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Advertising Signals of Product Quality*

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Title: Counteractive Advertising Signals of Product Quality

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Abstract:

A signalling model of advertising wars is studied. The product quality of an entrant is known to the entrant and, because of its expertise, by an incumbent firm. The entrant conducts an advertising campaign when it enters, in response to which the incumbent mounts a counteractive advertising campaign. Consumers use the advertising expenditures of both firms to draw inferences about the entrant's quality. This two-sender signalling game has ε-efficient equilibria in which a high quality entrant distinguishes itself by advertising only a small amount because it can rely upon the incumbent to foil – by counteractively advertising – any attempt by a low quality entrant to mislead consumers. Many equilibrium outcomes survive strong equilibrium refinement criteria; in particular, the "incumbent's Riley outcome" and the ε-efficient equilibria in which only the entrant signals (advertises).

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1. Introduction and Summary

This paper presents a signalling model of situations in which a firm's profits are affected by the actual or potential advertising of other firms.[1] The underlying idea is that a firm's advertising can convey not only information about its own product quality, but also about the product quality of its competitors.

The model is simple. Consumers are ignorant of the quality of a new experience good introduced by an entrant into an industry. But the entrant's product quality is known to itself and to an incumbent firm. The incumbent knows the entrant's quality because, for example, it has expertise in evaluating competing products, has obtained information from common suppliers, or has engaged in industrial espionage. The entrant conducts an advertising campaign when it enters, after which the incumbent, if it wishes, mounts a counteractive advertising campaign. Consumers treat the firms' advertising expenditures as signals; they base their inferences about the entrant's quality on the incumbent's as well as the entrant's advertising level.

Under a variety of assumptions, equilibria exist in which the incumbent advertises. A consistent signalling explanation is thus provided for why established firms of known quality (Coke, Chlorox) might advertise: incumbent firms possibly advertise in order to inform consumers about the product quality of new entrants.[2]

Equilibria in which only the entrant advertises also exist. Surprisingly, such separating equilibria exist even if the entrant's profit function does not satisfy the single-crossing property that is necessary for separation in other signalling models. The entrant's profit function satisfies single-crossing if a higher quality entrant type receives a greater marginal benefit from a given increase in demand. As is well known, this property is violated in a competitive setting if marginal production costs increase in quality (Kihlstrom and Riordan, 1984). Thus, the present model corrects

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1 A first version of the model is in Chapter 2 of Fertig's (1989) dissertation.

2 This explanation can hold even if the incumbent firm does not mention the entrant in its advertising messages. As usual in advertising signalling models, it is the size of the advertising expenditures rather than the literal meanings of the advertising messages that convey information.

the misconception that advertising by a price-taking entrant whose marginal production costs increase in quality cannot logically be interpreted as signalling.

The entrant can separate without satisfying single-crossing because its true quality can influence, via the incumbent's advertising reactions, the effect its advertising has on consumers. In particular, a low quality entrant may obtain no benefit from increasing its advertising because the incumbent is able and willing to counteract such "misleading" advertising.

We refer to a separating equilibrium in which the incumbent's advertising counteracts attempts by low quality entrants to fool consumers as a *counteractive equilibrium*. This definition is in terms of off-the-equilibrium-path behavior; the incumbent need not advertise on the equilibrium path.

The incombent in a counteractive equilibrium must find it worthwhile to counteract high advertising by a low quality entrant but not by a high quality entrant. This means that the incumbent's profit function must satisfy single-crossing: a given decrease in the demand for the entrant's product must be more valuable to the incumbent when the entrant's quality is low than when it is high. This is the case if the incumbent has decreasing marginal production costs, and some consumers know the entrant's true quality.

Given the incumbent's single-crossing condition, counteractive equilibria exist in which the equilibrium advertising expenditures are arbitrarily small. That is, the firms do not have to engage in (very much) wasteful advertising in order to reveal their private information. A high quality entrant can advertise just a small amount ε without fearing that its low quality alter ego will also advertise ε because the incumbent would advertise to counteract such masquerading.

The existence of such "ε-efficient" separating equilibria distinguishes this signalling model from all others we have seen.

Of course, real-world advertising expenditures are not usually infinitesimal. This can be explained in the present context either by noting that not all equilibria in the model are ε-efficient, or by noting that the model makes the simplifying but unrealistic assumption that consumers perfectly perceive all advertising (so that an ε advertising level cannot be overlooked).

The multiplicity of equilibria tempts us to identify "reasonable" equilibria. We apply two interesting refinement criteria that have been proposed for signalling games. The first criterion is based one the iterative elimination of dominated strategies and non-best replies. In one-sender signalling games satisfying single-crossing and monotonicity conditions, this criterion is extremely strong, eliminating all but the "Riley outcome" (a.k.a. the "Pareto dominant separating outcome").[3] In the present game this criterion is not so powerful; it is satisfied by some outcomes in which both firms advertise, and by some ε-efficient outcomes. The criterion rejects pooling outcomes, outcomes in which a high quality entrant advertises too much (when its single-crossing condition holds), and ε-efficient outcomes in which the incumbent advertises.

The second refinement criterion is the "unprejudiced criterion" proposed in Bagwell and Ramey (1989). This criterion has an intuitive basis (see Section 5.3), and it rejects all outcomes in which both firms separate.

Each of the criteria we consider is satisfied by the ε-efficient outcomes in which only the entrant advertises, and by the outcome in which the entrant does not advertise and the incumbent does advertise (a lot) when the entrant's quality is low (we refer to this outcome as "the incumbent's Riley outcome").

Before embarking, we remark that the assumption that the entrant advertises first is arbitrary. In particular, it does not reflect the fact that advertising is how consumers are informed of the entrant's presence. The assumption of a common knowledege prior about the entrant's quality implies that consumers must know of the entrant's existence before it advertises. The entrant's advertising expenditures, therefore, are to be interpreted as the amount it spends in excess of what is required to inform consumers of its presence. We could have assumed the incumbent advertises first, in which case the asymmetric features of the refinement results would be reversed.

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[3] See, e.g., Cho and Kreps (1987), Banks and Sobel (1987), Cho and Sobel (1988), and Ramey (1988).

The model is presented in Section 2. In Section 3 is presented a class of ε-efficient counteractive equilbria for the case in which the entrant's quality can take on only two values. In Section 4 the entrant's quality is drawn from a continuum, and we show that then ε-efficient equilibria still exist; that both firms can advertise at almost all values of the entrant's quality; and that neither firm's advertising expenditure is necessarily monotonic in the entrant's quality. Section 5 returns to the two-type case in order to apply the refinement criteria. A few comments on forward and backward induction are made in Section 6. The Appendix contains all proofs.

2. The Model

2.1 *The Reduced Form Game*

An industry currently contains one firm, the *incumbent*. Another firm, the *entrant*, will soon enter with a competing product. The quality of the product is q, a real number contained in a set Q. For now, only two quality levels are possible: $Q=\{q_L,q_H\}$, with $q_L<q_H$.

The entrant and the incumbent know the entrant's quality, but some consumers do not. These consumers have a prior on the entrant's quality according to which its expected value is $\bar{q} \in [q_L,q_H]$. They revise their beliefs after they perceive how much the firms advertise. The expected quality according to their revised beliefs is the *believed quality*, denoted as b.

The firms advertise in the pre-entry period, and they play some sort of market game in the post-entry period. Their equilibrium profits in the post-entry game depend on the entrant's quality and the consumers' beliefs about it at the start of the post-entry period. When the true quality is q and the believed quality is b, the profits of the entrant and the incumbent in the post-entry equilibrium are E(b,q) and I(b,q), respectively.[4] Both profit functions are twice continuously differentiable. The entrant's profit increases and the incumbent's profit decreases in the believed quality

[4] These profit functions depend only on the first moment, b, of the consumer's belief distribution on q. This is not a restriction when only two quality levels are possible. It is a restriction if more than two qualities are possible and attention is not restricted to separating equilibria.

of the entrant's product: $E_1(b,q)>0$ and $I_1(b,q)<0$. These monotonicity properties are what one would expect from a natural specification of the post-entry game; an example is presented at the end of this section.

In the pre-entry period, the entrant announces his product with an advertising campaign. The incumbent then responds with his advertising campaign. Their advertising expenditures are denoted a_E and a_I, both nonnegative. After they advertise, consumers form their beliefs and the post-entry game is played. If the true quality is q and the believed quality is b, the net profits of the entrant and the incumbent are $E(b,q) - a_E$ and $I(b,q) - a_I$.

The entrant advertises as a function of his quality, $a_E=S_E(q)$. The incumbent advertises as a function of the entrant's quality and advertising choice, $a_I=S_I(a_E,q)$. The consumers are the third 'player' in this reduced form game. A strategy for them is a belief function B: $\Re^2_+ \rightarrow [q_L,q_H]$ mapping each pair of advertising levels into a believed quality, $b=B(a_E,a_I)$.

Corresponding to a pair $\langle S_E,S_I\rangle$ of strategies is an *outcome*, which is a pair of advertising functions telling how much each firm advertises at each quality level: $A_E(q) \equiv S_E(q)$ and $A_I(q) \equiv S_I(A_E(q),q)$.

A *(pure strategy, sequential) equilbrium* is a pair of strategies and a belief function, $\langle S_E,S_I,B\rangle$, satisfying three conditions. First, consumers have correct beliefs on the equilibrium path: for all $q \in Q$,

$$\text{(E1)} \qquad B(A_E(q),\, A_I(q)) = \begin{cases} q & \text{if} \quad (A_E(q_L),\, A_I(q_L)) \neq (A_E(q_H),\, A_I(q_H)) \\ \bar{q} & \text{if} \quad (A_E(q_L),\, A_I(q_L)) = (A_E(q_H),\, A_I(q_H)). \end{cases}$$

If the top line of (E1) holds the equilibrium is *separating*, and it is *pooling* if the bottom line holds. Second, the incumbent advertises an amount that maximizes his profit given the advertising choice of the entrant and the way consumers form beliefs: for all $q \in Q$ and $a_E \geq 0$,

$$\text{(E2)} \qquad S_I(a_E,q) \in \underset{a_I \geq 0}{\operatorname{argmax}}\; I[B(a_E,a_I),\, q] - a_I\,.$$

Third, the entrant maximizes his profit given the way the incumbent reacts and consumers form beliefs: for all $q \in Q$,

(E3) $$S_E(q) \in \underset{a_E \geq 0}{\operatorname{argmax}} \, E[B(a_E, S_I(a_E,q)),q] - a_E .$$

We note for later use that even if Q contains more than two quality levels, a separating equilibrium is still defined by the top line of (E1), (E2), and (E3).

The first proposition tabulates an important property of separating equilibria: the entrant does not advertise if $q=q_L$ and the incumbent does not advertise if $q=q_H$. The argument for this is well known. The advertising level of the lowest quality entrant in a separating equilibrium induces consumers to correctly believe $b=q_L$. This is the most pessimistic belief consumers can have. If the equilibrium advertising level of the lowest quality entrant were positive, he could increase profits by deviating to a zero advertising level, thereby cutting advertising costs without lowering consumer beliefs. A similar argument holds for the incumbent when $q=q_H$. This proves the following proposition, which is stated to hold for a general set Q of possible qualities.

Proposition 2.1: *If* $\langle A_E, A_I \rangle$ *is a separating equilibrium outcome and* $q_L = \min(Q)$ *and* $q_H = \max(Q)$, *then* $A_E(q_L)=0$ *and* $A_I(q_H)=0$.

From this proposition, we can index the separating equilibrium outcomes by a pair of numbers, (e,c), where $e=A_E(q_H)$ and $c=A_I(q_L)$. We follow this convention when it should cause no confusion. We refer to a pooling equilibrium as a pair of constants (A_E, A_I), where this should cause no confusion.

2.2 *Single-Crossing Conditions*

In the usual Spence-style signalling model, higher sender types can distinguish themselves from lower types only if their marginal benefit from signalling exceeds that of lower types. To define this property for each firm, first define for each $q \in \Re$,

$$G_E(q) \equiv E(q_H,q) - E(q_L,q) \qquad \text{and} \qquad G_I(q) \equiv I(q_L,q) - I(q_H,q).$$

(Note that $G_E>0$ and $G_I>0$.) If the entrant's quality is q, his gain from convincing consumers that his quality is high rather than low is $G_E(q)$, and $G_I(q)$ is the incumbent's gain from changing consumers' beliefs in the opposite direction. The *single-crossing condition for the entrant* is satisfied iff $G_E(q_L) \leq G_E(q_H)$, i.e. iff a high quality entrant gains more than a low quality entrant from persuading consumers his quality is high rather than low. This condition holds if $E_{12}>0$. Similarly, the single-crossing condition is satisfied for the incumbent iff $G_I(q_H) \leq G_I(q_L)$, and it holds if $I_{12}>0$.

After Section 3, $I_{12}>0$ will be assumed, but not $E_{12}>0$.

2.3 *The Post-Entry Game*

We present here a trivial but reasonable post-entry game,[5] stating it generally to allow for a continuum of qualities.

In this specification the post-entry prices of the firms' products are fixed and equal. For example, their prices might be set by a regulator who, because he is imperfectly informed, fixes them at the same level. Think, perhaps, of a cable television market in which entry has been permitted. Prices which the firms do not choose cannot reveal their private information. This makes it sensible for consumers to base their inferences only upon advertising levels.[6]

The price of the firms' products is p. All consumers have the same preferences and incomes. A fraction $k \in (0,1)$ of them are knowledgeable; these consumers know, or have the expertise to ascertain, the quality of the entrant. Each knowledgeable consumer, when the entrant's quality is q, demands $x_I(q)$ of the incumbent's product and $x_E(q)$ of the entrant's product (the fixed price p has been dropped as an argument). Each ignorant consumer, when he believes the entrant's quality is b, demands $x_I(b)$ from the incumbent and $x_E(b)$ from the entrant. Normalizing the mass of consumers at unity, the aggregate demands are

$$X_I(b,q) = kx_I(q) + (1-k)x_I(b) \quad \text{and} \quad X_E(b,q) = kx_E(q) + (1-k)x_E(b).$$

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[5] Fertig (1989) presents a more complicated Cournot game.

[6] Even if the firms choose prices or other observable variables, separating equilibria would exist in which consumer inferences are based only on advertising choices. Allowing inferences to be based on additional variables only adds equilibria. However, it may then be that only equilibria in which beliefs depend on these other variables would satisfy various refinement criteria. See, e.g., Milgrom and Roberts (1986) and Ramey (1988).

The firms are required to supply these demands. The incumbent's cost function is C_I, and the entrant's is $C^E(\cdot,q)$. Their profits in the post-entry period are therefore

$$I(b,q) = pX_I - C_I(X_I) \qquad \text{and} \qquad E(b,q) = pX_E - C^E(X_E,q).$$

As an operational definition of "quality," assume demand for the entrant's product increases with its quality: $x_E'>0$. The entrant's product substitutes for the incumbent's as its quality increases: $x_I'<0$.[7] Assuming the price is set above the marginal costs, we have our desired slopes:

$$E_1(b,q) = (1-k)[p - C_1^E(X_E,q)]\, x_E'(b) > 0,$$

$$I_1(b,q) = (1\text{-}k)[p - C_I'(X_I)]\, x_I'(b) < 0.$$

The incumbent's single-crossing condition is satisfied if $C_I''<0$, for then

$$I_{12}(b,q) = -\, k(1\text{-}k)C_I''(X_I)x_I'(q)x_I'(b) > 0.$$

Changes in the believed quality of the entrant change the incumbent's profit proportionally to his price - marginal cost markup, which, given decreasing marginal cost, is greater when the entrant's true quality is low because then the incumbent sells a larger quantity.

The single-crossing condition for the entrant is satisfied if

$$E_{12}(b,q) = -\,(1\text{-}k)[kC_{11}^E(X_E,q)\, x_E'(q)\, x_E'(b) + C_{12}^E(X_E,q)\, x_E'(b)]$$

is positive, which it is if the entrant's marginal cost decreases in quantity and in quality. The latter is most problematic; marginal cost is likely to be greater for producing higher quality products.

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[7] For example, suppose the demands $x_I(q)$ and $x_E(q)$ maximize $u(x_E,x_I,y,q)$ subject to $px_E+px_I+y \leq \bar{y}$. Let $u(x_E,x_I,y,q) = v(x_E,x_I,q)+y$, and assume v is concave increasing in (x_E,x_I) and has second derivatives $v_{12}<0$, $v_{13}>0$, and $v_{23}=0$. Then $x_E'>0$ and $x_I'<0$. To make the demands of the ignorant consumers depend only on the expected quality even when it has a nondegenerate distribution, let $v_{33}=0$. A special, different kind of utility function that works as well is $u(x_E,x_I,y,q) = q \ln(x_E) + \ln(x_I) + \ln(y)$.

3. Counteractive Equilibria

In this section we define a counteractive equilibrium and present an example in the two-quality model. But first we discuss more familiar kinds of equilibria and their reliance on the single-crossing conditions.

In the absence of an incumbent, the single-crossing condition for the entrant, $G_E(q_H) \geq G_E(q_L)$, would be necessary and sufficient for a separating equilibrium to exist. The high quality entrant would advertise an amount high enough that his low quality alter ego would find it unprofitable to mislead consumers by advertising as much. The advertising level of a high quality entrant would exceed or equal $G_E(q_L)$. The "Riley outcome," $A_E(q_L)=0$ and $A_E(q_H)=G_E(q_L)$, would be the Pareto dominant separating equilibrium outcome.

Even when an incumbent is present, consumers may pay him no attention. The condition $G_E(q_H) \geq G_E(q_L)$ is still necessary and sufficient for the existence of an equilibrium of the following form: the entrant advertises precisely when his quality is high, the incumbent never advertises, and consumer beliefs are independent of the incumbent's advertising. Again, in such equilibria $G_E(q_L)$ is a lower bound on the high quality entrant's equilibrium advertising choice. Let us refer to the Pareto dominant outcome of this type as the *entrant's Riley outcome*, $(e,c) = (G_E(q_H), 0)$.

Similarly, the incumbent's single-crossing condition is necessary and sufficient for the existence of an equilibrium of the following form: the incumbent advertises an amount no less than $G_I(q_H)$ when the entrant's quality is low, the entrant never advertises, and consumer beliefs are independent of the entrant's advertising. Let us refer to the Pareto-dominant outcome of this type as the *incumbent's Riley outcome*, $(e,c) = (0, G_I(q_H))$.

In the Riley outcomes, total advertising expenditures exceed either $G_E(q_L)$ or $G_I(q_H)$ in some eventuality. This is the usual signalling inefficiency (relative to first best, assuming advertising is dissipative). However, if the incument's single-crossing condition holds, separating equilibria do exist in which the signalling inefficiency is insignificant.

The equilibria that are ε-efficient are "counteractive" in the sense that if a low quality entrant advertises the amount expected of a high quality entrant, the incumbent will respond by advertising enough to make sure the consumers are not fooled. That is, a separating equilibrium $\langle S_E, S_I, B \rangle$ of the two-quality model is *counteractive* if it satisfies the following condition:

$$B(a_E, S_I(a_E, q_L)) = q_L \quad \text{if} \quad a_E = S_E(q_H). \tag{3.1}$$

The following construction yields a counteractive equilibrium, given the incumbent's single-crossing condition. Let c and e be numbers such that $0 \le c < G_I(q_H)$ and $0 < e < G_E(q_H)$. We show that (e,c) is a separating equilibrium outcome (recall that $e{=}A_E(q_H)$ and $c{=}A_I(q_L)$).

Consider the following belief function, shown in Figure 3.1(a):

$$B(a_E, a_I) = \begin{cases} q_H & \text{if} \quad a_E < e \text{ and } a_I < c \\ q_L & \text{if} \quad a_E < e \text{ and } a_I \ge c \\ q_H & \text{if} \quad a_E \ge e \text{ and } a_I < G(q_H) \\ q_L & \text{if} \quad a_E \ge e \text{ and } a_I \ge G(q_H). \end{cases} \tag{3.2}$$

According to these beliefs, the incumbent can always convince consumers that the entrant's quality is low by advertising, at level c if $a_E{<}e$ and at level $G_I(q_H)$ if $a_E{\ge}e$.

Consider the incumbent's problem when consumers have this belief function and the entrant has advertised an amount a_E. His problem is to choose (a_I,b) subject to the constraint $b{=}B(a_E,a_I)$. Panels (b) and (c) of Figure 3.1 illustrate the two cases, $a_E{<}e$ and $a_E{\ge}e$. The curves labeled I_L and I_H are isoprofit curves of the incumbent when the entrant's quality is low or high, respectively. The I_H curve is steeper than the I_L curve because the incumbent's profit function satisfies the single-crossing condition. The incumbent's optimal choice in each panel when quality is high (low) is labeled H (L). As panel (b) indicates, c is so small that $a_I{=}c$ is the incumbent's optimal choice if $a_E{<}e$, regardless of the entrant's quality. If $a_E{\ge}e$, panel (c) indicates that $a_I{=}G_I(q_H)$ is the incumbent's optimal choice if $q=q_L$, but that $a_I=0$ is optimal if $q=q_H$.[8] Summarizing, the incumbent's best reply to B is

$$(3.3) \qquad S_I(a_E,q) = \begin{cases} c & \text{if} \quad a_E < e \\ G_I(q_H) & \text{if} \quad a_E \geq e \text{ and } q = q_L \\ 0 & \text{if} \quad a_E \geq e \text{ and } q = q_H \,. \end{cases}$$

Given (3.2) and (3.3), the entrant's problem is depicted in (d) or (e) of Figure 3.1depending on whether his quality is high or low. When it is low, the incumbent's counteractive advertising prevents the entrant from being able to persuade consumers that his quality is high, and his optimal choice is $a_E=0$. When his quality is high, he sets $a_E=e$ because e is small. The entrant's optimal strategy is thus

$$(3.4) \qquad S_E(q_L) = 0 \text{ and } S_E(q_H) = e.$$

If the firms play S_E and S_I, the beliefs given by (3.2) will be correct at the chosen advertising levels. Expressions (3.2) – (3.4) therefore define a separating equilibrium.

Three features of this equilibrium stand out:

(a) the entrant signals his quality by advertising, even though his profit function need not satisfy the single-crossing condition;

(b) the incumbent, on the equilibrium path, need never advertise (c=0 is allowed); and

(c) the sum of the equilibrium advertising levels, A_E+A_I, can be arbitrarily close to, but not equal to, zero (c and e can be arbitrarily small, but e must be positive).

An hypothesized signalling equilibrium cannot be rejected by observations to the effect that firms introducing high quality products advertise only trivial amounts, or that their profit functions violate single-crossing.

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8 The incumbent is indifferent between $a_I=0$ and $a_I=G_I(q_H)$ when $q=q_H$. He must choose the former for this to be an equilibrium. Any counteractive equilibrium satisfying the refinement criterion in which dominated strategies are iteratively eliminated has this "indifference property." See Section 5.

What observations could refute a signalling hypothesis? That is, what conditions are necessary for the existence of a separating equilibrium? Part (i) of Proposition 3.1 gives one condition: the single-crossing property must be satisfied for one firm at least. Part (ii) indicates that a counteractive equilibrium exists only if the incumbent's single-crossing property holds. The reason for this is that in a counteractive equilibrium, after the entrant has advertised a positive amount, the incumbent must advertise when the entrant's quality is low but not when it is high. (The Appendix contains all proofs.)

Proposition 3.1:

(i) *Separating equilibria exist iff* $G_E(q_H) \geq G_E(q_L)$ *or* $G_I(q_L) \geq G_I(q_H)$.

(ii) *Counteractive equilibria exist iff* $G_I(q_L) \geq G_I(q_H)$.

In view of (ii), we henceforth assume the incumbent's single-crossing condition, i.e., $I_{12}>0$. Otherwise counteractive equibria would not exist, and the incumbent would have no role.

Returning to counteractive and approximately efficient equilibria, what is their relationship and what are their telltale signs? The next proposition addresses these questions. Say that an equilibrium is *ε-efficient* for a given $\varepsilon>0$ if it is separating and its outcome satisfies $A_E(q)+A_I(q) < \varepsilon$ for all $q \in Q$.[9]

Proposition 3.2:

(i) *For* $\varepsilon < G_E(q_L)$, *every ε-efficient equilibrium is counteractive.*

(ii) *For* $\varepsilon < \min\{G_E(q_L), G_I(q_H)\}$, *every ε-efficient equilibrium satisfies* $e > 0$.

4. A Continuum of Qualities

We assume now that the set of possible quality levels is an interval, $Q=[q_L,q_H]$. Enlarging the number of possible qualities allows us to address new questions: Is it possible both firms to advertise in the same quality state? (This is not possible in the

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[9] We restrict the notion of ε-efficiency to separating equilibria because pooling equilibria may be inefficient. That is, the utility of consumers and even the expected profits of the industry may be greater when consumers are informed.

two-quality model, by Proposition 2.1.) When his quality is higher, does the entrant necessarily advertise more and the incumbent less? (This must be the case in the two-quality model, by Proposition 2.1.) Do ε-efficient equilibria exist when more than two quality levels are possible?

The definition of ε-efficiency remains the same: an equilibrium is *ε-efficient* for a given $\varepsilon>0$ if its outcome satisfies $A_E(q)+A_I(q) < \varepsilon$ for all $q \in Q$.

We show that ε-efficient equilibria exist. In some of them the firms' advertising levels are not monotonic in quality, and in some of them both firms advertise positively at every interior quality level.

We prove these claims by providing a sufficient set of conditions for a pair of advertising functions, $A_E: Q \to \Re_+$ and $A_I: Q \to \Re_+$, to be a separating equilibrium outcome. The conditions allow both advertising functions to be arbitrarily close to zero and nonmonotonic.

Three conditions apply to A_E:

(4.1a) $\quad A_E(q_L) = 0,$

(4.1b) $\quad A_E(q) - A_E(b) \leq E(q,q) - E(b,q) \quad \text{for all} \quad q_L \leq b < q \leq q_H,$

(4.1c) $\quad A_E(q) \neq A_E(\hat{q}) \quad \text{for all} \quad q, \hat{q} \in [q_L, q_H].$

Condition (4.1a) is required by Proposition 2.1. Condition (4.1b) requires that A_E not increase too rapidly, and (4.1c) requires it to be invertable. Observe that (4.1) implies neither continuity nor monotonicity.[10]

Two conditions apply to A_I:

(4.2a) $\quad A_I(q_H) = 0,$

(4.2b) $\quad A_I(b) \leq I(b,q_H) - I(q_H,q_H) \quad \text{for all} \quad b \in [q_L,q_H].$

Again, (4.2a) is required by Proposition 2.1. Condition (4.2b) requires that A_I be bounded above. Because $I_{12}>0$, (4.2b) implies

(4.2b') $\quad A_I(b) \leq I(b,q) - I(q_H,q) \quad \text{for all} \quad b, q \in [q_L,q_H].$

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[10] If A_E were continuous, it would be monotone by (4.1c).

Given an $\langle A_E, A_I \rangle$ satisfying (4.1) and (4.2), Figure 4.1 illustrates the construction of a corresponding equilibrium belief function B. The figure depicts the incumbent's problem after the entrant has chosen an advertising level $a_E = A_E(q)$. According to $B(a_E, \cdot)$, the believed quality is $b = q_H$ if the incumbent chooses $a_I < A_I(q)$, it is $b = q$ if $A_I(q) \leq a_I < \alpha$, and it is $b = q_L$ if $a_I \geq \alpha$. The figure indicates that the best reply of the incumbent is to choose $a_I = A_I(q)$ if the true quality exceeds q, but to choose $a_I = \alpha$ if the true quality is less than q. This is true for two reasons. First, by (4.2b') the incumbent will advertise to at least the level $a_I = A_I(q)$, regardless of the true quality. And second, because $I_{12} > 0$, the incumbent will choose the greater (lesser) of $A_I(q)$ and α if the quality is lower (higher) than the quality q at which he is indifferent. Consequently, if the actual quality is q and the entrant chooses $a_E = A_E(q)$, a best response for the incumbent is indeed $a_I = A_I(q)$, and the consumers' beliefs will be correct.

Now consider the entrant. No advertising choice on his part will convince consumers that his quality is greater than it truly is. Suppose it is q^-, but the entrant attempts to convince consumers it is $q > q^-$ by choosing $a_E = A_E(q)$. Then, because the true quality is lower than q, we see from Figure 4.1 that the incumbent will counteract the entrant's deceptive advertising by advertising at the high level α. This will convince consumers that the entrant's quality is the lowest possible, q_L.

Thus, the only feasible choices for the entrant of quality q^- are to reveal his true quality by choosing $a_E = A_E(q^-)$, or to fool consumers into believing his quality is at some lower level $q^{--} < q^-$ by choosing $a_E = A_E(q^{--})$ (or, if $q^{--} = q_L$, by choosing any $a_E > A_E(q^-)$). However, the entrant will not want to fool consumers into believing his quality is lower than it truly is: by (4.1b), he can persuade consumers that his quality is q^- rather than q^{--} by increasing only slightly his advertising expenditure.

This finishes the heuristic proof. Before stating it formally, we must deal with a few technical matters. We extend the inverse of A_E to $\Re_+$: for any $A_E: [q_L.q_H] \rightarrow \Re_+$ satisfying $A_E(q_L) = 0$, let

(4.3) $$\beta(a_E) \equiv \sup \{q \in [q_L, q_H] \mid A_E(q) \leq a_E\}.$$

If A_E is invertable and $a_E \in A_E(Q)$, then $a_E = A_E(q)$ iff $\beta(a_E) = q$. If consumers were to see a_E but not a_I, they would believe the quality is $\beta(a_E)$. Corresponding to this is a requirement that A_E be left-continuous. Finally, define $\alpha: \mathfrak{R}_+ \to \mathfrak{R}$ by

$$(4.4) \qquad \alpha(a_E) = I[q_L, \beta(a_E)] - I[\beta(a_E), \beta(a_E)] + A_I(\beta(a_E)).$$

Proposition 4.1: *Suppose* $I_{12} > 0$. *Let* A_E *and* A_I *be advertising functions satisfying* (4.1) *and* (4.2), *with* A_E *left-continuous. Then* $\langle A_E, A_I \rangle$ *is the outcome of an equilibrium given by*

$$(4.5) \qquad S_E = A_E;$$

$$(4.6) \qquad S_I(a_E, q) = \begin{cases} A_I(\beta(a_E)) & \text{if} \quad q \geq \beta(a_E) \\ \alpha(a_E) & \text{if} \quad q < \beta(a_E), \end{cases}$$

where β *and* α *are defined in* (4.3) *and* (4.4); *and*

$$(4.7) \qquad B(a_E, a_I) = \begin{cases} q_H & \text{if} \quad a_I \leq A_I(\beta(a_E)) \\ \beta(a_E) & \text{if} \quad A_I(\beta(a_E)) \leq a_I < \alpha(a_E) \\ q_L & \text{if} \quad \alpha(a_E) \leq a_I. \end{cases}$$

Although the consumer beliefs in Proposition 4.1 are discontinuous in a_I, they need not be. Assuming $I_{12} > 0$, any $\langle A_E, A_I \rangle$ satisfying the hypothesis is also the outcome of an equilibrium in which the belief function is continuous in a_I and, if A_E and A_I are continuous, in a_E as well. The beliefs in such an equilibrium are shown in Figure 4.2, and the argument is essentially the same as before.

The equilibria shown in Figures 4.1 and 4.2 have two arguably unpalatable features. First, the belief functions are kinked. Second, the incumbent's optimal response is discontinuous: a slight deviation by the entrant from an equilibrium a_E to an $\hat{a}_E$ which is the equilibrium choice of a higher quality entrant is met with a jump increase in the incumbent's advertising. This jump increase is what persuades consumers to believe that the entrant's quality is the lowest possible, so that any attempt he makes to mislead consumers will badly backfire. However, both of these features were assumed only to simplify the analysis. Proposition B in Appendix B shows that if A_E satisfies (4.1a), a slight strengthening of (4.1b), and is increasing, and if A_I satisfies (4.2), then $\langle A_E, A_I \rangle$ is the outcome of an equilibrium in which $B(a_E, a_I)$ is differentiable in a_I and $S_I(a_E, q)$ is differentiable in q. Both functions are differentiable in a_E as well if A_E is differentiable.

5. Refinements

In this section we apply versions of three refinement criteria that have been recently proposed: the iterative elimination of dominated strategies, the iterative elimination of non-best replies, and the unprejudiced criterion. For the sake of simplicity, we only consider the two-type model: $Q=\{q_L, q_H\}$. Of particular interest is whether the entrant's and the incumbent's Riley outcomes, and the ε-efficient outcomes, satisfy the criteria.

The results can be briefly summarized. First, the iterative elimination of dominated strategies merely puts an upper bound on how much the firms can advertise, and sometimes eliminates all pooling outcomes. Second, the iterative elimination of non-best replies and dominated strategies eliminates pooling outcomes, and separating outcomes in which the high quality entrant advertises too much (so that the entrant's Riley outcome is eliminated), and outcomes in which the incumbent advertises too much when the entrant's quality is low relative to how much the entrant advertises when his quality is high (so ε-efficient outcomes in which the incumbent advertises are eliminated). Third, the unprejudiced criterion eliminates outcomes in which both firms separate. The incumbent's Riley outcome satisfies all the criteria, as do ε-efficient outcomes in which only the entrant advertises. Figure 5.1 illustrates these results when $E_{12}>0$ (the case $E_{12}<0$ is similar).

5.1 *Eliminating Dominated Strategies*

A strategy for the entrant is dominated if it requires him when his quality is q to advertise an amount $a_E \geq G_E(q)$. A strategy for the incumbent is dominated if it ever requires him to advertise an amount $a_I \geq G_I(q)$. Equilibria in which dominated strategies are not played are *admissable*. As usual in signalling models, no distinct equilibrium outcomes are ruled out by the admissability criterion.

A stronger criterion is obtained by iteratively eliminating dominated strategies, as in, e.g., Milgrom and Roberts (1986). Its motivation is a presumption that the players commonly know that none of them will play a dominated strategy.[11]

To begin the process, first create a pruned game by removing the firms' dominated strategies. This causes some strategies of the consumers to become dominated. For example, consider an advertising pair (a_E,a_I) for which $G_E(q_L)\leq a_E<G_E(q_H)$ and $a_I<G_I(q_H)$. This advertising pair might occur in the pruned game, but only if the entrant's quality is high. Any action consumers might take in response to it is dominated unless it is optimal when the entrant's quality is high. We can therefore abuse terminology by referring to a belief function B as dominated if it does not assign q_H to this advertising pair. Similarly, B is dominated if $B(a_E,a_I)\neq q_L$ when $a_E<G_E(q_L)$ and $G_I(q_H)\leq a_I<G_I(q_L)$. Removing dominated belief functions gives us the following set (since $G_I(q_H)<G_I(q_L)$):

$$\begin{aligned}\mathcal{B}^1 \equiv \{ B: \Re_+^2\rightarrow[q_L,q_H] \mid\ & B(a_E,a_I)=q_L \ \text{ if } a_E<G_E(q_L) \text{ and } G_I(q_H)\leq a_I<G_I(q_L),\\ & B(a_E,a_I)=q_H \ \text{ if } G_E(q_L)\leq a_E<G_E(q_H) \text{ and } a_I<G_I(q_H),\\ & B(a_E,a_I)=q_L \ \text{ if } G_E(q_H)\leq a_E<G_E(q_L) \text{ and } a_I<G_I(q_L) \}.\end{aligned}$$

With consumer beliefs restricted to $\mathcal{B}^1$, a strategy for the entrant which ever requires him to advertise $a_E > \bar{G}_E$, where

$$\bar{G}_E \equiv \min\{G_E(q_L), G_E(q_H)\},$$

is dominated. Similarly dominated is a strategy for the incumbent which ever requires $a_I>G_I(q_H)$. Removing these strategies results in the following strategy sets:

$$\mathcal{S}_E^1 \equiv \{ S_E: Q\rightarrow[0, \bar{G}_E] \mid S_E(q)<G_E(q) \text{ for all } q\in Q \},$$

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[11] Although common knowledge of admissability is consistent with the iterative elimination of dominated strategies in the present game, this is not always true (Samuelson (1989)).

$$\mathcal{S}_I^1 \equiv \{ S_I\colon \Re_+ \times Q \to [0, G_I(q_H)] \mid S_I(\cdot, q_H) < G_I(q_H) \}.$$

This ends the process, as the previous deletion causes no beliefs to become dominated. Denote the game defined by $(\mathcal{S}_E^1, \mathcal{S}_I^1, \mathcal{B}^1)$ as Γ^1. Each equilibrium of Γ^1 is an equilibrium of the original game. Say that an equilibrium outcome of the original game satisfies the *DOM Criterion* iff some of the equilibria that achieve it are also equilibria of Γ^1.

Proposition 5.1: *If* $E_{12}>0$, *then a separating equilibrium outcome (e,c) satisfies the DOM Criterion iff*

(i) $0 < e \leq G_E(q_L)$ *and* $c \leq G_I(q_H)$, *or*

(ii) $e = 0$ *and* $c = G_I(q_H)$;

if $E_{12}<0$, *then* (i) *is replaced by*

(i′) $0 < e < G_E(q_H)$ and $c \leq G_I(q_H)$.

If $E_{12}>0$, *then a pooling equilibrium outcome* (A_E, A_I) *satisfies the DOM Criterion iff*

(iii) $A_E \leq G_E(q_L) - [E(q_H,q_H) - E(\bar{q},q_H)]$, *and*

(iv) $A_I \leq G_I(q_H) - [I(q_L,q_L) - I(\bar{q},q_L)]$;

if $E_{12}<0$, *then* (iii) *is replaced by*

(iii′) $A_E \leq E(\bar{q},q_H) - E(q_L,q_H)$.

This proposition should be compared to that which obtains in the incumbent's absence. Then, a separating equilibrium exists only if the entrant's single-crossing condition holds, in which case the only separating outcome to satisfy the DOM Criterion is the entrant's Riley outcome in which $A_E(q_H)=G_E(q_L)$. When an incumbent is present, the DOM Criterion merely requires that $G_E(q_L)$ be an upper bound to how much the high quality entrant advertises (part (i)).

The presence of an incumbent also affects the pooling equilibria. For example, if $E_{12}>0$ and no incumbent is present, a pooling outcome satisfies the DOM Criterion iff it satisfies (iii). Hence, no pooling outcome satisfies the criterion if the prior mean $\bar{q}$ is too near q_L. When an incumbent is present, a pooling equilibrium outcome is also ruled out if $\bar{q}$ is too near q_H, by (iv). For some profit functions, no pooling outcome satisifes the DOM criterion regardless of the value of $\bar{q}$.

Unsurprisingly, all ε-efficient equilibrium outcomes satisfy the DOM Criterion.

5.2 *Eliminating Non-Best Replies*

We consider now a criterion based on eliminating non-best replies. Such a criterion is viewed by Cho and Kreps (1987) as too strong.[12] In the one-sender signalling games for which they are defined, the Intuitive Criterion of Cho and Kreps and the various versions of the Divinity Criterion of Banks and Sobel (1987) are weaker than a criterion in which general non-best replies are eliminated.[13] Even so, such a criterion is weaker than the stability criterion of Kohlberg and Mertens (1986).

An intuition for the criterion can be put in terms of out-of-equilibrium beliefs. Given an equilibrium that is presumably being played, suppose an out-of-equilibrium message has been sent, and that the receiver believes it to have been sent by a particular sender type. This belief is reasonable if it is somehow "plausible" for the message to have been sent by that sender type. Arguably, that type could plausibly have sent the message only if there is *some* equilibrium giving rise to the same outcome in which that type would suffer no loss from sending the message. It is implausible for that sender type to have sent the message if doing so is a non-best reply to *every* equilibrium which has the same outcome as the putative equilibrium. The conviction in the outcome is maintained even though the equilibrium itself has been disproved; the receiver's secondary hypothesis is that the sender attempted to play some, perhaps different, equilibrium giving rise to the same outcome, but that a "tremble" may have caused him to play a non-equilibrium best response.

So a receiver should never have to believe that a sender played a strategy which is a non-best reply to the set of equilibria giving rise to the putative outcome. We regard this belief restriction as secondary to the restriction requiring a receiver to never have

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[12] There is disagreement: in Appendix B of the working paper version (1987) of van Damme (1989), it is argued that criteria based on eliminating non-best replies can be more intuitive than the Cho and Kreps's Intuitive Criterion.

[13] We could have chosen to apply the formulation of the Intuitive Criterion for general extensive form games developed by Cho (1987). The results and the arguments (applied to the incumbent) would have been similar to what they are for the two-type game Cho and Kreps analyze: Cho's criterion is satisfied by no pooling outcomes, but it is satisfied by all separating outcomes satisfying the DOM Criterion.

to believe a sender played a dominated strategy. Thus, we formulate the criterion by first removing the dominated strategies and beliefs, meaning that we start with the game Γ^1. An outcome will satisfy the criterion if it is an equilibrium outcome of any game obtained by removing from Γ^1 some of the firms' non-best replies to its set of equilibria that achieve the outcome, and then by removing the belief functions that become dominated after this removal.

Let $\mathcal{E}$ be a set of equilibria. A strategy $\hat{S}_E$ for the entrant is a *non-best reply* to $\mathcal{E}$ if $\hat{S}_E$ is not a best reply to any equilibrium in $\mathcal{E}$, i.e., if for each $\langle S_E,S_I,B\rangle \in \mathcal{E}$, there is some $q \in Q$, $\hat{a}_E=\hat{S}_E(q)$, and $a_E=S_E(q)$ such that

$$E[B(\hat{a}_E,S_I(\hat{a}_E,q),q)] - \hat{a}_E \;<\; E[B(a_E,S_I(a_E,q)),q] - a_E.$$

For the incumbent, a strategy $\hat{S}_I$ is a *non-best reply* to $\mathcal{E}$ if for any $\langle S_E,S_I,B\rangle \in \mathcal{E}$, some $q \in Q$ exists such that if $\hat{a}_I=\hat{S}_I(a_E,q)$ and $a_I=S_I(a_E,q)$, then

$$I(B(S_E(q),\hat{a}_I),q)) - \hat{a}_I \;<\; I(B(S_E(q),a_I),q) - A_I(q).$$

We need to extend the notion of a dominated belief. Given sets $\mathcal{S}_E$ and $\mathcal{S}_I$ of strategies for the firms, the set of types that could give rise to a pair (a_E,a_I) is

$$Q(a_E,a_I \mid \mathcal{S}_E,\mathcal{S}_I) \;\equiv\; \{q \in Q \mid (a_E,a_I) = (S_E(q), S_I(a_E,q)) \text{ for some } S_E \in \mathcal{S}_E,\, S_I \in \mathcal{S}_I\}.$$

A belief function B is *dominated relative to* $\mathcal{S}_E$ *and* $\mathcal{S}_I$ if $Q(a_E,a_I \mid \mathcal{S}_E,\mathcal{S}_I)$ is nonempty but $B(a_E,a_I)$ is not contained in its convex hull. Thus, an undominated belief function must assign quality q to an advertising pair which can arise, according to the given strategy sets, precisely when q is the actual quality.

Suppose $\langle A_E,A_I\rangle$ is an equilibrium outcome, and let $\mathcal{E}(A_E,A_I)$ be the set of equilibria of Γ^1 that give rise to it.[14] Then, the outcome $\langle A_E,A_I\rangle$ satisfies the *NBR Criterion* provided that $\mathcal{E}(A_E,A_I)$ contains an equilibrium of any game constructed from Γ^1 by first deleting from $\mathcal{S}_E^1$ and $\mathcal{S}_I^1$ some non-best replies to $\mathcal{E}(A_E,A_I)$ to form strategy

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[14] To make sure the criterion we are defining is implied by stability in generic finite extensive form games, $\mathcal{E}(A_E,A_I)$ should contain all equilibria giving rise to $\langle A_E,A_I\rangle$, including ones which are mixed or non-sequential. But an argument in Cho and Kreps (1987) applies here to allow us to consider only sequential equilibria, and there is no difference in the results below if $\mathcal{E}(A_E,A_I)$ is assumed to contain all mixed strategy sequential equilibria giving rise to $\langle A_E,A_I\rangle$.

sets $\mathcal{S}_E$ and $\mathcal{S}_I$, and then by deleting from $\mathcal{B}^1$ the belief functions that are not dominated relative to $\mathcal{S}_E$ and $\mathcal{S}_I$.

The DOM Criterion is trivially satisfied by outcomes satisfying the NBR Criterion. The following lemmas establish four other necessary conditions. The first one shows that the NBR Criterion rules out pooling equilibria.[15] The argument is similar to that of Cho and Kreps for why the Intuitive Criterion rules out pooling in a one-sender game, applied to the incumbent.

Lemma 5.1: *No pooling equilibrium outcome satisfies the NBR Criterion.*

The next lemma shows that the NBR Criterion puts an upper bound on how much the high quality entrant can advertise. It comes from a restriction that sequentiality puts on out-of-equilibrium beliefs, once the dominated strategies and beliefs are removed. In Γ^1 the incumbent can always persuade consumers that $q=q_L$ by setting $a_E=G_I(q_H)$. Thus, regardless of what the entrant does, the incumbent will never allow consumers to believe $b>\hat{b}$ when in fact $q=q_L$, where $\hat{b}$ is defined by

(5.1) $\quad I(\hat{b},q_L) \equiv I(q_L,q_L) - G_I(q_H)$.[16]

The low quality entrant consequently finds it impossible, rather than just unprofitable, to persuade consumers that his quality is greater than $\hat{b}$. If a_E is so large that $E(\hat{b},q_L) - a_E < E(q_L,q_L)$, then a strategy prescribing a_E for the low quality entrant is a non-best reply to *every* equilibrium of Γ^1. That is, advertising more than

(5.2) $\quad \hat{e} \equiv E(\hat{b},q_L) - E(q_L,q_L)$

is a non-best reply for the low quality entrant. After such strategies have been removed, a belief function is dominated unless it specifies $b=q_H$ when $a_E>\hat{e}$. After these dominated beliefs have been removed, the high quality entrant need advertise no more than $\hat{e}$ to reveal his type.

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15 The logic of Lemma 4.1 also rules out partially pooling outcomes. (We have neglected these outcomes by ignoring mixed strategy equilibria.)

16 Notice that $q_L<\hat{b}<q_H$, since $I_{12}>0$.

Lemma 5.2: *A separating outcome* (e,c) *satisfies the NBR Criterion only if* $e \le \hat{e}$.

The upper bound established in Lemma 5.2 binds if the entrant's single-crossing condition holds. This is because $E_{12}>0$ implies that $\hat{e}<G_E(q_L)<G_E(q_H)$. Thus, if $E_{12}>0$, the entrant's Riley outcome fails the NBR Criterion. (And if $E_{12}<0$, the entrant's Riley outcome is not an equilibrium outcome.)

The next lemma establishes another upper bound on e. It is not particularly interesting because it binds only if E_{12} changes signs. Define $\beta(\cdot)$ to be the function whose graph is the isoprofit curve of the low quality entrant containing $(a_E,b)=(0,q_L)$: $\beta(a_E)=q_H$ if $a_E \ge G_E(q_L)$, and otherwise

$$E(\beta(a_E),q_L) - a_E \equiv E(q_L,q_L). \tag{5.3}$$

Lemma 5.3: *A separating outcome* (e,c) *satisfies the NBR Criterion only if* $e \le \bar{e}$, *where*

$$\bar{e} \equiv \max \{ e \ge 0 \mid E(q_H,q_H) - e \ge E(\beta(a_E),q_H) - a_E \text{ for all } a_E \ge 0 \}.[17] \tag{5.4}$$

The next lemma shows that if a separating outcome (e,c) satisfies the NBR Criterion and c is positive, then c is high if e is low. This has an intuitive explanation. In an equilibrium with $c>0$, if the entrant does not advertise, the incumbent must advertise in order to persuade consumers that $q=q_L$. The consumers must therefore think it plausible that the entrant's quality is high even when he does not advertise. By the NBR formulation of "plausibility," this requires that in some equilibrium giving rise to (e,c), the high quality entrant is indifferent between $a_E=e$, his equilibrium advertising level, and $a_E=0$. For the high quality entrant to be indifferent in this way, his setting $a_E=0$ must cause consumers to believe that his quality is sufficiently high; the lower is e, which is the cost to the high quality entrant of convincing consumers that $q=q_H$, the higher must be the believed quality b that he would induce by choosing $a_E=0$. If c is low, it costs the incumbent little to convince consumers that $q=q_L$ when $a_E=0$; hence, in order for him to allow consumers to hold the belief b when $a_E=0$ and $q=q_H$, b must be low if c is low. So c must be high if e is low.

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[17] If $E_{12}<0$ everywhere, then $\bar{e}=G_E(q_H)=\bar{G}_E$. If $E_{12}>0$ everywhere, then $\bar{e}=G_E(q_L)=\bar{G}_E$.

Definitions are required. Define ψ_E: $[0, G_E(q_H)] \rightarrow [q_L, q_H]$ by

$$E(\psi_E(e), q_H) = E(q_H, q_H) - e. \tag{5.5}$$

Then $b=\psi_E(e)$ is the believed quality necessary when $a_E=0$ for the high quality entrant to be indifferent between $a_E=0$ and $a_E=e$. Observe that ψ_E is decreasing, $\psi_E(0)=q_H$, and $\psi_E(G_E(q_H)) = q_L$. Similarly, define ψ_I: $[0, G_I(q_H)] \rightarrow [q_L, q_H]$ by

$$I(\psi_I(c), q_H) = I(q_L, q_H) - c. \tag{5.6}$$

The incumbent, when the entrant's quality is q_H, is indifferent between advertising $a_I=c$ and $a_I=0$ if the former convinces consumers that $q=q_L$ and the latter convinces them that $q=\psi_I(c)$. Observe that ψ_I is increasing, $\psi_I(0)=q_L$, and $\psi_I(G_I(q_H))=q_H$.

Lemma 5.4: *A separating equilibrium outcome* (e,c) *satisfies the NBR Criterion only if* $c=0$ *or* $\psi_I(c) \geq \psi_E(e)$.

The necessary conditions established by Lemmas 5.1-5.4 are also sufficient.

Proposition 5.2: *The equilibrium outcomes satisfying the NBR Criterion are the separating outcomes* (e,c) *which satisfy the DOM Criterion,*

(i) $e \leq \text{Min}(\hat{e}, \bar{e})$, *and*

(ii) $c=0$ *or* $\psi_I(c) \geq \psi_E(e)$.

5.3 *The Unprejudiced Criteron*

Bagwell and Ramey (1989) propose a refinement criterion which applies to the present game. Its motivation is a notion of "reasonable beliefs"off the equilibrium path, which we shall take as an informal definition (the formal definition is in terms of "unprejudiced trembles"). The intuition is that out-of-equilibrium beliefs should be rationalized, if possible, by the secondary hypothesis that only a minimal number of players made the "mistake" of defecting from the putative equilibrium.

Thus, say that an equilibrium $\langle S_E, S_I, B\rangle$ of the present game is *unprejudiced* if whenever (a_E, a_I) cannot be generated by the strategy pair $\langle S_E, S_I\rangle$, but could be generated by a strategy pair of the form $\langle \hat{S}_E, S_I\rangle$ or $\langle S_E, \hat{S}_I\rangle$ iff the true quality is q, then $B(a_E, a_I)=q$.

An outcome satisfies the *UP Criterion* if it is the outcome of an unprejudiced equilibrium.

This criterion eliminates separating equilibria in which both firms advertise. The argument is simple. Let $\langle S_E, S_I, B\rangle$ be a separating equilibrium achieving an outcome (e,c) such that e>0 and c>0. Suppose that when the entrant's quality is low, the incumbent deviates by advertising an amount $\hat{c}<c$ instead of c, where $\hat{c}\neq S_I(0,q_H)$. Then consumers see $(a_E,a_I)=(0,\hat{c})$. If consumers were to infer that the entrant's quality is high, they would be presuming that both firms had deviated, since e>0. If they were to infer that the entrant's quality is low, they would be presuming that only the incumbent had deviated. Thus, for the equilibrium to be unprejudiced, consumers must infer from $(0,\hat{c})$ that the entrant's quality is low. But then the incumbent should not advertise $c>\hat{c}$ when $a_E=0$, and the equilibrium is not unprejudiced.

To borrow a concept from Bagwell and Ramey (1989), the unprejudiced criterion requires the incumbent to "informationally free-ride" on the entrant. Given that a high quality entrant is expected to advertise, consumers should infer that a non-advertising entrant is low quality. The incumbent's advertising is not needed to confirm this inference when consumers rationalizie deviations by the "minimality rule," since then they believe the quality of a non-advertising entrant is low regardless of what the incumbent does.

We state, without further proof,

Proposition 5.3: *An equilibrium outcome satisfies the UP Criterion if and only if it is a pooling outcome or a separating outcome in which* e=0 *or* c=0.

6. Refinement Postscript

Consider the incumbent's Riley outcome, $(e,c) = (0, G_I(q_H))$. The high quality entrant does not advertise, but lets consumers infer that his quality is high from the incumbent's lack of advertising. Such passivity by a high quality entrant strikes us as unintuitive; we would not predict that this outcome would be played.

Yet, the incumbent's Riley outcome satisfies all the refinement criteria posed in the previous section. Furthermore, it seems to be uniquely selected by some combination of forward and backward induction criteria.[18]

To see this, let us present an informal "hypothetical speech argument." Suppose an equilibrium in which the high quality entrant advertises is supposed to be played. Consider what might happen if the high quality entrant, instead of advertising, surprises the other players by making the following speech:

> "Although I am not advertising, do not infer that my quality is low. Instead, wait to see what the incumbent does. If my quality is low, he will advertise a lot. In fact, if you apply the arguments of Cho and Kreps (1987) to the subform that starts with the incumbent's move, you should conclude that he will not advertise if my quality is high, and that he will advertise $a_I=G_I(q_H)$ if my quality is low."

If the speech is believed, the entrant will have successfully switched the equilibrium from one in which he advertises to one in which he does not. Both types of entrant would be better off (the high type strictly), so that the speech is arguably credible. It need not be spoken if it is sufficiently compelling; in this case we should not see an equilibrium in which the entrant advertises. The only separating equilibrium in which the entrant does not advertise is the incumbent's Riley outcome.

Still, on intuitive grounds we find the incumbent's Riley outcome "odd;" we are likely to question a criterion that selects it.

It is beyond our present scope to pursue this line of inquiry. Formal criteria that combine forward and backwards induction are delicate to formulate, and they may or may not be implied by stability (Kohlberg and Mertens, 1986; Cho, 1989). The refinement criteria we have applied have selected the incumbent's Riley outcome and the ε-efficient outcomes in which the entrant but not the incumbent advertises. We leave for future research the task of determining criteria which select among these outcomes.

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[18] Cho (1989) studies a refinement criterion called STABAC which combines foreward and backward induction. This criterion is defined for two-sender one-receiver games similar to ours, except that the sender's types are independent rather than perfectly correlated, and the action sets of all three players are finite.

Appendix A: Proofs for Section 3

If (e,c) is the outcome of a separating equilibrium $\langle S_E, S_I, B\rangle$, then

(A1) $S_E(q_L) = 0$ and $S_E(q_H) = e$,

(A2) $S_I(0,q_L) = c$ and $S_I(e,q_H) = 0$.

On the equilibrium path beliefs are correct:

(A3) $B(0,c) = q_L$ and $B(e,0) = q_H$.

The equilibrium is counteractive according to (3.1) iff

(A4) $B(e, S_E(e,q_L)) = q_L$.

If $\langle S_E, S_I, B\rangle$ is a counteractive equilibrium, a_E=e, and $q=q_H$, then the incumbent could persuade consumers that $q=q_L$ by advertising $a_I=S_I(e,q_L)$. As he prefers $a_I=0$, which does not mislead consumers, $I(q_H,q_H) \geq I(q_L,q_H)-S_I(e,q_L)$. Rearranging this yields the first of the following two inequalities that necessarily hold for a counteractive equilibrium (the second follows from sequential rationality):

(A5) $G_I(q_H) \leq S_I(e,q_L) \leq G_I(q_L)$.

Proof of Proposition 3.1:

(i) If $G_E(q_H) \geq G_E(q_L)$, the following is a separating equilibrium: $S_E(q_L)=0$ and $S_E(q_H)=G_E(q_L)$, $S_I \equiv 0$, and $B(a_E,a_I)=q_L$ if $a_E<G_E(q_L)$ and $B(a_E,a_I)=q_H$ if $a_E \geq G_E(q_L)$. A similar construction reversing the roles of the entrant and the incumbent, or (3.2)-(3.4), sufficies if $G_I(q_L) \geq G_I(q_H)$.

Now assume $G_E(q_H)<G_E(q_L)$ and $G_I(q_L)<G_I(q_H)$. Suppose $\langle S_E, S_I, B\rangle$ is a separating equilibrium giving rise to outcome (e,c).

Assume e=0. From (A3), $B(e,c)=q_L$. So when the $q=q_H$ and a_E=e, the incumbent could convince consumers that $q=q_L$ by setting a_I=c. As he instead chooses a_I=0 in these circumstances,

$$I(q_H,q_H) \geq I(q_L,q_H) - c.$$

When $q=q_L$ the incumbent prefers to inform consumers of this by setting $a_I=c$, rather than to let them believe $q=q_H$ by setting $a_I=0$:

$$I(q_L,q_L) - c \geq I(q_H,q_L).$$

These inequalities imply that $G_I(q_H) \leq c \leq G_I(q_L)$, contrary to $G_I(q_H) > G_I(q_L)$.

Assume now $e>0$. If $q=q_H$, the entrant prefers $a_E=e$ to $a_E=0$:

(A6) $$E(q_H,q_H) - e \geq E[B(0,S_I(0,q_H)), q_H].$$

As $e>0$, this implies $B(0,S_I(0,q_H)) = q_L$. So from (A6) and $G_E(q_H) < G_E(q_L)$,

(A7) $$e \leq G_E(q_H) < G_E(q_L).$$

When $q=q_L$, the entrant prefers $a_E=0$ to $a_E=e$:

$$E(q_L,q_L) \geq E[B(e,S_I(e,q_L)), q_L] - e.$$

In view of (A7), this implies $B(e,S_I(e,q_L)) = q_L$. So the equilibrium is counteractive and (A5) holds, contrary to $G_I(q_H) > G_I(q_L)$.

(ii) The sufficiency of $G_I(q_L) \geq G_I(q_H)$ for a counteractive equilibrium was established by (3.2)–(3.4). Necessity follows from (A5). ///

Proof of Proposition 3.2:

(i) Suppose $\langle S_E,S_I,B\rangle$ is an ε-efficient equilibrium for some $\varepsilon<G_E(q_L)$. Then $e<G_E(q_L)$. Because a low quality entrant prefers $a_E=0$ to $a_E=e$,

$$E(q_L,q_L) \geq E[B(e,S_I(e,q_L)), q_L] - e.$$

This inequality would contradict $e \geq G_E(q_L)$ if $B(e,S_I(e,q_L)) = q_H$. Therefore $B(e,S_I(e,q_L)) = q_L$, and the equilibrium is counteractive.

(ii) Assume $\langle S_E,S_I,B\rangle$ is ε-efficient for $\varepsilon < \min\{G_E(q_L), G_I(q_H)\}$. Then $c < G_I(q_H)$ and, because the equilibrium is counteractive, (A5) holds. If $e=0$, then $S_I(e,q_L) = S_I(0,q_L) = c$, and (A5) implies $c \geq G_I(q_H)$. Thus, $e > 0$. ///

Appendix B: Proofs for Section 4

Proof of Proposition 4.1:

First, note that the outcome of $\langle S_E, S_I, B\rangle$ is indeed $\langle A_E, A_I\rangle$. By (4.5) if q is the true quality, the entrant sets $a_E = A_E(q)$. Because A_E is invertable, $a_E = A_E(q)$ implies $\beta(a_E) = q$. Therefore, by (4.6), the incumbent chooses $a_I = A_I(q)$.

The requirement (E4) that beliefs be correct on the equilibrium path is also satisfied, because we have just shown that the middle line of (4.4) holds on the equilibrium path, with $\beta(a_E) = q$. The requirement (E3) that the believed quality be contained in $Q = [q_L, q_H]$ follows from (4.3) and (4.5).

We now show that (4.6) gives the incumbent's best response to $\langle S_E, B\rangle$, i.e, that (E1) holds. Suppose q is the true quality, and the entrant has chosen a_E. To simplify notation, let $\alpha = \alpha(a_E)$ and $\beta = \beta(a_E)$. Because of (4.7), the incumbent's best response is $a_I = 0$, $a_I = A_I(\beta)$, or $a_I = \alpha$. If he chooses $a_I = 0$, consumers will believe $b = q_H$. If he chooses $a_I = A_I(\beta)$, consumers will believe $b = \beta$. The incumbent therefore prefers $a_I = A_I(\beta)$ to $a_I = 0$, since (4.2b') implies

$$I(\beta,q) - A_I(\beta) \geq I(q_H,q).$$

Now, from (4.4),

$$\begin{aligned}[I(\beta,q) - A_I(\beta)] - [I(q_L,q) - \alpha] &= I(\beta,q) - I(\beta,\beta) + I(q_L,\beta) - I(q_L,q) \\ &= \int_{\beta}^{q} \int_{q_L}^{\beta} I_{12}(x,y)\,dx\,dy.\end{aligned}$$

As $I_{12} > 0$, the last expression is nonegative if $q \geq \beta$ and nonpositive if $q < \beta$. The incumbent therefore prefers $a_I = A_I(\beta)$ if $q \geq \beta$, but $a_I = \alpha$ if $q < \beta$. So (4.6) gives his best response to $\langle S_E, B\rangle$.

Lastly, we show that $S_E = A_E$ is the entrant's best response to $\langle S_I, B\rangle$, establishing (E2). Let q be the true quality. If the entrant chooses $a_E = A_E(q)$, the believed quality will be $b = \beta(a_E) = q$, as was shown above. And his profit will be $E(q,q) - A_E(q)$.

Suppose the entrant chooses an a_E satisfying $\beta(a_E)\leq q$. Then the incumbent chooses $a_I=A_I(\beta(a_E))$, and consumers believe $b=\beta(a_E)$. Now, by (4.3) and the left-continuity of A_E, $A_E(\beta(a_E))\leq a_E$. Therefore, his profit from choosing this a_E is

$$\begin{aligned} E(b,q) - a_E &= E(\beta(a_E),q) - a_E \\ &\leq E(\beta(a_E),q) - A_E(\beta(a_E)) \\ &\leq E(q,q) - A_E(q), \end{aligned}$$

where the last inequality follows from (4.1b) and $\beta(a_E)\leq q$. So the entrant cannot not gain by choosing such an a_E instead of $A_E(q)$.

Finally, suppose the entrant chooses an a_E satisfying $\beta(a_E)>q$. Then the incumbent chooses $a_I=\alpha(a_E)$, and consumers believe $b=q_L$. The entrant's profit is

$$\begin{aligned} E(b,q) - a_E &= E(q_L,q) - a_E \\ &\leq E(q_L,q) \\ &\leq E(q,q) - A_E(q), \end{aligned}$$

where the last inequality follows from (4.1a) and (4.1b). So the entrant cannot gain by choosing such an a_E instead of $A_E(q)$: $S_E=A_E$ is his best response to $\langle S_I,B\rangle$. ///

Proposition B: *Suppose* $I_{12}>0$, *and let* A_I *satisfy* (4.2). *Suppose* A_E *is strictly increasing and satisfies* (4.1a) *and the following strengthening of* (4.1b):

(4.1b*) *for some differentiable* $\tau: \Re^2\rightarrow\Re$ *satisfying* $\tau(q,b)=q$ *if* $q\leq b$, $b<\tau(q,b)<q$ *and* $\tau_1(q,b)>0$ *if* $q>b$, *and* $\tau_1(q,b)\rightarrow 1$ *as* $q\rightarrow b+$, $A_E(q) - A_E(b) \leq E(q,q) - E(\tau(q,b),q)$ *for all* $q_L \leq b < q \leq q_H$.[19]

Let $\sigma: \Re_+\times Q\rightarrow\Re$ *be defined by*

$$\text{(B1)} \qquad \sigma(a_E,q) = A_E(\beta(a_E)) + \int_\beta^q \tau_1(x,\beta(a_E))I_1(\tau(x,\beta(a_E)),x)dx.$$

Then $\langle A_E,A_I\rangle$ *is the outcome of an equilibrium* $\langle S_E,S_I,B\rangle$ *defined by*

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[19] For example, let $\tau(q,b) = b + (q-b)\exp[\min\{0,(b-q)/(q_H-q_L)\}]$.

(B2) $\quad S_E = A_E$;

(B3) $\quad S_I(a_E,q) = \max\{0,\sigma(a_E,q)\}$; *and*

(B4)
$$B(a_E,a_I) = \begin{cases} q_H & \text{if} \quad q_H < \tau(\sigma^{-1}(a_E,a_I),\beta(a_E)) \\ \tau(\sigma^{-1}(a_E,a_I),\beta(a_E)) & \text{if} \quad q_L \le \tau(\sigma^{-1}(a_E,a_I),\beta(a_E)) \le q_H \\ q_L & \text{if} \quad \tau(\sigma^{-1}(a_E,a_I),\beta(a_E)) < q_L . \end{cases}$$

where for each a_E, $\sigma^{-1}(a_E,\cdot)$ *is the inverse function of* $\sigma(a_E,\cdot)$.

Proof of Proposition B: The proof that $\langle A_E,A_I\rangle$ is the outcome of $\langle S_E,S_I,B\rangle$ is straightforward. So is the proof that $\langle S_E,S_I,B\rangle$ satisfies (E3) and (E4), the equilibrium belief conditions.

We now show (E1), i.e., we show that (B3) gives the incumbent's best response to $\langle S_E,B\rangle$. Suppose q is the true quality, and the entrant has chosen a_E. Let $\beta=\beta(a_E)$. We first show that the incumbent's best response is an advertising level a_I satisfying $q_L \le \tau(\sigma^{-1}(a_E,a_I),\beta) \le q_H$.

If $\tau(\sigma^{-1}(a_E,a_I),\beta) < q_L$, then $\tau(\sigma^{-1}(a_E,\hat{a}_I),\beta) < q_L$ for some $\hat{a}_I<a_I$, since $\tau_1>0$ and $\partial\sigma^{-1}(a_E,a_I)/\partial a_I < 0$. Both advertising levels therefore induce the same believed quality, $B(a_E,a_I)=q_L$. This shows that a_I cannot be the incumbent's best response, since the lower advertising level must be more profitable.

If $q_H < \tau(\sigma^{-1}(a_E,a_I),\beta)$, the believed quality is $B(a_E,a_I)=q_H$ and the incumbent's profit is $I(q_H,q)-a_I$. Now,

$$\begin{aligned} I(q_H,q) - a_I &\le I(q_H,q) \\ &\le I(\beta,q) - A_I(\beta) \\ &= I(B(a_E,A_I(\beta)),q) - A_I(\beta), \end{aligned}$$

where the first inequality follows from $a_I\ge 0$, the second follows from (4.2b'), and the equality follows from (B1), (B3), (B4), and $t(\beta,\beta)=\beta$. Thus, the incumbent receives lower profit by choosing a_I than by choosing $A_I(\beta)$. Because $\tau(\sigma^{-1}(a_E,A_I(\beta)),\beta) = \beta \le q_H$, this shows that the incumbent's optimal a_I satisfies $\tau(\sigma^{-1}(a_E,a_I),\beta) \le q_H$.

We have shown that the incumbent's problem can be reduced to

(B5) $$\underset{a_I \geq 0}{\text{Maximize}}\ I(B(a_E,a_I),q) - a_I \quad \text{such that} \quad q_L \leq \tau(\sigma^{-1}(a_E,a_I),\beta) \leq q_H.$$

For each a_I in the constraint set, $B(a_E,a_I) = \tau(\sigma^{-1}(a_E,a_I),\beta)$. As this mapping is continuous and invertable, with $a_I = \sigma(a_E,B(a_E,a_I))$, we can change the choice variable from a_I to $b=B(a_E,a_I)$:

(B6) $$\underset{b}{\text{Maximize}}\ I(b,q) - \sigma(a_E,b) \quad \text{such that} \quad q_L \leq b \leq \tau(\sigma^{-1}(a_E,0),\beta).$$

From (B1),

$$\begin{aligned}\frac{\partial[I(b,q) - \sigma(a_E,b)]}{\partial b} &= \tau_1(b,\beta)I_1(\tau(b,\beta),q) - \tau_1(b,\beta)I_1(\tau(b,\beta),b) \\ &= \tau_1(b,\beta)\int_b^q I_{12}(\tau(b,\beta),x)dx.\end{aligned}$$

This expression, since $I_{12}>0$, is positive if $b<q$ and negative if $b>q$. The solution of (B6) is therefore $b=q$ if $q \leq \tau(\sigma^{-1}(a_E,0),\beta)$, and $b = \tau(\sigma^{-1}(a_E,0),\beta)$ otherwise. Inverting, this means the solution of (A5) is $a_I = \sigma(a_E,q)$ if $\sigma(a_E,q)\geq 0$, and $a_I=0$ otherwise. Thus (E1).

It remains to show (E2), i.e. that $S_E=A_E$ is the entrant's best response to $\langle S_I,B\rangle$. Again, let q be the true quality. If the entrant chooses $a_E=A_E(q)$, then $\beta(a_E)=q$ and the incumbent will choose $a_I=\sigma(a_E,q)=A_I(q)$. The believed quality will be $B(a_E,a_I)=\tau(\sigma^{-1}(a_E,a_I),q)=\tau(q,q)=q$. Consequently, the entrant's profit from $a_E=A_E(q)$ is $E(q,q) - A_E(q)$.

Suppose the entrant chooses a_E such that $\beta\leq q$, where $\beta=\beta(a_E)$. Then the incumbent chooses $a_I=S_I(a_E,q)\geq\sigma(a_E,q)$. Hence $\sigma^{-1}(a_E,a_I) \leq q$, so that $B(a_E,a_I)\leq\tau(q,\beta)$. The entrant's profit is thus

$$E(B(a_E,a_I),q) - a_E \leq E(\tau(q,\beta),q) - a_E.$$

Now, the left-continuity of A_E implies $A_E(\beta(a_E))\leq a_E$. Therefore,

$$\begin{aligned}E(\tau(q,\beta),q) - a_E &\leq E(\tau(q,\beta),q) - A_E(\beta) \\ &\leq E(q,q) - A_E(q),\end{aligned}$$

where the last inequality follows from (4.1b*) and $\beta\leq q$. Combining the inequalities yields

$$E(B(a_E,a_I),q) - a_E \leq E(q,q) - A_E(q),$$

showing that the entrant cannot not gain by choosing a_E instead of $A_E(q)$.

Finally, suppose the entrant chooses an a_E satisfying $\beta(a_E)>q$. Then the incumbent chooses $a_I=\sigma(a_E,q)$. Hence $\sigma^{-1}(a_E,a_I) = q$, so that $B(a_E,a_I)=\tau(q,\beta(a_E))=q$. The entrant's profit is therefore

$$\begin{aligned} E(B(a_E,a_I),q) - a_E &= E(q,q) - a_E \\ &\leq E(q,q) - A_E(\beta) \\ &\leq E(q,q) - A_E(q), \end{aligned}$$

where the first inequality follows from $A_E(\beta(a_E))\leq a_E$, and the second inequality holds because A_E is increasing and $\beta>q$. So the entrant cannot gain by choosing this a_E instead of $A_E(q)$. Thus, $S_E=A_E$ is his best response to $\langle S_I,B\rangle$. ///

Appendix C: Proofs for Section 5

Proof of Proposition 5.1:

We give the proof for the case $E_{12}>0$. The reverse case is similar.

Assume (e,c) is a separating equilibrium outcome of Γ^1. The definitions of $\mathcal{S}_E^1$ and $\mathcal{S}_I^1$ imply $0 \le e \le G_E(q_L)$ and $0 \le c \le G_I(q_H)$. Thus, to show that (i) or (ii) holds, we must show that e=0 implies $c=G_I(q_H)$. If e=0, then $B(0,c)=q_L$ and $B(0,0)=q_H$. Therefore, since the incumbent prefers $a_I=0$ over $a_I=c$ when $q=q_H$, $I(q_H,q_H) \ge I(q_L,q_H)-c$, i.e., $c \ge G_I(q_H)$. The definition of $\mathcal{S}_I^1$ implies the opposite, so $c=G_I(q_H)$.

Now suppose (e,c) is a separating equilibrium outcome satisfying (i) or (ii). If (ii), the following is easily verified to be an equilibrium of Γ^1 having outcome (e,c): $S_E\equiv 0$, $S_I(\cdot,q_L)=G_I(q_H)$ and $S_I(\cdot,q_H)=0$, and $B(a_E,a_I)=q_L$ if $a_I \ge G_I(q_H)$ and $B(a_E,a_I)=q_H$ otherwise.

If (i) is satisfied with $c<G_I(q_H)$, then (3.2)–(3.4) define an equilibrium of the original game and of Γ^1. The firms' strategies are in $\mathcal{S}_E^1$ and $\mathcal{S}_I^1$ because $0<e<G_E(q_H)$ and $0 \le c<G_I(q_H)$. The belief function is in $\mathcal{B}^1$ because $e \le G_E(q_L) < G_E(q_H)$ (see Figure 3.1).

Suppose (i) is satisfied with $c=G_I(q_H)$. Let $\psi_E(e)$ be defined as in (5.4): $E(\psi_E(e), q_H) \equiv E(q_H,q_H)-e$. Because $0<e<G_E(q_H)$ (from (i) and $E_{12}>0$), $q_L<\psi_E(e)<q_H$. Let $\hat{c} = \min\{a_I \ge 0 \mid I(q_L,q_L)-G_I(q_H) \ge I(\psi_E(e),q_L)-a_I\}$. Note that $\hat{c}<G_I(q_H)$. Define $\langle S_E,S_I,B\rangle$ by $S_E=A_E$, $S_I(\cdot,q_L)=G_I(q_H)$, $S_I(a_E,q_H)=\hat{c}$ if $a_E<e$, $S_I(a_E,q_H)=0$ if $a_E \ge e$, and

$$B(a_E,a_I) = \begin{cases} q_H & \text{if} \quad a_E < e \quad \text{and} \quad a_I < \hat{c} \\ \psi_E(e) & \text{if} \quad a_E < e \quad \text{and} \quad \hat{c} \le a_I < G_I(q_H) \\ q_H & \text{if} \quad a_E \ge e \quad \text{and} \quad a_I < G(q_H) \\ q_L & \text{if} \quad a_I \ge G(q_H)\,. \end{cases}$$

From (i) and the single-crossing conditions, $\langle S_E,S_I,B\rangle$ is a profile in Γ^1. Its outcome is $\langle A_E,A_I\rangle$. The beliefs specified by B at this outcome are correct. Given $\langle S_I,B\rangle$, the high quality entrant is indifferent between $a_E=0$ and $a_E=e$, and the low quality entrant prefers $a_E=0$; hence, S_E is a best reply to $\langle S_I,B\rangle$. Given B and $a_E \ge e$, $I_{12}>0$ implies that $a_I=0$ ($a_I=G_I(q_H)$) is optimal for the incumbent when $q=q_H$ ($q=q_L$). Because $I_{12}>0$, $I(q_L,q_L) - G_I(q_H) > I(q_H,q_L)$. Also, $I(q_L,q_L)\ G_I(q_H) \ge I(\psi_E(e),q_L) - \hat{c}$, from the definition of $\hat{c}$.

Hence, in response to B and $a_E<e$, the incumbent prefers $a_I=G_I(q_H)$ to both $a_I=0$ and $a_I=\hat{c}$ when $q=q_L$. When $q=q_H$, both $a_I=0$ and $a_I=G_I(q_H)$ give the incumbent $I(q_H,q_H)$, and he prefers $a_I=\hat{c}$: if $\hat{c}=0$ then $\psi_E(e)<q_H$ implies that $I(\psi_E(e),q_H) - \hat{c} > I(q_H,q_H)$, and if $\hat{c}>0$ then its definition and $I_{12}>0$ imply

$$
\begin{aligned}
I(\psi_E(e),q_H) - \hat{c} &= I(\psi_E(e),q_H) - [\, I(\psi_E(e),q_L) - I(q_L,q_L) + G_I(q_H)\,] \\
&= I(q_H,q_H) + [I(q_L,q_L) - I(\psi_E(e),q_L)] - [I(q_L,q_H) - I(\psi_E(e),q_H)] \\
&> I(q_H,q_H).
\end{aligned}
$$

Thus, S_I is a best reply to B. Hence, $\langle S_E,S_I,B\rangle$ is an equilibrium of Γ^1 achieving (e,c).

Now suppose (A_E,A_I) is a pooling equilibrium outcome. In Γ^1 the entrant (incumbent) can make consumers believe $b=q_H$ ($b=q_L$) by setting $a_E=G_E(q_H)$ ($a_I=G_I(q_L)$). Hence, if (A_E,A_I) is an equilibrium outcome of Γ^1, (iii) and (iv) hold because the high type entrant and the low type incumbent, respectively, prefer their equilibrium advertising levels to so convincing consumers. Conversely, if (A_E,A_I) satisfies (iii) and (iv), the following is an equilibrium of Γ^1 having outcome (A_E,A_I): $S_E=A_E$, $S_I(a_E,\cdot)=A_I$ if $A_E \le a_E<G_E(q_L)$ and $S_I(a_E,\cdot)=0$ otherwise, and

$$
B(a_E,a_I) = \begin{cases} q_L & \text{if} \quad a_E < A_E \text{ or } a_I \ge G_I(q_H) \\ \bar{q} & \text{if} \quad A_E \le a_E < G_E(q_L) \text{ and } A_I \le a_I < G_I(q_H) \\ q_H & \text{otherwise.} \end{cases}
$$

///

Proof of Lemma 5.1:

Let (A_E,A_I) be a pooling outcome of Γ^1. Since $I_{12}>0$, some a_I exists such that

(C1) $\qquad A_I + I(q_L,q_H) - I(\bar{q},q_H) < a_I < A_I + I(q_L,q_L) - I(\bar{q},q_L).$

From the first inequality in (C1), for any $\langle S_E,S_I,B\rangle \in \mathcal{E}(A_E,A_I)$,

$$
\begin{aligned}
I(B(A_E,a_I),q_H) - a_I &\le I(q_L,q_H) - a_I \\
&< I(\bar{q},q_H) - A_I.
\end{aligned}
$$

But $I(\bar{q},q_H)-A_I$ is the high type incumbent's payoff in any equilibrium in $\mathcal{E}(A_E,A_I)$. Thus, any strategy $\hat{S}_I$ such that $\hat{S}_I(A_E,q_H)=a_I$ is a non-best reply to $\mathcal{E}(A_E,A_I)$. Remove them from $\mathcal{S}_I^1$ to define a strategy set $\mathcal{S}_I$. The belief functions that are dominated relative to $\mathcal{S}_E^1$ and $\mathcal{S}_I$ are those which do not assign q_L to (A_E,a_I). Remove them from $\mathcal{B}^1$ to obtain a set $\mathcal{B}$. Denote as Γ the game defined by $\mathcal{S}_E^1$, $\mathcal{S}_I$, and $\mathcal{B}$. From (C1),

$$\begin{aligned} I(B(A_E,a_I),q_L) - a_I &\le I(\bar{q},q_L) - A_I \\ &< I(q_L,q_L) - a_I. \end{aligned}$$

This shows that for every $\langle S_E,S_I,B\rangle \in \mathcal{E}(A_E,A_I)$, $B(A_E,a_I) \neq q_L$ and, hence, $B \notin \mathcal{B}$. Therefore, no equilibrium of Γ is in $\mathcal{E}(A_E,A_I)$. ///

Proof of Lemma 5.2:

Let $\langle S_E,S_I,B\rangle$ be an equilibrium of Γ^1, $a_E < \bar{G}_E$, and $b = B(a_E, S_I(a_E,q_L))$. Recall that $a_I = G_I(q_H)$ persuades consumers to believe $q = q_L$. As $a_I = S_I(a_E,q_L)$ is the incumbent's equilibrium advertising level after the entrant has chosen a_E when $q = q_L$,

$$\begin{aligned} I(b,q_L) - S_I(a_E,q_L) &\ge I(q_L,q_L) - G_I(q_H) \\ &= I(\hat{b},q_L), \end{aligned}$$

where the equality comes from (5.1). So $I(b,q_L) \ge I(\hat{b},q_L)$, which implies $b \le \hat{b}$.

Consequently, from (5.2), every $\hat{S}_E$ such that $\hat{S}_E(q_L) > \hat{e}$ is a non-best reply to any equilibrium of Γ^1. Remove these strategies from Γ^1. The belief functions in $\mathcal{B}^1$ which this removal causes to become dominated are those that do not assign q_H to every $(a_E,a_I) \in (\hat{e},\bar{G}_E) \times [0,G_I(q_H))$. Remove them as well, to obtain a game Γ. If $a_E > \hat{e}$, then $B(a_E, S_I(a_E,q_H)) = q_H$ for any $\langle S_I,B\rangle$ in Γ. Thus, in no equilibrium of Γ will the high quality entrant advertise more than $\hat{e}$. The same is therefore true of any separating outcome satisfying the NBR Criterion. ///

Proof of Lemma 5.3:

Let (e,c) be a separating equilibrium outcome of Γ^1 such that $e > \bar{e}$. Then $\bar{a}_E \ge 0$ exists such that $E(q_H,q_H) - e < E(\beta(\bar{a}_E),q_H) - \bar{a}_E$. Notice that $\bar{a}_E < e$. Let $\mathcal{E}$ be the set of equilibria of Γ^1 achieving (e,c). Let $\langle S_E,S_I,B\rangle \in \mathcal{E}$, $b_L = B(\bar{a}_E, S_I(\bar{a}_E,q_L))$, and $b_H = B(\bar{a}_E, S_I(\bar{a}_E,q_H))$. The usual revealed preference argument shows that $b_L \le b_H$, as $I_{12} > 0$. Since

$$\begin{aligned} E(\beta(\bar{a}_E),q_H) - \bar{a}_E &> E(q_H,q_H) - e \\ &\ge E(b_H,q_H) - \bar{a}_E\,, \end{aligned}$$

we know that $\beta(\bar{a}_E)>b_H$. Hence $\beta(\bar{a}_E)>b_L$, and (5.3) gives us

$$\begin{aligned} E(q_L,q_L) &= E(\beta(\bar{a}_E),q_L) - \bar{a}_E \\ &> E(b_L,q_L) - \bar{a}_E . \end{aligned}$$

Thus, since $E(q_L,q_L)$ is the low quality entrant's payoff in every equilibrium in $\mathcal{E}$, any $\hat{S}_E$ such that $\hat{S}_E(q_L)=\bar{a}_E$ is a non-best reply to $\mathcal{E}$. Remove these strategies from Γ^1. The belief functions in $\mathcal{B}^1$ which this removal causes to become dominated are those that do not assign q_H to every $(\bar{a}_E,a_I)$ with $a_I<G_I(q_H)$. Remove them to obtain a game Γ. For any $\langle S_I,B\rangle$ feasible in Γ, $B(\bar{a}_E, S_I(\bar{a}_E,q_H))=q_H$. Thus, the high quality entrant does not advertise more than $\bar{a}_E$ in any equilibrium of Γ. Since $\bar{a}_E<e$, this shows that $\mathcal{E}$ contains no equilibrium of Γ. ///

Proof of Lemma 5.4:

Let (e,c) be a separating equilibrium outcome of Γ^1 for which $\psi_I(c)<\psi_E(e)$ and $c>0$. Let $\mathcal{E}$ be the set of equilibria of Γ^1 achieving (e,c). For some $\langle S_E,S_I,B\rangle\in \mathcal{E}$, let $b=B(0, S_I(0,q_H))$. In this equilibrium, if $a_E=0$ and $q=q_H$, the incumbent prefers $a_I=S_I(0,q_H)$ to $a_I=c$, even though the latter would persuade consumers that $q=q_L$. Hence, using (5.6),

$$\begin{aligned} I(b,q_H) &\geq I(b,q_H) - S_I(0,q_H) \\ &\geq I(q_L,q_H) - c \\ &= I(\psi_I(c), q_H). \end{aligned}$$

So $b\leq\psi_I(c)$, which implies $b<\psi_E(e)$. From (5.5),

$$\begin{aligned} E(b,q_H) &< E(\psi_E(e), q_H) \\ &= E(q_H,q_H) - e. \end{aligned}$$

Thus, since $E(q_H,q_H)-e$ is the high quality entrant's payoff in every equilibrium in $\mathcal{E}$, that any strategy $\hat{S}_E$ for which $\hat{S}_E(q_H)=0$ is a non-best reply to $\mathcal{E}$. The belief functions in $\mathcal{B}^1$ which this removal causes to become dominated are those that do not assign q_L to every $(0,a_I)$. Remove them to obtain a game Γ. In no equilibrium of this game does the incumbent advertise when $a_E=0$. This shows, since $c>0$, that $\mathcal{E}$ contains no equilibrium of Γ. ///

Proof of Proposition 5.2:

Because of Lemmas 5.1-4, we need only show that the listed conditions are sufficient. Let (e,c) be a separating equilibrium outcome satisfying the DOM Criterion, (i), and (ii). Let $\mathcal{E}$ be the corresponding equilibrium set of Γ^1. Suppose $\mathcal{S}_E$ and $\mathcal{S}_I$ are strategy sets obtained by deleting from $\mathcal{S}_E^1$ and $\mathcal{S}_I^1$ some non-best replies to $\mathcal{E}$. Let $\mathcal{B}$ consist of the belief functions in $\mathcal{B}^1$ that are not dominated relative to $\mathcal{S}_E$ and $\mathcal{S}_I$. Let Γ be the game defined by $\mathcal{S}_E$, $\mathcal{S}_I$, and $\mathcal{B}$. We show that $\mathcal{E}$ contains an equilibrium of Γ.

Note that $c \leq G_I(q_H)$ and $e < \bar{G}_E$: the DOM Criterion implies $c \leq G_I(q_H)$ and $e < G_E(q_H)$, and $e \leq \hat{e} < G_E(q_L)$ implies $e < G_E(q_L)$.

Let $Q(a_E,a_I) \equiv Q(a_E,a_I \mid \mathcal{S}_E, \mathcal{S}_I)$, which is equal to either $\varnothing$, $\{q_L\}$, $\{q_H\}$, or $\{q_L,q_H\}$. Recall that the set of advertising pairs that can arise in Γ^1 is $[0,\bar{G}_E] \times [0,G_I(q_H)]$. Note that if $a_E < \bar{G}_E$, then $Q(G_I(q_H),a_E) = \{q_L\}$.

Lemma C1:

(i) *If* $a_E \notin \{0,e\}$, $a_I < G_I(q_H)$, *and* $q \in Q$, *then* $q \in Q(a_E,a_I)$ *iff* $S_E \in \mathcal{S}_E$ *exists such that* $S_E(q) = a_E$.

(ii) *If* $e>0$ *and* $a_I < G_I(q_H)$, *then* $q_H \in Q(0,a_I)$ *iff* $S_E \in \mathcal{S}_E$ *exists such that* $S_E(q_H) = 0$.

Proof: This follows from the observation that changing a best reply S_I at an off-the-equilibrium-path point (a_E,q) will not convert it to a non-best reply. ///

Define a mapping τ of belief functions into belief functions by

$$\text{(C2)} \qquad \tau(B)(a_E,a_I) = \begin{cases} q_L & \text{if} \quad Q(a_E,a_I) = \{q_L\} \\ q_H & \text{if} \quad Q(a_E,a_I) = \{q_H\} \\ B(a_E,a_I) & \text{otherwise.} \end{cases}$$

Then $\tau(B) \in \mathcal{B}$ for all $B \in \mathcal{B}^1$. Also, if $B \in \mathcal{B}^1$, then $\tau(B)(a_E,a_I) = B(a_E,a_I)$ if $a_E = \bar{G}_E$ or $a_I = G_I(q_H)$.

Henceforth, S_E denotes the strategy defined by $S_E(q_L)=0$ and $S_E(q_H)=e$. The proof breaks into cases: (a) $c=0$; (b) $\psi_E(e) \leq \psi_I(c)$ and $e>0$; and (c) $\psi_E(e) \leq \psi_I(c)$ and $e=0$.

Case (a): $c=0$.

Thus, $e>0$. Recall that the graph of $\beta(a_E)$ is the isoprofit curve of the low quality entrant through $(a_E,b)=(0,q_L)$. Define γ: $[0,G_I(q_H)]\rightarrow[q_L,q_H]$ to be the function whose graph is the isoprofit curve of the high type incumbent through $(a_I,b)=(0,q_H)$:

(C3) $\qquad I(\gamma(a_I), q_H) - a_I \equiv I(q_H,q_H)$.

Observe that γ is decreasing, $\gamma(0)=q_H$, and $\gamma(G_I(q_H))=q_L$.

Let $\mathcal{B}^a$ be the set of belief functions in $\mathcal{B}^1$ that satisfy the following properties for all $a_E \in [0, \bar{G}_E)$ and a_I, $\hat{a}_I \in [0, G_I(q_H))$:

(p1) $B(0,0) = q_L$,

(p2) $B(0,a_I) \in \{q_L,q_H\}$,

(p3) $B(a_E,\hat{a}_I) = B(a_E,a_I) \in \{\beta(a_E), q_L\}$ if $0<a_E<e$,

(p4) $B(e,a_I) \in \{\gamma(a_I), q_H\}$,

(p5) $B(a_E,\hat{a}_I) = B(a_E,a_I) \in \{q_L,q_H\}$ if $a_E>e$.

Lemma C2: *If* $B\in \mathcal{B}^a$, *then* $\langle S_E,S_I,B\rangle\in \mathcal{E}$, *where* S_E *is defined above and*

$$\text{(C4)} \qquad S_I(a_E,q) = \begin{cases} G_I(q_H) & \text{if} \quad a_E = e \quad \text{and} \quad q = q_L \\ G_I(q_H) & \text{if} \quad a_E > e,\ q = q_L,\ B(a_E,a_I) = q_H \ \text{for}\ a_I < G_I(q_H), \\ & \qquad \text{and}\ B(a_E,G_I(q_H)) = q_L \\ 0 & \text{otherwise.} \end{cases}$$

Proof: Though straightforward, two details should be noted. First, $a_I=0$ is a best reply to B when $0<a_E<e$ because $e\leq\hat{e}$; the proof of this for $q=q_L$ uses $B(a_E,0)\leq\beta(a_E)\leq\beta(\hat{e})=\hat{b}$, and the proof for $q=q_H$ then follows from the incumbent's single-crossing property. Second, $a_E=e$ is a better reply for the entrant than any $0<a_E<e$ if $q=q_H$ because $e\leq\bar{e}$; the proof of this uses $\gamma(0)=q_H$ and $E(q_H,q_H)-\bar{e} \geq E(\beta(a_E),q_H)-a_E$ for $a_E<e$. ///

Let B^a be a belief function in $\mathcal{B}^a$ such that $B^a(a_E,a_I)=\beta(a_E)$ in (p3) and $B^a(e,a_I)=\gamma(a_I)$ in (p4). Let S_I^a be defined from B^a by (C4), and let $\pi^a=\langle S_E,S_I^a,B^a\rangle$. Then $\pi^a\in \mathcal{E}$.

Let $B=\tau(B^a)$. We finish the proof of this case by showing that $B\in \mathcal{B}^a$. For then, from Lemma C2, $\mathcal{E}$ contains an equilibrium of the form $\langle S_E,S_I,B\rangle$, where S_I is given by

(C4). Therefore S_E and S_I are best replies to $\mathcal{E}$, which implies $S_E \in \mathcal{S}_E$ and $S_I \in \mathcal{S}_I$. This implies, since $B=\tau(B^a) \in \mathcal{B}$, that $\langle S_E, S_I, B\rangle$ is an equilibrium of Γ^2.

We show that $B \in \mathcal{B}^a$ by showing that B satisfies (p1)–(p5) for $a_E < \bar{G}_E$ and $a_I, \hat{a}_I < G_I(q_H)$.

(p1) $q_L \in Q(0,0)$ because $(0, 0)=(S_E(q_L), S_I^a(S_E(q_L),q_L))$ and both $S_E \in \mathcal{S}_E$ and $S_I^a \in \mathcal{S}_I$. Hence, because $B^a(a_E,a_I)=q_L$, (C2) implies $B(a_E,a_I)=q_L$.

(p2) Since $B^a(0,a_I) \in \{q_L,q_H\}$, the same must be true of $B(0,a_I)$, by (C2).

(p3) Assume $0<a_E<e$. Lemma C1(i) implies $Q(a_E,\hat{a}_I)=Q(a_E,a_I)$. Since also $B^a(a_E,\hat{a}_I)=B^a(a_E,a_I)$, (C1) implies $B(a_E,\hat{a}_I)=B(a_E,a_I)$. Now, if the low type entrant chooses a_E in the equilibrium π^a, consumers will believe $b=\beta(a_E)$ and his payoff will be $E(\beta(a_E),q_L)-a_E = E(q_L,q_L)$, by (5.3). As this is his equilibrium payoff, the $\hat{S}_E$ defined by $\hat{S}_E(q_L)=a_E$ and $\hat{S}_E(q_H)=e$ is an undominated best reply to $\hat{\pi}$. Thus, $\hat{S}_E \in \mathcal{S}_E$ and, from Lemma C1(i), $q_L \in Q(a_E,a_I)$. So from (C2), either $B(a_E,a_I)=q_L$ or $B(a_E,a_I)=B^a(a_E,a_I)=\beta(a_E)$.

(p4) We know $S_E \in \mathcal{S}_E$. In the equilibrium π^a, if $q=q_H$ and the entrant has chosen $a_E=e$, the choice $a_I<G_I(q_H)$ gives the incumbent his equilibrium payoff, since (C3) implies that $I(B^a(e,a_I),q_H)-a_I = I(\gamma(a_I),q_H)-a_I = I(q_H,q_H)$. So if we let S_I be the same as S_I^a except that $S_I(e,q_H)=a_I$, $S_I \in \mathcal{S}_I$. This shows that $q_H \in Q(e,a_I)$. Thus, from (C2), either $C(e,a_I)=q_H$ or $C(e,a_I)=C^a(e,a_I)=\gamma(a_I)$.

(p5) Assume $a_E>e$. Then Lemma C1(i) implies $Q(a_E,\hat{a}_I)=Q(a_E,a_I)$, and because $B^a(a_E,\hat{a}_I)=B^a(a_E,a_I)$, (C2) implies $B(a_E,\hat{a}_I)=B(a_E,a_I)$. Since $B^a(a_E,a_I) \in \{q_L,q_H\}$, (C2) also implies $B(a_E,a_I) \in \{q_L,q_H\}$.

Case (b): $\psi_E(e) \le \psi_I(c)$ and $e>0$.

Let α: $[0,c] \to [q_L,q_H]$ be the function whose graph is the isoprofit curve of the high type incumbent through $(a_I,b)=(c,q_L)$:

(C5) $$I(\alpha(a_I), q_H) - a_I \equiv I(q_L,q_H) - c.$$

Observe that α is decreasing, $\alpha(0)=\psi_I(c)$, and $\alpha(c)=q_L$.

Define $\bar{a}_I$ by $\alpha(\bar{a}_I)=\psi_E(e)$. Then $0 \le \bar{a}_I \le c$, because $q_L \le \psi_E(e) \le \psi_I(c)$. Actually, $\bar{a}_I<c$: if $\bar{a}_I=c$, then $\psi_E(e)=\alpha^a(c)=q_L$, and (5.5) would imply the contradiction $e=G_E(q_H)$.

Let $\mathcal{B}^b$ be the set of belief functions in $\mathcal{B}^1$ which, for $a_E \in [0, \bar{G}_E)$ and $a_I, \hat{a}_I \in [0, G_I(q_H))$, satisfy the properties (p3)–(p5) and

(p6) $B(0,a_I) \in \{\alpha(a_I), q_H\}$.

Lemma C3: *If* $B \in \mathcal{B}^b$, *then* $\langle S_E, S_I, B\rangle \in \mathcal{E}$, *where*

$$\text{(C6)} \qquad S_I(a_E,q) = \begin{cases} c & \text{if} \quad a_E = 0 \\ G_I(q_H) & \text{if} \quad a_E = e \ \text{ and } \ q = q_L \\ G_I(q_H) & \text{if} \quad a_E > e,\ q = q_L,\ B(a_E,a_I) = q_H \ \text{ for } \ a_I < G_I(q_H), \\ & \qquad \text{and } \ B(a_E,G_I(q_H)) = q_L \\ 0 & \text{otherwise.} \end{cases}$$

Proof: The details noted in the proof of Lemma **C2** also apply here. Also, note that if $a_E=0$, the incumbent may be indifferent between $a_I=c$ and some $a_I<c$ if $q=q_H$, but he strictly prefers $a_I=c$ if $q=q_L$ (by single-crossing). ///

Let B^b be a belief function in $\mathcal{B}$ such that $B^b(a_E,a_I)=\beta(a_E)$ in (p3), $B^b(e,a_I)=\gamma(a_I)$ in (p4), and $B^b(0,a_I)=\alpha(a_I)$ in (p6). Let S_I^b be the belief function defined as in (C6), with B^b replacing B on the right side, except that for $a_E=0$ it is defined by,

$$\text{(C7)} \qquad S_I^b(0,q) = \begin{cases} \bar{a}_I & \text{if} \quad q = q_H \\ c & \text{if} \quad q = q_L. \end{cases}$$

Then S_I^b is a best reply to B^b, since the incumbent is indifferent among all choices $a_I \in [0,c]$ when the belief function is B^b, $a_E=0$, and $q=q_H$. Hence, $\pi^b \equiv \langle S_E, S_I^b, B^b\rangle \in \mathcal{E}$.

Now let $B=\tau(B^b)$. As in case (a), the proof is finished by showing that $B \in \mathcal{B}^b$. The proofs that B satisfies (p3)–(p5) are the same as in case (a). To show (p6), let $a_I<G_I(q_H)$. In the equilibrium π^b, if the high type entrant chooses $a_E=0$ he still gets his equilibrium payoff, since $E(B^b(0,S_I^b(0,q_H),q_H) = E(\alpha(\bar{a}_I),q_H) = E(\psi_E(e),q_H) = E(q_H,q_H)-e$. This shows, using Lemma C1(ii), that $q_H \in Q(0,a_I)$. Hence, from (C2), $B(0,a_I)=q_H$ or $B(0,a_I)=B^b(0,a_I)=\alpha(a_I)$.

Case (c): $\psi_E(e) \le \psi_I(c)$ and e=0.

In this case, from Proposition 5.1, e=0 and $c=G_I(q_H)$. Let $\delta(a_I)$ be the function whose graph is the isoprofit curve of the high type incumbent through $(a_I,b)=(0,q_H)$. Hence, $\delta(0)=q_H$, $\delta(G_I(q_H))=q_L$, and δ is decreasing. Let B^c be a belief function in $\mathcal{B}^1$ satisfying, for $a_E \in [0, \bar{G}_E)$ and $a_I, \hat{a}_I \in [0, G_I(q_H))$,

$$\text{(C8)} \qquad B^c(a_E,a_I) = \begin{cases} \delta(a_I) & \text{if} \quad a_E = 0 \\ q_L & \text{otherwise.} \end{cases}$$

For each (a_E,q), let $S_I^c(a_E,q)$ by the q-type incumbent's best advertising response to $B^c(a_E,\cdot)$ in which $S_I^c(0,q_H)=0$. (Note that $S_I^c(0,q_L)=G_I(q_H)$, and $S_I^c(a_E,q_L)=S_I^c(a_E,q_H)=0$ for $0<a_E<\bar{G}_E$.) Then $\pi^c \equiv \langle S_E,S_I^c,B^c \rangle \in \mathcal{E}$. In this equilibrium, when the entrant has chosen $a_E=0$, the high type incumbent is indifferent among all $a_I \in [0,G_I(q_H)]$. This shows that $q_H \in Q(0,a_I)$ and, hence,

$$\text{(C9)} \qquad \tau(B^c)(0,a_I) \in \{\delta(a_I),q_H\} \text{ for all } a_I \le G_I(q_H).^{20}$$

In every equilibrium in $\mathcal{E}$, the high type entrant gets his highest possible payoff without advertising; hence, any $a_E>0$ is a non-best reply by him to $\mathcal{E}$. Therefore, $q_H \notin Q(a_E,a_I)$ for $0<a_E<\bar{G}_E$. This and (C2) imply that

$$\text{(C10)} \qquad \tau(B^c)(a_E,a_I)=B^c(a_E,a_I) \text{ for } a_E>0.$$

From (C9) and (C10), $\langle S_E,S_I^c,\tau(B^c)\rangle \in \mathcal{E}$. Since $\tau(B^c) \in \mathcal{B}$, this shows that $S_E \in \mathcal{S}_E$ and $S_I^c \in \mathcal{S}_I$. Consequently, $\langle S_E,S_I^c,\tau(B^c)\rangle$ is an equilibrium of Γ^2. ///

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[20] Actually, $\tau(B^c)(0,a_I)=q_H$ for $a_I<G_I(q_H)$.

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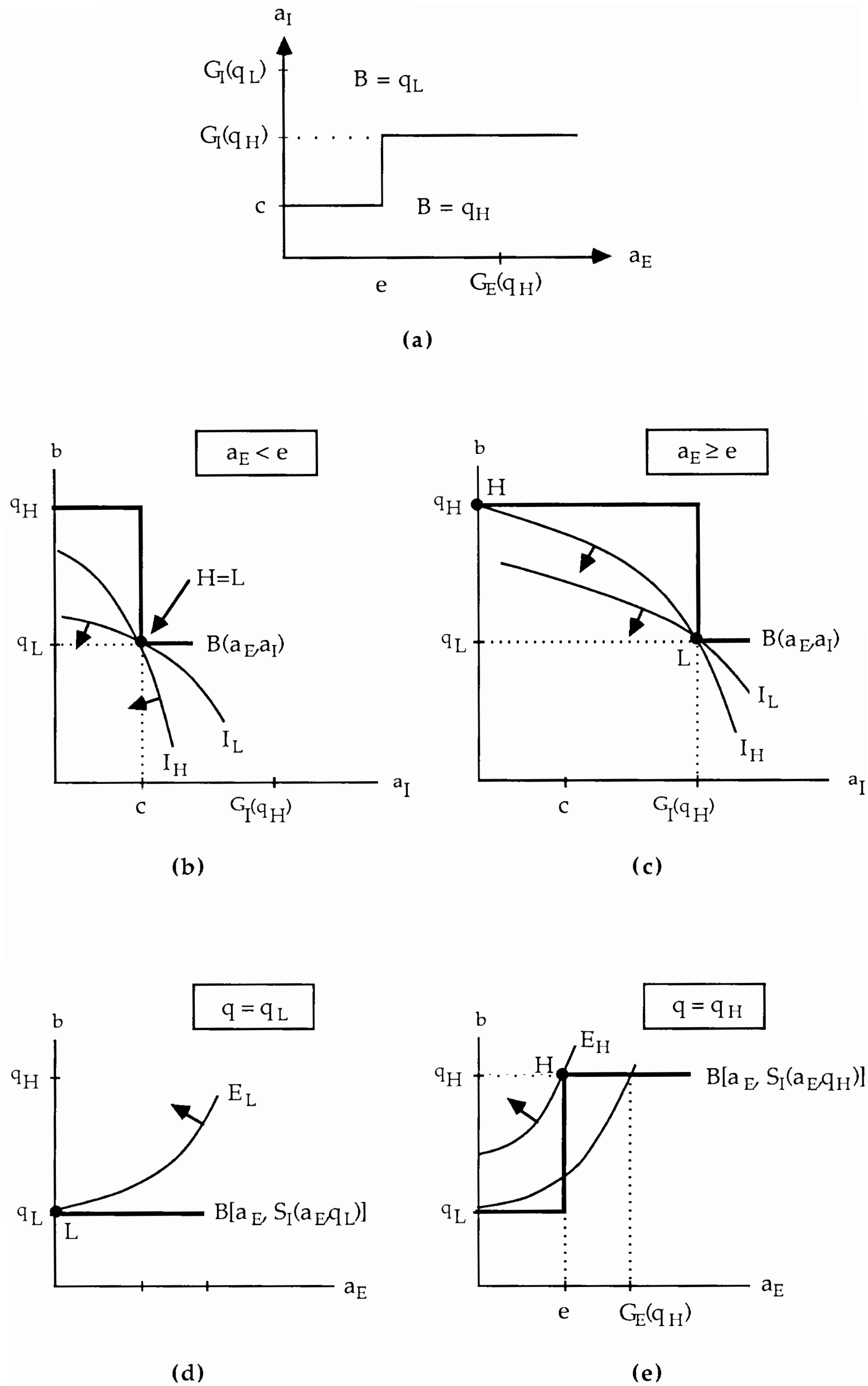

a_I
G_I(q_L)
B = q_L
G_I(q_H)
c
B = q_H
a_E
e
G_E(q_H)
(a)
b
a_E < e
q_H
H=L
q_L
B(a_E,a_I)
I_L
I_H
a_I
c
G_I(q_H)
(b)
b
a_E ≥ e
H
q_H
q_L
B(a_E,a_I)
L
I_L
I_H
a_I
c
G_I(q_H)
(c)
b
q = q_L
q_H
E_L
q_L
B[a_E, S_I(a_E,q_L)]
L
a_E
(d)
b
q = q_H
E_H
q_H
H
B[a_E, S_I(a_E,q_H)]
q_L
a_E
e
G_E(q_H)
(e)

Figure 3.1

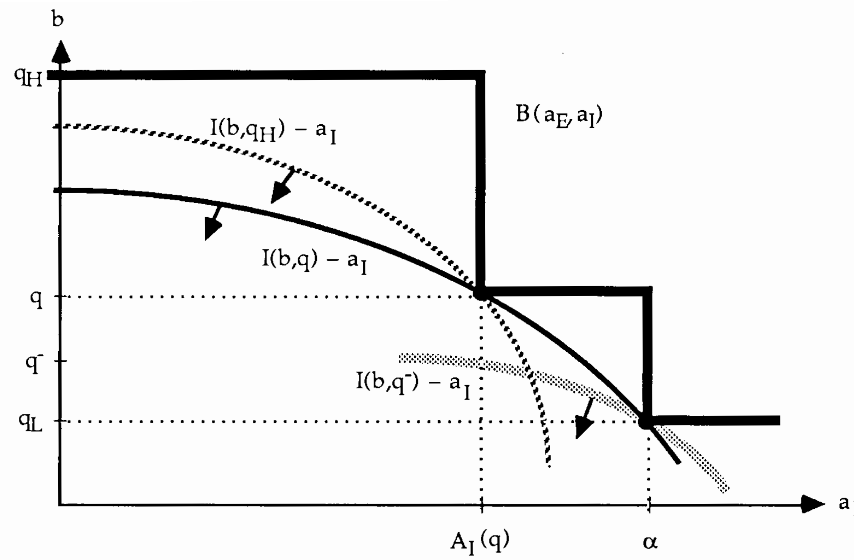

Figure 4.1

The incumbent's problem if the entrant chooses $a_E=A_E(q)$.

In the notation of Proposition 4.1, $q=\beta(a_E)$ and $\alpha=\alpha(a_E)$.

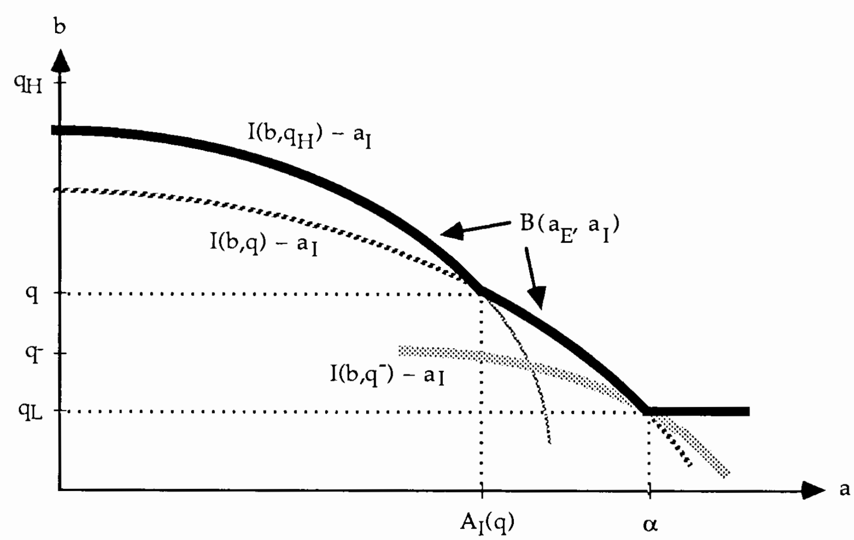

Figure 4.2

The incumbent's problem with a belief function continuous in a_I.

The entrant has chosen $a_E=A_E(q)$, and $\alpha=\alpha(a_E)$ is as defined in (4.6).

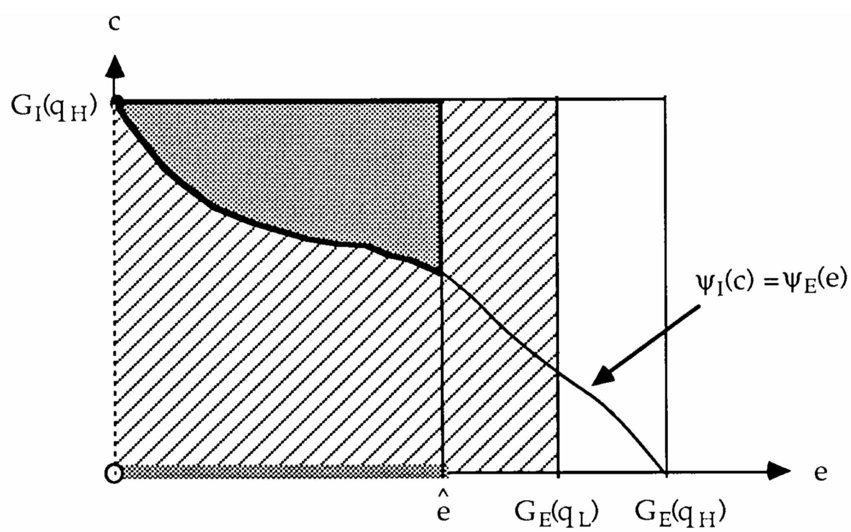

Figure 5.1

Separating outcomes (e,c) satisfying refinement criteria when $E_{12}>0$.

Outcomes in the lined region satisfy the DOM Criterion.

Outcomes in the shaded regions satisfy the NBR Criterion.

Outcomes on the e-axis and the point $(0, G_I(q_H))$ satisfy the UP Criterion.