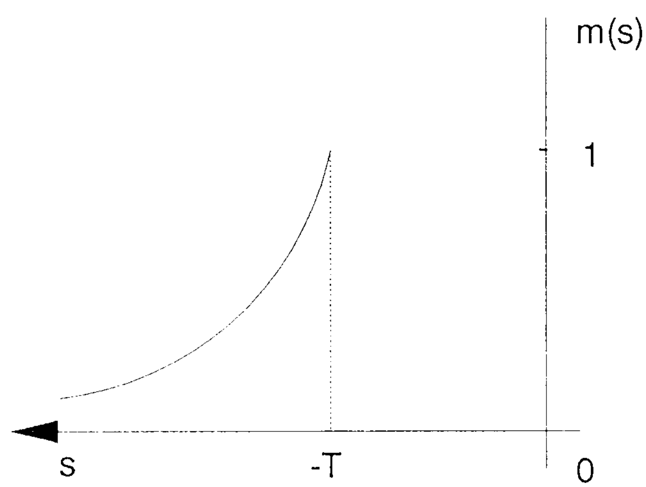

Figure 6.8

In this case the characteristic equation is given by

$$\frac{e^{-zT}}{z+\gamma} = 0 \tag{6.19}$$

so that $S = \{0\}$. The dynamic behavior is therefore very similar to the exponential decay case. In fact, it is easy to check that the optimal policy is given as follows. At time 0, with capital stock k_0, consider the value

$$K(T) = \int_{-\infty}^{0} e^{\gamma s} k(s)\, ds \equiv C_1 .$$

Let $C(\mu) = \int_{-\infty}^{0} e^{\gamma s} d\mu(s)$. If $C_1 > C(\mu)$, then set $c(t) \equiv f(K(t))$ for any $t \leq t_0$,

where $t_0 = \inf\left(t' \geq 0 : \; k(t'+T) > C(\mu)\right)$, and then for $t \geq t_0$ set $c(t) \equiv c^*$, $c^* + k^* = f(C(\mu))$. (We call this the steady state policy.)

If $C_1 < C(\mu)$, then set $c(t) \equiv 0$ until $K(t+T) = C(\mu)$ and then turn to the steady state policy. In conclusion, the capital stock $K(t)$ converges to the optimal level with no oscillations. Note also that the above example can be modified to allow for maintenance costs during the initial unproductive phase $(-T, 0]$. In such a case $m(s)$ would be negative during the unproductive phase.

Oscillations appear, as in the case of no gestations, if there is a truncation, i.e., if machines disappear after a finite time. In this case we have

$$m(s) = e^{\gamma(s+T)}\chi_{\{s\in[-T_2,-T_1]\}} \tag{6.20}$$

with $T_2 > T_1$. The characteristic equation is

$$e^{-zT_1}[1 - e^{-(T_2-T_1)(z+\gamma)}](z+\gamma)^{-1} = 0 \tag{6.21}$$

so that $S = \{0\} \bigcup \{-\gamma \pm (2\pi i)k\Delta^{-1} \mid k = 0 \pm 1, \ldots\}$ where $\Delta \equiv T_2 - T_1$. The analysis of the dynamic behavior of the equilibrium path is therefore very similar to the one described in example 2 above.

7. The Non-Linear Utility Case.

An interesting application of the analysis of the linear utility case developed above can now be given for the case of non-linear utility.

We recall that interior optimal paths are characterized as solutions of the equation:

$$-e^{-rt}U'(c(t)) + \int_t^{+\infty} e^{-rs}u'(C(s))f'(K(s))m(t-s)\,ds = 0 \tag{7.1}$$

where $c(s) \equiv f(K(s)) - k(s)$, for $s \geq 0$.

We have seen in the analysis of the linear utility case that as the slope, b, of the depreciation function $m(s) = (a+bs)\chi_{[-1,0]}(s)$ passes from positive to neagative values, the eigenvalues of the characteristic equation cross the imaginary axis. The Hopf bifurcation is "critical" and therefore degenerate because the equation is linear. In this section we shall analyze this transition in the non-linear model (7.1) above.

We shall first linearize the equation (7.1) at the steady state function (c^*, k^*), and compute the associated characteristic equation. This is done by differentiating the first order condition (7.1) above with respect to the vintage produced at time u, k(u), and then integrating over $\mathbb{R}$ the product of this derivative with the function e^{zu}. When (as in the case we are interested) the support of the function m is bounded, all the computations formally performed are justified. Also, it is clearly enough (by the stationarity of the problem) to consider the equation (7.1) above at $t = 0$.

The computations which we have outlined will give a characteristic equation $T(z) = 0$, where T is the sum of the different terms T_1, T_2, T_3 defined below:

$$T_1(z) = U''(c^*)[1-f'(K^*)m(-\cdot)(z)]$$

and

$$T_2(z) = U''(c^*)f'(K^*) \int_0^{+\infty} e^{(z-r)u}[f'(k^*)C(\mu) - m(-u)]\,du .$$

The integral in T_2 will converge on the region $\{Re z < \gamma\}$. Since the asymptotic behavior of the roots of $\hat{m}(-\cdot)(z)$ satisfies $Re z = \log\left(1 - \frac{b}{a}\right)$, this restriction creates no problems for small enough b. Recall $C(\mu) \equiv \int_{-\infty}^{0} e^{zs} m(s)\, ds$. Finally the third term is given by

$$T_3 = m(-\cdot)(z) \int_0^{+\infty} e^{(z-\gamma)u} m(-u) f''(K^*)\, du .$$

Notice that the first two terms can be made, in compact subsets of the complex plane, arbitrarily small. The critical Hopf bifurcation in the case of linear utility becomes a non critical bifurcation here. (For an analysis of Hopf bifurcation for integral equations, see Dieckmann and van Gils [1984]; see also Rustichini [1989].)

We conclude therefore that persistent oscillations in investment that are robust can occur with non-linear utility functions when we allow for some of "learning by doing". Such persistent oscillations in continuous time are different in nature from the multisector cycles obtained by Benhabib and Nishimura [1979] which arise from factor intensity relationships in production. They represent another departure from the classical turnpike results studied by McKenzie [1986].

References

Bellman, R. and Cooke, K. L. [1963], Differential Difference Equations, Academic Press, New York–London.

Benhabib, J. and A. Rustichini [1989], "A Vintage Capital Model of Investment and Growth: Theory and Evidence," C.V. Starr Center for Applied Economics, 89–26, New York University.

Diekmann, O. and van Gils, S. A. [1984], "Invariant Manifolds for Volterra Integral Equations of Convolution Type," Journal of Differential Equations, 54, 139–180.

Hale, J. K. [1977], Theory of Functional Differential Equations, Springer Verlag, New York–Heidelberg–Berlin.

Mitra, T. and H. Wan, Jr. [1985], "Some Theoretical Results on the Economics of Forestry," Review of Economic Studies, 52, 263–282.

Mitra, T, and H. Wan, Jr. [1986], " On the Faustmann Solution to the Forest Management Problem," Journal of Economic Theory, 40, 229–249.

McKenzie, L. [1986], "Optimal Economic Growth and Turnpike Theorems," Handbook of Mathematical Economics, 3 , K. Arrow and M. Intrilligator (eds.), North Holland (New York).

Rustichini, A. [1989], "Hopf Bifurcation for Functional Differential Equations of the Mixed Type," Journal of Dynamics and Differential Equations, 1.

Appendix

Proof of Theorem 4.1

Given the assumption of absolute continuity of μ (i.e., $d\mu(s) = m(s)d(s)$), the proof is immediate by differentiation. More precisely, if we define $I_{t_0,h} \equiv \chi_{[t_0-h,t_0+h]}(t)$ and $k_\epsilon(t) = k(t) + \epsilon I_{t_0,h}$ as a perturbation of the optimal path k, and then differentiating with respect to ϵ the function of ϵ given by $\int_0^{+\infty} e^{-rs} \left\{ f\left(\int_{-\infty}^{0} k_\epsilon(s+u)\, d\mu(u) \right) - k_\epsilon(s) \right\} ds$, we derive, at $\epsilon = 0$:

$$r \int_0^{+\infty} e^{-rs} f'(K(s))\psi(t,h,s)\, ds - e^{-rt}(e^{-rh} - e^{rh}) = 0 \qquad (4.5)$$

for every $t \geq 0$, where we have defined

$$\psi(t,h,s) = \begin{cases} 0 & s < t-h \\ \mu(-s+[t-h,s]) & t-h \leq s < t+h \\ \mu(-s+[t-h,t+h]) & s \geq t+h \end{cases}$$

or, after dividing by h and taking the limit as $h \to 0$,

$$\int_t^{+\infty} e^{-rs} f'(K(s))\, m(t-s)\, ds - e^{-rt} = 0, \quad \text{for every } t \geq 0. \qquad \square$$

Note that if $K(s) = C$, a constant, then the above equation is satisfied if and only if $C \equiv f^{-1}\left(C(\mu)^{-1}\right)$ and $C(\mu) \equiv \int_{-\infty}^{0} e^{rs} d\mu(s)$.

Proof of Lemma 5.2

Let $k \in \overline{M}$ be such that

1. $k|_{[0,+\infty)}$ is continuous

2. $k(t) = m(0)^{-1} \int_{-\infty}^{0} k(t+s)m'(s)ds \quad t\geq 0; \quad k(0) = C.$

If $k(t) \equiv \int_{-\infty}^{0} k(t+s)m(s)ds$ we want to prove $K(t) = C$ for $t\geq 0$. Fix any $T > 0$ and consider a family $\{k^{\epsilon}\}_{\epsilon>0}$ of functions in M such that $k_0^{\epsilon} = \psi$, $k_{[0,T]}^{\epsilon} \to k_{[0,T]}$ uniformly, $k_{[0,+\infty)}^{\epsilon} \in C^{\infty}(\mathbb{R}_+)$. These define $K^{\epsilon}(t) \equiv \int_{-\infty}^{0} k^{\epsilon}(t+s)m(s)ds$. It suffices to show $\lim_{\epsilon\to 0} K^{\epsilon}(t) = C$ for every $t \in [0, T]$. Using condition 2 above, we have

$$3.\quad k^{\epsilon}(t) = m(0)^{-1}\int_{-\infty}^{0} k^{\epsilon}(t+s)m'(s)ds + A(\epsilon, t)$$

with $A(\epsilon, \cdot) \to 0$ as $\epsilon \to 0$, uniformly in $t \in [0, T]$.

(Note $A(\epsilon, t) = \left\{ k^{\epsilon}(t) - k(t) + \int_{-\infty}^{0} [k(t+s) - k^{\epsilon}(t+s)]m'(s)ds \right\} m(0)^{-1}$ satisfies

that condition.) But, integrating by parts we have

$$\frac{d}{dt}K^{\epsilon}(t) = k^{\epsilon} - m(0)^{-1}\int_{-\infty}^{0} k^{\epsilon}(t+s)m'(s)ds = A(\epsilon, t), \quad t \in [0, T]$$

and our claim follows. ▯